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Master's Thesis

Evaluation of Frame Synchronization Algorithms for Wireless Communications

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Ich versichere hiermit wahrheitsgemäß, die Arbeit bis auf die dem Aufgabensteller bereits bekannte Hilfe selbständig angefertigt, alle benutzten Hilfsmittel vollständig und genau angegeben und alles kenntlich gemacht zu haben, was aus Arbeiten anderer unverändert oder mit Abänderung entnommen wurde.

München, 15.03.2013

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Ort, Datum (Juan Tárrega Alonso)

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Abstract

In this document, we study the synchronization between the sent frame and the received frame in a communications system for a good recovery of the data and its probability of success. It is a important due to the channel introduces impairments, and the signal is different when arrives to the finish.
Studying Massey algorithm for real sequences, and improving it for complex sequences used in Long Term Evolution, we see success of a correct detection or not.
The results show for big $SNRs$ a great detection, with a very high percentage of success, but adding the different impairments produced during the transmission in the channel, the errors grow up, so in a perfect situation, the algorithm works for LTE.

1 Introduction

One problem in a communications system is to know where the data starts, we know when the data is sent, but never when it arrives, for that reason, we need something to know where the data starts and we can recover. That is why we need a synchronization word at the beginning of the data, and this synchronization word has to have specific properties that we can use to determine where exactly is in all the symbols we receive.

Now we know we need a synchronization word, but not the length of it and if it depends on total length frame or the signal power. These are aspects we will show in this study, so using a sight window of a length equal to the total frame length, previously known, and considering the synchronization word is the same in each frame, we can move this window, symbol per symbol, and determine, with a rate error, the probability of success in the correct position.

But in real communications systems, we have different impairments that hinder the detection, as noise, and others that we will study in this document, and affirm, if we correct these impairments, how we can detect the correct start position of the frame.

So we will study a typical communications system scheme composed by a transmitter, a channel and a receiver, where we will generate our own data word, completly random and binary and then apply the Massey Algorithm, that James Massey designed in 1972, for real sequences. Then, we improve it for complex sequences, like Zadoff-Chu sequences used in Long Term Evolution (LTE) in the present day, and study how the error rate changes depending which impairment we use and adding to create the more closely situation of a real communications system.

The Massey algorithm uses the property of cross correlation, so these synchronization words should have a good autocorrelation properties to detect with the maximum easily.

After studying the Massey algorithm as he did in his study in 1972, we will modify for complex sequences and study them only with the time delay and noise, introduced by the channel, as unique impairments. Later we will add the phase impairment due to the reflections and refractions of the signal in the channel, and then, we will finish our study with the Doppler effect suffered in the communications channel, developing the algorithm using the property of the coherence length.

In each study we will show the results in plots and their comments about them, provided

by referring to the signal to noise ratio and different frame lengths and lengths of synchronization word, because is what we can play with to obtain a success ratio as better as possible, the rest of the parameters are totally random.
At the end, we will show the conclusion and possible utility in a future research and application.

2 System model

We consider a point-to-point communication system with a BPSK Modulation to evaluate the performance of frame synchronization algorithms. The system consists of 3 Parts that we describe below:

2.1 Transmitter

We consider the transmitter which aim is to generate the frame, composed by the synchronization word and the data word.
We assume that a transmitted frame consists of a synchronization word of length L, $\mathbf{s} = [s_0, s_1, ..., s_{L-1}]$ and a data word of length N-L, $\mathbf{d} = [d_L, d_{L+1}, ..., d_{N-1}]$, such that, the whole frame is N symbols long.
After generating the sync-word and data word, we concatenate both in one simply vector and send into the channel.
This vector will be

$$\mathbf{sd} = [\mathbf{sd}] = [s_0, s_1, ..., s_{L-1}, d_L, d_{L+1}, ..., d_{N-1}] \tag{2.1}$$

2.2 Channel

As in many communications systems, the channel introduces Additive White Gaussian Noise (AWGN) which is a real noise with variance $N_0/2$ and zero mean of length N.

$$\mathbf{n} = [n_0, n_1, ..., n_{N-1}] \tag{2.2}$$

In the channel, we introduce the different types of impairments, like phase shift and frequency shift, and the time delay, depending the focus point we study, but in all the cases we study the time delay.
We will define a vector $\mathbf{r}$ of length N as the output of the channel

$$\mathbf{r} = \sqrt{E}T^m(\mathbf{sd}) + \mathbf{n} \tag{2.3}$$

where $T^m()$ operator is a cyclic shift by m symbols to simulate the time delay.

2.3 Receiver

In the receiver, we only focus on the frame synchronization and try to find where the sequences start. We will measure the probability of errors in estimation of the frame start due to the noise and different delays. We do not focus on the demodulation, rather, the first step to start the data recovery.
To obtain the estimated m, start index of our frame, we use the *Maximum Likehood Method* [BPV08]. We consider $\boldsymbol{\rho}$ as the actual value assumed by the random vector $\mathbf{r}$, which length is also N, $\boldsymbol{\rho} = [\rho_0, \rho_1, ..., \rho_{N-1}]$, and μ as the estimation of m, $0 \leq \mu < N$ which maximize $S_1 = Pr[m = \mu|\mathbf{r} = \boldsymbol{\rho}]$ and mixed with the Bayes' Theorem for events and Random Variables, so, the estimation problem should be like $\max_m S_1$ such that

$$S_1 = \frac{p_r(\boldsymbol{\rho}|m = \mu)Pr[m = \mu]}{p_r(\rho)} \tag{2.4}$$

The $Pr[m = \mu] = \frac{1}{N}$ for all μ, and is a constant, thus the maximum problem can be formulated as $\max_m S_2$

$$S_2 = p_r(\boldsymbol{\rho}|m = \mu) \tag{2.5}$$

Now we assume δ as the possible values of random vector data, $\mathbf{d}$, so, $\boldsymbol{\delta} = [\delta_L, \delta_{L+1}, ..., \delta_{N-1}]$, so S_2 is the same as

$$S_2 = \sum_{\forall\delta} p_r(\boldsymbol{\rho}|\mathbf{d} = \boldsymbol{\delta}, m = \mu)Pr(\mathbf{d} = \boldsymbol{\delta}) \tag{2.6}$$

Since $Pr(\mathbf{d} = \boldsymbol{\delta}) = \frac{1}{2^{N-L}}$, and again, it is constant, the problem can be reformulated as $\max_m S_3$ with S_3 equal:

$$S_3 = \sum_{\forall\delta} p_r(\boldsymbol{\rho}|\mathbf{d} = \boldsymbol{\delta}, m = \mu) \tag{2.7}$$

which upon making use of (2.3) becomes

$$S_3 = \sum_{\forall\delta} p_n(\boldsymbol{\rho} - \sqrt{E}T^{\mu}(\mathbf{s}\boldsymbol{\delta})) \tag{2.8}$$

In the next chapters, depending the focus of the study, we will see how to obtain the estimate of m according to the received vector $\mathbf{r}$.

3 Synchronization Algorithms for Real Sequences

In this chapter we will focus how detect the correct slot position of the received frame for real sequences.

3.1 Massey Algorithm

In this section, we briefly show the Massey algorithm [Mas72].
As we saw in formula (2.8),

$$S_3 = \sum_{\forall\delta} p_n(\rho - \sqrt{E}T^{\mu}(s\delta)) \tag{3.1}$$

and the Gaussian assumption on noise, we consider

$$p_n(\rho - \sqrt{E}T^{\mu}(s\delta)) = (2\pi)^{-N/2} \sum_{\forall\delta} ([\prod_{i=0}^{L-1} e^{-(\rho_{i+\mu}-\sqrt{E}s_i)^2/N_0}] \prod_{i=L}^{N-1} e^{-(\rho_{i+\mu}-\sqrt{E}\delta_i)^2/N_0}) \tag{3.2}$$

Now, focusing only in the two products, we define $A = (\rho_{i+\mu} - \sqrt{E}s_i)^2$ and $B = (\rho_{i+\mu} - \sqrt{E}\delta_i)^2$, and develop both terms, considering $\delta_i = -1$ or $\delta_i = 1$ equiprobable:

$$A = \rho_{i+\mu}^2 + Es_i^2 - 2E\rho_{i+\mu}s_i \tag{3.3}$$

considering just the term depending in μ due to $\rho_{i+\mu}^2$ and Es_i^2 are constants, we can ignore them and redefine A as $A_2 = -2\sqrt{E}\rho_{i+\mu}s_i/N_0$. Same process for term B and renaming, we define $B_2 = -2\sqrt{E}\rho_{i+\mu}\delta_i/N_0$. This is equivalent to maximize $\max_m S_4$, where S_4 is

$$S_4 = \sum_{\forall\delta} ([\prod_{i=0}^{L-1} e^{-A_2}] \prod_{i=L}^{N-1} e^{-B_2}) \tag{3.4}$$

$$S_4 = \sum_{\forall\delta} ([\prod_{i=0}^{L-1} e^{2\sqrt{E}\rho_{i+\mu}s_i/N_0}] \prod_{i=L}^{N-1} e^{2\sqrt{E}\rho_{i+\mu}\delta_i/N_0}) \tag{3.5}$$

Carrying out the term of the summation in all delta and remembering, δ just take +1 or -1, and defining the second product as B_3, it will be

$$B_3 = \sum_{\forall\delta} (\prod_{i=L}^{N-1} e^{2\sqrt{E}\rho_{i+\mu}\delta_i/N_0}) = \prod_{i=L}^{N-1} e^{2\sqrt{E}\rho_{i+\mu}/N_0} + e^{-2\sqrt{E}\rho_{i+\mu}/N_0} \tag{3.6}$$

and the property of cosinus hyperbolicus $\frac{1}{2}(e^a + e^{-a}) = cosh(a)$

$$B_3 = \prod_{i=L}^{N-1} 2cosh(2\sqrt{E}\rho_{i+\mu}/N_0) \tag{3.7}$$

so finally, S_4 is

$$S_4 = [\prod_{i=0}^{L-1} e^{2\sqrt{E}\rho_{i+\mu}s_i/N_0}] \cdot B_3 \tag{3.8}$$

$$S_4 = [\prod_{i=0}^{L-1} e^{2\sqrt{E}\rho_{i+\mu}s_i/N_0}] \prod_{i=L}^{N-1} 2cosh(2\sqrt{E}\rho_{i+\mu}/N_0) \tag{3.9}$$

Applying logarithms and disregarding the constants, is equal to maximize the expression $\max_m S_5$

$$S_5 = \sum_{i=0}^{L-1} 2\sqrt{E}\rho_{i+\mu}s_i/N_0 + \sum_{i=L}^{N-1} ln(cosh(2\sqrt{E}\rho_{i+\mu}/N_0)) \tag{3.10}$$

If we define $\sum_{i=0}^{N-1} ln(cosh(2\sqrt{E}\rho_{i+\mu}/N_0))$ as the summation over all components of ρ, independent of μ, we can subtract to S_5 without affecting the maximization, so, is equal to maximize $\max_m S_6$

$$S_6 = \sum_{i=0}^{L-1} 2\sqrt{E}\rho_{i+\mu}s_i/N_0 - \sum_{i=0}^{L-1} ln(cosh(2\sqrt{E}\rho_{i+\mu}/N_0)) \tag{3.11}$$

And finally, disregarding the constants, we obtain the Optimum Rule for locating the Synchronization Word as Massey defined

$$S = \sum_{i=0}^{L-1} \rho_{i+\mu} s_i - \sum_{i=0}^{L-1} f(\rho_{i+\mu}) \tag{3.12}$$

where

$$f(x) = ln(cosh(2\sqrt{E}x/N_0)) \tag{3.13}$$

The first summation in (3.12) is the correlation between the synchronization word we generate in the transmitter, and L bits from the received frame, and the second summation is a kind of energy correction to account the random data surrounding the sync-word.

Moreover, we can make approximations for higher and lower SNRs. Considering *Low* SNR when the $SNR < 0dB$ and *High* SNR for $SNR \geq 15dB$ [Mas72].

For *High* SNR, the argument in the *cosh* is greater than 1, so, we can approximate the $cosh(x)$ as $\frac{1}{2}e^{|x|}$, and we can obtain

$$f(x) = ln(cosh(2\sqrt{E}x/N_0)) = ln(\frac{1}{2}e^{\left|2\sqrt{E}x/N_0\right|}) = ln(\frac{1}{2}) + ln(e^{\left|2\sqrt{E}x/N_0\right|}) \tag{3.14}$$

We can disregard the first term since it is a constant and affects all terms, the constants of the second term, and replacing x for $\rho_{i+\mu}$. Finally, the S is

$$S_{highSNR} = \sum_{i=0}^{L-1} \rho_{i+\mu} s_i - \sum_{i=0}^{L-1} |\rho_{i+\mu}| \tag{3.15}$$

For *Low* SNR, the argument of the *cosh* is smaller than one, so, we can approximate the *cosh* with the Maclaurin series [Bab09]

$$cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}, \forall x \tag{3.16}$$

And the Taylor series [Bab09] for exponential function is

$$e^x = \sum_{n=0}^{\infty} \frac{1}{(n)!} x^n, \forall x; n \in N_0 \tag{3.17}$$

Combining (3.16) and (3.17), the correction term is

$$ln(cosh(2\sqrt{E}\rho_{i+\mu}/N_0)) = \frac{2\sqrt{E}}{N_0}\frac{1}{2}{\rho_{i+\mu}}^2 = \frac{\sqrt{E}}{N_0}{\rho_{i+\mu}}^2 \tag{3.18}$$

$$S_{lowSNR} = \sum_{i=0}^{L-1}\rho_{i+\mu}s_i - \frac{\sqrt{E}}{N_0}\sum_{i=0}^{L-1}{\rho_{i+\mu}}^2 \tag{3.19}$$

Finally we have the Optimum Rule, now we will use some training sequences what have some good correlation properties like Barker Sequences [Bar53], Neuman-Hofman Sequence [NH71] and Willard Sequences [Wil61], and also for random sequences.

3.1.1 Barker Sequences

In this section, we apply the algorithm after receiving a sequence ρ and estimate the position μ. As the training sequence, we use Barker sequence of length 7 and 13 [Bar53]:

```
L=7       s=(1, -1, 1, 1, -1, -1, -1)
L=13      s=(1, 1, 1, 1, 1, -1, -1, 1, 1, -1, 1, -1, 1)
```

In the study we use different frame lengths to see how it influences.

3.1.2 Neuman-Hofman Sequence

Another different training sequence is Neuman-Hofman sequence of length 13 [NH71]:

```
L=13       s=(-1, -1, -1, -1, -1, -1, 1, 1, -1, -1, 1, -1,1)
```

And, as in Barker sequences, the same idea for frame's length.

3.1.3 Willard Sequences

We use Willard sequences to see the comparison with Barker and Neuman-Hofman, due to these sequences do not appear in the Massey's results [Mas72], and see what happens for these sequences of lengths 7 and 13 [Wil61]:

```
L=7         s=(1, 1, 1, -1, 1, -1, -1)
L=13        s=(1, 1, 1, 1, 1, -1, -1, 1, -1, 1, 1, -1, -1)
```

And also, we use same frame lengths to compare them.

3.1.4 Random Sequences

In this point we will see what happens if the synchronization word is not a specific known sequence, what are going to be the results if both, synchronization word and data word, are random, but only taking values of +1 or -1.
Also here, we will see for different lengths of synchronization word and frame's length, not only $L = 7$ and $L = 13$, and what are their theoretical result.

3.2 Results

3.2.1 Barker and Neuman-Hofman Sequences Results

We show what happens using the algorithm for the Barker and Neuman-Hofman sequence described before. We divide this section in 2 different cases, where we have the fixed L, explained before and a fixed N as in [Mas72], and for different length of N to see how the error grows.

Fixed L and Fixed N

We fix $N = 91$ for largest Barker sequence and Neuman-Hofman sequence, and $N = 28$ for smallest Barker sequence [Mas72].
In Figure 3.1 and Figure 3.2, we observe the results, after computing the algorithm using Matlab. The flat lines, without $*$, are the results after applying the algorithm, and the $*$ lines are the reference given in [Mas72] without the correction term. The axes are, in abscissa the SNR in dB, and the ordinates the percentage of errors done in the receiver by our detector.
The computed results are quiet similar with the reference, so we can affirm the algorithm is correctly programmed. On the other hand, the results are very few, so we enlarge the SNR vector to obtain a huge vision of the errors. The axes in Figure 3.3, as earlier, in abscissa we have the SNR in dB, and in ordinates, the number of errors in a logarithm scale. From this point, all the plots of this chapter will have the same axes and structure. So, Figure 3.3, in we have a big sight of how the sequences behave, and we can observe

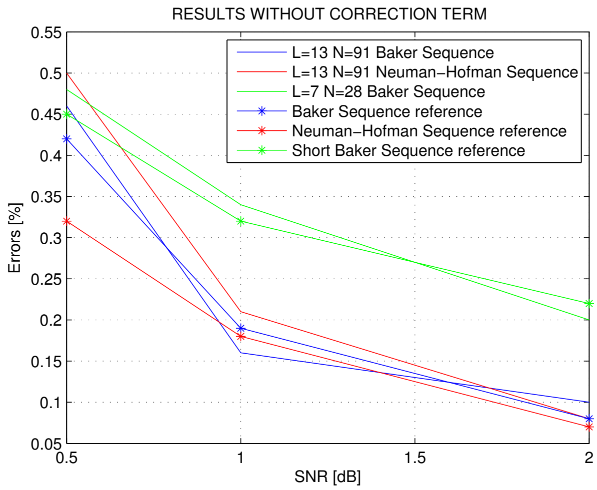

Figure 3.1: Reference without correction term

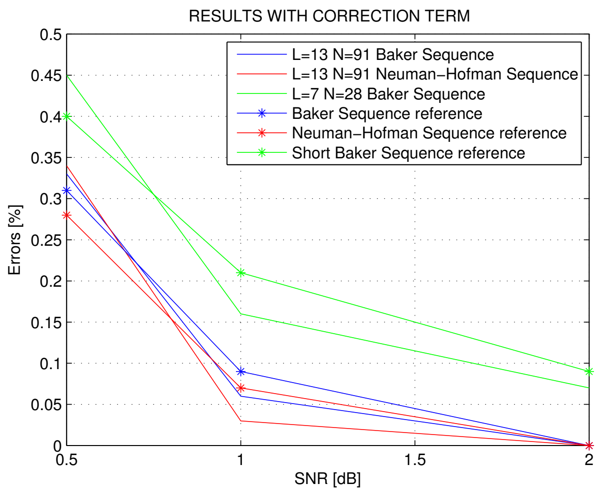

Figure 3.2: Reference with correction term

a gain, in all cases, more or less, between $SNR = -5\ dB$ and $SNR = 11\ dB$, from the cross correlation (the algorithm without the correction term) to the correction term.

There is a valley in Neuman-Hofman sequence without correction that goes so deep in the graph, that is produced because the simulation is done with 2.000 experiments, if we enlarge this number until 100.000, this valley will be converge above the line with the correction term.

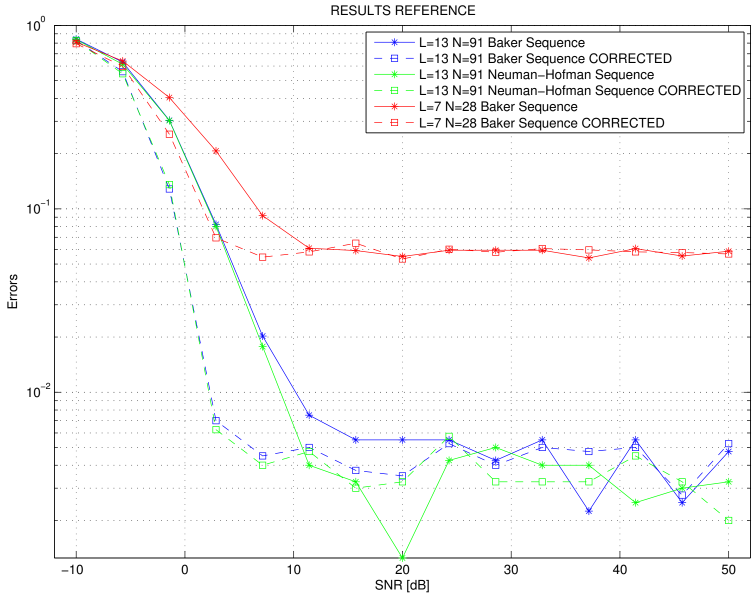

Figure 3.3: Reference with larger SNR

Fixed L and Larger N

Here we will see different results with larger N and how the errors will increase. In this point, we will observe the results when the frame length, N, is the double or nearby this value.

In Figure 3.4, we observe the results when the frame length is greater than the double, and we observe, again, this gain between $SNR = -5\ dB$ and $SNR = 11\ dB$, but if we compare this Figure 3.4, and Figure 3.3, we can observe a better flatness and soft curve in Figure 3.4, specially in Barker sequence, in the other 2 will happen the same if we had done the simulation with more experiments, and also, we can observe, in Figure 3.3, the number of errors are less, and that has sense, because if we enlarge the length of our total frame, but we keep the length of the synchronization word, the probability to found the synchronization word inside the data vector, is higher than if we have another frame with the total length smaller.

Later we will see how is this probability look like when the synchronization word and data are completely random.

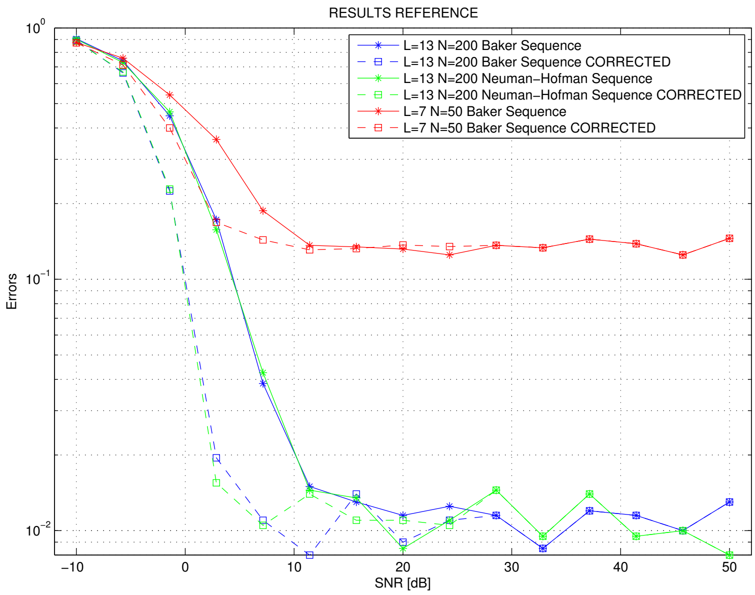

Figure 3.4: Reference with larger N

3.2.2 Willard Sequences Results

After studying the Barker and Neuman-Hofman sequences, we just compare Willard sequences and Barker sequences for different lengths of N. As we can see, in Figure 3.5, the gain is again between $SNR = -5\ dB$ and $SNR = 10\ dB$ and it is represented for different lengths of total frame, and we can observe, how the number of errors is growing up with the size of the total frame. In Figure 3.7 we can observe the same result for larger $SNRs$, but the main difference is, the gain is bigger, that means, the difference between the results without the correction term, and the correction term, when they start to stabilize, is bigger in the Willard Sequences than in the Barker Sequences.

3.2.3 Random Sequences Results

In random sequences we have, both, synchronization word and data word, random, as we saw before, and we made the study for different lengths of synchronization word, no just $L = 7$ and $L = 13$.

In all figures, we can observe the gain between the cross correlation term without the correction term and with it, and also, how we enlarge the synchronization word, for the same total length frame, the number of errors are decreasing.

For example, focusing $N = 750$, it is in all 4 plots, we can see for Figure 3.9 the results are practically zero, they are a huge rate of error, it starts over 95% and decrease till

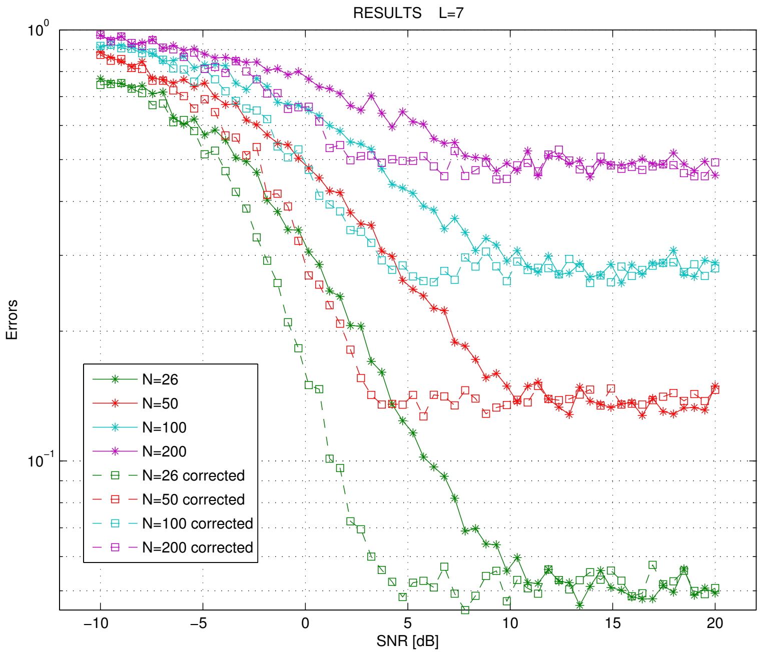

Figure 3.5: Barker Sequence L=7

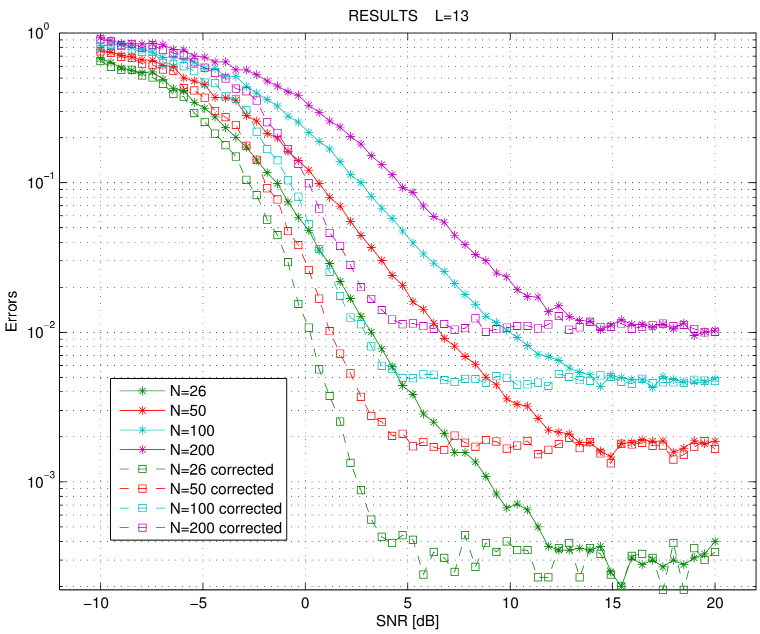

Figure 3.6: Barker Sequence L=13

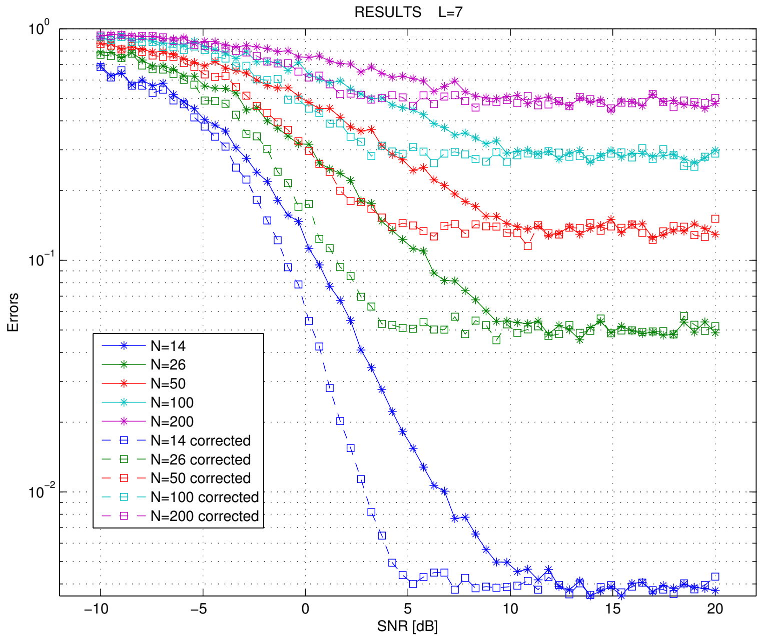

Figure 3.7: Willard Sequence L=7

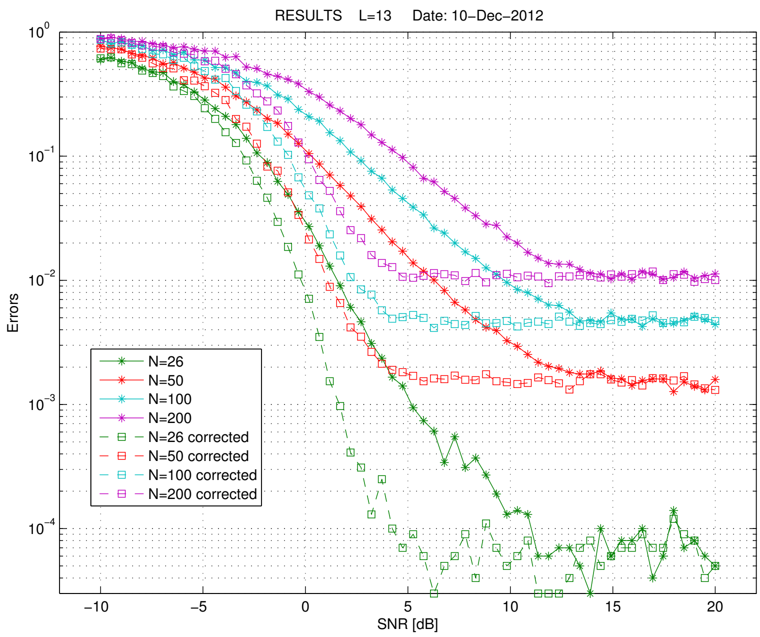

Figure 3.8: Willard Sequence L=13

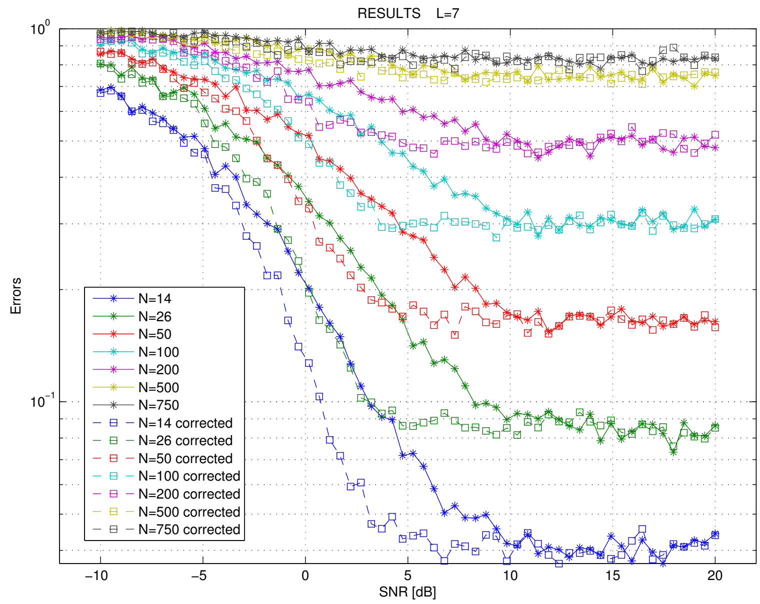

Figure 3.9: Random Sequence L=7

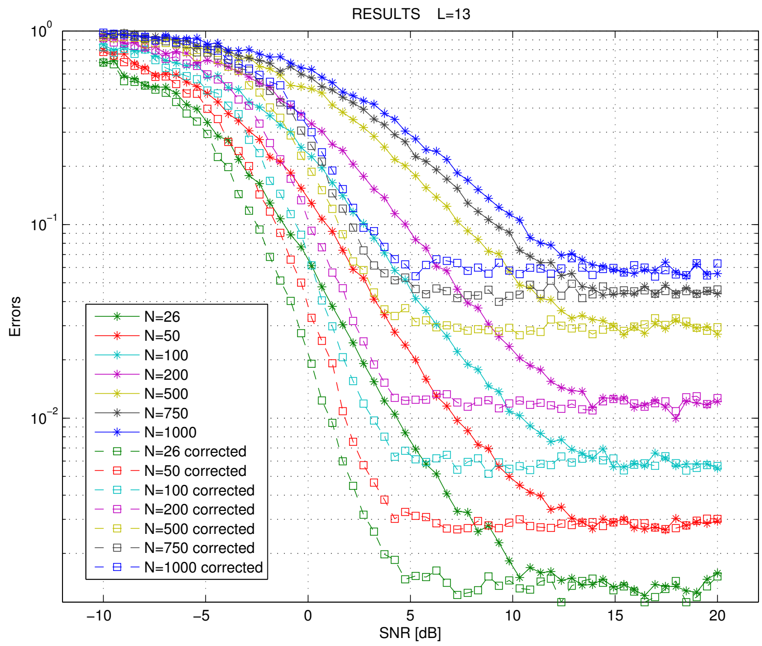

Figure 3.10: Random Sequence L=13

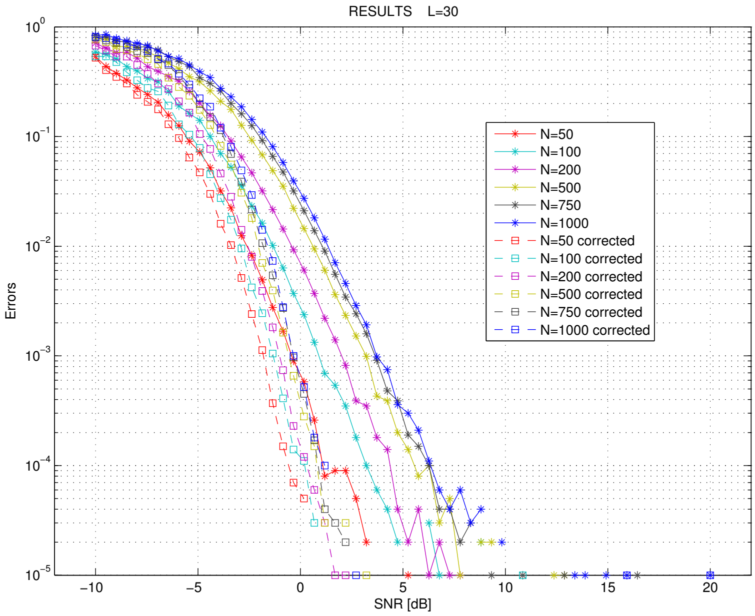

Figure 3.11: Random Sequence L=30

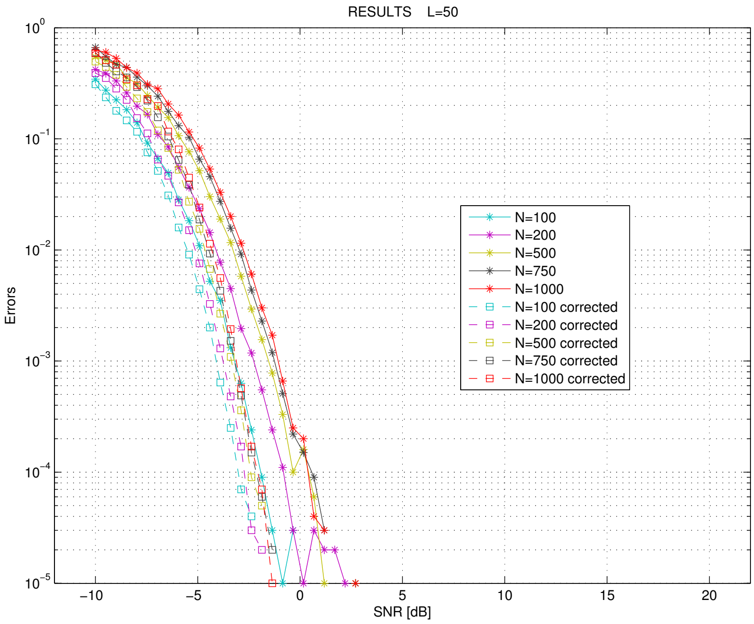

Figure 3.12: Random Sequence L=50

80%, but it continues being a very huge rate of error, and we can't observe the difference gain due to the correction term.
In Figure 3.10, the error for big $SNRs$ is so much smaller than Figure 3.9 going down the 10% of error rate, and just doubling the length of the synchronization word. The gain is also more clear now, and we can observe a difference gain due to the correction term of 8 dB.
In Figure 3.11 and In Figure 3.12 there are some rate error where their values are zero, because we can not see the the corresponding value, so we consider no errors when the plot ends. In both figures, the curves decrease faster than other 2 plots and their gain are, in Figure 3.11 a gain of 7.5 dB and for Figure 3.12 a gain of 3 dB but here the receiver does it correctly with the correction term, even when the noise is 10 times greater than our signal power.

Probability of error without noise

Now we will see if our simulated results are corrected obtaining the probability of error without noise, and then, compare to the value of $SNR = 20$
All the symbols are equiprobable, so $Pr(x = 1) = Pr(x = -1) = \frac{1}{2}$, and, as the synchronization word has length L, the probability to detect it correctly is

$$Pr(r_{0:L-1} = s) = \left(\frac{1}{2}\right)_0 \cdot \left(\frac{1}{2}\right)_1 \cdots \left(\frac{1}{2}\right)_{L-1} = \left(\frac{1}{2}\right)^L \tag{3.20}$$

Then, the probability of success is $\left(\frac{1}{2}\right)^L$, but we are looking for the probability of error. There are N different possibilities to find the synchronization word inside the frame, do not forget, our window sight is length N and the first one is between 0 and $N - 1$, the second one is between N and $2N - 1$ and so on, but the whole study for one window sight, includes also the values between 0 and $[(N - 1) + (L - 2)]$ because the start position could be in the last part of the frame, the frame is cyclic shifted, so, we have N different possibilities to find inside the frame, but only one is corrected, so there are $N - 1$ possibilities erroneous, so

$$Pr(e) = (N - 1)\left(\frac{1}{2}\right)^L \tag{3.21}$$

This is the probability of error in case there is no noise. For Figure 3.12, and using (3.21), the probability of error for $N = 500$ and $L = 50$ is $Pr(e) = 4.432010314303625e - 13$, nearly zero, and in our plot, does not appear result because it is zero, so we can affirm it is correct.

Another example, in Figure 3.10 and $N = 500$ and $L = 13$, the probability of error is $Pr(e) = 0.060913085937$, and in our plot, the error is $10^{-1.7} = 0.019952623149689$ better result than theoretical result, but increasing the number of experiments, the result will converge to the theoretical value.

4 Frame Synchronization Algorithms for Complex Sequences

In this chapter we will modify our previous algorithm for complex sequences, in this case we will use Zadoff-Chu sequences [Pop10] because they are used in Long Term Evolution (LTE) systems in the Primary Synchronization Signal (PSS) [PB08], and see what happens for $High$ and $Low\ SNRs$.

4.1 Method references

Zadoff-Chu Sequence is a complex vector sequence which is parametrised by a root u defined as:

$$x_u(n) = e^{-i\frac{\pi u n(n+1)}{N_{ZC}}} \tag{4.1}$$

where N_{ZC} is the sequence length.
These sequences have different properties interesting for our study:

1. They are periodic with period N_{ZC} if N_{ZC} is odd.
2. The product of 2 prime length Zadoff-Chu sequences with a cyclically shifted version of itself is zero.

We will consider as synchronization word the Zadoff-Chu Sequence with root $u = 25$ and length $N_{ZC} = 63$.
That was respect the synchronization word, now, we focus on the data word, which possible values for 1 bit or symbol are

$$x_k = \pm\frac{1}{\sqrt{2}} \pm \frac{1i}{\sqrt{2}}$$

Each x_k is normalized with $\frac{1}{\sqrt{2}}$ to have $|x_k| = 1$
Starting in (3.1) and remembering ρ and s are complex now, we will use absolute values,

$$p_n(\rho - \sqrt{E}T^{\mu}(s\delta)) = (2\pi)^{-N/2}\sum_{\forall\delta}[\prod_{i=0}^{L-1} e^{-\left|\rho_{i+\mu}-\sqrt{E}s_i\right|^2/N_0}]\prod_{i=L}^{N-1} e^{-\left|\rho_{i+\mu}-\sqrt{E}\delta_i\right|^2/N_0} \tag{4.2}$$

Defining $A = \left|\rho_{i+\mu} - \sqrt{E}s_i\right|^2$, and $B = \left|\rho_{i+\mu} - \sqrt{E}\delta_i\right|^2$

$$A = (\rho_{i+\mu} - \sqrt{E}s_i)(\rho_{i+\mu} - \sqrt{E}s_i)^* = |\rho_{i+\mu}|^2 - \rho_{i+\mu}\sqrt{E}{s_i}^* - s_i\sqrt{E}{\rho_{i+\mu}}^* + E\,|s_i|^2 \quad (4.3)$$

In complex domain, we have:

$$a + a^* = Re\{a\} + iIm\{a\} + Re\{a\} - iIm\{a\} = 2Re\{a\} \quad (4.4)$$

And using this with (4.3)

$$A = (\rho_{i+\mu} - \sqrt{E}s_i)(\rho_{i+\mu} - \sqrt{E}s_i)^* = |\rho_{i+\mu}|^2 + E\,|s_i|^2 - 2\sqrt{E}Re\{\rho_{i+\mu}{s_i}^*\} \quad (4.5)$$

And following the same process, we obtain

$$B = (\rho_{i+\mu} - \sqrt{E}\delta_i)(\rho_{i+\mu} - \sqrt{E}\delta_i)^* = |\rho_{i+\mu}|^2 + E\,|\delta_i|^2 - 2\sqrt{E}Re\{\rho_{i+\mu}{\delta_i}^*\} \quad (4.6)$$

Disregarding in term A and B the absolute values of s_i and $\rho_{i+\mu}$ (since they are constants) and focusing on terms depending on μ, A and B become the following form:

$$A = -2\sqrt{E}Re\{\rho_{i+\mu}{s_i}^*\}$$
$$B = -2\sqrt{E}Re\{\rho_{i+\mu}{\delta_i}^*\}$$

And now we have

$$S_4 = \sum_{\forall\delta}[\prod_{i=0}^{L-1} e^{-A/N_0}]\prod_{i=L}^{N-1} e^{-B/N_0} \quad (4.7)$$

$$S_4 = \sum_{\forall\delta}[\prod_{i=0}^{L-1} e^{2\sqrt{E}Re\{\rho_{i+\mu}{s_i}^*\}/N_0}]\prod_{i=L}^{N-1} e^{2\sqrt{E}Re\{\rho_{i+\mu}{\delta_i}^*\}/N_0} \quad (4.8)$$

As mentioned of the beginning of the chapter, the four different possible values of 1 bit, but disregarding the $\frac{1}{\sqrt{2}}$ to make easier the calculation, we replace δ_i for these values, and defining $C = \prod_{i=0}^{L-1} e^{2\sqrt{E}Re\{\rho_{i+\mu}{s_i}^*\}/N_0}$, and $D = \prod_{i=L}^{N-1} e^{2\sqrt{E}Re\{\rho_{i+\mu}{\delta_i}^*\}/N_0}$

$$
\begin{aligned}
D = \prod_{i=L}^{N-1} & e^{2\sqrt{E}Re\{\rho_{i+\mu}(1-1i)\}/N_0} + e^{2\sqrt{E}Re\{\rho_{i+\mu}(1+1i)\}/N_0} \\
& + e^{2\sqrt{E}Re\{\rho_{i+\mu}(-1-1i)\}/N_0} + e^{2\sqrt{E}Re\{\rho_{i+\mu}(-1+1i)\}/N_0} \\
= \prod_{i=L}^{N-1} & e^{2\sqrt{E}Re\{\rho_{i+\mu}\}/N_0} e^{2\sqrt{E}Im\{-\rho_{i+\mu}\}/N_0} + e^{2\sqrt{E}Re\{\rho_{i+\mu}\}/N_0} e^{2\sqrt{E}Im\{\rho_{i+\mu}\}/N_0} \\
& + e^{2\sqrt{E}Re\{-\rho_{i+\mu}\}/N_0} e^{2\sqrt{E}Im\{-\rho_{i+\mu}\}/N_0} + e^{2\sqrt{E}Re\{-\rho_{i+\mu}\}/N_0} e^{2\sqrt{E}Im\{\rho_{i+\mu}\}/N_0} \\
= \prod_{i=L}^{N-1} & 2cosh(2\sqrt{E}Re\{\rho_{i+\mu}\}/N_0) e^{-2\sqrt{E}Im\{\rho_{i+\mu}\}/N_0} \\
& + 2cosh(2\sqrt{E}Re\{\rho_{i+\mu}\}/N_0) e^{2\sqrt{E}Im\{\rho_{i+\mu}\}/N_0} \\
= \prod_{i=L}^{N-1} & 2cosh(2\sqrt{E}Re\{\rho_{i+\mu}\}/N_0)[e^{2\sqrt{E}Im\{\rho_{i+\mu}\}/N_0} + e^{-2\sqrt{E}Im\{\rho_{i+\mu}\}/N_0}] \\
= \prod_{i=L}^{N-1} & 4cosh(2\sqrt{E}Re\{\rho_{i+\mu}\}/N_0) cosh(2\sqrt{E}Im\{\rho_{i+\mu}\}/N_0)
\end{aligned} \tag{4.9}
$$

Now, combining the cross-correlation (term C) and the term D

$$S_4 = C \cdot D \tag{4.10}$$

Applying logarithms and disregarding constants

$$S_5 = \sum_{i=0}^{L-1} 2\sqrt{E}Re\{\rho_{i+\mu} {s_i}^*\}/N_0$$

$$+ \sum_{i=L}^{N-1} ln[cosh(2\sqrt{E}Re\{\rho_{i+\mu}\}/N_0) cosh(2\sqrt{E}Im\{\rho_{i+\mu}\}/N_0)] \tag{4.11}$$

Using the same condition like in (3.11),

$$S_6 = \sum_{i=0}^{L-1} 2\sqrt{E}Re\{\rho_{i+\mu} {s_i}^*\}/N_0$$

$$- \sum_{i=0}^{L-1} ln[cosh(2\sqrt{E}Re\{\rho_{i+\mu}\}/N_0)cosh(2\sqrt{E}Im\{\rho_{i+\mu}\}/N_0)] \tag{4.12}$$

And finally, the optimum rule for complex sequences is

$$S = \sum_{i=0}^{L-1} Re\{\rho_{i+\mu}s_i{}^*\} - \sum_{i=0}^{L-1} f(\rho_{i+\mu}) \tag{4.13}$$

where $f(x)$ is

$$f(x) = \frac{N_0}{2\sqrt{E}} ln[cosh(2\sqrt{E}Re\{x_{i+\mu}\}/N_0)cosh(2\sqrt{E}Im\{x_{i+\mu}\}/N_0)] \tag{4.14}$$

Now we see the approximations for $High$ and $Low\ SNRs$ using the same conditions as in real sequences

For $High\ SNR$, the argument in the $cosh$ is greater than 1, so, we can approximate the $cosh(x)$ as $\frac{1}{2}e^{|x|}$, and we can obtain, disregarding the constants

$$S = \sum_{i=0}^{L-1} Re\{\rho_{i+\mu}s_i{}^*\} - \sum_{i=0}^{L-1} |Re\{\rho_{i+\mu}\}| + |Im\{\rho_{i+\mu}\}| \tag{4.15}$$

For $Low\ SNR$, the argument of the $cosh$ is smaller than one, so, we can approximate the $cosh$ with the Maclaurin series as we saw in chapter 3,

$$f(x) = \frac{N_0}{2\sqrt{E}}(ln[cosh(2\sqrt{E}Re\{x_{i+\mu}\}/N_0)] + ln[cosh(2\sqrt{E}Im\{x_{i+\mu}\})]) \tag{4.16}$$

$$f(x) = \frac{N_0}{2\sqrt{E}} \left(\frac{1}{2}(2\sqrt{E}Re\{x_{i+\mu}\}/N_0)^2 + \frac{1}{2}(2\sqrt{E}Im\{x_{i+\mu}\}/N_0)^2 \right) \tag{4.17}$$

Disregarding the constants

$$f(x) = \frac{N_0}{2\sqrt{E}} \frac{E}{{N_0}^2} (Re\{x_{i+\mu}\}]^2 + Im\{x_{i+\mu}\}]^2) \tag{4.18}$$

$$f(x) = \frac{\sqrt{E}}{N_0}|x_{i+\mu}|^2 \tag{4.19}$$

Finally

$$S = \sum_{i=0}^{L-1} Re\{\rho_{i+\mu}{s_i}^*\} - \frac{\sqrt{E}}{N_0}\sum_{i=0}^{L-1}|\rho_{i+\mu}|^2 \tag{4.20}$$

4.2 Results

As we saw in the theory of this chapter, we will use for our study a Zadoff-Chu Sequence [Pop10], with root $u = 25$ and length $N_{ZC} = 63$, so first we will see how is this Sequence its real and imaginary part look like, because it is a complex sequence.

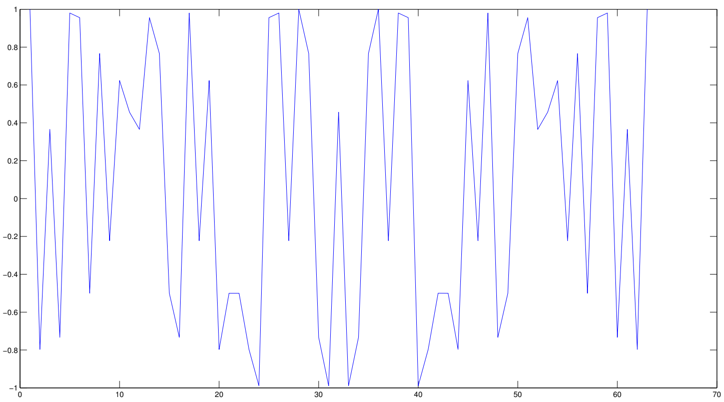

Figure 4.1: Real Part Zadoff-Chu Sequence $u = 25$ and $N_{ZC} = 63$

As we can see in Figure 4.1 and Figure 4.2, the amplitude is bounded and if we put a mirror in the middle of the image, we can see a periodicity but reflected.
Now we will see the number of errors versus the SNR for different lengths of total frame N and the length of our Zadoff-Chu sequence $L = 63$. We can observe in Figure 4.3 the difference in number of errors when we enlarge the length of total frame, we have more errors in a frame of $N = 5000$ than in one of $N = 500$. We can appreciate also, the difference between the cross correlation term with and without the correction term, we always obtain better results using this correction term.

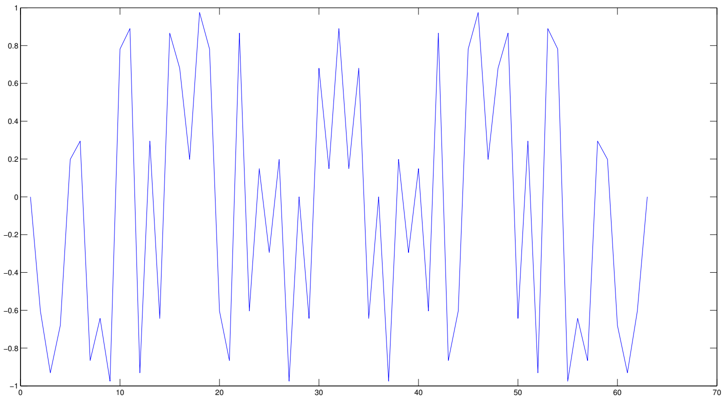

Figure 4.2: Imaginary Part Zadoff-Chu Sequence $u = 25$ and $N_{ZC} = 63$

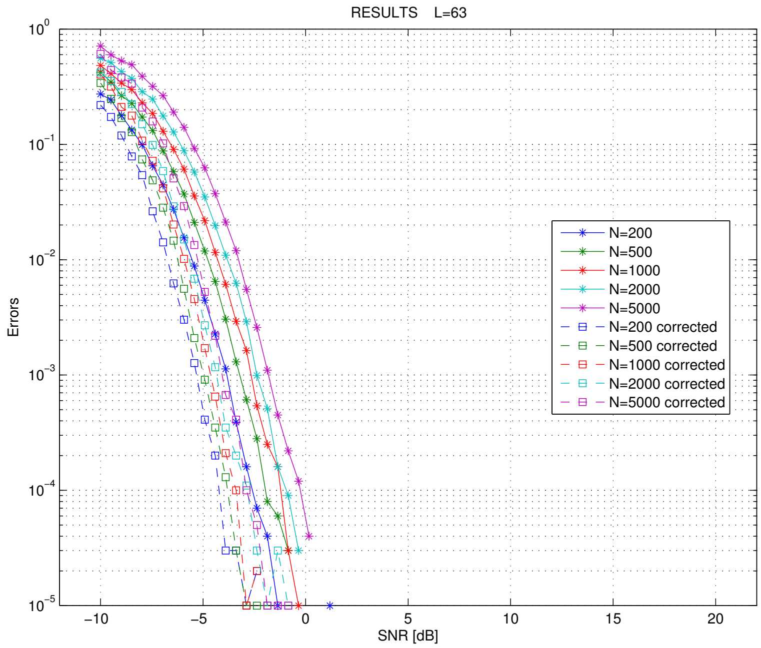

Figure 4.3: Error rate in a Zadoff-Chu Sequence

Moreover, we can see, the curves converge faster if we compare with the Real Sequences in last chapter, and the first zero value appears for $N = 200$ and $N = 500$, the blue and green squared lines, in $SNR - 3\ dB$, where the noise is even greater than signal power. We can not consider $N = 1000$ also in that evaluation, because, if we continue seeing, the next point, for $SNR = -2.5\ dB$, appears an error, and then, for $SNR = -2\ dB$

goes again to zero and now stabilized in zero. Than means, the receiver didn't detect any error in $SNR = -3\ dB$, but, if we increase the number of experiments, we will see an error in $SNR = -3\ dB$ and then in $SNR = -2\ dB$ the receiver detects always correct. As a last note, we can observe for $N = 5000$ we have the bigest difference between the cross correlation with and without the correction term, concretely, this value is $G_{N=5000} = 2.5$ dB.

5 Frame Synchronization Algorithms for Complex Sequences with Phase Shift

In this chapter we will see how a constant phase shift introduced in the channel influences the performance of frame synchronization. We will use the same scheme but adding the phase shift, φ, to the formula in (2.3).

$$\mathbf{r} = \sqrt{E} \cdot e^{j\varphi} \cdot T^m(\mathbf{sd}) + \mathbf{n} \tag{5.1}$$

5.1 Method references

Using (5.1) and starting with (4.8) but adding the phase shift, we continue with the process

$$S_4 = \sum_{all\delta} [\prod_{i=0}^{L-1} e^{2\sqrt{E}Re\{\rho_{i+\mu}s_i{}^* e^{-j\varphi}\}/N_0}] \prod_{i=L}^{N-1} e^{2\sqrt{E}Re\{\rho_{i+\mu}\delta_i{}^* e^{-j\varphi}\}/N_0} \tag{5.2}$$

Having a look in the process, s_i is conjugated, as well the phase, that's why appears with the negative and in (5.1) is positive.
We use the same synchronization word and same values for 1 bit data, which are

$$x_k = \pm\frac{1}{\sqrt{2}} \pm \frac{1i}{\sqrt{2}}$$

For the calculation, we disregard the term $\frac{1}{\sqrt{2}}$ to make easier the derivation.
Defining $C = \prod_{i=0}^{L-1} e^{2\sqrt{E}Re\{\rho_{i+\mu}s_i{}^* e^{-j\varphi}\}/N_0}$ and $D = \prod_{i=L}^{N-1} e^{2\sqrt{E}Re\{\rho_{i+\mu}\delta_i{}^* e^{-j\varphi}\}/N_0}$

$$D = \prod_{i=L}^{N-1} e^{2\sqrt{E}Re\{\rho_{i+\mu}(1-1i)e^{-j\varphi}\}/N_0} + e^{2\sqrt{E}Re\{\rho_{i+\mu}(1+1i)e^{-j\varphi}\}/N_0} +$$

$$+ e^{2\sqrt{E}Re\{\rho_{i+\mu}(-1-1i)e^{-j\varphi}\}/N_0} + e^{2\sqrt{E}Re\{\rho_{i+\mu}(-1+1i)e^{-j\varphi}\}/N_0} \tag{5.3}$$

And following the same steps as we saw in chapter four,

$$D = \prod_{i=L}^{N-1} 4cosh(2\sqrt{E}Re\{\rho_{i+\mu}e^{-j\varphi}\}/N_0)cosh(2\sqrt{E}Im\{\rho_{i+\mu}e^{-j\varphi}\}/N_0) \tag{5.4}$$

$$S_4 = C \cdot D \tag{5.5}$$

Applying logarithms to S_4,

$$S_5 = \sum_{i=0}^{L-1} Re\{\rho_{i+\mu}{s_i}^* e^{-j\varphi}\}$$

$$+ \sum_{i=L}^{N-1} 4cosh(2\sqrt{E}Re\{\rho_{i+\mu}e^{-j\varphi}\}/N_0)cosh(2\sqrt{E}Im\{\rho_{i+\mu}e^{-j\varphi}\}/N_0) \tag{5.6}$$

In complex domain, defining x as a complex number

$$x = |x|\, e^{i\angle x}$$

$$\begin{aligned} Re\{x\} &= Re\{|x|\, e^{i\angle x}\} \\ &= Re\{|x|\}Re\{e^{i\angle x}\} \end{aligned} \tag{5.7}$$

The $Re\{|x|\}$ is constant, so $Re\{|x|\} = |x|$, and the $Re\{e^{i\angle x}\}$ we can use the Euler formula [Mos02].

$$e^{i\angle x} = cos(\angle x) + isin(\angle x) \tag{5.8}$$

Applying the real part for $e^{i\angle x}$

$$Re\{e^{i\angle x}\} = cos(\angle x) \tag{5.9}$$

Applying (5.7) with the earlier equation,

$$Re\{x\} = |x|\, cos(\angle x) \tag{5.10}$$

So the first term would be

$$\sum_{i=0}^{L-1} Re\{\rho_{i+\mu}{s_i}^* e^{-j\varphi}\} = \left|\sum_{i=0}^{L-1} \rho_{i+\mu}{s_i}^*\right| cos(\angle\sum_{i=0}^{L-1} \rho_{i+\mu}{s_i}^* - \hat{\varphi}) \tag{5.11}$$

And now we are actually not interested in φ, so we must to eliminate from (5.11), so we need to estimate a $\hat{\varphi}$ which maximize the summation and the cosinus in (5.11). The maximum value of a cosinus is $max\{|cos(x)|\} = 1$, so we have to equate $cos(\angle\sum_{i=0}^{L-1} \rho_{i+\mu}{s_i}^* - \hat{\varphi})$ to 1. The cosinus is 1 when the cosinus argument is 0, so

$$cos(\angle\sum_{i=0}^{L-1} \rho_{i+\mu}{s_i}^* - \hat{\varphi}) = 1 \tag{5.12}$$

$$\angle\sum_{i=0}^{L-1} \rho_{i+\mu}{s_i}^* - \hat{\varphi} = 0 \tag{5.13}$$

$$\hat{\varphi} = \angle\sum_{i=0}^{L-1} \rho_{i+\mu}{s_i}^* \tag{5.14}$$

Changing the limits in the summation of second term, as we did in the other chapters, and disregarding the constants, the optimum rule is

$$S = \left|\sum_{i=0}^{L-1} \rho_{i+\mu}{s_i}^*\right| - \sum_{i=0}^{L-1} f(\rho_{i+\mu}, \hat{\varphi}) \tag{5.15}$$

where

$$f(x, \hat{\varphi}) = \frac{N_0}{2\sqrt{E}} ln[cosh(2\sqrt{E}Re\{x_{i+\mu}e^{-j\hat{\varphi}}\}/N_0)cosh(2\sqrt{E}Im\{x_{i+\mu}e^{-j\hat{\varphi}}\}/N_0)] \tag{5.16}$$

Now we derive the approximations for larger and lower SNR to the correction term as we did in all the chapters earlier, the cross correlation term remains equal.

For $SNR \geq 15dB$, using the same conditions as in the other two chapters, $cosh(y) = \frac{1}{2}e^{|y|}$ and disregarding the constants

$$f_{highSNR}(x, \hat{\varphi}) = \frac{N_0}{2\sqrt{E}} \cdot \frac{1}{2}|2\sqrt{E}Re\{x_{i+\mu}e^{-j\hat{\varphi}}\}/N_0|$$

$$+\frac{N_0}{2\sqrt{E}} \cdot \frac{1}{2}|2\sqrt{E}Im\{x_{i+\mu}e^{-j\hat{\varphi}}\}/N_0| \tag{5.17}$$

$$f_{highSNR}(x,\hat{\varphi}) = \left(|Re\{x_{i+\mu}e^{-j\hat{\varphi}}\}| + |Im\{x_{i+\mu}e^{-j\hat{\varphi}}\}|\right) \tag{5.18}$$

And for $SNR < 0dB$, and the Maclaurin series, $ln(cosh(y)) = \frac{1}{2}y^2$,

$$f_{lowSNR}(x,\hat{\varphi}) = \frac{N_0}{2\sqrt{E}}\frac{\sqrt{E}}{N_0} \cdot \frac{1}{2}(2\sqrt{E}Re\{x_{i+\mu}e^{-j\hat{\varphi}}\}/N_0)^2$$
$$+\frac{N_0}{2\sqrt{E}}\frac{\sqrt{E}}{N_0} \cdot \frac{1}{2}(2\sqrt{E}Im\{x_{i+\mu}e^{-j\hat{\varphi}}\}/N_0)^2 \tag{5.19}$$

$$f_{lowSNR}(x,\hat{\varphi}) = \frac{N_0}{2\sqrt{E}}\frac{4E}{{N_0}^2}\left(Re\{x_{i+\mu}e^{-j\hat{\varphi}}\}^2 + Im\{x_{i+\mu}e^{-j\hat{\varphi}}\}^2\right) \tag{5.20}$$

And finally

$$f_{lowSNR}(x,\hat{\varphi}) = \frac{\sqrt{E}}{N_0}\left|x_{i+\mu}e^{-j\hat{\varphi}}\right|^2 \tag{5.21}$$

5.2 Results

To see how the phase affects to the signal detection for different $SNRs$, we will compare the same synchronization word with a Zadoff-Chu Sequence with root $u = 25$ and $N_{ZC} = 63$, but for 2 different total frame lengths, for $N = 200$ and $N = 2000$. As we can see here, the plot is a bit different from the others. In Figure 5.1 we see represented in ordinates the errors, as earlier, but in the abscissa axis we have represented the phase shift in radians, and plotted the different $SNRs$.
In the plot are only represented from $SNR = -10\ dB$ until $SNR = 0\ dB$ because the others $SNRs$ used all the experiments in the simulation, so, first point we can affirm, the phase shift is not a problem always we have a signal power greater than noise power, even, if we use the correction term in the cross correlation term, we can have a sensitivity of 0 dB and always detects correct.
According with the second part of this affirmation, the cross correlation term, without

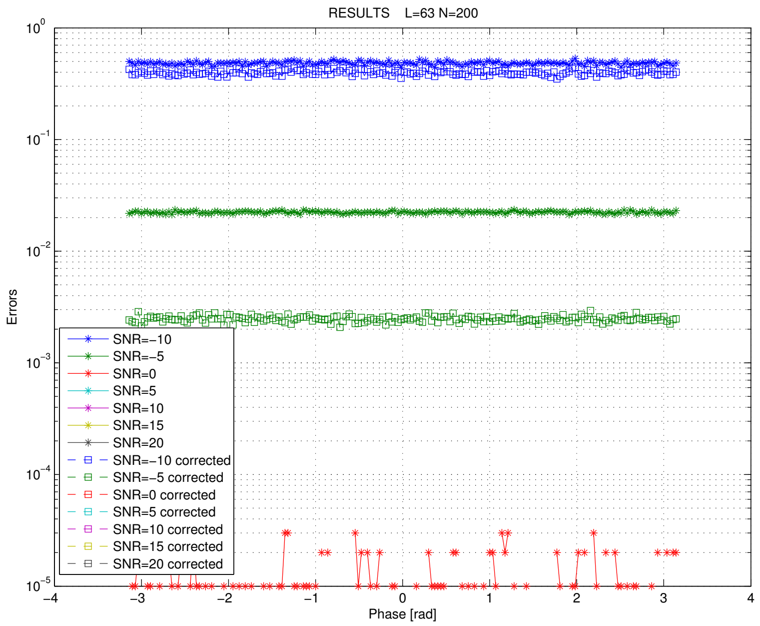

Figure 5.1: $ZC_{N_{ZC}=63}$ Sequence with $N = 200$ adding phase shift

the correction term, shows some errors for $SNR = 0\ dB$, so that means, the receiver detects 10 errors per each 1 million of experiments. In Figure 5.2 we can observe the same pattern as in In Figure 5.1, but the difference, as we expected, is the error rate. Here, for a larger total frame length, the error rate is bigger, but the receiver continues doing a good detection for big $SNRs$, so if we have a good sensitivity, we do not have to worry about it.

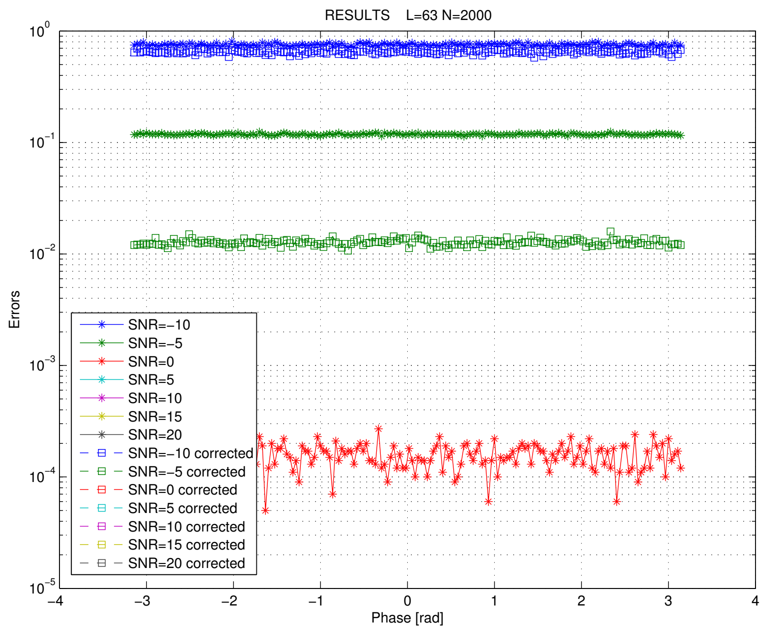

Figure 5.2: $ZC_{N_{ZC}=63}$ Sequence with $N = 2000$ adding phase shift

6 Frame Synchronization Algorithms for Complex Sequences with Frequency Shift

In this last chapter, we will study the Doppler effect after solving the phase shift problem.

6.1 Method references

Remembering the vector we have in the channel output, (5.1), and adding the Doppler effect.

$$\mathbf{r} = \sqrt{E} \cdot e^{j\varphi}[e^{j2\pi f' T_a 0} e^{j2\pi f' T_a 1} \dots e^{j2\pi f' T_a (N-1)}] \cdot T^m(\mathbf{sd}) + \mathbf{n} \tag{6.1}$$

where T_a is the Sampling period and f' is the Doppler shift.
After solving the phase problem as we saw in chapter 5, we have

$$S_{opt}(m, f') = \left| \sum_{i=0}^{L-1} \rho_{i+\mu} {s_i}^* e^{-j2\pi f' T_a i} \right| - \sum_{i=0}^{L-1} f(\rho_{i+\mu}, \hat{\varphi}, \hat{f}'_n) \tag{6.2}$$

where

$$f(\rho_{i+\mu}, \hat{\varphi}, \hat{f}'_n) = ln[cosh(2\sqrt{E} Re\{\rho_{i+\mu} e^{-j\hat{\varphi}} e^{-j2\pi \hat{f}'_n T_a i}\}/N_0)$$

$$\cdot cosh(2\sqrt{E} Im\{\rho_{i+\mu} e^{-j\hat{\varphi}} e^{-j2\pi \hat{f}' T_a i}\}/N_0)] \tag{6.3}$$

Now, we need to solve the Doppler effect in the cross correlation term (first term in S_{opt}) and estimate the $\hat{f}'_n$ for the correction term (second term in S_{opt}).

The idea is to chop the whole length frame N in small sections of length L_{coh} [AE07] where we suppose, in each new segment, the phase is constant through the channel. The criteria to choose the length of L_{coh}:

1. $L_{coh} \ll L$
2. L_{coh} is a multiple of L i.e. $mod(\frac{L}{L_{coh}}) = 0$

First, we define $\Gamma = \left|\sum_{i=0}^{L-1} \rho_{i+\mu} {s_i}^* e^{-j2\pi f' T_a i}\right|$ [PVVC+10], and we can approximate:

$$2\pi f' T_a i \approx 2\pi f' T_a \lfloor \frac{i}{L_{coh}} \rfloor L_{coh} \tag{6.4}$$

Using (6.4) and the definition of Γ,

$$\Gamma \approx \sum_{k=0}^{K-1} \sum_{i=kL_{coh}}^{(k+1)L_{coh}-1} \rho_{i+\mu} {s_i}^* e^{-j2\pi f' T_a k L_{coh}} =$$

$$= \sum_{k=0}^{K-1} e^{-j2\pi f' T_a k L_{coh}} \sum_{i=kL_{coh}}^{(k+1)L_{coh}-1} \rho_{i+\mu} {s_i}^* \tag{6.5}$$

where $k = \lfloor \frac{i}{L_{coh}} \rfloor$.

Defining $x_{k,\mu} = \sum_{i=kL_{coh}}^{(k+1)L_{coh}-1} \rho_{i+\mu} {s_i}^*$ we can continue with our development.

$$\Gamma = \sum_{k=0}^{K-1} e^{-j2\pi f' T_a k L_{coh}} x_{k,\mu} \tag{6.6}$$

Next step, we make absolute value squared, easier to work with complex vectors.

$$|\Gamma|^2 = \sum_{k=0}^{K-1} x_{k,\mu} e^{-j2\pi f' T_a k L_{coh}} \sum_{h=0}^{K-1} {x_{h,\mu}}^* e^{j2\pi f' T_a h L_{coh}} =$$

$$= \sum_{k=0}^{K-1} |x_{k,\mu}|^2 + \sum_{n=1}^{K-1} 2Re\{\sum_{k=n}^{K-1} x_{k,\mu} x_{k-n,\mu}^* e^{-j2\pi f' T_a n L_{coh}}\} \tag{6.7}$$

Defining $\Lambda_0(\mu) = \sum_{k=0}^{K-1} |x_{k,\mu}|^2$ and $\Lambda_n(\mu, f') = 2Re\{\sum_{k=n}^{K-1} x_{k,\mu} x_{k-n,\mu}^* e^{-j2\pi f' T_a n L_{coh}}\}$, we can see, the formula (6.7) as

$$|\Gamma|^2 = \Lambda_0(\mu) + \sum_{n=1}^{K-1} \Lambda_n(\mu, f') \tag{6.8}$$

Now we need to estimate $\hat{f}'_n$ applying the same process as (5.9)

$$\hat{f}'_n = arg\max_{f'} \Lambda_n(\mu, f'_n) = \frac{\angle \sum_{k=n}^{K-1} x_{k,\mu} x^*_{k-n,\mu}}{2\pi n L_{coh} T_a} \tag{6.9}$$

Then, we can substitute the estimation of $\hat{f}'_n$ into f'

$$\Lambda_n(\mu, f') \approx \Lambda_n(\mu, \hat{f}'_n) \tag{6.10}$$

And then, applying this last approximation into (6.8),

$$|\Gamma| = \sqrt{\sum_{k=0}^{K-1} |x_{k,\mu}|^2 + \sum_{n=1}^{K-1} 2 \left| \sum_{k=n}^{K-1} x_{k,\mu} x^*_{k-n,\mu} \right|} \tag{6.11}$$

Finally, S_{opt} is

$$S_{opt} = |\Gamma| - \sum_{i=0}^{L-1} f(\rho_{i+\mu}, \hat{\varphi}, \hat{f}'_n) \tag{6.12}$$

where

$$x_{k,\mu} = \sum_{i=kL_{coh}}^{(k+1)L_{coh}-1} \rho_{i+\mu} {s_i}^* \tag{6.13}$$

and

$$f(\rho, \hat{\varphi}, \hat{f}'_n) = \frac{N_0}{2\sqrt{E}} \cdot ln[cosh(2\sqrt{E} Re\{\rho_{i+\mu} e^{-j\hat{\varphi}} e^{-j2\pi \hat{f}'_n T_a i}\}/N_0)$$

$$\cdot cosh(2\sqrt{E} Im\{\rho_{i+\mu} e^{-j\hat{\varphi}} e^{-j2\pi \hat{f}'_n T_a i}\}/N_0)] \tag{6.14}$$

Now we derive the approximations for larger and lower SNR to the correction term as we did in all the chapters earlier, the cross correlation term remains equal.

For $SNR \geq 15dB$, using the same conditions as in all the chapters, $cosh(y) = \frac{1}{2}e^{|y|}$ and disregarding the constants

$$f_{highSNR}(\rho, \hat{\varphi}, \hat{f}'_n) = \frac{N_0}{2\sqrt{E}} \left| 2\sqrt{E} Re\{\rho_{i+\mu} e^{-j\hat{\varphi}} e^{-j2\pi \hat{f}'_n T_a i}\}/N_0 \right|$$

$$+ \frac{N_0}{2\sqrt{E}} \left| 2\sqrt{E} Im\{\rho_{i+\mu} e^{-j\hat{\varphi}} e^{-j2\pi \hat{f}'_n T_a i}\}/N_0 \right| \tag{6.15}$$

$$f_{highSNR}(\rho, \hat{\varphi}, \hat{f}'_n) = \left(|Re\{\rho_{i+\mu} e^{-j\hat{\varphi}} e^{-j2\pi \hat{f}'_n T_a i}\}| + |Im\{\rho_{i+\mu} e^{-j\hat{\varphi}} e^{-j2\pi \hat{f}'_n T_a i}\}| \right) \tag{6.16}$$

And for $SNR < 0dB$, and the Maclaurin series, $ln(cosh(y)) = \frac{1}{2}y^2$,

$$f_{lowSNR}(\rho, \hat{\varphi}, \hat{f}'_n) = \frac{N_0}{2\sqrt{E}} \frac{1}{2} 4E Re\{\rho_{i+\mu} e^{-j\hat{\varphi}} e^{-j2\pi \hat{f}'_n T_a i}\}^2 / {N_0}^2$$

$$+ \frac{N_0}{2\sqrt{E}} \frac{1}{2} 4E Im\{\rho_{i+\mu} e^{-j\hat{\varphi}} e^{-j2\pi \hat{f}'_n T_a i}\}^2 / {N_0}^2 \tag{6.17}$$

$$f_{lowSNR}(\rho, \hat{\varphi}, \hat{f}'_n) = \frac{\sqrt{E}}{N_0} Re\{\rho_{i+\mu} e^{-j\hat{\varphi}} e^{-j2\pi \hat{f}'_n T_a i}\}^2$$

$$+ \frac{\sqrt{E}}{N_0} Im\{\rho_{i+\mu} e^{-j\hat{\varphi}} e^{-j2\pi \hat{f}'_n T_a i}\}^2 \tag{6.18}$$

And finally

$$f_{lowSNR}(\rho, \hat{\varphi}, \hat{f}'_n) = \frac{\sqrt{E}}{N_0} \left| \rho_{i+\mu} e^{-j\hat{\varphi}} e^{-j2\pi \hat{f}'_n T_a i} \right|^2 \tag{6.19}$$

6.2 Results

For the Doppler simulation, we will fix some parameters to do the simulation faster. First, we fix the phase shift as a perfect phase estimation $\varphi = 0$ and see what happens for a Zadoff-Chu synchronization word of length $L = N_{ZC} = 63$ and two total frame length of

$N_1 = 200$ and $N_2 = 2000$ to compare them.
We will see also what happens with the L_{coh} parameter, so we will do, for same $L = N_{ZC} = 63$ and total frame length of $N_1 = 200$ and $N_2 = 2000$, but for $L_{coh1} = 1$ and $L_{coh2} = 7$.
The figures will represent the number of errors in a logarithm scale in the ordinates, and the frequency shift in Hz in the abscissa.

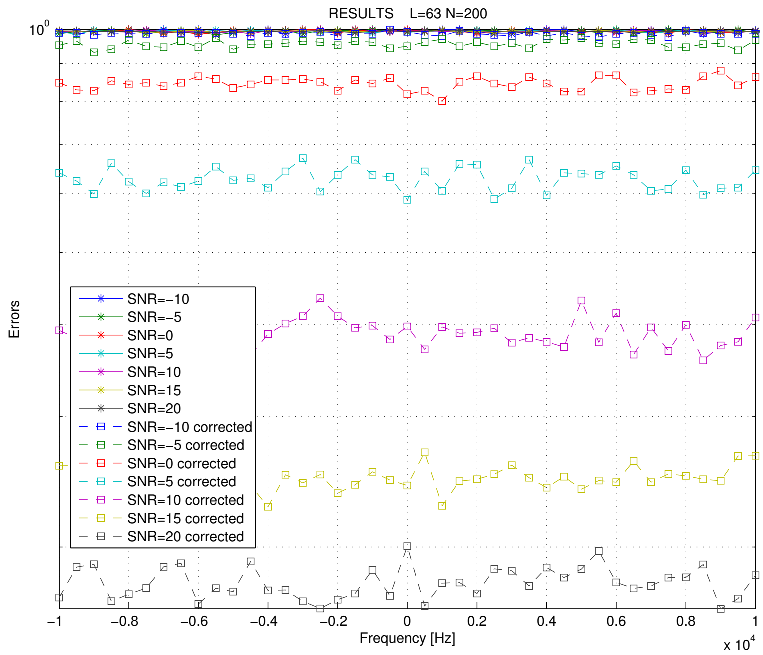

Figure 6.1: $ZC_{N_{ZC}=63}$ Sequence $N = 200$ $L_{coh} = 1$ and frequency shift

In Figure 6.1 we represent in the different coloured lines the $SNRs$, and we observe can not dispense of the correction term, because we have an error of 99% or even 100% only with the correlation term, so we need this correction term.
For negative $SNRs$ we still continue having a big rate of error, but, when the signal power is greater then the noise, this rate decreases, and arrives, for the case of $SNR = 20$ dB and no frequency shift, $f_n = 0$ Hz, the error rate is $P(error) = 10^{-0.7} = 0.1995$ it is a high rate if we compare with the other results where we consider only the phase shift and time delay, but considering, each symbol was affected by a different frequency, we can affirm, it is a good solution and a good rate. It could be better, using the Secondary Synchronization Signal (SSS) [KHRC09], but this is not our objective, we only focus on the Primary Synchronization Signal (PSS) [PB08] using the Zadoff-Chu sequence.
In Figure 6.2 we observe a huge error rate even with the correction term, so we can affirm

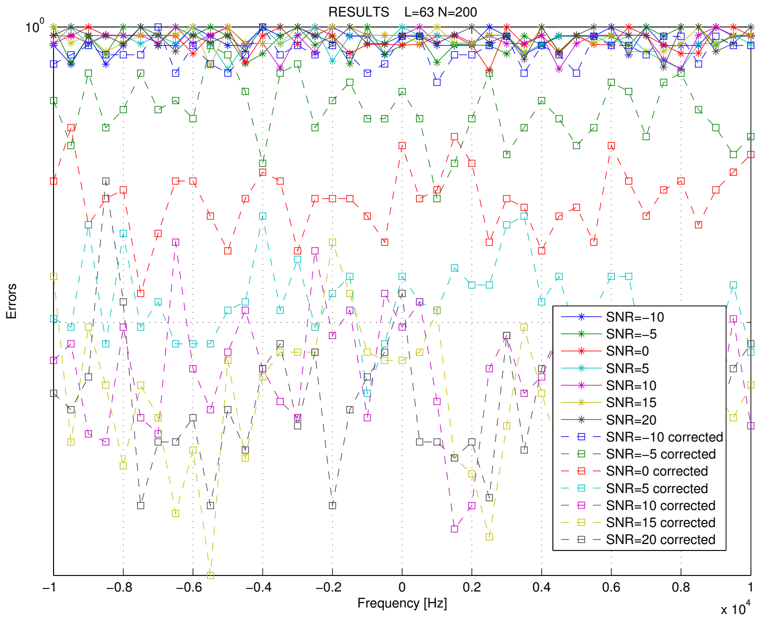

Figure 6.2: $ZC_{N_{ZC}=63}$ Sequence $N = 200$ $L_{coh} = 7$ and frequency shift

is better with a $Lcoh = 1$ in case of a total length frame of $N = 200$.

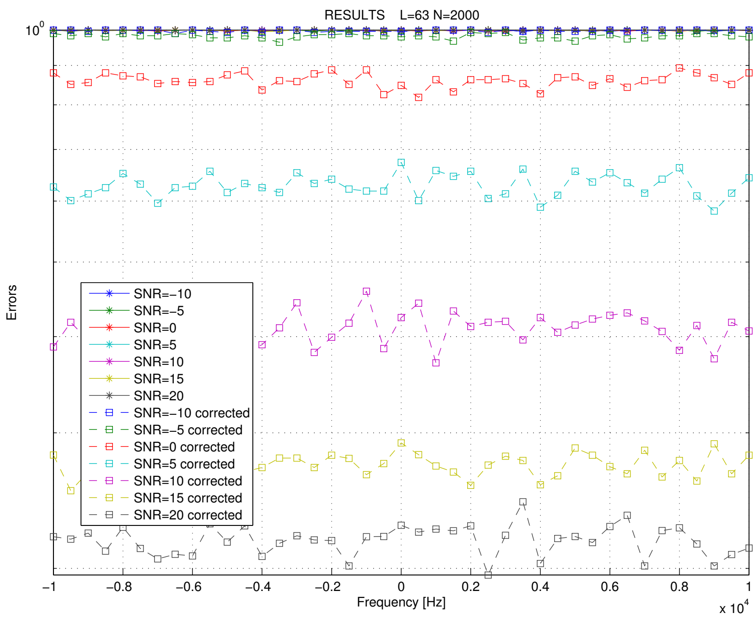

Figure 6.3: $ZC_{N_{ZC}=63}$ Sequence $N = 2000$ $L_{coh} = 1$ and frequency shift

As we can see in Figure 6.3, as expected, the results are worse than Figure 6.1, but we still need the correction term, to minimize the errors, and a good SNR.
These results could be better if we use the Secondary Synchronization Signal again.

7 Conclusion

We have the problem of data recovery in a communications system, we know when we send the information, but not when it reaches the destination, the message suffers noise interference introduced by the communication channel, phase impairment due to reflections and Doppler effect, so we need an algorithm which can detects correctly the data, and then start the demodulation of the signal, in a modern communications system.
We have focused in Massey Algorithm, tested in real sequences, applied larger SNRs and total length frame, then we improved the algorithm for complex sequences and used the Zadoff-Chu sequences as example. We study the time delay and noise impairment, and later, we added the phase impairment and the Doppler effect.
In the best case of correction of phase shift and Doppler effect, we observed a gain between the simple cross correlation term and the correction term so we can use a receiver with bad sensitivity for example 0 dB instead of a one with good sensitivity, for example 30 dB, because the second one is more expensive, but we need the correction term always to decrease the error ratio. The total length frame is also a point of view, if we are going to use large frames, it is good use large Synchronization words too, and backwards.
Only considering the phase shift impairment, we saw that having a high Signal-to-Noise Ratio, we can detect with a very high probability correctly the starting position of our frame.
In the Doppler effect, we saw a rate of error we did not have in the other cases, so with a good sensitivity and using always our correction term we are going to need more methods, like Secondary Synchronization Signal, to decrease this rate.
If we have a best case of correction of the Doppler effect, we can affirm that using a receiver with a good Signal-to-Noise Ratio is enough to detect correctly the beginning of the frame and start the data recovery in further processes.

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