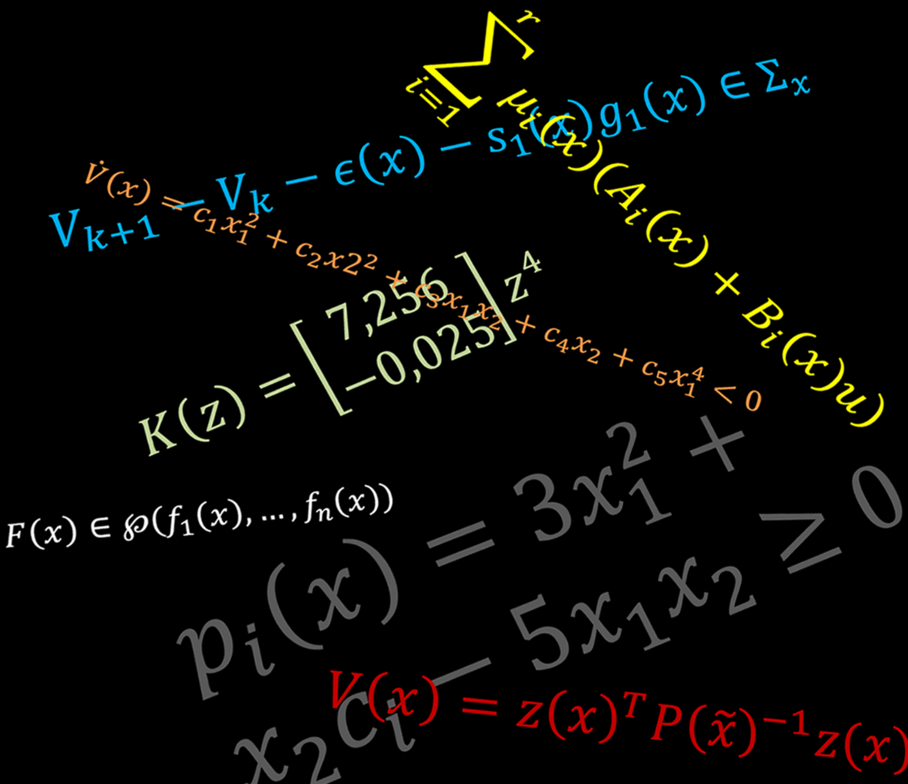
Contributions to Fuzzy Polynomial Techniques for Stability Analysis and Control



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PH.D. IN CONTROL, ROBOTICS AND INDUSTRIAL COMPUTING

PROGRAMA DE DOCTORADO EN AUTOMÁTICA, ROBÓTICA E INFORMÁTICA INDUSTRIAL

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Abstract

The present thesis employs fuzzy-polynomial control techniques in order to improve the stability analysis and control of nonlinear systems. Initially, it reviews the techniques in the field of Takagi-Sugeno fuzzy systems, as well as the more relevant results about polynomial and fuzzy polynomial systems. The basic framework uses fuzzy polynomial models by Taylor series and sum-of-squares techniques (semidefinite programming) in order to obtain stability guarantees.

The contributions of the thesis are:

- Improved domain of attraction estimation of nonlinear systems for both continuous-time and discrete-time cases. An iterative methodology based on invariant-set results is presented for obtaining polynomial boundaries of such domain of attraction.

- Extension of the above problem to the case with bounded persistent disturbances acting on a nonlinear system. Different characterizations of inescapable sets with polynomial boundaries are determined.

- State estimation: extension of the previous results in literature to the case of fuzzy observers with polynomial gains, guaranteeing stability of the estimation error and inescapability in a subset of the zone where the model is valid.

- Proposal of a polynomial Lyapunov function with discrete delay in order to improve some polynomial control designs from literature. Preliminary extension to the fuzzy polynomial case.

The last chapters present a preliminary experimental work in order to check and validate the theoretical results on real platforms.

Resumen

La presente tesis emplea técnicas de control polinomial borroso para mejorar el análisis de estabilidad y control de sistemas no lineales. Inicialmente revisa las técnicas existentes en el ámbito de sistemas borrosos Takagi-Sugeno, así como los resultados más relevantes de los sistemas polinomiales y borrosos polinomiales. El marco base del trabajo utiliza modelos borrosos polinomiales por serie de Taylor y técnicas de suma de cuadrados (programación semidefinida) para obtener garantías de estabilidad.

Las aportaciones de la tesis son:

- Estimación mejorada del dominio de atracción de sistemas no lineales, tanto para el caso continuo como discreto. Se presenta una metodología iterativa para la obtención de fronteras polinomiales de dicho dominio de atracción basada en resultados sobre conjuntos invariantes.

- Extensión del problema anterior al caso de perturbaciones acotadas persistentes. Se determinan distintas caracterizaciones de conjuntos inescapables con frontera polinomial.

- Estimación del estado: extensión de los resultados de literatura previa al caso de observadores borrosos con ganancias polinomiales, garantizando estabilidad del error de estimación e inescapabilidad de un subconjunto de la zona donde el modelo es válido.

- Propuesta de una función de Lyapunov polinomial con retardo discreto para mejorar determinados diseños de controladores polinomiales en literatura. Extensión preliminar al caso borroso polinomial.

Los últimos capítulos presentan trabajo experimental preliminar para poder probar y validar los resultados teóricos en plataformas reales.

Resum

La present tesi empra tècniques de control polinomial borrós per a millorar l'anàlisi d'estabilitat i control de sistemes no lineals. Inicialment revisa les tècniques existents en l'àmbit de sistemes borrosos Takagi-Sugeno, així com els resultats més rellevants dels sistemes polinomials i borrosos polinomials. El marc base del treball utilitza models borrosos polinomials per sèrie de Taylor i tècniques de suma de quadrats (programació semidefinida) per a obtindre garanties d'estabilitat.

Les aportacions de la tesi son:

- Estimació millorada del domini d'atracció de sistemes no lineals, tant per al cas continu com discret. Es presenta una metodologia iterativa per a l'obtenció de fronteres polinomials d'aquest domini d'atracció basada en resultats sobre conjunts invariants.

- Extensió del problema anterior al cas de pertorbacions acotades persistents. Es determinen distintes caracteritzacions de conjunts inescapables amb frontera polinomial.

- Estimació de l'estat: extensió dels resultats de literatura prèvia al cas d'observadors borrosos amb ganancies polinomials, garantint estabilitat de l'error d'estimació i inescapabilitat d'un subconjunt de la zona on el model es vàlid.

- Proposta de una funció de Lyapunov polinomial amb retard discret per a millorar determinats dissenys de controladors polinomials en literatura. Extensió preliminar al cas borrós polinomial.

Els últims capítols presenten treball experimental preliminar per a poder provar i validar els resultats teòrics en plataformes reals.

Contents

Chapter 1

Introduction

"Wheel was a great idea,
relativity was a great idea.
This one is an idea,
and a pretty loose one I should say."
Dr. Sheldon Cooper

Automatic control is present in industrial processes since the 1940s. In this framework, "automatic" means "without the need of any operator intervention".

Traditionally, early model-based design of automatic control systems used input-output models and linear control techniques, such as PID controllers.

Those techniques would be enough if all processes in the real world were linear. Unfortunately, real processes are always of high order (or even infinite) and have some non-linearities, which, in their major part, are not negligible.

In order to solve, or at least to mitigate this problem, different methodologies have been proposed with the objective of effectively controlling nonlinear systems. From the Jacobian linearization to gain scheduling (based on the linearization on many points), to fuzzy control (say, by exact sector-nonlinearity modelling), to pure nonlinear control (backstepping, adaptive, etc.), there is a wide choice of tools and methodologies to select them when faced with a control problem.

Using pure nonlinear control techniques may ensure, theoretically, that the system is going to stable in all possible state range of the process. However, this way may be complex to analyze and designing a controller might be even impossible. Therefore, excluding simple nonlinear cases in which designing a controller is not so difficult, the option is usually restricting the problem to modelling the nonlinear system around a desired operation point and applying linear techniques.

1.1 Motivation and background

When the designer tries to address a control problem, many questions related to modelling accuracy, controller complexity, kind of design approach, etc, arise in mind and, therefore, many choices have to be made.

1.1.1 Linear VS nonlinear

A nonlinear system can be any system in which the state dynamics is governed by a set of nonlinear functions (polynomials, powers, exponentials, logarithmic, sinusoidal, etc). Nonlinear systems exhibit complex behaviours:

- Several equilibrium points (there is only one in linear systems).
- Limit cycles, bifurcations and chaos.
- Finite escape time: the unstable system state can go to infinity in finite time.

As a result, global stability analysis and controller design for those kind of systems may be very complicated, in a general case.

By contrast, a linear dynamical system, as it is well-known, is a system whose dynamical equations are linear, i.e, the state variables and inputs only appear with degree one and multiplied by (possibly time-dependent) coefficients in the state-evolution equations. The linear time-invariant (LTI, constant coefficients) and linear time-varying (LTV, time-dependent coefficients) systems are included in this group. There exist a well-defined theory and powerful tools for analyzing and controlling such systems, particularly the LTI ones.

Usually, LTV systems are understood as those linear systems in which the time variation of the coefficients is known beforehand. On the other hand, a so-called linear parameter-varying (LPV) system is a family of linear plants which are indexed by a “scheduling parameter”; they differ from LTV systems in that the time variation (i.e., the scheduling parameter) is unknown a priori but it can be measured upon operation of the system (or, at least, bounded somehow).

Classically, LPV systems are not considered as linear systems due to the uncertainty about the future values of the scheduling parameters (which might be state-dependent), but they are "close" to them because its family of dynamical equations are still linear, so some tools derived from linear systems can be applied to them (perhaps conservatively).

In this wide framework, the so-called fuzzy systems provide an intermediate option for analyzing and controlling nonlinear systems. Fuzzy systems are a time-varying convex combination of various low-complexity vertex models. In particular, the set of fuzzy systems formed by linear vertex models are a class of LPV systems denoted as Takagi-Sugeno (TS) fuzzy systems. Moreover, those TS fuzzy systems in which one or more of the scheduling parameters are the states of the system, are included in the so-called class of "quasi-LPV" systems. Suitable definitions of all the above concepts appear on Chapter 2.

Recently, extensions of some linear analysis and control tools have been developed for polynomial dynamic systems. Hence, some of these extensions can apply to fuzzy systems with polynomial vertex models, to be denoted as "fuzzy polynomial" systems, to be discussed on Chapter 3.

One key advantage of the fuzzy approach to nonlinear control is the fact that there exists a systematic methodology to model a smooth nonlinear system as a fuzzy one (sector-nonlinearity approach) or as a polynomial one (Taylor series approach, being the sector-nonlinearity a particular case of it).

1.1.2 Control problems in nonlinear systems

Control has the mission of achieving a desired dynamic behavior on a system. In order to do that, the control structure is in charge of transforming the original system dynamics.

Following Khalil (2002), nonlinear control techniques can be grouped in two broad approaches:

1. Feedback linearization (input-output, input to state or exact): Tries to introduce an auxiliar nonlinear feedback in order to make the resultant system be linear from the control action point of view.

2. Based on Lyapunov stability theory (backstepping, sliding mode, adaptive control, Lyapunov redesign).

In order to deal with the control problem in a easier way, the set of fuzzy-modelling methodologies, which are based on expressing the original nonlinear system as a convex combination of various low-complexity vertex models, is an interesting option. Those methodologies have the advantage of extending the use of linear tools, at the price of conservativeness.

In this framework, knowledge of the achievable stable operating region is very important: this determines the safe operating zone. If the system goes out of it, perhaps it cannot go back to the desired operating point. This involves determining the *domain of attraction* (DA) of the nonlinear system around the desired equilibrium/operating point. Nevertheless, exact determination of such region is only possible in very simple cases. Usually, a computation of an inner "estimate" of such DA is the only possible, meaning the computation of a (possible small) subset of the DA.

1.2 Scope and objectives

At sight of the presented control problems, the motivation of this research work is using fuzzy model-based techniques in order to improve (meaning by this to reduce conservativeness holding a tractable complexity) analysis and control of nonlinear systems. Fuzzy techniques may provide useful advantages over pure nonlinear ones (using linear tools, well-defined methodologies, etc) and over pure linear ones (possibility of giving guarantee of stability in a non-infinitesimal region, etc).

The basic framework in this work is using fuzzy-polynomial techniques and the available "sum-of-squares" semidefinite-programming tools in order to search for theoretical stability guarantees.

In particular, this thesis is focused in contributing to the following problems:

- Stability analysis: domain of attraction estimation. The local-stability analysis can be addressed in a more promising way by fuzzy-polynomial techniques, which provide more degrees of freedom in order to fit the actual nonlinear dynamics, i.e., in sense of obtaining proven estimates which are closer to the real DA of the nonlinear system.

- Stability analysis under disturbances: analyzing the effect of bounded disturbances over a system and, based on the developments for DA estimation, computing reachable sets via inescapable-set considerations.

- State estimation: designing fuzzy observers with polynomial gains, including information about local regions on the state and the estimation error. Moreover, guaranteeing inescapable regions and/or performance indexes in systems with presence of disturbances.

- Control design: reducing existent conservativeness in fuzzy-polynomial control-synthesis literature for nonlinear systems by using a more general class of Lyapunov functions and relaxing design conditions with local information about modelling regions and input constraints.

In the last stages of the work, modelling and experimental data collection from real plants has been undertaken. The state estimation methodologies have been validated in practice, and there is ongoing work on testing the other theoretical results from this thesis on these platforms.

1.3 Structure of the thesis

The manuscript is organized in three blocks:

Part I summarizes the most relevant results existent in literature which are related to the objective of this thesis. In this way, next chapter summarizes the main results within the fuzzy TS framework. Similarly is done in Chapter 3, with the fuzzy polynomial literature.

Contributions are presented in Part II. First, in Chapter 4, new developments in local domain of attraction estimation are presented in both continuous-time and discrete-time cases. Fuzzy-polynomial models, polynomial Lyapunov functions and invariant/inescapable set considerations are used as the basis over which existent results in literature are improved.

The extension to the stability analysis in which disturbances are present is also addressed in Chapter 5, discussing computation of the so-called *inescapable sets*. In addition, classical control designs like $\mathcal{H}_\infty$ for continuous-time nonlinear systems are analyzed in presence of nonvanishing disturbances and a design methodology is proposed in order to ensure the validity of the fuzzy polynomial models by inescapable-set considerations.

Chapter 6 proposes a state-observer design for nonlinear systems in presence of disturbances. The bases are fuzzy polynomial modelling, sum of squares tools and inescapable-set considerations. Two approaches are considered: first, with vanishing disturbances and, second, with nonvanishing disturbances plus measurement noise.

Chapter 7 presents a novel control design for discrete-time polynomial systems which propose the use of Lyapunov functions and controller gains depending on present and past states. Such increase of complexity is proven to overcome existent results when information about input saturation is available. Then, this idea is also extended preliminarily to a more general class of nonlinear systems by using fuzzy polynomial models.

Part III presents two experimental applications developed in the last stages of the research work. The applications consider observer design for two prototypes: a three degree-of-freedom fixed quadrotor (Chapter 8) and a turbocharged internal combustion spark-ignition engine (Chapter 9).

The thesis ends with a summary of the main conclusions extracted from the research work and some ideas about interesting problems for future work. In addition, some appendices summarize well-known results and tools in semidefinite programing (linear matrix inequalities) as well as those in sum-of-squares polynomials, which are widely used throughout this thesis.

1.4 Publications

The research work performed within the framework of this thesis has led to several publications, listed below.

Book chapters

PITARCH, J. L., SALA, A. and ARIÑO, C. Polynomial fuzzy systems: Stability and control. In *Fuzzy Modeling and Control: Theory and Applications* (edited by F. Matía, G. N. Marichal and E. Jiménez), vol. 9 of *Atlantis Computational Intelligence Systems*, pages 95–115. Atlantis Press, 2014a. ISBN 978-94-6239-081-2

Referred journal papers

PITARCH, J. L., SALA, A. and ARIÑO, C. V. Closed-form estimates of the domain of attraction for nonlinear systems via fuzzy polynomial models. *IEEE Transactions on Cybernetics*, vol. 44(4), pages 526–538, 2014b

PITARCH, J. L. and SALA, A. Multicriteria fuzzy-polynomial observer design for a 3 DoF nonlinear electromechanical platform. *Engineering Applications of Artificial Intelligence*, vol. 30, pages 96–106, 2014

PITARCH, J. L., SALA, A., LAUBER, J. and GUERRA, T. M. Control synthesis for polynomial discrete-time systems under input constraints via delayed-state lyapunov functions. *International Journal of Systems Science*, vol. 47(5), pages 1176–1184, 2016

PITARCH, J. L., SALA, A., ARIÑO, C. V. and BEDATE, F. Estimación del dominio de atracción de sistemas no lineales mediante modelos borrosos polinomiales. *Revista Iberoamericana de Automática e Informática Industrial RIAI*, vol. 9(2), pages 152 – 161, 2012b

PITARCH, J. L., SALA, A. and ARIÑO, C. V. Estabilidad de sistemas Takagi-Sugeno bajo perturbaciones persistentes: Estimación de conjuntos inescapables. *Revista Iberoamericana de Automática e Informática industrial RIAI*, vol. 12(4), pages 457–466, 2015

SALA, A. and PITARCH, J. L. Optimisation of transient and ultimate inescapable sets with polynomial boundaries for nonlinear systems. *Automatica*, vol. 73, pages 82 – 87, 2016

Conference papers

SALA, A., PITARCH, J. L., BERNAL, M., JAADARI, A. and GUERRA, T. M. Fuzzy polynomial observers. *Proc. of the 18^{th} IFAC World Congress, Milano, Italy*, pages 12772–12776, 2011

PITARCH, J. L., ARIÑO, C. V., BEDATE, F. and SALA, A. Local fuzzy modeling: Maximising the basin of attraction. In *2010 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, pages 1–7. Barcelona, Spain, 2010. ISSN 1098-7584

PITARCH, J. L. and SALA, A. Discrete fuzzy polynomial observers. In *20th Mediterranean Conference on Control Automation (MED)*, pages 819 –823. Barcelona, Spain, 2012

PITARCH, J. L., ARIÑO, C. V. and SALA, A. Estimating domains of attraction of fuzzy polynomial systems. In *Proceedings of the 7th Conf. of the European Society for Fuzzy Logic and Technology (EUSFLAT-LFA)*, Advances in Intelligent Systems Research, pages 680–685. Atlantis Press, Aix-les-Bains, France, 2011. ISBN 9789078677000

PITARCH, J. L., SALA, A., BEDATE, F. and ARIÑO, C. V. Inescapable-set estimation for nonlinear systems with non-vanishing disturbances. In *3rd IFAC Inter. Conf. on Intelligent Control and Automation Science (ICONS)*, pages 457–462. Chengdu, China, 2013

PITARCH, J. L., SALA, A. and ARIÑO, C. V. Spherical domain of attraction estimation for nonlinear systems. In *1st IFAC Conf. on Embedded Systems, Comp. Int. and Tel. in Control (CESCIT)*, pages 108–114. Wüzburg, Germany, 2012a

BERNAL, M., SOTO-COTA, A., CORTEZ, J., PITARCH, J. L. and JAADARI, A. Local non-quadratic $\mathcal{H}_\infty$ control for continuous-time Takagi-Sugeno models. In *2011 IEEE International Conference on Fuzzy Systems (FUZZ)*, pages 1615–1620. Taipei, Taiwan, 2011b. ISSN 1098-7584

PITARCH, J. L. and SALA, A. Síntesis de observadores y controladores para sistemas no lineales mediante técnicas borrosas polinomiales. In *IX Simposio CEA de control inteligente*, pages 87–92. Tenerife, Spain, 2013b. ISBN 978 84 695 8152 0

PITARCH, J. L. and SALA, A. Estabilidad local y control de sistemas no lineales mediante técnicas borrosas polinomiales. In *XI Simposio CEA de Ingeniería de Control*, pages 74–82. Valencia, Spain, 2013a. ISBN 978 84 695 7298 6

Part I

State of the Art

Chapter 2

Takagi-Sugeno Fuzzy Systems

Like a plucked and skinny goose,
I asked myself with unsteady voice
if of all the stuff I read,
I'll ever make the slightest use.

James Clerk Maxwell

ABSTRACT: This chapter gives a brief outline about the existent literature on the use of fuzzy Takagi-Sugeno techniques in order to deal with analysis and control of nonlinear systems. Sector nonlinearity approach and linear matrix inequalities (LMI) are the basis of the results summarized on this chapter. Methodologies for analysis/design are provided in both, continuous-time and discrete-time cases. The cases of disturbance rejection and performance evaluation are also addressed.

Fuzzy control originated more than 40 years ago (Zadeh, 1965), but nowadays it is a mature control methodology with many alternatives based on LMI's. They allow to analyze nonlinear systems and design stable controllers with a desired performance in terms of decay rate or disturbance attenuation (Tanaka and Wang, 2001; Liu and Zhang, 2003; Sala, Guerra and Babuška, 2005; Feng, 2006). Furthermore, a large class of nonlinear systems can be modelled as fuzzy Takagi-Sugeno (TS) systems via the sector nonlinearity approach (later presented) or via identification paradigms (Babuška, 1998), which are out of the scope of this thesis.

Hence, fuzzy control is nowadays a viable option for control of many nonlinear systems in practice (Guelton, Delprat and Guerra, 2008; Lee, Park, Joo, Lin and Ham, 2010; Mohammad, Guerra, Grobois and Hecquet, 2011), although there are some sources of conservativeness with respect to an *ideal* nonlinear controller (Sala, 2009). For instance, most processes are subject to constraints on input and/or state in practice (actuator saturation or rate variation are the most common ones). However controllers designed with classical TS approaches actually never reach those limits, except perhaps at the initial instant or under strong disturbances.

Those classical approaches, joint with further works which propose modifications for reducing conservatism, are outlined on the following sections.

This chapter organizes as follows: next section presents the Takagi-Sugeno fuzzy modelling methodology by the well-known sector nonlinearity approach, Section 2.2 reviews the results of Lyapunov-based stability and its application to fuzzy TS systems, Section 2.3 deals with the most employed controller and observer design methodologies in such kind of systems, in addition Section 2.4 discusses the most used performance indexes and, finally, Sections 2.5 and 2.6 provide a discussion about the existent problems/limitations and some conclusions of the TS systems approach respectively.

2.1 Takagi-Sugeno fuzzy modelling

This thesis is focused on fuzzy models of nonlinear systems for which a mathematical model is available, for instance by first principles. In those systems, there exist some nonlinear terms which may be complicated to deal with from a control point of view. In this framework, Takagi-Sugeno (TS) fuzzy modelling (Takagi and Sugeno, 1985) gives a fairly simple and well-defined methodology to deal with a large class of such nonlinear systems. Let us first recall important definitions of dynamical systems from literature.

Definition 2.1 (Chen (1998)). *A **linear time-varying** system (LTV) is a dynamical system whose evolution is defined through state-space equations of the form*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ denotes the system state, $u(t) \in \mathbb{R}^b$ denotes the system inputs, $y(t) \in \mathbb{R}^c$ denotes the system outputs, and where $\dot{x}(t) = \frac{dx}{dt}(t)$ denotes the derivative of x with respect to time.

In those systems the time variations are known beforehand. A particular case of LTV systems are the *linear time-invariant* (LTI) ones, in which A, B, C are constant matrices.

Definition 2.2 (Shamma (1988))**.** *A **linear parameter-varying** (LPV) system is defined as a linear system whose coefficients depend on an exogenous time-varying parameter $\theta(t)$:*

$$\begin{aligned} \dot{x}(t) &= A(\theta(t))x(t) + B(\theta(t))u(t) \\ y(t) &= C(\theta(t))x(t) \end{aligned} \tag{2.2}$$

*The "scheduling" parameters θ take values in some prescribed set $\theta(t) \in \Theta$. If the scheduling parameters are actually endogenous to the state dynamics (e.g. $\theta \equiv x$) the system is named **quasi-LPV**.*

Now consider a nonlinear system represented by

$$\begin{aligned} &\dot{x} = f(x) + b(x, u) \\ &y = c(x) \end{aligned} \tag{2.3}$$

where $f(x)$, $c(x)$ and $b(x, u)$ can be any set of linear and nonlinear functions of the states and/or inputs respectively.

The objective of the fuzzy TS modelling is to express the existent nonlinearities (usually in a region of interest $\Omega(x, u)$) between a convex combination of LTI vertex models. In this way, the nonlinearity is always proven to lie in some point between the linear systems. Each rule associates a LTI model at the consequent part of a weighting function obtained from the premises (supposedly known variables). The R_i fuzzy rules of the TS model are of the form

$$\begin{array}{rl} IF & z_1(t) \ is \ M_{i1} \ and...and \ z_p(t) \ is \ M_{ip} \\ THEN & \begin{cases} \dot{x}(t) = A_i x(t) + B_i u(t) \\ y(t) = C_i x(t) \end{cases} \qquad i = 1, \ldots, r \end{array} \tag{2.4}$$

where M_{ij} is the fuzzy set, r is the number of fuzzy rules and $z_1(t)...z_p(t)$ are the premise variables which can be function of the state, external inputs and/or time.

Definition 2.3 (Tanaka and Wang (2001))**.** *A **Takagi-Sugeno fuzzy** model of a* *nonlinear system* (2.3) *can be represented in the form*

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} \mu_i(z(t))(A_i x(t) + B_i u(t)) \\ y(t) &= \sum_{i=1}^{r} \mu_i(z(t)) C_i x(t) \end{aligned} \tag{2.5}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^b$ are the inputs, μ_i are the ***membership functions*** *being its arguments z the* ***premise variables*** *of the fuzzy rules, and the output $y \in \mathbb{R}^c$ is a linear combination of the rule consequents. The membership functions μ_i are designed by taking into account the original nonlinearities and they should verify the convex sum property:*

$$\sum_{i=1}^{r} \mu_i(z) = 1, \quad \mu_i(z) \geq 0 \, \forall z, \; i : 1, \ldots, r \tag{2.6}$$

Further details and the discrete-time case can be consulted in Tanaka and Wang (2001, Chap. 2).

In this way, by taking the membership functions $\mu(z(t))$ as the scheduling variables $\theta(t)$, i.e., $A(\theta) = \sum_{i=1}^{r} \mu_i(z) A_i$, $B(\theta) = \sum_{i=1}^{r} \mu_i(z) B_i$ and $C(\theta) = \sum_{i=1}^{r} \mu_i(z) C_i$ in (2.2), the TS fuzzy systems can be interpreted as a quasi-LPV form (Guerra, Kruszewski and Lauber, 2009).

In the following, this chapter focus on the form (2.5) of TS models. Two methods are generally possible to obtain (2.5) from (2.3). The first one is based on the linearization around several set points of the nonlinear plant. In this case, the resulting model is only an approximation and the $\mu_i(z(t))$ are chosen as triangular or sigmoid functions (Tanaka and Wang, 2001). The second one is the sector nonlinearity approach which allows representing *exactly* the nonlinear model (2.3) in a compact set of the state variables. This thesis focus on that second methodology, detailed below.

2.1.1 Sector Nonlinearity approach

Defining a *global* fuzzy model for a nonlinear system is usually a difficult task (sometimes impossible, i.e., if the partial derivatives are unbounded for instance). Moreover, even if possible, may be very conservative. Therefore, the usual way to proceed is making a *local* fuzzy model which is valid within an expected region Ω of the state space.

In order to perform that local modelling, the well-known *Sector Nonlinearity* approach is employed. It appeared first in Kawamoto, Tada, Ishigame and Taniguchi (1992) and the idea is that, given a nonlinearity $f(x)$ which fulfills $f(0) = 0$, defining r linear systems $A_i x(t)$ within an interval of interest $\Omega = \{x \in -d \leq x \leq d\}$. This defines a sector where the nonlinearity will always lie in (see Figure 2.1). This technique guarantees a fuzzy model which represents the original nonlinear system in an *exact* way, inside the modelling region Ω of course.

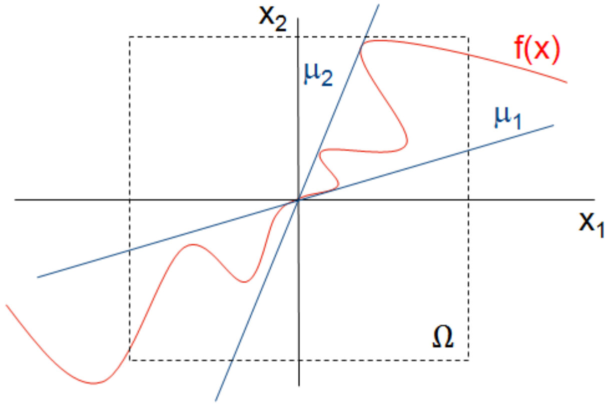


Figure 2.1: Sector nonlinearity modelling.

In order to obtain the membership functions μ_i, the maximum and minimum values of the nonlinearities present in $f(x)$ are computed for the modelling region Ω. Then, jointly with the convex sum property (2.6), $f(x)$ can be represented by the memberships and that max/min values. Examples of modelling can be found in Tanaka and Wang (2001, Chap. 2). Details are omitted for brevity as the issue has been deeply explored in prior literature and, furthermore, this will be a particular case of polynomial fuzzy modelling to be addressed in next chapter.

2.2 Stability analysis

The stability analysis for TS fuzzy systems is based on the Lyapunov theory (Kalman and Bertram, 1960) and Linear Matrix Inequalities (LMI) developments (Boyd, Ghaoui, Feron and Balakrishnan, 1994).

Lyapunov stability theory analyzes the stability of a system respect to an equilibrium point. Variations to it allow analyzing BIBO stability, as later discussed.

2.2.1 Lyapunov stability

The following definition introduces several types of stability.

Definition 2.4. *Consider the general nonlinear autonomous dynamical system $\dot{x}(t) = f(x(t))$ and let x^e be an equilibrium point for it.*

1. *The equilibrium point x^e is* ***Lyapunov stable*** *if, for every neighborhood U of x^e, there is a region $V \subset U$, $x^e \in V$ such that every solution $x(t)$ starting in V ($x(0) \in V$) remains in U for all $t \geq 0$. See Figure 2.2a. Note that $x(t)$ does not need to approach x^e.*

2. *The equilibrium point is* locally ***asymptotically stable*** *if it is Lyapunov stable and, additionally, V can be chosen such that $\|x(t) - x^e\| \to 0$ when $t \to \infty$ for all $x(0) \in V$. See Figure 2.2b. Moreover, if for all $x(0) \in \mathbb{R}^n$, $\lim_{t\to\infty} x(t) = x^e$, the equilibrium point is* globally *asymptotically stable.*

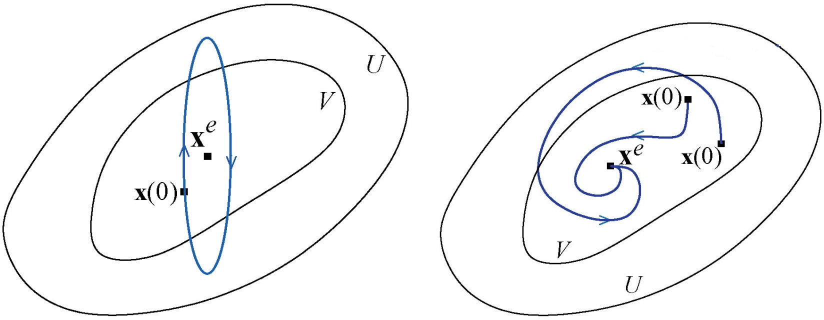
3. *The equilibrium point x^e is* locally ***exponentially stable*** *if there is a neighborhood W of x^e and positive constants β, α such that $\|x(t) - x^e\| < \beta \|x(0) - x^e\| e^{-\alpha t}$ for all $t \geq 0$ and $x(0) \in W$. Moreover, if $W = \mathbb{R}^n$, the equilibrium point is* globally *exponentially stable.*

4. *Finally, an equilibrium point x^e is* ***unstable*** *if it is not Lyapunov stable.*

Exponentially stable equilibria are also asymptotically stable, and hence Lyapunov stable. For more detailed explanation the reader is referred to the works Haddad and Chellaboina (2008, Chap. 3,4) and Khalil (2002, Chap. 3,4).

Lyapunov functions

Aleksandr Mikhailovich Lyapunov, in his original Ph.D. thesis (1892), proposed two methods for demonstrating stability. The second method is almost universally used nowadays. It makes use of a *Lyapunov function* $V(x)$ which has an analogy to the energy function in mechanical systems.



(a) Stable equilibrium (b) Asymptotically stable equilibrium

Figure 2.2: Lyapunov stability

Theorem 2.1 (Lyapunov (1992))**.** *Consider a system having an equilibrium point at* $x = 0$ *and let* $\Omega \subset \mathbb{R}^n$ *be a domain containing* $x = 0$*. If a continuously differentiable candidate function* $V(x) : \Omega \to \mathbb{R}$ *is found fulfilling the following conditions*

$$\begin{aligned} V(0) &= 0 \\ V(x) &> 0 \quad \forall \quad x \in \Omega,\ x \neq 0 \\ \dot{V}(x) = \frac{dV(x)}{dt} &< 0 \quad \forall \quad x \in \Omega,\ x \neq 0 \end{aligned} \tag{2.7}$$

then the equilibrium point is asymptotically stable in the sense of Lyapunov. If $\Omega \equiv \mathbb{R}^n$ *and the additional condition for* $V(x)$ *being radially unbounded is required, global asymptotic stability is proven.*

This ensures stability in the Lyapunov sense because the function $V(x)$ always decreases until reaching the equilibrium point. The discrete-time case is almost identical to that for continuous-time systems, replacing the condition $\dot{V}(x) < 0$ by the one-step discrete increment $\Delta V = V_{k+1}(x) - V_k(x) < 0$.

Let us denote the trajectory of a certain nonlinear dynamic system $\dot{x} = f(x)$ as $x(t) = \psi(t, x_0)$, where x_0 are the initial conditions at $t = 0$. Theorem 2.1 implies that, when a trajectory crosses a Lyapunov level surface $V(x) = c$, it moves inside the set $\{x \in \mathbb{R}^n : V(x) \leq c\}$ and never can come out again. Due to condition (2.7), the trajectory moves from one Lyapunov surface to an inner one with a smaller c. As c decreases, the Lyapunov surface $V(x) = c$ shrinks to the origin. Therefore, computing Lyapunov surfaces are very related to the domain (or region) of attraction estimation problem (Gordillo, 2009).

Next, a definition of the domain of attraction of the equilibrium point $x(t) = 0$ of the nonlinear system is given.

Definition 2.5. *The **domain of attraction** of a system* (2.3)*, denoted as $\mathcal{D}$, is the set of points belonging to the state space whose trajectory $x(t) = \psi(t, x_0)$ ends in the asymptotically stable equilibrium point $x = 0$ (Khalil, 2002, Chap. 4).*

$$\mathcal{D} = \left\{ x \in \mathbb{R}^n | \psi(t, x) \in \Omega \forall t \geq 0, \lim_{t \to \infty} \psi(t, x) = 0 \right\} \tag{2.8}$$

Introducing the notation $V_c = \{x : V(x) < c\}$ to denote the level sets of $V(x)$, we have:

Lemma 2.1 (Khalil (2002))**.** *If $V(x) \geq 0$ and $\dot{V}(x) < 0$ in Ω, then $V_c \subset \Omega$ implies $V_c \subset \mathcal{D}$.*

Then, in many literature references, the estimated DA is given by V_{c^*} where c^* is the largest c such that $V_c \subset \Omega$.

Conditions on the Lyapunov function $V(x)$ in Theorem 2.1 can be relaxed. In particular, the strict negative-definiteness condition on the Lyapunov function derivative can be relaxed while ensuring asymptotic stability.

Definition 2.6. *A set $\mathcal{N}$ is said to be a **positively-invariant set** with respect to system* (2.3) *if*

$$x(0) \in \mathcal{N} \Rightarrow x(t) \in \mathcal{N}, \quad \forall t \geq 0$$

That is, if a solution belongs to $\mathcal{N}$ at some instant, then it belongs to $\mathcal{N}$ for all future time.

If a $V(x)$, defined on a compact invariant set with respect to the nonlinear system, can be constructed such that its derivative is negative semidefinite and no system trajectories can stay indefinitely at points where $\dot{V}(x)$ vanishes, except at $x = 0$, then the system is asymptotically stable on the equilibrium point $x = 0$. This result follows from the LaSalle invariance principle.

Theorem 2.2 (LaSalle (1960))**.** *Assume that $\Omega \subset \mathbb{R}^n$ is a compact positively invariant set with respect to system* (2.3) *and assume there exists a continuously differentiable function $V : \Omega \to \mathbb{R}$ such that $\dot{V}(x) \leq 0$, $x \in \Omega$. Let $\mathcal{R} = \{x \in \Omega : \dot{V}(x) = 0\}$ and let $\mathcal{N}$ be the largest invariant set contained in $\mathcal{R}$. If $x(0) \in \Omega$, then $x(t) \to \mathcal{N}$ as $t \to \infty$.*

There are plenty of applications of the above ideas in generic nonlinear control (Albea, Gordillo and de Wit, 2013; Freeman and Kokotovic, 2008;

Khalil, 2002). However, the motivation of this thesis is to focus on Takagi-Sugeno systems (or its polynomial enhancements), so the discussion will now be devoted to such class of systems.

2.2.2 Fuzzy systems' stability

A nonlinear system can be analyzed around an equilibrium point $x(0) = 0$ by using its equivalent fuzzy TS model. Given a fuzzy TS model derived from a set of rules (2.4), in the form (2.5), the goal is to analyze its stability when there is no input.

Theorem 2.3 (Tanaka and Wang (2001))**.** *Consider a quadratic candidate Lyapunov function $V(x) = x^T Px$. The equilibrium $x = 0$ of the continuous fuzzy system* (2.5)*, with $u(t) = 0$, is globally asymptotically stable if there exists a matrix P such that*

$$P \succ 0 \tag{2.9}$$

$$A_i^T P + PA_i \prec 0, \quad i : 1, ..., r \tag{2.10}$$

that is, a common P has to exist for all vertex subsystems.

And this is an LMI problem (see Appendix A.1) in which a matrix $P \succ 0$ has to be found fulfilling the second condition for each linear subsystem conforming the fuzzy model. The result for discrete-time systems can be obtained in a similar way by replacing (2.10) by the discrete increment $A_i^T PA_i - P \prec 0$. Details omitted for brevity.

The Lyapunov function $V(x)$ provides a set of initial conditions of the state space which belong to the domain of attraction of the nonlinear system around the equilibrium point $x = 0$. That set of initial conditions is defined by the largest Lyapunov level set of $V(x)$, say $V \leq c$, which is contained in the modelling region Ω, i.e., $\{x \in \mathbb{R}^n : V(x) \leq c\} \subset \Omega$. See Figure 2.3.

If $V(x)$ is defined by a quadratic function as in Theorem 2.3, the provable domain of attraction is an ellipsoid and it is also called by "basin of attraction" in the quadratic TS literature (Cuesta, Gordillo, Aracil and Ollero, 1999; Wang, Tanaka and Griffin, 1996).

Remark 2.1. Proving *global* stability of a fuzzy TS model means proving *local* stability of the original nonlinear system, because the TS model is *only* valid inside the chosen modelling region Ω.

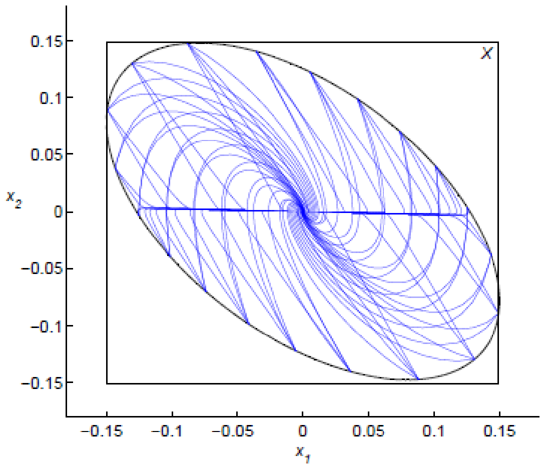


Figure 2.3: Domain of attraction and trajectories of a nonlinear system in the phase plane.

2.3 Control design

In order to control nonlinear systems modelled in TS form (2.5), a control law $u(t)$ must be designed. The most extended methodology for control design of TS systems is the Parallel Distributed Compensator (PDC) and its subsequent modifications, whose ideas and design methodologies are summarized below.

2.3.1 Parallel distributed compensator

In this controller, each control law is designed from the corresponding fuzzy rule of the TS model and shares the same fuzzy sets and memberships with the model (Wang, Tanaka and Griffin, 1995; Li, Niemann, Wang and Tanaka, 1999).

The controller rules are built in the following form, analogous to (2.4):

$$\begin{array}{rll} IF & z_1(t)\ is\ M_{i1}\ and...and\ z_p(t)\ is\ M_{ip} & \\ THEN & u(t) = -K_i x(t), & i = 1, \ldots, r. \end{array} \tag{2.11}$$

Finally the complete controller is expressed as

$$u(t) \;\; = \;\; -\sum_{i=1}^{r} \mu_i(z(t)) K_i x(t), \tag{2.12}$$

so the closed-loop system becomes:

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(z(t))\mu_j(z(t))\,(A_i - B_iK_j)\,x(t) \tag{2.13}$$

The design problem is reduced to finding the gains K_j which fulfill the Lyapunov stability conditions (2.7). The synthesis problem can be solved in a similar way to Theorem 2.3 by taking $A_i - B_iK_j$ instead of A_i and then performing the change of variable $X = P^{-1}$, $M_j = K_jX$ in order to put the problem in convex form. For details see Tanaka and Wang (2001, Chap. 3). In this way, the PDC design reduces to solve an LMI problem from which the values of $P = X^{-1}$ and $K_j = M_jP$ are obtained. However, this is very conservative because double fuzzy-summation terms appear, see below.

In addition, constraints on the inputs/outputs as well as performance guarantees (see Section 2.4) can be included on the design phase. See Tanaka and Wang (2001, Chap. 3) for details on the procedure.

Fuzzy summations

In the above design, as in many fuzzy control problems, a double fuzzy summation appears in the resulting LMI's due to the term $\sum_i^r \sum_j^r \mu_i(z)\mu_j(z)B_iK_j$.

First, in order to obtain LMI's from expressions such as

$$\sum_i^r \sum_j^r \mu_i(x)\mu_j(x)M_{ij} > 0$$

where the memberships μ_i are state dependent (premise variables $z \equiv x$), a conservative step transforms $\mu_i(x)$ into an arbitrary scalar μ_i in the standard simplex $\mu_i \in \Delta$ (state independent). Hence, the original problem has been translated to a co-positivity (Wang, Tanaka and Griffin, 1996) one:

$$\sum_i^r \sum_j^r \mu_i\mu_j M_{ij} > 0 \quad \forall \mu_i \in \Delta$$

In this way, stability can be proved by setting LMI's with the vertex models.

Remark 2.2. Note that due to the above transformation, Theorem 2.3, if feasible, proves stability for the particular nonlinear system and for all the rest which can be represented by the same vertices.

However, checking all j gains for each i condition is useless in the sense of obtaining fuzzy controller gains: they will never be better than a single gain. Therefore, sum relaxations are required to reduce this conservatism. A basic sufficient solution for this fuzzy control problem was carried out in Wang, Tanaka and Griffin (1996). A refinement of that approach was proposed in Tuan, Apkarian, Narikiyo and Yamamoto (2001). Later, those approaches were generalized by the works Sala and Ariño (2007a); Kruszewski, Sala, Guerra and Ariño (2009) on which a progressive asymptotically exact reduction of conservatism is proven by increasing the complexity and/or number of LMI conditions. See the cited references for further details.

2.3.2 Further developments

In the classical PDC design, the Lyapunov function is quadratic and the fuzzy control rules have linear consequents, state feedback in this case. Nevertheless, other kind of controllers can be used, for instance static and dynamic output feedback ones (Bouarar, Guelton and Manamanni, 2013; Tanaka and Wang, 2001, Chap. 12,13). In the work Lam and Leung (2007) a similar idea is presented and it is more focused on finding control laws which ensure some dynamical specifications by iterative LMI algorithms (Appendix A.1.4).

Non-quadratic results with PDC based control laws are also stated by using fuzzy Lyapunov functions

$$V(x) = x^T \left(\sum_{i=1}^{r} \mu_i(z) P_i \right) x$$

assuming a priori bounds on the membership function's derivatives $|\dot{\mu}_i(z)| \leq \phi_i$ (Tanaka, Hori and Wang, 2003; Mozelli, Palhares and Avellar, 2009), or on its partial derivatives (Jaadari, 2013) by expressing

$$\sum_{i=1}^{r} \dot{\mu}_i(z) = \sum_{i=1}^{r} \left(\frac{\partial \mu_i}{\partial z} \right)^T \dot{z} \quad , \quad \left| \frac{\partial \mu_i}{\partial z} \right| \leq \beta_i.$$

Furthermore, the non-PDC control laws introduced in Guerra and Vermeiren (2004) for the discrete-time case, whose ideas are further developed in Jaadari (2013) for the continuous-time one

$$u = \sum_{i=1}^{r} \mu_i(z) K_i \left(\sum_{j=1}^{r} \mu_j(z) P_j \right)^{-1} x,$$

can also be used to improve over classical PDC designs by adding information on the membership derivatives. See details on the cited sources.

In addition, the works presented in the references Sala and Ariño (2007b); Bernal, Guerra and Kruszewski (2009), focus on the problem of conservativeness when assuming state independence in the membership functions, mentioned in the above section, and propose ideas in order to add the membership information to the LMI problem.

2.3.3 Observer design

Most control designs include total or partial state feedback but, unfortunately, not all the states are measurable. For those cases, a good state estimation is required. The problem of observer design for TS systems has been widely addressed on literature (Tanaka and Wang, 2001; Lendek, Guerra, Babuška and De Schutter, 2010). This section summarises the main approaches.

An observer under no disturbances and no modelling error must satisfy

$$x(t) - \hat{x}(t) \overset{t\to\infty}{\rightarrow} 0$$

where $\hat{x}(t)$ are the estimated states by the observer. This condition guarantees that the estimated states will converge to the real ones in stationary state.

An idea for designing a fuzzy observer is to follow a similar structure to the PDC controller:

$$\begin{array}{rl} IF & z_1(t)\ is\ M_{i1}\ and ...and\ z_p(t)\ is\ M_{ip} \\ THEN & \left\{ \begin{array}{l} \dot{\hat{x}}(t) = A_i\hat{x}(t) + B_iu(t) + L_i(y(t) - \hat{y}(t)),\ i = 1, \ldots, r \\ \hat{y}(t) = C_i\hat{x}(t) \end{array} \right. \end{array} \tag{2.14}$$

The fuzzy observer has linear observation laws as consequents in this case.

The observer gains L_i can be obtained again from solving an LMI problem. However, the design procedure varies considerably with the dependence of the premise variables $z(t)$ on the states $x(t)$. Two alternatives can be presented:

1. The premise variables depend on measurable quantities (outputs or measurable inputs):

$$\begin{aligned} \dot{\hat{x}}(t) &= \sum_{i=1}^{r} \mu_i(z(t)) \left[A_i\hat{x}(t) + B_iu(t) + L_i(y(t) - \hat{y}(t))\right] \\ \hat{y}(t) &= \sum_{i=1}^{r} \mu_i(z(t)) C_i\hat{x}(t) \end{aligned} \tag{2.15}$$

2. The observer premises depend on the estimated states:

$$\begin{aligned} \dot{\hat{x}}(t) &= \sum_{i=1}^{r} \mu_i(\hat{z}(t)) \left[A_i \hat{x}(t) + B_i u(t) + L_i (y(t) - \hat{y}(t)) \right] \\ \hat{y}(t) &= \sum_{i=1}^{r} \mu_i(\hat{z}(t)) C_i \hat{x}(t) \end{aligned} \quad (2.16)$$

The design procedure guaranteeing stability for case 1 is much simpler than for case 2. This is because the membership functions of the model and the observer are the same and depend on known variables in every moment. The procedure for case 1, both in continuous-time and discrete-time, is done by solving an LMI problem on the estimation error $e(t) = x(t) - \hat{x}(t)$. See Tanaka and Wang (2001, Cap. 4) for details.

However, for case 2 design becomes much more complicated. This is because the membership functions of the model depend on the "unknown" states and the observer ones on the estimated states. Therefore, both memberships can take different values at same time, i.e., $\mu_i(z) \neq \mu_i(\hat{z})$ because $z \neq \hat{z}$ in general. In this case, the design developments for case 1 are no longer valid.

Some conservative proposals have been presented recently to deal with case 2. For instance, the observer-model mismatch must fulfill a Lipschitz-like bound (Ichalal, Marx, Ragot and Maquin, 2010; Lendek, Guerra, Babuška and De Schutter, 2010)

$$\left\| \sum_{i=1}^{r} (\mu_i(z) - \mu_i(\hat{z})) A_i x \right\| \leq \sigma \left\| e \right\|, \quad (2.17)$$

or bounding its derivatives and expressing it as a TS model too (Ichalal, Marx, Maquin and Ragot, 2012a), or also bounding the derivatives of the memberships $|\dot{\mu}_i(\hat{z})| < \rho_i$ ensuring input-to-state stability (ISS) (Ichalal, Marx, Ragot and Maquin, 2012b), in order to proceed further and set up LMI's.

2.4 Nominal performance and LMI's

In this section, the results on dissipative systems and linear matrix inequalities presented on the above sections are used to characterize a number of relevant performance criteria for dynamical systems.

Consider a dynamical fuzzy TS system in the form (2.5) where $u \in \mathbb{R}^b$ are external inputs (can be control or disturbances). It is assumed throughout this section that the system is asymptotically stable. If not, a controller must be designed to make the closed-loop system fulfill this assumption.

There exist several indicators for nominal performance, for instance the energy in the impulse response of the system, the percentage overshoot or speed of convergence in step responses, minimizing or maximizing the effect of u (depending on the control or disturbance nature of u) on the output y (error indicator), etc. On the following, the most common indicators with LMI approaches existent in literature are summarized.

2.4.1 $\mathcal{H}_2$ performance

Generally, definitions of $\mathcal{H}_2$ for exponentially stable LTV systems have been proposed based on system response to either stationary white noise or a unit impulse.

Stochastical interpretation. The first interpretation of the $\mathcal{H}_2$ norm makes use of stochastics. Assume that the components of the input u are independent zero-mean white noise processes with identity covariance matrix. According to Stoorvogel (1993), the $\mathcal{H}_2$ norm of an exponentially stable LTI system T, with $x(0) = 0$, can be defined by

$$\|T\|_2^2 := \lim_{h\to\infty} \mathbb{E}\left\{\frac{1}{h}\int_0^h y(t)^t y(t)dt\right\} \tag{2.18}$$

where $y(t) \in \mathbb{R}^c$ is the objective output and $\mathbb{E}$ denotes the mathematical expectation.

Then, $\mathcal{H}_2$ performance can be interpreted in LTI systems as the *output variance* when subject to that input (Scherer and Weiland, 2004).

The time-varying version of the $\mathcal{H}_2$ norm, for an LTV system T_{tv}, given by $\dot{x} = A(t)x + B(t)u$, $y = C(t)x$ is defined in Stoorvogel (1993) by:

$$\|T_{tv}\|_2^2 := \lim_{h\to\infty} \sup \mathbb{E}\left\{\frac{1}{h}\int_0^h y(t)^t y(t)dt\right\} = \lim_{h\to\infty} \sup\ Trace\left\{\frac{1}{h}\int_0^h B(t)^T L(t,h)B(t)dt\right\} \tag{2.19}$$

where L is the unique bounded matrix function satisfying the Lyapunov differential equation:

$$\begin{aligned} -\dot{L}(t,h) &= A^T(t)L(t,h) + L(t,h)A(t) + C^T(t)C(t) \\ L(h,h) &= 0 \end{aligned}$$

The "lim sup" guarantees existence of this limit (the integral is bounded). However, it is only a semi-norm, (2.19) can be 0 for nonzero systems (e.g. $B(t) = 0, t > 1000$).

In a general case, let us consider computing a bound for the state variance. Consider a matrix function of the state $V(x(t))$ for which we want to compute its mean, for instance $V(x(t)) = x(t)x^T(t)$, with the goal of obtaining $\mathbb{E}(V(x(t))$. Then, by following Itô's lemma derivations (Itô, 1944) we can state for system $dx = A(z)xdt + B(z)db$, $y = C(z)x$

$$\begin{aligned} dV(x(t)) = (A(z)x(t)x^T(t) + x(t)x^T(t)A(z)^T + B(z)^T B(z))dt \\ + B(z)x(t)db = (A(z)V(x(t)) + V(x(t))A(z)^T \\ + B(z)^T B(z))dt + B(z)x(t)db \end{aligned} \quad (2.20)$$

where functions $A(z) = \sum_{i=1}^{r} \mu_i(z)A_i$, $B(z) = \sum_{i=1}^{r} \mu_i(z)B_i$ and $C(z) = \sum_{i=1}^{r} \mu_i(z)C_i$ are nonlinear functions of the premise variables z and db represents the Brownian motion.

Afterwards, taking variable expectations in (2.20), two possibilities arise:

1. *z is statistically independent to x.* In this case, the expectation $\mathbb{E}(V(x(t)))$ is the solution of the matrix differential equation:

$$\frac{d}{dt}\mathbb{E}(V(x(t))) = A(z)\mathbb{E}(V(x(t))) + \mathbb{E}(V(x(t)))A(z)^T + B(z)^T B(z) \quad (2.21)$$

   Then, if a matrix $P \succ 0$ can be found fulfilling

$$A(z)P + PA(z)^T + B(z)^T B(z) < 0 \quad (2.22)$$

   we have that P is an upper bound for the considered expectation, i.e., $P > \mathbb{E}(V(x(t)))$ (Stoorvogel, 1993; Xie, 2005). Consequently, the output variance leads to

$$\begin{aligned} \mathbb{E}(y^T y) = \mathbb{E}(x^T C(z)^T C(z)x) = \mathbb{E}\,\mathrm{trace}(C(z)xx^T C(z)^T) \\ = \mathrm{trace}(C(z)\mathbb{E}(xx^T)C(z)^T) \le \mathrm{trace}(C(z)PC(z)^T) \end{aligned} \quad (2.23)$$

   which can be cast as an LMI optimization problem. Less conservative results can be obtained by using a parameter-dependent matrix $P(z)$ instead of a constant one. See Xie (2005) for details.

2. *z depends on x.* This is the case of the TS fuzzy models, where $z \equiv x$. In this case, the expectation $\mathbb{E}(dV(x(t)))$ cannot be expressed as a function of $\mathbb{E}(V(x(t)))$ because $\mathbb{E}(A(x)V(x)) \neq A(x)\mathbb{E}(V(x))$.

   Therefore, although similar LMI conditions to LPV systems can be posed, the output-variance meaning of the $\mathcal{H}_2$ norm bound can only be interpreted as it when the membership functions of the TS system remain almost constant during a long time.

Note. In LTI systems, the exact solution for (2.19) or, equivalently, (2.21) can be computed and it is called the *observability/controllability gramian* respectively (Scherer and Weiland, 2004)[Chap. 3]. Therefore, in this case the $\mathcal{H}_2$ norm is *exactly* computing the asymptotic output variance of the system when it is excited by white-noise input signals.

Impulsive behaviour. The second interpretation of the $\mathcal{H}_2$ norm of a system is related to its *impulsive behaviour*, i.e, the interest is only focused in minimizing the impulse responses of the system ($u(t) = \delta(t)e_i$)[1]. Assuming there are a total of b inputs, in this case, the $\mathcal{H}_2$ norm of an LTV system T_{tv} is defined in Stoorvogel (1993) by:

$$\|T_{tv}\|_2^2 := \sum_{i=1}^{b} \|y_i\|_2^2 \tag{2.24}$$

See the cited reference for details in the LTV case. For continuous-time TS systems, the values of the state-dependent membership functions may change within the impulsive response. So, up to the author's knowledge, the problem of computing a bound of this $\mathcal{H}_2$ interpretation for TS systems has not been posed in LMI form. Instead of it, the guaranteed-cost performance is used, see Section 2.4.4.

Remark 2.3. Under this point of view, the $\mathcal{H}_2$ performance of a nonlinear system is understood as a speed of convergence index, because it evaluates the system's output evolution starting from the particular set of initial conditions given by unit impulse inputs.

Equivalent developments can be addressed for discrete-time systems in which D is not forced to be zero (typical measurement noise present on discrete-time systems). See De Caigny, Camino, Oliveira, Peres and Swevers (2010) for details in the LPV case and Zhou, Doyle and Glover (1996)[Chap. 21] in the LTI one.

[1] e_i is the i-th basis vector in the standard basis of the input space $\mathbb{R}^b$ ($i : 1, \ldots, b$).

2.4.2 $\mathcal{H}_\infty$ performance

Consider an asymptotically stable system (2.5) denoted by T. The $\mathcal{H}_\infty$ norm for system T is defined by the *worst case gain*:

$$\|T\|_\infty := \sup_{0<\|u\|_2<\infty} \frac{\|y\|_2}{\|u\|_2} \tag{2.25}$$

In LTI systems this means that the $\mathcal{H}_\infty$ norm of a transfer function is the supremum of the maximum singular value of the system's frequency response (Scherer and Weiland, 2004)[Chap. 3]:

$$\|T\|_\infty := \sup_{j\omega\in\mathbb{C}^+} \sigma_{max}(T(j\omega)) < \infty \tag{2.26}$$

The following proposition gives a bound $\sqrt{\gamma}$ for the worst case gain of system (2.5) in terms of linear matrix inequalities.

Proposition 2.1 (Tanaka and Wang (2001))**.** *Let the system* (2.5) *be asymptotically stable and assume that the initial conditions are equal to zero* $x(0) = 0$*. Then the system is strictly dissipative with respect to the supply function* $S = \gamma\|u\|^2 - \|y\|^2$ *and* $\|T\|_\infty < \sqrt{\gamma}$ *if the following LMI problem is feasible:*

Minimize γ *subject to*

$$\gamma > 0, \quad P \succ 0, \tag{2.27}$$

$$\begin{bmatrix} -A_i^T P - PA_i & C_i^T & PB_i \\ C_i & I & 0 \\ B_i^T P & 0 & \gamma I \end{bmatrix} \succ 0 \quad i:1,\dots,r \tag{2.28}$$

Proof omitted for brevity, see Appendix B.1.

Equivalent result can be obtained for discrete-time systems by replacing the derivative $\dot{V}$ by its discrete increment $\Delta V = V_{k+1} - V_k$ and using discrete sums $\sum_{k=0}^{T} V_k$ instead of the integrals. See Tanaka and Wang (2001, Chap. 3.7) for details.

Remark 2.4. The $\mathcal{H}_\infty$ performance of a nonlinear system can be understood as a robustness index because it evaluates the effect of the worst case input (external signal or even a state-dependent modelling error). The proof is based on the small-gain theorem, see Khalil (2002, Chap. 5) for details.

For example, given a TS model of a nonlinear system under disturbances, the objective is to find a PDC controller with a good disturbance attenuation.

The closed loop will be on the form

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} \mu_i(z(t))((A_i - B_i K_i)x(t) + E_i v(t)), \\ y(t) &= \sum_{i=1}^{r} \mu_i(z(t)) C_i x(t), \end{aligned} \tag{2.29}$$

where $v(t)$ are now the considered disturbances to reject, i.e., $(A_i - B_i K_i) \equiv A_i$ and $E_i \equiv B_i$ on Proposition 2.1.

The controller gains K_i have to be designed in order to achieve a minimum variation on the output $y(t)$ when a disturbance $v(t)$ is acting. It can be expressed as:

$$\sup_{||v(t)||_2 \neq 0} \frac{||y(t)||_2^2}{||v(t)||_2^2} \leq \sqrt{\gamma}. \tag{2.30}$$

This synthesis problem is very popular within LTI, LPV and LTV literature and can be cast as an LMI optimization problem following Proposition 2.1 which, in case of TS systems and PDC controller, can be consulted in Tanaka and Wang (2001, Chap. 3).

2.4.3 Decay Rate

The so-called *decay rate* of a system is the largest time constant α of a negative exponential ($e^{-\alpha t}$) for which the system is proven to decrease equal or faster (related to exponential stability according with Definition 2.4) (Haddad and Chellaboina, 2008, Chap. 3). Therefore it is a measure of system's speed.

This decay-rate guarantee can be proven by adding a quadratic constraint to the stability LMI's, forcing the derivative of the Lyapunov function V to be more negative than $-2\alpha V$ (Ichikawa, 1993):

$$\dot{V}(x(t)) \leq -2\alpha V(x(t)) \tag{2.31}$$

For instance, in case of having TS systems and quadratic Lyapunov functions $V(x)$, the LMI's (2.10) are modified as follows:

$$A_i^T P + P A_i \leq -2\alpha P \tag{2.32}$$

Nevertheless, (2.32) is not a convex problem due to the product of decision variables α and P. Fortunately it is a quasiconvex problem denoted GEVP (Appendix A.1.4.1) which can be solved by bisection, fixing α and solving the remaining LMI problem in each step. In this way α can be maximized. Detailed development, also for discrete-time case, can be found in Tanaka and Wang (2001, Chap. 3).

2.4.4 Guaranteed cost

Guaranteed cost fuzzy control under LMI's was presented in Tanaka and Wang (2001, Chap. 6). Consider a quadratic performance index in the form

$$J = \int_0^\infty (y^T(t)Wy(t) + u^T(t)Ru(t))dt \tag{2.33}$$

where y is the output, u is the control action and W, R are user-defined weighting matrices. The designed controller must minimize an upper bound on it, defined by a positive definite function $V = x^T P x$, so that:

$$J < x_0^T P x_0 \tag{2.34}$$

This bound holds if the following index J_d is negative:

$$J_d = y^T W y + u^T R u + \dot{x}^T P x + x^T P \dot{x} < 0 \tag{2.35}$$

The proof arises easily by integrating (2.35) between 0 and ∞, see the above cited reference for more details.

Then, substituting in (2.35) the expression of $\dot{x}$ and y as stated in (2.5), and defining u by a PDC control law (2.12), the following double-summation condition is obtained after application of Schur Complement (Appendix A.1.1)

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(z)\mu_j(z)x^T Q_{ij} x \leq 0 \tag{2.36}$$

with

$$\begin{aligned} Q_{ij} &= \frac{1}{2}\begin{pmatrix} 2D_{ij} & XC_i^T & -M_j & XC_j^T & -M_i \\ C_iX & -W^{-1} & 0 & 0 & 0 \\ -M_j & 0 & -R^{-1} & 0 & 0 \\ C_jX & 0 & 0 & -W^{-1} & 0 \\ -M_i & 0 & 0 & 0 & -R^{-1} \end{pmatrix} \\ D_{ij} &= A_iX + XA_i^T - B_iM_j - M_j^T B_i^T \end{aligned}$$

and $X = P^{-1}$, $K_i = M_iP$.

The optimization problem to solve is:

Maximize λ^{-1} subject to

$$\lambda^{-1} I < X \tag{2.37}$$

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(z)\mu_j(z) x^T Q_{ij} x \leq 0 \tag{2.38}$$

in order to obtain a lower bound for the cost (2.33) given by $J \leq \lambda x_0^T x_0$. For details see Tanaka and Wang (2001, Chap. 6).

Note. The above expression (2.38) is not directly an LMI. There are many well-known procedures in order to transform it into an LMI problem, see Section 2.3.1 on fuzzy summations.

2.5 Discussion

The reviewed Takagi-Sugeno fuzzy approaches for stability analysis and control of nonlinear systems are obviously conservative (comparing to other pure nonlinear control techniques). The main drawbacks are listed below:

- The LMI optimization problems derived from Lyapunov theory, if feasible, are proving stability for all possible nonlinear systems represented by the same vertices. However, knowledge of the shape of the membership functions may allow to lift some conservativeness. For instance, Example 6 in Tanaka and Wang (2001, Chap. 2) shows that the basin of attraction for fuzzy systems may be membership dependent: the same vertices but different memberships give rise to different (nonlinear) basins of attraction. Anyway, unless fuzzy-Lyapunov functions are used (Tanaka, Hori and Wang, 2003; Jaadari, 2013), the basins of attraction proved with LMI's are not membership dependent. This drawback has been addressed in some works like Sala and Ariño (2007b); Bernal, Guerra and Kruszewski (2009), which add membership function information to the LMI problem. Nevertheless, this is an inherent price to pay in extending linear tools for doing nonlinear control.

- There are stable or stabilizable systems which do not necessarily have a quadratic Lyapunov function (Sala, 2009). This issue has been addressed also in literature with several works which propose different non-quadratic candidate Lyapunov functions: piecewise (Feng, 2004; Feng, Chen, Sun and Zhu, 2005), polynomial (Prajna, Papachristodoulou and Wu, 2004b; Tanaka, Yoshida, Ohtake and Wang, 2009c), line-integral (Rhee and Won, 2006) or fuzzy ones (Tanaka, Hori and Wang, 2003; Guerra and Vermeiren, 2004; Jaadari, 2013). Despite the amount of existent literature, this issue is still an open problem because all works present their own drawbacks in addressing the general case.

- The type of TS model: the model representation is not unique (Sala, 2009) and this fact can lead to a significant difference on the required number of fuzzy rules, with their consequent increase of LMI vertex conditions to check in the optimization problem. The more number of LMI's is the high computational resources are (Boyd, Ghaoui, Feron and Balakrishnan, 1994). For instance, the TS descriptor representation is usually more efficient for mechanical systems than the form (2.5), see Tanaka and Wang (2001, Chap. 10).

2.6 Conclusions

This chapter has reviewed the existent fuzzy TS-based approaches for control of nonlinear systems. They have been demonstrated to be a valid way to analyze and control systems for which its original nonlinear model is available. Moreover, the use of powerful tools on semidefinite programming allow managing a large amount of decision variables and fuzzy rules for proving stability analysis and stabilization.

However, there exist some sources of conservativeness in the modelling phase (forced to embed nonlinearities between linear vertices), choice of the Lyapunov functions, locality of the results, conditioning issues, growing computational requirements to approach pure nonlinear control, and issues arising with non-measurable premise variables.

In summary, the conservativeness of LMI fuzzy control as a tool for general nonlinear control can be significantly reduced using some “fuzzy” sufficient conditions (asymptotically) and exploiting some nonlinear knowledge. This later fact motivates research in the use of fuzzy-polynomial modelling methodologies and tools with the objective of decreasing part of such conservativeness.

Chapter 3

Fuzzy Polynomial Systems

The best structure does not guarantee neither results nor performance. However, the wrong structure is a guarantee of failure.

Peter Drucker

ABSTRACT: This chapter summarizes the existent literature on local stability analysis and stabilization of nonlinear systems by using fuzzy polynomial techniques. Results in sum-of-squares polynomials and available semidefinite programming tools are the basis for the results of this chapter. Methodologies for analysis and design are provided in both continuous-time and discrete-time cases. Developments in case of existence of bounded disturbances are also addressed.

The previous chapter reviewed a structured way to address stability analysis and controller synthesis for nonlinear systems by expressing it as a fuzzy combination of linear vertex models. In this chapter, the stability analysis over polynomial systems is presented. This methodology was presented first by Papachristodoulou and Prajna (2002) and provided a way to analyze systems with only polynomial nonlinearities, by checking if the resulting polynomial Lyapunov conditions $V(x) > 0$, $-\dot{V}(x) > 0$ hold.

The core idea in this chapter is putting together the advantages of both fuzzy TS and polynomial methodologies, in order to address the control problem of nonlinear systems in a less conservative way. In order to do this, an extension of the sector-nonlinearity modeling technique, based on the Taylor series decomposition (Sala and Ariño, 2009; Chesi, 2009), is employed. With this technique, a family of progressively more precise[1] fuzzy polynomial models can be obtained. In this way, generalized *fuzzy polynomial models* from nonlinear systems are obtained. Importantly, Takagi-Sugeno models become a particular case of the fuzzy polynomial models. Following this methodology, some stability and stabilization problems can be successfully solved for the resulting fuzzy polynomial systems (Sala, 2009; Tanaka, Yoshida, Ohtake and Wang, 2007b,a; Tanaka, Ohtake and Wang, 2009b; Tanaka, Yoshida, Ohtake and Wang, 2009c), by extending the seminal methodologies in Prajna, Papachristodoulou and Wu (2004b) to the fuzzy case.

The presented methodology allows asymptotically exact results for smooth nonlinear systems: if there is a smooth Lyapunov function for it, there will exist a polynomial Lyapunov function and a fuzzy polynomial model with a finite degree, which will allow proving stability of the original system with some extra assumptions (Sala and Ariño, 2009). Asymptotic exactness applies *only* in compact regions of interest where the Taylor series approximation of the nonlinearities, as well as those of a valid Lyapunov function and its derivatives, converge *uniformly*. Control synthesis, however, requires an affine-in-control structure as well as some additional artificial variables, which introduce some conservativeness (Sala, 2009).

A set of examples will show that fuzzy polynomial modeling is able to reduce conservativeness with respect to standard TS approaches as the degrees of the involved polynomials increase.

The chapter is structured as follows: next section presents the fuzzy polynomial modelling methodology, the stability analysis for fuzzy polynomial systems is addressed in Section 3.2, the stabilization and observer designs are outlined on Section 3.3, a discussion on the main drawbacks and open problems is given in Section 3.4 and, finally, a conclusion section closes the state of the art review.

3.1 Fuzzy polynomial modelling

Sector-nonlinearity modelling methodology (Section 2.1.1) can be extended to a polynomial case. Indeed, there exists the possibility to bound a nonlinearity

[1]The sector-nonlinearity technique is exact (no approximation involved) for any polynomial degree, including the classical TS models. *Precise* above must be understood as consequent models fitting more closely the nonlinearity being modeled.

within two polynomials (instead of two *constants*): the result would be a so-called fuzzy polynomial system representation, but more general than the TS. This is the objective of this subsection, where $f(x)$ can correspond to any of the non-polynomial nonlinearities in a system's dynamics. Nevertheless, the sector-nonlinearity technique represents exactly the nonlinear system in a modelling region (no approximation involved) for any polynomial degree, including the classical TS models.

3.1.1 Taylor-series based polynomial fuzzification

Lemma 3.1 (Sala and Ariño (2009); Chesi (2009)). *Consider a sufficiently smooth function of one real variable, $f(x)$, so that its Taylor expansion of degree "n" exists (Apostol, 1967), i.e., there exists an intermediate point $\psi(x) \in [0, x]$, so that:*

$$f(x) = \sum_{i=0}^{n-1} \frac{f^{[i]}(0)}{i!} x^i + \frac{f^{[n]}(\psi(x))}{n!} x^n \tag{3.1}$$

where $f^{[i]}(x)$ denotes the i-th derivative of f and $f^{[0]}(x)$ is defined, plainly, as $f(x)$. Assume also that $f^{[n]}(x)$ is continuous in a compact region of interest Ω. Denoting the Taylor reminder by $f(x) - f_n(x) = T_n(x)x^n$ and computing its maximum and minimum values in Ω

$$\psi_1 := \sup_{x \in \Omega} T_n(x), \quad \psi_2 := \inf_{x \in \Omega} T_n(x)$$

the reminder can be expressed as a fuzzy term in the form

$$T_n(x) = (\mu_1(x) \cdot \psi_1 + \mu_2(x) \cdot \psi_2) \tag{3.2}$$

with the memberships

$$\mu_1(x) = \frac{T_n(x) - \psi_2}{\psi_1 - \psi_2}, \quad \mu_2 = 1 - \mu_1 \tag{3.3}$$

Then, an equivalent fuzzy representation of (3.1) *exists in the form:*

$$f(x) = \mu_1(x) \cdot p_1(x) + \mu_2(x) \cdot p_2(x) \quad \forall x \in \Omega \tag{3.4}$$

where $\mu_1(x) + \mu_2(x) = 1$ and $p_1(x)$, $p_2(x)$ are polynomials of degree n. Furthermore, $p_1 - p_2$ is a monomial of degree n.

Proof omitted for brevity, see Appendix B.2.

Note. As in TS modeling, the representation (3.4) is *exact*, i.e., there is equality (no approximation involved) and there is no uncertainty in the membership functions, defined in (3.3).

Conceptually, the resulting membership functions can be thought of as entities capturing "*the nonlinearity which cannot be described by a polynomial of a prescribed degree*". The idea generalizes the interpretation of classical Takagi-Sugeno memberships (they captured "all" the nonlinearity between some linear sector boundings).

Remark 3.1. If $f(0) = 0$, setting $n = 1$ in the developments in Lemma 3.1 we obtain the usual sector-nonlinearity methodology (Section 2.1.1).

As a conclusion, using the Taylor-based modeling for non-polynomial nonlinearities (say, trigonometric, exponential, etc.), any smooth nonlinear system can be exactly expressed as a fuzzy polynomial one in a compact domain Ω. On the following, some illustrative examples are provided.

Example 3.1.1. Consider a nonlinear system

$$\dot{x} = \begin{cases} -\tanh(x_1^3) - (0.05 + 0.95\sin^2(x_1))x_1x_3^2 \\ -(1+\sin^2(x_1))x_2 - x_1x_2 \\ -x_3 + 3x_1^2x_3 - 3\sin(x_1)x_3 \end{cases} \tag{3.5}$$

to be modeled in the zone $\Omega = \{-1 \le x_1 \le 1\}$. Then, by computing maximum and minimum values in Ω we have

$$\sin(x_1) = 0.8415\mu_{11} + (-0.8415)\mu_{12}, \mu_{11} + \mu_{12} = 1$$

$$\sin^2(x_1) = 0.708122\mu_{11}^2 - 1.41624\mu_{11}\mu_{12} + 0.708122\mu_{12}^2$$

and, also, as $\tanh(x_1^3) = ((\tanh(x_1^3))/x_1^3) \cdot x_1^3$ we may model:

$$\tanh(x_1^3)/x_1^3 = 0.7616\mu_{21} + 1 \cdot \mu_{22}$$

so $\tanh x_1^3 = \mu_{21} \cdot 0.7616x_1^3 + \mu_{22}x_1^3$, $\mu_{21} + \mu_{22} = 1$. In this way, a polynomial fuzzy model results by replacing the above expressions in the original equations (details omitted for brevity). Note that the modeling is not unique: for instance, $\tanh x_1^3$ may also be approximated by $(0.7616\mu_3 + 0 \cdot \mu_4)x_1$ (in fact, this would be the standard sector nonlinearity approach by bounding $0 \le \tanh(x_1^3)/x_1 \le 0.7617$ for $-1 \le x_1 \le 1$).

Example 3.1.2. Consider the Taylor series of the cosine around $x = 0$:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

then, we define, from the notation in Section 3.1.1:

$$f_1 = 1, \quad T_1(x) = (\cos(x) - 1)/x$$

so by finding the maximum and minimum of $T_1(x)$ in [-1,1], which are 0.45961 and -0.45961, we may express:

$$\cos(x) = \mu_1(x)(1 + 0.45961x) + \mu_2(x)(1 - 0.45961x) \tag{3.6}$$

which, in fact, coincides with the standard sector-nonlinearity TS modelling, reviewed in the previous chapter.

By considering higher order terms in the Taylor series development, the fuzzy model will be more precise when fitting $\cos(x)$, in the terms discussed in the footnote on page 34. Table 3.1 summarizes the tested different modeling choices. Note that the two bounding polynomials are closer and closer as their degree increases, as the maximum and the integral of their difference express.

Order	$p_1(x)$	$p_2(x)$
1^{st}	$1 + 0.45961x$	$1 - 0.45961x$
3^{th}	$1 - 0.5x^2 - 0.0403x^3$	$1 - 0.5x^2 + 0.0403x^3$
5^{th}	$1 - 0.5x^2 + \frac{x^4}{24} + 0.001364x^5$	$1 - 0.5x^2 + \frac{x^4}{24} - 0.001364x^5$
	$\max \lvert p_1 - p_2 \rvert$	$\int_{-1}^{1} \lvert p_1 - p_2 \rvert$
1^{st}	0.91922	0.91922
3^{th}	0.0806	0.0403
5^{th}	0.002728	0.0009093

Table 3.1: Expressing $\cos(x)$ as a fuzzy combination of two polynomials $\mu_1(x)p_1(x) + \mu_2(x)p_2(x)$

In fact, both 3^{rd}- and 5^{th}-order bounds are very accurate (maximum error 0.0806 and 0.002728, respectively) in the chosen interval. In order to illustrate the differences between the various polynomial model choices, a similar procedure using as region of interest the wider interval $[-3.1, 3.1]$ has also been carried out. The results are the upper and lower approximations depicted in Figure 3.1, for linear, 3^{rd} and 5^{th} order polynomials.

For more details on the procedure and examples the reader is referred to Sala and Ariño (2009).

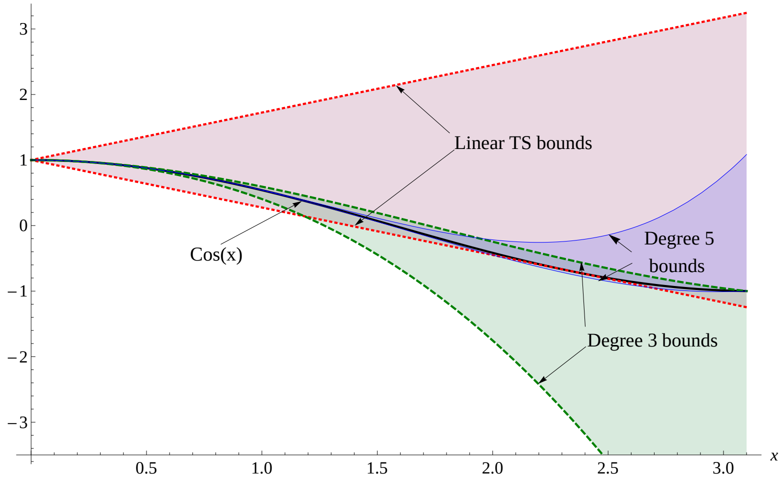


Figure 3.1: Linear (TS), 3^{rd} and 5^{th} order fuzzy models of $f(x) = \cos(x)$ for $x \in [-3.1, 3.1]$. Only the positive interval is depicted due to symmetry.

The Taylor-series approach above can be applied to any function which can be written as an expression tree with functions of one variable, addition and multiplication. See Example 6 in Sala and Ariño (2009).

Note. As each nonlinearity results in a two-rule polynomial fuzzy description, the number of rules will still be a power of 2, keeping a final tensor-product structure (Ariño and Sala, 2007). Details are analogous to those in classical sector-nonlinearity models, omitted for brevity.

Homogeneous expression of fuzzy polynomial models.

Consider a continuous-time fuzzy polynomial system expressed as

$$\dot{x}(t) = p(x(t), \mu(t)) \tag{3.7}$$

or its discrete-time equivalent

$$x_{k+1} = p(x_k, \mu_k) \tag{3.8}$$

where $k \in \mathbb{N}$ is the sample number, fulfilling $t = kT_s$ and T_s is the considered sample time.

For later operational purposes (Sala, 2009), the polynomials should be made homogeneous in the membership functions μ. It is easy to see that an arbitrary fuzzy polynomial system can be expressed without loss of generality as:

$$\dot{x} = \sum_{i=1}^{r} \mu_i(z) p_i(x), \tag{3.9}$$

Or, in the discrete case, as:

$$x_{k+1} = \sum_{i=1}^{r} \mu_i(z) p_i(x_k) \tag{3.10}$$

Example 3.1.3. Consider a nonlinear system

$$\dot{x} = \begin{cases} -x_1^3 - (0.05 + 0.95 \sin^2(x_1)) x_1 x_3^2 \\ -(1 + \sin^2(x_1) + x_1^2) x_2 \\ (3x_1^2 - 4) x_3 \end{cases} \tag{3.11}$$

which, using the methodology presented on Section 3.1, can be modelled as:

$$\dot{x} = \begin{cases} -x_1^3 - \mu_1(x_1) x_1 x_3^2 - 0.05 \mu_2(x_1) x_1 x_3^2 \\ -\mu_1(x_1) x_2 - 2\mu_2(x_1) x_2 - x_1^2 x_2 \\ 3x_1^2 x_3 - 4x_3 \end{cases} \tag{3.12}$$

where the sector-nonlinearity methodology is used to model $\sin^2(x_1)$ expressed as an interpolation between 0 and 1. Multiplying the terms x_1^3, $x_1 x_2$, $x_1^2 x_3$ and x_3 by $\mu_1 + \mu_2$ the system is expressed as a degree-1 homogeneous polynomial in the memberships.

3.1.2 State growing by recasting nonlinearities

An alternative polynomial modelling methodology is recasting the non polynomial nonlinearities to new auxiliary state variables in order to get a pure polynomial extended model.

The procedure, adapted from the one explained in Savageau and Voit (1987), would be as follows:

1. Create new state variables z_i for each elemental non-polynomial nonlinear function (sinus, cosinus, logarithm, exponential, etc) $f_i(x)$, or combination of them, and assign $z_i = f_i(x)$.

2. Compute, using the chain rule, the derivative of the new state variables $\dot{z}_i = \frac{df_i(x)}{dt}$ and replace each $f_i(x)$ by the new z_i in the whole system's model.

3. As a result of the above step, new nonlinearities might appear in $\dot{z}_i$. Then, repeat the above steps with the new extended dynamic equations until obtaining a totally polynomial model.

4. Additional information, if provided, can be added as algebraic constraints over the new variables z_i.

At the end of the procedure, a polynomial model is obtained in the form

$$\begin{aligned} \dot{\tilde{x}}_1 &= f_1(\tilde{x}_1, \tilde{x}_2, u), \\ \dot{\tilde{x}}_2 &= f_2(\tilde{x}_1, \tilde{x}_2, u), \end{aligned} \tag{3.13}$$

such that

$$\begin{aligned} G_1(\tilde{x}_1, \tilde{x}_2) &= 0 \\ G_2(\tilde{x}_1, \tilde{x}_2) &\geq 0 \end{aligned} \tag{3.14}$$

where $\tilde{x}_1 = (x_1, ..., x_n)$ are the original state variables and $\tilde{x}_2 = (z_1, ..., z_n)$ are the new artificial variables product of recasting non-polynomial nonlinearities.

Example 3.1.4. Consider the whirling pendulum in (Furuta, Yamakita and Kobayashi, 1992) with length l_p and whose suspension end is attached to a rigid arm of length l_a and a mass m_b is attached to its free end. The arm rotates with angular velocity $\dot{\theta}_a$. The pendulum can oscillate with angular velocity $\dot{\theta}_p$ in a plane normal to the arm, making an angle θ_p with the vertical in the instantaneous plane of motion. Using $x_1 = \theta_p$ and $x_2 = \dot{\theta}_p$ as state variables, a simplified model for the system is:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \dot{\theta}_a^2 \sin(x_1)\cos(x_1) - \frac{g}{l_p}\sin(x_1) \end{aligned} \tag{3.15}$$

When the condition

$$\dot{\theta}_a^2 < \frac{g}{l_p} \tag{3.16}$$

is satisfied, the only equilibria in the system are $(x1, x2)$ satisfying $\sin(x_1) = 0$, $x_2 = 0$. One equilibrium corresponds to $x_1 = 0$, i.e., the pendulum is hanging vertically downward (stable), and the other equilibrium corresponds to $x_1 = \pi$, i.e., the vertically upward position (unstable).

Since the vector field is not polynomial, a transformation to a polynomial one must be performed before stability analysis. For this purpose, introduce new variables $z_1 = \sin(x_1)$ and $z_2 = \cos(x_1)$ to get the extended model:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \dot{\theta}_a^2 z_1 z_2 - \frac{g}{l_p} z_1 \\
\dot{z}_1 &= x_2 z_2 \\
\dot{z}_2 &= -x_2 z_1
\end{aligned}$$

In addition, the trigonometric constraint $\sin^2(x) + \cos^2(x) = 1$ can be added by the algebraic constraint:

$$z_1^2 + z_2^2 - 1 = 0$$

Wide details on this and more examples can be consulted in the work Papachristodoulou and Prajna (2005).

Note. The extended model (3.13) is *not* a fuzzy model. However, this technique of recasting can be combined with the sector nonlinearity (Section 3.1.1) in order to fuzzify a new non-polynomial nonlinearity involving any z_i. This avoids the introduction of a new variable z with its corresponding dynamical equation. In this way, an extended fuzzy polynomial model is obtained.

3.2 Stability analysis via sum of squares

The techniques outlined in the above section allow obtaining polynomial fuzzy models of arbitrary degree for any smooth nonlinearity present on the first-principle equations of a physical system (say, exponentials, trigonometric functions, etc). Once such local models are available, fuzzy stability analysis and control design techniques can be explored on them.

As the derivatives (increments in discrete-time case) of functions of the state will involve polynomial terms, stability analysis and control design (to be discussed in next sections) will require proving positiveness or negativeness of some polynomials (in several variables, actually state-space ones). These issues have been developed in literature at the beginning of the past decade, conforming what it is now called sum of squares (SOS) programming and optimization. Appendix A.2.4 summarizes the main results and software used in this thesis.

Notation

The following notation about polynomials will be used within the rest of the thesis:

Arbitrary degree polynomials in variables "z" will be denoted by $\mathcal{R}_z$. For instance, the polynomials in $x = (x_1, x_2, x_3)$ such as $p(x) = x_1^4 x_2 + x_2^6 - 2x_1^3 + 5 - x_3$ will be said to verify $p(x) \in \mathcal{R}_x$. The n-dimensional vectors of polynomials in variables z will be denoted by $\mathcal{R}_z^n$. For instance, the right-hand side of state-space equations of a polynomial 3rd-order system will be a polynomial vector in $\mathcal{R}_x^3$.

In matrices, the corresponding element of a symmetric expression will be omitted and denoted as $(*)$. For instance, $(*)$ denotes $B(z)$ on both of the expressions below:

$$\begin{bmatrix} A(z) & (*)^T \\ B(z) & C(z) \end{bmatrix}, \quad (*)^T P B(z)$$

Polynomials in some variables z which can be decomposed as a sum of squares of other polynomials will be denoted by Σ_z.

Given polynomials $\{k_1, ..., k_d\}$, where d denotes the number of them, $\mathcal{M}$ will denote the **multiplicative monoid**, $\wp$ denotes the **cone**, and $\mathcal{I}$ the **ideal** generated by the set of k_d's. See Appendix A.2.2 for such formal definitions. In order to shorten notation, the ideal generated by a vector of polynomials $P \in \mathcal{R}_z^n$ will be defined as the ideal generated by its elements.

3.2.1 Global stability

In fuzzy polynomial systems (3.7), as the memberships are always positive, they may be described by the change of variable $\mu_i = \sigma_i^2$, resulting in a polynomial model:

$$\dot{x} = \tilde{p}(x, \sigma) \tag{3.17}$$

Identically for discrete-time systems (3.8):

$$x_{k+1} = \tilde{p}(x_k, \sigma_k) \tag{3.18}$$

Example 3.2.1. For instance, (3.12) can be written as:

$$\dot{x} = \begin{cases} -(\sigma_1^2 + \sigma_2^2)x_1^3 - \sigma_1^2 x_1 x_3^2 - 0.05\sigma_2^2 x_1 x_3^2 \\ -\sigma_1^2 x_2 - 2\sigma_2^2 x_2 - (\sigma_1^2 + \sigma_2^2) x_1^2 x_2 \\ (\sigma_1^2 + \sigma_2^2)(3x_1^2 x_3 - 4x_3) \end{cases} \tag{3.19}$$

where $\mu_i = \sigma_i^2$.

Once the systems are in the above form, the following well-known results are derived from Lyapunov stability theory (Tanaka, Yoshida, Ohtake and Wang, 2007b).

Lemma 3.2 (Continuous-time)**.** *Global stability of a system* (3.17) *will be proved if a polynomial Lyapunov function $V(x)$ can be found verifying*

$$V(x) - \epsilon_1(x) \in \Sigma_x \tag{3.20}$$

$$-\frac{dV}{dx}\tilde{p}(x,\sigma) - \epsilon_2(x) \in \Sigma_{x,\sigma} \tag{3.21}$$

where $\epsilon_1(x)$ and $\epsilon_2(x,\sigma)$ are radially unbounded (in the state x) positive polynomials, usually $\epsilon_1(x) = \sum_{i=1}^n x_i^2$ and $\epsilon_2(x) = \epsilon_1(x)$ (but not necessarily so), in order to avoid trivial solutions $V(x) = 0$ and to ensure asymptotic stability.

Replacing the derivative of the Lyapunov function $\dot{V}$ by its discrete increment $\Delta V = V_{k+1} - V_k$, the above lemma can be adapted to prove stability for discrete-time systems.

Lemma 3.3 (Discrete-time)**.** *Global stability of a system* (3.18) *will be proved if a polynomial Lyapunov function $V(x)$ can be found such that* (3.20) *holds and*

$$V(x) - V(\tilde{p}(x,\sigma)) - \epsilon_2(x) \in \Sigma_{x,\sigma} \tag{3.22}$$

Indeed, setting $V(x)$ to be an arbitrary degree polynomial in the state variables (but *not* in the memberships, in order to avoid the need of its derivatives), $\frac{dV}{dx}$ is also a vector of polynomials in the variables x and σ. Hence $\frac{dV}{dt}$ in (3.21) and ΔV in (3.22) are polynomial. If V is linear in some decision variables (the natural choice are the polynomial coefficients), expressions (3.20), (3.21) and (3.22) can be introduced into sum-of-squares programming packages (Appendix A.2.5) in order to get values of the decision variables fulfilling the above constraints. Also, expressions (3.21) and (3.22) need to be modified for actual computations, making it homogeneous in the memberships (i.e., all the monomials must have the same degree in σ). It can be achieved by multiplying anything by $\sum_{i=1}^r \sigma_i^2$ (which is equal to one, anyway) as many times as needed. Details omitted for brevity, as the procedure is well known (Sala, 2009).

Example 3.2.2. Recall system (3.19) and consider now a Lyapunov function $V(x)$ in the form of a fourth degree polynomial in the state variables x_1, x_2, x_3 and tolerances:

$$\epsilon_1(x) = 0.01(x_1^4 + x_2^4 + x_3^4) \quad \epsilon_2(x)(\sigma_1^2 + \sigma_2^2) = \epsilon_1(x)(\sigma_1^2 + \sigma_2^2)$$

Then, conditions of Lemma 3.2 can be introduced in a SOS programming package in order to check their positiveness. See the Appendix for details.

Note that the derivative of the Lyapunov function will then be non-positive for any value of σ_i, i.e., for any non-negative μ_i.

Using YALMIPSOS and SeDuMi, the solver finds a 4^{th}-degree Lyapunov function given by (terms lower than 10^{-7} set to zero):

$$V(x) = 24.49x_1^2 + 6.453x_3^2 + 50.875x_1^4 + 19.18x_2^2 + 19.532x_2^4 + 0.0182x_3^4 + 33.555x_1^2x_2^2 + 6.138x_1^2x_3^2 + 1.683x_2^2x_3^2$$

which is, evidently, SOS and whose time derivative with changed sign is also SOS. Luckily, the proposed system is asymptotically stable in all state space (as it is easy to derive from system dynamic equations). However, $\dot{V}$, or the equivalent discrete ΔV, often are not SOS for all $x \in \mathbb{R}^n$ (for instance if $\dot{V}$ is not an even degree polynomial). Therefore, proving global stability for fuzzy polynomial systems is usually difficult.

Remark 3.2. Despite setting up global conditions, as Taylor-series fuzzy polynomial models are only valid locally in most cases, stability or decay-rate performance is not proved in the whole state space where the SOS conditions hold (unless $\Omega = \mathbb{R}^n$). The actually proved domain of attraction (DA) is the largest invariant set $V(x) \leq \gamma$, γ constant, contained in Ω. The set can actually be a very small fraction of Ω. This problem will be widely addressed on Chapter 4.

Remark 3.3. Note that, as there are many positive polynomials which are not SOS (Section A.2.1), even if (asymptotically) there is no conservativeness in many fuzzy polynomial modeling cases, conservativeness remains in the SOS approach to fuzzy polynomial system analysis.

Unfortunately, many nonlinear systems of interest are not globally stable[2]. Hence, refinements to the above conditions are needed in order to obtain a DA estimate (ideally, as large as possible).

3.2.2 Local stability

Consider a region of the state space $\Omega = \{x : g_1(x) \geq 0, \ldots, g_i(x) \geq 0, h_1(x) = 0, \ldots, h_j(x) = 0\}$, being $g_i(x)$, $h_j(x)$ a set of d and l known polynomials respectively.

[2]Even if they are, maybe a low-degree polynomial Lyapunov function might not be enough to prove such global stability.

If the global stability problem renders infeasible, a local stability condition can be posed, based on standard invariant-set arguments (Theorem 2.2).

In order to apply Lemma 2.1 to polynomial systems, the *Positivstellensatz* theorem (Appendix A.2.2) enables checking local positiveness conditions with SOS programming (sufficient conditions). It will be used to modify conditions (3.20) and (3.21) or (3.20) and (3.22), in order to make them hold locally in Ω, as follows:

Lemma 3.4 (Chesi (2011)). *If polynomials* $(s_0(x), s_i(x)) \in \Sigma_x$, $z_j(x) \in \mathcal{R}_x$, *can be found fullfilling*

$$-s_0(x)(\frac{\partial V}{\partial x}\tilde{p}(x,\sigma)+\epsilon(x))-\sum_{i=1}^{d} s_i(x)g_i(x)+\sum_{j=1}^{l} z_j(x)h_j(x) \in \Sigma_{x,\sigma} \quad (3.23)$$

then $\dot{V}(x)$ *is locally negative in* Ω *except at the origin and, hence, its level sets* $V_\gamma \subset \Omega$ *belong to the DA of the origin if* $V(x) \geq 0$ *(Lemma 2.1).*

The above lemma applies a simplified version of the original *Positivstellensatz* result in which (3.23) would be replaced by the less conservative expression:

$$F_1(x,\sigma) + F_2(x,\sigma) \in \Sigma_{x,\sigma} \quad (3.24)$$

where F_1 belongs to the polynomial cone $\wp(-\frac{\partial V}{\partial x}\tilde{p}(x,\sigma), g_1(x), ..., g_d(x))$ and F_2 belongs to the ideal $\mathcal{I}(h_1(x), .., h_l(x))$.

Note. Conditions (3.23) are not linear in decision variables if both s_0, and V have to be found. However, the problem becomes convex if either $V(x)$ is fixed, proposed in Chesi (2011, Chap. 4)[3], or $s_0(x)$ is fixed, for instance to $s_0(x) = 1$, as proposed in Tanaka, Yoshida, Ohtake and Wang (2007b). Once $V(x)$ is found, a bound for the maximum γ fulfilling Lemma 2.1 can be also easily found via SOS techniques. Note also that (3.23) should be made homogeneous in σ for actual computations, as stated in the above section.

In order to avoid ill-shaped solutions, additional SOS constraints may be added to find the Lyapunov function level set containing the largest region with a particular predefined shape (circle, hypercube, ...), or maximising an approximation to the volume based on the maximum-volume formula for a

[3] Chesi (2011) uses a Lyapunov function from the linearized system, and take $\Omega \equiv V_\gamma$. If γ is maximized, it can be recast as a generalized SOS problem (Appendix A.2.4), a kind of SDP quasiconvex problem.

quadratic form (Chesi, 2011). A variation of this methodology is addressed in Khodadadi, Samadi and Khaloozadeh (2014) in which a BMI iterative procedure is proposed by forcing to compute a new Lyapunov level set containing the largest possible region defined by the quadratic part of the previously computed Lyapunov function. Nevertheless, BMI problems are nonconvex and are out of the scope of this thesis.

Discrete systems. Equivalent result can be proved for discrete-time systems using (3.22) instead of (3.21) in Lemma 3.4. However, the resulting condition involves a polynomial whose degree is that of $\tilde{p}$ plus that of V in the state variables as well as two plus the degree of V in the auxiliary variables σ; it also needs the algebraic manipulations to make the inequality homogeneous in σ, see Section 3.1 for details. Hence, the degree of the polynomials and the number of decision variables may be high even for simple local stability problems.

Next example illustrates the local-stability analysis procedure.

Example 3.2.3. Consider the system:

$$\begin{aligned} \dot{x}_1 &= -3x_1 + 0.5x_2 \\ \dot{x}_2 &= (-2 + 3\sin(x_1))x_2 \end{aligned} \tag{3.25}$$

with the objective of finding a decreasing Lyapunov function in the square region

$$\Omega = \{x : x_1^2 - \pi^2 \leq 0,\ x_2^2 - \pi^2 \leq 0\} \tag{3.26}$$

ensuring maximal decay rate. Under usual TS modeling, as $\sin(x_1)$ ranges between -1 and 1, we have $-5x_2 \leq (-2 + 3\sin(x_1))x_2 \leq x_2$, so we would obtain $\dot{x} = \sum_{i=1}^{2} \mu_i A_i x$, with:

$$A_1 = \begin{pmatrix} -3 & 0.5 \\ 0 & -5 \end{pmatrix} \quad A_2 = \begin{pmatrix} -3 & 0.5 \\ 0 & 1 \end{pmatrix}$$

and, as A_2 is not stable, the TS approach would fail in proving stability.

However, using the 1^{th}-order Taylor expansion of $\sin(x_1)$, we can show that there exists a fuzzy model so that $\sin(x_1) = \mu_1 \cdot 0 \cdot x_1 + \mu_2 \cdot 1 \cdot x_1$. In this way, we obtain the fuzzy-polynomial model:

$$\begin{aligned} \dot{x}_1 &= -3x_1 + 0.5x_2 \\ \dot{x}_2 &= -2x_2 + 3\mu_2 x_1 x_2 \end{aligned} \tag{3.27}$$

Replacing $\mu_1 = \sigma_1^2$, $\mu_2 = \sigma_2^2$, and noting that $\sigma_1^2 + \sigma_2^2 = 1$, we get:

$$\begin{aligned} \dot{x}_1 &= (-3x_1 + 0.5x_2)(\sigma_1^2 + \sigma_2^2) \\ \dot{x}_2 &= -2x_2(\sigma_1^2 + \sigma_2^2) + 3\sigma_2^2 x_1 x_2 \end{aligned} \tag{3.28}$$

Looking for a quadratic Lyapunov function

$$V(x) = v_1 x_1^2 + v_2 x_1 x_2 + v_3 x_2^2$$

with $v_i \in \mathbb{R}$ decision variables, and a decay-rate bound α, condition

$$Q(x, \sigma) = -\dot{V} - 2\alpha V(\sigma_1^2 + \sigma_2^2) > 0$$

must be proved (see Boyd, Ghaoui, Feron and Balakrishnan (1994); Tanaka and Wang (2001)) in the region defined by (3.26). Hence, using Corollary A.1, a SOS program is set up in order to find decision variables proving

$$V(x) - \epsilon(x) \in \Sigma_x \tag{3.29}$$

$$Q(x, \sigma) - s_1(x, \sigma)(\pi^2 - x_1^2) - s_2(x, \sigma)(\pi^2 - x_2^2) \in \Sigma_{x,\sigma} \tag{3.30}$$

where $(s_1, s_2) \in \Sigma_{x,\sigma}$ are polynomial Positivstellensatz multipliers, homogeneous in σ, to be found. As $Q(x, \sigma)$ is degree 3 in x and degree 2 in σ, we chose multipliers of of degree 4 in x and σ which proved powerful enough to obtain a feasible result. The tolerance $\epsilon(x)$ is set to $0.01(x_1^2 + x_2^2)$.

As a result, tools like SOSTOOLS+SeDuMi are able to find a Lyapunov function

$$V(x) = 3.8982x_1^2 - 0.0096x_1x_2 + 0.2087x_2^2$$

proving a decay rate of $\alpha = 0.272$. The actual code for the example appears in Appendix A.2.6. A larger value of α resulted into numerical problems or infeasibility.

If a third-order approximation of the sinusoid is used:

$$x_1 - 0.16667x_1^3 \le \sin(x_1) \le x_1 - 0.1012x_1^3 \quad \forall\, x_1 \in [-\pi, \pi] \tag{3.31}$$

we get a decay $\alpha = 0.309$ with also a quadratic Lyapunov function.

Hence, this example has illustrated that fuzzy polynomial modeling may provide satisfactory results in situations where classical TS methods fail, and that increasing the precision (degree of Taylor expansion) of the fuzzy polynomial modeling may improve the achieved results.

3.3 Fuzzy-Polynomial Stabilization

Once stability analysis for fuzzy polynomial systems has been addressed, the stabilization problem is the next step to be solved. In this, controller gains are required to be designed in order to close the loop with the nonlinear system and to fulfill some required specifications.

3.3.1 Controller design

Consider a modelling region Ω defined by $g_i(x)$ polynomial boundaries:

$$\Omega = \{x : g_1(x) > 0, \dots, g_k(x) > 0\} \tag{3.32}$$

Given an affine-in-control continuous-time fuzzy polynomial model, computed in (3.32), in the form

$$\dot{x} = \sum_{i=1}^{r} \mu_i(x, w)(A_i(x)z(x) + B_i(x)u) \tag{3.33}$$

or a discrete-time equivalent one

$$x_{k+1} = \sum_{i=1}^{r} \mu_i(x_k, w_k)(A_i(x_k)z(x_k) + B_i(x_k)u_k) \tag{3.34}$$

where $z(x) \in \mathbb{R}_x^v$ is a vector of monomials of x and $w \in \mathbb{R}^d$ are known external inputs and/or time, a first approach to design a stabilizing control law could be extending the well-known ideas of parallel-distributed compensator (PDC) to the polynomial framework (Tanaka, Yoshida, Ohtake and Wang, 2007a) (an adaptation to the fuzzy case of those in Prajna, Papachristodoulou and Wu (2004b)):

$$u = \sum_{i=1}^{r} \mu_i(x, w)K_i(x)z(x) \tag{3.35}$$

Define also a polynomial candidate Lyapunov function in the form

$$V(x) = z(x)^T P(\tilde{x})z(x) \tag{3.36}$$

where $\tilde{x}$ are the state variables whose time derivative does not depend on u, i.e., $(\partial \dot{x}_i/\partial u) = 0$ (the corresponding row of all B_i in (3.33) is zero).

Then, the following theorem, which is an adaptation with Positivstellensatz of the one presented in Tanaka, Yoshida, Ohtake and Wang (2007a), can be used to design a polynomial PDC control law locally in Ω.

Theorem 3.1. *If matrices $X(\tilde{x})$, $M_i(x)$ can be found fulfilling:*

$$\rho^T \left(X(\tilde{x}) - \epsilon I\right) \rho \in \Sigma_{x,\rho} \tag{3.37}$$

$$- 2\rho^T R(x) \left(A_i(x)X(\tilde{x}) - B_i(x)M_i(x)\right) \rho + \rho^T \frac{dX(\tilde{x})}{dx} \left(A_i(x)z(x)\right) \rho - \sum_{c=1}^{k} s_{ic}(x,\rho)g_c(x) \in \Sigma_{x,\rho} \; i: 1,\dots,r \tag{3.38}$$

$$- 2\rho^T R(x) \left(A_i(x)X(\tilde{x}) - B_i(x)M_j(x) + A_j(x)X(\tilde{x}) - B_j(x)M_i(x)\right) \rho + \rho^T \frac{dX(\tilde{x})}{dx} A_i(x)z(x)\rho - \sum_{c=1}^{k} s_{ijc}(x,\rho)g_c(x) \in \Sigma_{x,\rho} \quad \begin{matrix} i: 1,\dots,r \\ r \geq j > i \end{matrix} \tag{3.39}$$

where $\epsilon > 0$ acts as a tolerance, $R(x) = \frac{dz(x)}{dx} \in \mathcal{R}_x^{v\times n}$ and $s_i(x) \in \Sigma_x$, $(s_{ic}(x,\rho), s_{ijc}\ (x,\rho)) \in \Sigma_{x,\rho}$, then controller (3.35) *stabilizes system* (3.33) *in a region of the state space $V_c \subset \Omega$ by Lemma 2.1. Controller gains can be obtained by $K_j(x) = M_j(x)X(\tilde{x})^{-1}$.*

Proof omitted for brevity, see Appendix B.3.

Remark 3.4. Note that conditions (3.39) may be relaxed via dimensionality expansion or via artificial decision variables by using Polya's theorem (Sala and Ariño, 2007a).

Other state-feedback design criteria (such as decay rate, $\mathcal{H}_\infty$, etc) may also be adapted to the fuzzy polynomial case. Details and examples omitted for brevity, see Prajna, Papachristodoulou and Wu (2004b); Tanaka, Ohtake and Wang (2009b).

The discrete-time case can be addressed by using the discrete-time fuzzy polynomial model (3.34) and replacing conditions (3.38) and (3.39) by

$$\rho^T \begin{pmatrix} X(\tilde{x}) - \sum_{c=1}^{k} s_{ic}(x,\rho)g_c(x)I & (*) \\ T(\tilde{A}(x)x)(A_i(x)X(\tilde{x}) - B_i(x)M_i(x)) & X(\tilde{A}(x)x) \end{pmatrix} \rho \in \Sigma_{x,\rho} \quad i: 1,\dots,r \tag{3.40}$$

$$\rho^T \begin{pmatrix} X(\tilde{x}) - \sum_{c=1}^{k} s_{ijc}(x,\rho)g_c(x)I & (*) \\ \frac{1}{2}T(\tilde{A}(x)x)(\Xi_{ij}(x) + \Xi_{ji}(x)) & X(\tilde{A}(x)x) \end{pmatrix} \rho \in \Sigma_{x,\rho} \quad \begin{matrix} i: 1,\dots,r \\ r \geq j > i \end{matrix} \tag{3.41}$$

respectively, where $\Xi_{ij}(x) = A_i(x)X(\tilde{x}) - B_i(x)M_j(x)$, $(s_i(x), s_{ic}(x,\rho), s_{ijc}(x,\rho))$ are as in Theorem 3.1, $\tilde{x}$ are the states whose time derivative does not depend on u and does not contain non-polynomial nonlinear terms, $\tilde{A}(x)$ is a matrix formed by the rows of $\dot{\tilde{x}}$ and $T(\tilde{x})$ is a polynomial matrix defined by $z(x) = T(\tilde{x})x$. Proof omitted for brevity, for details see Tanaka, Ohtake and Wang (2008).

3.3.2 Observer design

State-feedback controllers, like the one proposed in the above section, require the complete knowledge of the whole system state. However, measuring all the states is not often possible, so a state observer is needed in order to estimate the unmeasurable states.

Observers for nonlinear systems have been developed in literature by different ways (Koenig, 2006; Howell and Hedrick, 2002; Arcak and Kokotovic, 2001). Those achieve better performance than classical LTI designs but they are only focused on some kind of nonlinear systems and the obtained gains are still constant matrices.

The work in Ichihara (2009a) presented an observer design methodology for non-fuzzy polynomial systems under vanishing disturbances. In such approach, the observer gains are allowed to depend polynomially on the measurements and estimated states, see below.

Consider the polynomial system:

$$\dot{x} = p(x) + E(x)w \quad y = C(x) \tag{3.42}$$

where $x \in \mathbb{R}^n$ is the state vector and $w \in \mathbb{R}^d$ are process disturbances in this case. Then, a state observer in order to estimate state x from the measurement y, is on the form

$$\dot{\hat{x}} = p(\hat{x}) + L(y,\hat{x})(y - \hat{y}) \tag{3.43}$$

where $\hat{x}$ denotes the estimated state and $L(y,\hat{x})$ is the set of polynomial observer gains, on the measured output and estimated state, to be computed.

Also make the following assumptions:

- Consider vanishing disturbances bounded by $\|w\|_2^2 < \beta$, i.e., limited energy.
- Initial estimation error is equal to initial state, as $\hat{x}(0) = 0$ is freely assignable.

- System trajectories are ensured to remain in an operating region $\chi_x = \{x : 1 - x^T S_x x > 0, S_x \succ 0\}$. The system is assumed stable or there exist a control law stabilizing it.

- There exist a maximum allowed region where the estimation error $e = x - \hat{x}$ must remain in:

$$\chi_e = \{e : 1 - e^T S_e e > 0, S_e \succ 0\}, \quad \chi_e(0) \subset \chi_x$$

  Holding the error inside this region is one of the objectives of the observer design.

Consider a quadratic candidate Lyapunov function on the error $V(e) = e^T Q e$ and consider the initial error starting from points inside the region defined by

$$\chi_e(0) = \{e \in \mathbb{R}^n : V(e) \leq \alpha,\ \alpha > 0\}$$

Then the following result can be applied.

Theorem 3.2 (Ichihara (2009a)). *If the following SOS problem is feasible with* $(s_{21}, s_{22}) \in \Sigma_{x,\hat{x}}$ *and* $s_{11}, s_{12}, s_{31} \in \Sigma_x$

$$x^T P x - \epsilon(x) - s_{11}\chi_x - s_{12}\chi_e \in \Sigma_x \tag{3.44}$$

$$\begin{bmatrix} \Psi(x,\hat{x}) & (*)^T \\ E(x)^T Q e & I \end{bmatrix} \in \Sigma_{x,\hat{x}} \tag{3.45}$$

$$x^T Q x - \alpha - \beta + s_{31}\chi_x \in \Sigma_x \tag{3.46}$$

where

$$\Psi(x,\hat{x}) = -2e^T \left(Q(p(x) - p(\hat{x})) - H(y,\hat{x})(C(x) - C(\hat{x}))\right) - s_{21}\chi_x - s_{22}\chi_e$$

then the polynomial observer gains can be obtained by $L(y,\hat{x}) = Q^{-1}H(y,\hat{x})$ *and trajectories of the estimation error starting from* $\chi_e(0)$ *will remain in the region* $\{e : V(e) \leq \alpha + \beta\}$.

Proof and details omitted for brevity. The reader is referred to the cited source.

Once fuzzy polynomial models are available, design methodologies for TS systems, previously summarized on Chapter 2, can be combined with the above polynomial ones in order to obtain fuzzy polynomial observers. A first approach was presented by Tanaka, Ohtake, Wada, Wang and Ying-Jen (2009a) designing the controller and observer simultaneously under some assumptions about system's structure.

Consider the fuzzy polynomial system:

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{r} \mu_i(z)(A_i(y)x + B_i(y)u) \\ y &= \sum_{i=1}^{r} \mu_i(z)C_i x \end{aligned} \tag{3.47}$$

where u is the control input, z are *known* premise variables and the matrices A_i, B_i are assumed to depend only on the measurements. In addition, consider the following polynomial PDC control law

$$u = -\sum_{i=1}^{r} \mu_i(z)K_i(y)\hat{x} \tag{3.48}$$

instead of the general (3.35).

Then, a state observer in order to estimate state x from the measurement y, is proposed of the form

$$\dot{\hat{x}} = \sum_{i=1}^{r} \mu_i(z)(A_i(y)\hat{x} + B_i(y)u + L_i(y)(y - \hat{y})) \tag{3.49}$$

where $\hat{x}$ denotes the estimated state and L_i is the set of polynomial observer gains on the measured output to be computed.

Thanks to the above assumptions, the separation principle can be applied in order to compute the controller and observer gains together, ensuring stability on the whole closed loop.

Consider a candidate Lyapunov function:

$$V(\hat{x}, e) = \begin{bmatrix} \hat{x} & e \end{bmatrix} \begin{bmatrix} \lambda Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} \hat{x} \\ e \end{bmatrix}, \quad \lambda > 0$$

Theorem 3.3 (Tanaka, Ohtake, Wada, Wang and Ying-Jen (2009a))**.** *The overall control system consisting of* (3.47)*,* (3.48) *and* (3.49)*, is stable and the estimated states converge to the real ones if the following SOS problem is feasible*

$$x^T(X_1 - \epsilon_1 I)x \in \Sigma_x \tag{3.50}$$

$$x^T(Q_2 - \epsilon_1 I)x \in \Sigma_x \tag{3.51}$$

$$-2v^T(A_i(y)X_1 - B_i(y)M_i(y) - \epsilon_2(y)I)v \in \Sigma_{y,v}\ \forall i \tag{3.52}$$

$$-2v^T(Q_2 A_i(y) - N_i(y)C_i - \epsilon_2(y)I)v \in \Sigma_{y,v}\ \forall i \tag{3.53}$$

$$- 2v^T(A_i(y)X_1 - B_i(y)M_j(y) + A_j(y)X_1 \\ - B_j(y)M_i(y) - \epsilon_2(y)I)v \in \Sigma_{y,v} \; i < j \quad (3.54)$$

$$- 2v^T(Q_2A_i(y) - N_i(y)C_j + Q_2A_j(y) \\ - N_j(y)C_i - \epsilon_2(y)I)v \in \Sigma_{y,v} \; i < j \quad (3.55)$$

where $v \in \mathbb{R}^n$ are variables independent of y, $\epsilon_1 > 0$ and $\epsilon_2(y) > 0$ when $y \neq 0$ and is 0 when $y = 0$. The polynomial controller and observer gains can be obtained by $K_i(y) = M_i(y)X_1^{-1}$ and $L_i(y) = Q_2^{-1}N_i(y)$ respectively.

Proof and details omitted for brevity. The reader is referred to the cited source.

This preliminary result has been extended recently in Tanaka, Ohtake, Seo, Tanaka and Wang (2012), where both controller and observer can be designed together only requiring matrices B_i to be independent from unmeasurable states

$$\dot{x} = \sum_{i=1}^{r} \mu_i(z) \{A_i(x)x + B_i(\xi)u\} \quad (3.56)$$

where ξ is a measurable time-varying vector that may be external variables, outputs, and/or time. The general case where $A_i(x)$, $B_i(x)$ is addressed too but, as in Ichihara (2009a), the controller has to be designed first assuming that all states are known. Afterwards, the observer needs to be designed guaranteeing the stability for the whole closed loop. So, sequential design might be suboptimal.

3.4 Discussion

The use of fuzzy polynomial techniques allows reducing conservativeness (comparing to fuzzy TS ones) in the modelling phase as well as the stability analysis: the Lyapunov function lifts the strong constraint of being defined by a quadratic curve. However, some issues are still inherited from TS methodologies: the shape of the membership functions is often forgotten in the resulting SOS stability conditions and the derivative of the Lyapunov function is required to be strictly negative in the whole modelling region, so there cannot be other equilibrium points or limit cycles inside the considered region. Recent literature focuses on some of those problems: Narimani and

Lam (2010) introduce membership information in the SOS conditions by approximating those functions with polynomials in subregions of the modelling region; Lam, Narimani, Li and Liu (2013) divide also the modelling region in user-defined subspaces and they use piecewise polynomial Lyapunov functions in order to reduce conservativeness, and Chesi (2011) checks stability conditions ($V > 0, \dot{V} < 0$) only in the largest Lyapunov level set to be found inside the modelling region, but leading to a nonconvex BMI problem.

The case of synthesis is even more misleading, except for only-polynomial systems. The polynomial controller allows a more complex structure to control the nonlinear system but, due to the changes of variable required in Schur complement in order to put the problem in convex form, the information about the states in $z(x)$ is lost on SOS conditions. As a consequence, the local results developed for stability analysis cannot be applied directly to the synthesis problem for obtaining domain-of-attraction guarantees, but only in a posteriori analysis. In addition, the Lyapunov functions used for design cannot be of arbitrary degree in the full-state variables, if the problem is desired to remain convex (Lam, Narimani, Li and Liu, 2013; Tanaka, Ohtake and Wang, 2008). Therefore, the great advantage of fuzzy-polynomial techniques over classical TS ones for stability analysis of nonlinear systems often vanishes for controller synthesis.

The review on polynomial observer design literature shows that the problem is still addressed preliminarily: Ichihara (2009a) uses Positivstellensatz theorem to state local SOS design conditions but the approach is only for polynomial systems and vanishing disturbances. On the other hand, Tanaka, Ohtake, Seo, Tanaka and Wang (2012) address the problem for fuzzy polynomial systems but the controller/observer joint design forces taking strong assumptions on system's structure. In addition, the approach does not consider neither disturbances nor local conditions, so the observer, if feasible, is valid in all the state space. Even this might be an advantage, the usual case is obtaining global infeasibility, because proving a polynomial being SOS for all x is difficult (see Appendix A.2.1 and A.2.2). For those reasons, the approach of Tanaka, Ohtake, Seo, Tanaka and Wang (2012) is limited when trying to obtain a successful solution in most practical cases.

3.5 Conclusions

In this chapter, a fuzzy-modelling methodology where the consequents are polynomial vertex models is used in order to represent nonlinear systems in an exact way. Those fuzzy-polynomial modelling techniques have been proved to generalize the classical TS ones but having the advantage of obtaining vertex models which fit more precisely the existent nonlinearities. Therefore, the existent conservativeness associated to the use of those models for analysis is reduced as polynomial degrees increase.

Stability analysis for such models has been addressed from both, global and local points of view. Also common methodologies in order to give a local estimate of the domain of attraction using fuzzy polynomial systems were outlined. In addition, typical controller and observer designs from fuzzy TS literature have been translated to the fuzzy polynomial framework.

The conclusion from the review of the state of the art is that there are still some open problems without an extensive analysis in both, local domain-of-attraction estimation and controller/observer design for nonlinear systems using fuzzy polynomial techniques, as discussed in the above section. Some of these open issues are the starting point for the contributions presented in this thesis. In particular, the problems of

- lifting the strictly-negativeness constraint of the Lyapunov-function derivative inside the whole modelling region for stability analysis,
- using more general Lyapunov functions for controller synthesis,
- and employing local conditions to present the fuzzy-polynomial observer design under disturbances in an unified way,

will be further addressed on the following chapters. Moreover, the stability analysis for nonlinear systems under disturbances is quite disregarded in present fuzzy-polynomial literature. This thesis will also address this problem, albeit in a preliminary way because not all the results of interest are finally cast as convex problems.

Part II

Contributions

Chapter 4

Domain of Attraction Estimation

True stability results when presumed order and presumed disorder are balanced. A truly stable system expects the unexpected, is prepared to be disrupted, and waits to be transformed.

Tom Robbins

ABSTRACT: A common approach for stability and performance analysis in TS/polynomial fuzzy models is to solve an LMI problem which proves global or local stability and an upper bound of a given performance function for any shape of membership functions. However, fuzzy models are local and, obviously, a larger modelling region yields lower performance bounds. This chapter analyzes the common problems in local stability and domain of attraction estimation literature. Then, several ideas are presented and developed in order to relax classical constraints, thus reducing conservativeness and obtaining better estimates.

Many results in present literature address the stability analysis problem for nonlinear systems using fuzzy TS or polynomial-modelling techniques (TS are simply a special case of polynomials). In particular, Lyapunov-based methods are very extended in order to prove local stability of the original nonlinear system and to obtain a *conservative* estimate of the domain of attraction. However in many of the literature results (usually those which are addressing the stabilization problem), once a feasible Lyapunov function is found, the analysis of the proved domain of attraction is forgotten. The case of DA estimation with disturbance analysis is even more disregarded in literature.

However, domain of attraction estimation is an important problem when a precise stability analysis is to be carried out. Indeed, this problem has recently received an increasing interest within the polynomial framework (Chesi, 2009; Chakraborty, Seiler and Balas, 2011; Chesi, 2011; Hancock and Papachristodoulou, 2013; Khodadadi, Samadi and Khaloozadeh, 2014). In particular, analyzing the *proven* performance obtained in a controller synthesis is an interesting point to be taken into account. For instance, an ill-shaped Lyapunov function might result in state trajectories with an overshoot greater than the initial conditions. Obtaining a simple *closed form* characterizing the domain of attraction is also required if the obtained estimate is going to be used for carrying out some further computations or designs.

In this chapter, some novel contributions to the DA estimation are presented and developed. They address some problems which usually arise from existent approaches in literature (see Section 3.4). The work has led to several conference and journal contributions, which are summarized on the following sections.

This chapter is organized as follows: next section summarizes briefly the existent drawbacks in literature related to DA estimation and highlights which of them are going to be addressed in this chapter, Section 4.2 presents some stability analysis results with the objective of finding the largest prefixed-shape DA estimate, an unified iterative approach to enlarge the DA estimate is presented on Section 4.3, some examples are included on Section 4.4 in order to show the achieved improvement with the proposed results and, finally, Section 4.5 gives the conclusions of this work.

4.1 Preliminaries

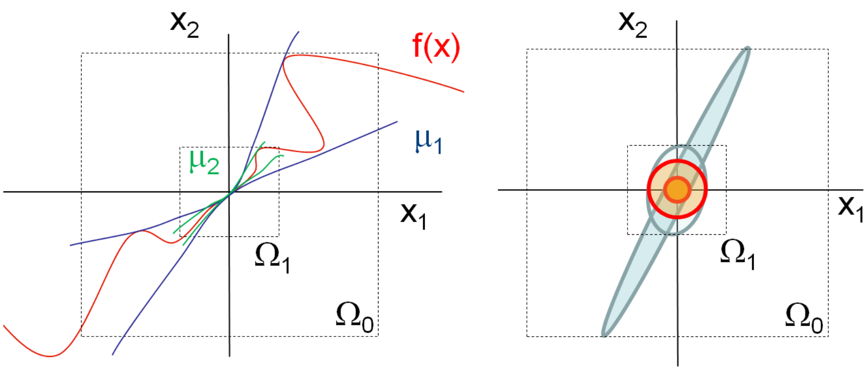
In the major part of *level-set* based DA estimation contributions in literature, LMI and SOS stability conditions are stated in order to prove global (or local) stability and performance of fuzzy systems (Lemmas 3.2, 3.3 and 3.4 or similar). However, such techniques give only a positive or negative response to the stability analysis problem but they often forget to analyze the region of validity for those results. In case of unsatisfactory Lyapunov-function search (but the linearized system at the origin is stable, 1^{st} Lyapunov theorem), the ways to explore proposed in literature are:

- Try with a higher-degree global Lyapunov function. This can quickly exhaust computational resources (Prajna, Papachristodoulou and Wu, 2004b).

- Remodel the nonlinear system in a smaller region Ω in order to reduce the "distance" between vertex models (see Figure 4.1a: modelling in Ω_1 gives closer bounds than those in Ω_0) and try to find a new Lyapunov function (Sala and Ariño, 2006).

- Add shape information, if it is possible, by using polynomial constraints: splitting the state space in regions Ω_d where a set of polynomial constraints $r_d(x, \mu)$ hold. They can be added to Lyapunov conditions as explained for TS systems in Sala and Ariño (2008) or in the polynomial framework (bounds on partial derivatives) in Bernal, Sala, Jaadari and Guerra (2011a).

- Prove local stability in a region $\Omega = \{x : g_i(x) \geq 0\}$ defined by "d" known polynomial boundaries $g_i(x)$, as discussed on Section 3.2.2.



(a) Remodelling (b) Conditioning

Figure 4.1: Conditioning and model-conservativeness associated problems in local stability study.

In particular, despite of obtaining feasible results with local stability approaches for fuzzy models, there exist still some important sources of conservativeness to discuss:

1. **Existence of larger invariant sets in Ω.** There exist invariant sets in Ω which are *not* level sets of a low-degree polynomial Lyapunov function. Indeed, although a decreasing Lyapunov function $V(x)$ has been found, local stability in the modelling region Ω has not been proved, but only in the largest invariant set $V_\gamma = \{x : V(x) \leq \gamma\}$ (Lyapunov level set) such that $V_\gamma \subset \Omega$ (see Figure 4.1b: the found ellipsoid for Ω_0 is worse conditioned than the one found in Ω_1). Ill-shaped Lyapunov functions may be of little use for practical results if the goal is to certify a stable "large" enough zone around the equilibrium point.

2. **Choice of modelling region.** If the modelling region Ω is chosen too large, the associated Lyapunov conditions may render infeasible (consequents separate too much). From classical Lyapunov theorems, if the linearised system is stable, a "small enough" modelling region will render a feasible problem[1]. The problem, then, is how to choose which is the right modelling region to obtain the largest DA estimate.

3. **Conservative sign conditions.** One of the reasons for infeasibility is requiring $V(x)$ and $-\dot{V}(x)$ (increment of V in the discrete case) to be positive in all Ω. In fact, it would be needed only inside a suitable level set. For instance, if there is more than one equilibrium point in the modelling region Ω, all the theorems and corollaries from Chapter 3 fail as there is one $x \neq 0$ where $\dot{V}(x) = 0$ for any choice of $V(x)$.

Some of these above issues have been addressed partially in literature. For instance, the first one gives rise to piecewise Lyapunov functions in the form $V(x) = \min_i V_i(x)$, etc (Feng, 2004).

For completeness, note that, apart from Lyapunov methods, DA estimation can be done numerically (Genesio, Tartaglia and Vicino, 1985).

4.1.1 Problem statement

Assuming that global stability of the nonlinear system is not feasible, the objective of the stability analysis phases should be obtaining the largest region in which a particular performance bound is attained.

Consider a continuous-time nonlinear dynamic system without disturbances denoted by:

$$\dot{x}(t) = f(x(t)) \tag{4.1}$$

[1] Indeed, if the linearised system is stable, a quadratic $V(x)$ will suffice for small Ω.

or equivalently in discrete-time

$$x_{k+1} = f(x_k) \tag{4.2}$$

where $t = kT_s$ and T_s is the sampling time, $x \in \mathbb{R}^n$ is the state vector and $f(x)$ is a nonlinear function of the state.

The trajectory of a nonlinear dynamic system (4.1) is denoted as $x(t) = \psi(t, x_0)$, where x_0 are the initial conditions at $t = 0$.

A continuous-time fuzzy polynomial model, in a region Ω, from a nonlinear system (4.1) is expressed in the form:

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(z(t))p_i(x(t)) \tag{4.3}$$

or equivalently for a discrete-time system:

$$x_{k+1} = \sum_{i=1}^{r} \mu_i(z_k)p_i(x_k) \tag{4.4}$$

where $x \in \mathbb{R}^n$ is the state vector, $z \in \mathbb{R}^p$ are the premise variables of the membership functions and $p_i(x) \in \mathcal{R}_x^n$ are the polynomial vertex models. The origin $x = 0$ is a *stable* equilibrium point by assumption.

As the objective of this section is local stability analysis, $u = 0$ has been supposed in (4.3), (4.4) and further developments. However, the presented results could estimate the "a posteriori" domain of attraction of closed-loop fuzzy polynomial systems whose controllers were previously designed with the approaches mentioned in Chapter 3. Indeed, if u in (3.33) is replaced by a control law $u = K(x)$, allowing $K(x)$ to have constant or polynomial terms fulfilling $f(0, K(0)) = 0$ (i.e., the origin is an equilibrium point), then the local analysis methodology proposed in this chapter applies.

The local domain of attraction for fuzzy polynomial systems in a region Ω can be defined as follows.

Definition 4.1. *The* ***Local Robust Domain of Attraction*** *(LRDA) of system* (4.3) *or* (4.4)*, referred to a region Ω, will be denoted by $\mathcal{D}_\Omega$. It is defined as the set of the state-space initial conditions whose trajectory $x(t) = \psi(t, x_0)$ never escapes Ω for any x_0, and ends in the asymptotically stable equilibrium point $x = 0$:*

$$\mathcal{D}_\Omega = \left\{ x_0 \in \Omega : \begin{array}{l} \psi(t, x_0, \mu) \in \Omega \forall t \geq 0, \forall \mu \in \Gamma \\ \lim_{t\to\infty} \psi(t, x_0, \mu) = 0, \forall \mu \in \Gamma \end{array} \right\} \tag{4.5}$$

Note that the condition for the trajectories not leaving Ω above is needed as (4.3) and (4.4) are, by assumption, not valid outside Ω.

The term "robust" in the definition is due to the fact that $\mathcal{D}_\Omega$ is defined considering "all" possible μ in the standard simplex Γ and not the particular, possibly non-polynomial, $\mu(z)$ giving actually exact equivalence with the nonlinear system. That allows polynomial techniques to be used at the price of conservativeness. Indeed, based on the above, as (4.3) and (4.4) include the present in-study nonlinear system in Ω (plus many other systems), then obviously, $\mathcal{D}_\Omega \subset \mathcal{D}$: by finding the LRDA from a fuzzy polynomial model, we have found an inner estimate of the DA of the original nonlinear system, as defined in (Chesi, 2011, Chap. 6).

As the actual LRDA may be a very complicated region and hardly characterizable, the goal of this chapter is to obtain an estimate of such LRDA, as large as possible, with a *closed-form* expression given by a low-degree polynomial boundary. The polynomial degree will be chosen depending on the available computing resources. The presented results are proved to outperform existent results in Lyapunov literature.

The main ideas are:

1. Exploring with different modelling regions Ω_d to obtain the largest DA estimate.

2. Setting "regions of interest" smaller than the modelling region Ω in local stability conditions.

3. Lifting the restriction of the DA estimate being a Lyapunov level set.

4. Allowing for more than one equilibrium point (also saddle points or limit cycles) inside the modelling region Ω.

The proposed contributions following those guidelines are presented on next sections. All the results are recast as SOS optimization problems, solvable by standard convex SDP optimization tools.

4.2 Prefixed-shape DA estimation

This work is based on, once a fuzzy polynomial model is obtained from a nonlinear system in a large region Ω, exploring how to obtain local results in a smaller region of interest without the need of remodelling. This idea was first addressed by Pitarch, Ariño, Bedate and Sala (2010) in a preliminary way for

TS systems. This section generalizes those results and gives a deeper study on them by employing fuzzy-polynomial tools.

In this section, the idea is applied to stability analysis and it is shown that ensuring the largest *prefixed-shape* domain of attraction around the origin requires an iterative procedure in the size of the modelling region. The idea applies to sophisticated Lyapunov functions, performance inequalities or to non-polynomial (polytopic) target shapes.

As long as the Jacobian linearization at the origin is (strictly) stable, a local result will be obtained with a polynomial Lyapunov function for a small enough region of interest. If the Jacobian is not strictly stable, the method may fail (as stability of the nonlinear system is not guaranteed in this case, neither the existence of a local Lyapunov function).

In addition, the existence of larger invariant sets inside Ω belonging to the DA, which are not bounded by a low-degree Lyapunov level set is addressed. A first approach in order to lift this restriction is proposed by allowing the Lyapunov level set to leave Ω by some zones of its boundary.

The results presented on this section have been published on several conference papers (Pitarch, Ariño, Bedate and Sala, 2010; Pitarch, Ariño and Sala, 2011; Pitarch, Sala and Ariño, 2012a) and in the journal paper Pitarch, Sala, Ariño and Bedate (2012b).

4.2.1 Local fuzzy polynomial models

In order to analyze local stability of a previously computed fuzzy polynomial model within a region, such original model can be modified using the information of the membership functions.

The idea was presented first by Sala and Ariño (2006) for TS fuzzy models and the goal is avoiding the need of remodelling by expressing the the membership functions $\mu(z)$ as a convex sum of some vectors v_j. This section extends it to the fuzzy polynomial case.

Lemma 4.1. *If the membership functions $\mu(z)$ of a fuzzy polynomial system described as* (4.3) *in a region Ω, can be themselves expressed as a convex sum of some vertex vectors*[2] $v_j \in \mathbb{R}^n,\ j:1,\dots,n_v$, *of elements* $v_{ji},\ i:1,\dots,n$

$$\mu(z) = \sum_{j=1}^{n_v} \beta_j(z) v_j, \quad \forall\, z \in \Omega \tag{4.6}$$

[2] The vertices v_j can be easily obtained computing the minimum and maximum bounds of μ_i in Ω. See Note 4.2.1.

where

$$\mu(z) = [\mu_1(z), \mu_2(z), \ldots, \mu_n(z)]$$

$$\sum_{j=1}^{n_v} \beta_j(z) = 1, \ \beta_j(z) > 0 \, \forall z \in \Omega \quad j : 1, \ldots, n_v$$

then, for polynomial fuzzy models, the system can be transformed to

$$\dot{x} = \sum_{j=1}^{n_v} \beta_j(z) p_j(x)^*, \quad \forall z \in \Omega \tag{4.7}$$

where the new vertex polynomial vectors are:

$$p_j(x)^* = \sum_{i=1}^{n} v_{ji} p_j(x) \tag{4.8}$$

Proof. The expression (4.6) can be substituted in the system equation (4.3):

$$v_j = [v_{j1}, v_{j2}, \ldots, v_{jn}] \quad \rightarrow \quad \mu_i(z) = \sum_{j=1}^{n_v} \beta_j(z) v_{ji}$$

$$\dot{x} = \sum_{i=1}^{n} \sum_{j=1}^{n_v} \beta_j(z) v_{ji} p_i(x) \quad \rightarrow \quad \dot{x} = \sum_{j=1}^{n_v} \beta_j(z) \sum_{i=1}^{n} v_{ji} p_i(x)$$

so the local representation of the system in Ω is (4.7). □

An equivalent result can be applied with discrete-time systems (4.4).

The advantage of this result is to express a fuzzy polynomial model in a region $\Omega_d \subset \Omega$, i.e., smaller than the original, without the need of remodelling again from the nonlinear system. The convex-combination conditions for the membership functions required in the above lemmas are easy to meet. Indeed μ_i are assumed to be known in fuzzy systems. Then, the result below may be applied to obtain a (possibly conservative) vertex set.

Note. Let us consider a region $\Omega_d \subset \Omega$. If bounds μ_i^M and μ_i^m on the extreme values of the membership functions in Ω_d can be computed, in such a way that:

$$\mu_i^M \geq \max_{z \in \Omega_d} \mu_i(z) \quad \mu_i^m \leq \min_{z \in \Omega_d} \mu_i(z) \tag{4.9}$$

then there exist a set of $\beta_j(z)$, $j = 1, \ldots, n_v$ so that the vector of membership functions

$$\mu(z) = [\mu_1(z), \mu_2(z), \ldots, \mu_n(z)]$$

may be expressed in Ω as (4.6).

Indeed, the linear restrictions $\mu_i^M \geq \mu_i \geq \mu_i^m$, $\sum_i \mu_i = 1$ describe a bounded polytope with a finite number of vertices, so well-known linear-programming-related methods to obtain the membership-vector vertices may be used (related to the obtention of the basic feasible solutions in an LP problem). See Luenberger (2003) for details.

4.2.2 Local stability analysis

Local stability results may be obtained by stating the lemmas presented in Section 3.2 with the transformed models discussed above.

Lemma 4.2. *The region* $\Theta \subset \Omega_d$ *defined as:*

$$\Theta = \{x : V(x) \leq V_M,\ V(x) > 0\} \tag{4.10}$$

belongs to the LRDA of the equilibrium point $x = 0$ *of the system* (4.3) *if*

$$V_M \quad \leq \quad min\{V(x) : x \in\ \partial\Omega_d\} \tag{4.11}$$

where $\partial\Omega_d$ *denotes the boundary of* Ω_d *and* $V(x)$ *verifies Lyapunov stability conditions:*

$$V(x) > 0\ \forall x \in \Omega_d \backslash \{x = 0\} \tag{4.12}$$

$$-\frac{dV}{dx} p_j(x)^* > 0\ \forall x \in \Omega_d \backslash \{x = 0\} \quad j : 1, \ldots, n_v \tag{4.13}$$

i.e., all trajectories with initial state in Θ *converge asymptotically to* $x = 0$.

Proof. As, by Lemma 4.1, the system can be expressed in Ω_d as (4.7), if conditions (4.12) and (4.13) are feasible for a function $V(x)$, then a Lyapunov function proving global asymptotic stability for (4.3) is found, and Θ is an invariant set (Khalil, 2002, Chap. 4.2).

As the expression of the local system (4.7) is not valid outside Ω_d, then the local stability can only be proved in the biggest region Θ with boundary $V(x) = V_M$ contained in Ω_d. This region will be defined by a value of V_M equal to the minimum value of $V(x)$ in the boundary of Ω_d ($\partial\Omega_d$):

$$V_M = \min_{x \in \partial\Omega_d} V(x).$$

□

The following lemma is useful in order to determine when a region characterized by a polynomial curve (for instance a Lyapunov level set) contains a prefixed-shape region Φ.

Lemma 4.3. *Let Φ be a region defined by the intersection of previously-fixed n_r polynomials $k_i(x)$*

$$\Phi = \{x : k_i(x) \geq 0, \quad i = 1, \ldots, n_r\}, \tag{4.14}$$

such that the interior of Φ contains the equilibrium point $x = 0$ and its frontier is the union of sets defined by:

$$B_i = \{k_i(x) = 0, k_j(x) \geq 0 \; i \neq j\} \quad i : 1, \ldots, n_r \tag{4.15}$$

Let $\mathcal{M}(k_1, \ldots, k_{n_r})$ be the multiplicative monoid generated by the constraints B_i, let $\mathcal{I}(k_i)$ be the ideal generated by the polynomials $k_i(x)$ associated to B_i and let $\wp(k_1, \ldots, k_{n_r})$ be the cone generated by the set of polynomials which define the region Φ. Then, the region $\Theta = \{x : q(x) \leq 1\}$ contains the frontier of Φ if the following n_r SOS conditions hold:

$$-(q(x) - 1) - F_i(x) - G_i(x) \in \Sigma_x \quad i : 1, \ldots, n_r \tag{4.16}$$

Where $F_i(x) \in \wp(k_1, \ldots, k_{n_r})$ and $G_i(x) \in \mathcal{I}(k_i)$.

Proof. If there exist polynomials $F_i(x) \in \wp(k_1, \ldots, k_{n_r})$, $G_i(x) \in \mathcal{I}(k_i)$ and $g \in \mathcal{R}_x$ verifying

$$(q(x) - 1) + F_i(x) + G_i(x) + g^2 = 0$$

then the set of values $\{x : q(x) - 1 > 0\} \cap B_i$ is empty by Theorem A.1. This can be proved by performing a sign change, if (4.16) is feasible. □

Definition 4.2. *Given a region with predefined shape defined as a set of polynomial constraints*

$$\Phi_1 = \{x : k_1(x) \geq 0, \ldots, k_{n_r}(x) \geq 0\}, \tag{4.17}$$

the set $\Phi_2 = \{x : k_1(\lambda^{-1}x) \geq 0, \ldots, k_{n_r}(\lambda^{-1}x) \geq 0\}$ will be named as scaled region *and λ will be denoted as* ***scale factor***. *For simplicity, the notation $\Phi_2 = \lambda\Phi_1$, $\bar{k}_i(x) = k_i(\lambda^{-1}x)$ will be sometimes used in later developments.*

The overall problem will consider the full region of interest Ω, where the model is valid, plus a predefined-shape one to be enlarged as much as possible in its interior. This scalable predefined-shape region, stated in Definition 4.2, must belong to the DA of the nonlinear system for the maximum achievable scale factor. In order to ensure this objective, Lemma 4.3 will be applied to force the predefined-shape region to be contained in a Lyapunov level set. In addition, a similar condition to the considered in (4.16) will be used in order to force this Lyapunov level set to be contained in $\Omega_d \subset \Omega$. Let us detail the procedure below.

Consider a region of the state space Ω_d defined by n_g polynomial inequalities as follows:

$$\Omega_d = \{x : g_1(x) \geq 0, \ldots, g_{n_g}(x) \geq 0\} \tag{4.18}$$

Theorem 4.1. *Consider system* (4.3) *and let $V(x)$ be a predefined-degree candidate polynomial Lyapunov function. The region with prefixed shape $\Phi_2 = \{x : k_i(\lambda^{-1}x) \geq 0\}$ and scale factor λ, according to Definition 4.2, belongs to the local robust domain of attraction of $x = 0$ in a region of interest Ω_d, defined on* (4.18)*, if the following SOS problem is feasible:*

$$1 - V(x) - F_{1m}(x) + G_{1m}(x) \in \Sigma_x \quad m : 1, \ldots, n_r \tag{4.19}$$

$$V(x) - 1 - F_{2i}(x) + G_{2i}(x) \in \Sigma_x \quad i : 1, \ldots, n_g \tag{4.20}$$

$$V(x) - \epsilon(x) - T_0(x) \in \Sigma_x \tag{4.21}$$

$$-\frac{dV(x)}{dx} p_j(x)^* - \epsilon(x) - T_j(x) \in \Sigma_x \quad j : 1, \ldots, n_v \tag{4.22}$$

Where $F_{1m}(x) \in \wp(\bar{k}_i, \ldots, \bar{k}_{n_r})$, $G_{1m}(x) \in \mathcal{I}(\bar{k}_m)$, $(F_{2i}, T_0(x), T_j(x)) \in \wp(g_i, \ldots, g_{n_g})$, $G_{2i}(x) \in \mathcal{I}(g_i)$ and $\epsilon(x)$ is a radially unbounded positive polynomial. Moreover, the region $\Theta = \{x \in \Omega_d : V(x) \leq 1\}$ belongs also to the LRDA of $x = 0$.

Proof. Conditions (4.21)-(4.22) imply (4.12) and (4.13) locally in Ω_d by Theorem A.1 and the fact that $\beta_i \geq 0$, $\sum_{i=1}^{r} \beta_i = 1$. Therefore, $V(x)$ is a Lyapunov function by Lemma 3.2 and, jointly with conditions (4.20), the region

$$\Theta = \{x \in \Omega_d : V(x) \leq 1\} \subset \mathcal{D}_\Omega$$

because conditions (4.20) force $V(x) \geq 1$ over the boundary of Ω_d by Theorem A.1. This is achieved, in a similar way to Lemma 4.3, by forcing $V(x) \geq 1$ in each $g_i(x) = 0, g_1(x) \geq 0, \ldots, g_{n_g}(x) \geq 0$, $i : 1, \ldots, n_g$.

By condition (4.19), the region Φ_2 with size λ is contained in the region $\Theta = \{x \in \Omega_d : V(x) \leq 1\}$ by Lemma 4.3. Again by conditions (4.20), the region $\Phi_2 \subset \Theta \subset \Omega_d$, and being $V(x)$ a Lyapunov function, $\Phi_2 \subset \mathcal{D}_\Omega$. □

The discrete-time case can be also addressed with Theorem 4.1 by using system (4.4) instead of (4.3) and replacing (4.22) by:

$$V(x) - V(p_j(x)^*) - \epsilon(x) - T_j(x) \in \Sigma_x \quad j : 1, \ldots, n_v \tag{4.23}$$

However degree in the involved polynomials increases considerably by composition of functions in $V(p_j(x)^*)$, instead of the product of them in the continuous-time term $-\frac{dV(x)}{dx} p_j(x)^*$.

Note that optimization problem presented in Theorem 4.1 is a convex SOS one with λ fixed a priori.

The above result can be used in order to obtain a "well-shaped" estimate of the RLDA by solving a generalized SOS problem (Appendix A.2.4) (which are closely related to the well-known procedure of GEVP optimization problems in the LMI case), maximizing the scale factor λ by bisection search or other similar methods. Higher considered degrees for $V(x), F(x), G(x)$ and $T(x)$ will give larger estimates, as more degrees of freedom are provided to the optimization problem.

Despite of maximizing the size of the chosen prefixed-shape region Φ_2, obtaining the largest provable DA estimate (for fixed maximum degree of $V(x), F(x), G(s)$ and $T(x)$ in Theorem 4.1) depends also on the modelling-region size. Indeed, the smaller Ω_d is, the smaller the provable DA is. However, the larger Ω_d is, the more conservative the fuzzy model becomes, so the provable DA might be small too (large Ω_d might result in infeasible SOS conditions).

If the objective is to obtain the largest DA estimate of the original nonlinear system, an exploration in the modelling-region size is required, as first noted in Pitarch, Ariño, Bedate and Sala (2010). Examples 4.4.1 and 4.4.2 in Section 4.4 illustrate this nonlinear behaviour in the modelling-region size.

Iterative algorithm

The results in previous sections may be combined in order to obtain an algorithm to compute, for example, an estimate of the largest spherical region around $x = 0$ with radius λ, $\Phi = \{x : (x^T \lambda^{-2} x \leq 1\}$, for which attraction is ensured. This can be checked, for instance, by choosing a modelling region defined by a symmetric polytope containing $x = 0$

$$\Omega = \{x : (a_i^T\, x)^2 \leq 1 \quad i : 1, \ldots, n_g\} \tag{4.24}$$

and a set of smaller regions of interest Ω_d with scale factor ρ:

$$\Omega_d = \left\{ x : \left(\frac{a_i^T}{\rho}\, x \right)^2 \leq 1 \quad i : 1, \ldots, n_g \right\} \tag{4.25}$$

As Φ must be contained in Ω_d, the maximum radius λ_{max} can be computed "a priori" as a LMI problem in the more general case (Boyd, Ghaoui, Feron and Balakrishnan, 1994). This is a particular case of Theorem 4.1, nevertheless other kind of geometries for Ω_d and Φ can be applied.

Basically, the procedure will first check the extreme cases; [1] checking for feasibility of SOS problems as stated in Section 3.2, [2] checking for stability of the linearised model around $x = 0$.

Then, in case that [1] will not be feasible or it will lead in poor results but [2] will be stable or marginally stable (the nonlinear model might be locally stable (Khalil, 2002, Chap. 8)), the following algorithm is proposed to obtain better results.

Algorithm 4.1. *Consider the system* (4.3) *obtained in a large region* Ω.

1. *Define a new region* $\{\Omega_d \subset \Omega : \Omega_d = \rho\Omega\}$ *with a initial* $\rho = \rho_0 \cong 0$ *and express the system* (4.3) *in* Ω_d *as* (4.7) *using Lemma 4.1.*
2. *Fix a maximum degree for the polynomial Lyapunov function* $V(x)$.
3. *Solve the following problem:*

   Maximize λ, $0 \leq \lambda \leq \rho\lambda_{max}$, subject to

$$V(x) - \epsilon(x) + \sum_{i=1}^{n_g} \upsilon_i(x) \left(\left(\frac{a_i^T}{\rho}\, x \right)^2 - 1 \right) \in \Sigma_x \tag{4.26}$$

$$V(x) - 1 + \phi_{ii}(x) \left(\left(\frac{a_i^T}{\rho}\, x \right)^2 - 1 \right) + \sum_{j=1}^{n_g} \phi_{ij}(x) \left(\left(\frac{a_j^T}{\rho}\, x \right)^2 - 1 \right) \in \Sigma_x \qquad \begin{matrix} i : 1, \ldots, n_g \\ j \neq i \end{matrix} \tag{4.27}$$

$$1 - V(x) + \chi_0(x)(x^T\lambda^{-2}x - 1) \in \Sigma_x \tag{4.28}$$

$$-\frac{dV}{dx}p_j(x)^* - \epsilon(x) + \sum_{i=1}^{n_g}\psi_{ij}\left(\left(\frac{a_i^T}{\rho}x\right)^2 - 1\right) \in \Sigma_x \quad j: 1, \ldots, n_v \tag{4.29}$$

with multipliers $(v_i, \phi_{ij}, \psi_{ij}) \in \Sigma_x$ *and* $(\phi_{ii}, \chi_0) \in \mathcal{R}_x$.

4. *Increase the zoom factor* $\rho = \rho + \Delta\rho$ *if* $\rho < 1$ *and return to step 2. In case of reaching* $\rho = 1$ *or if the problem was infeasible, the algorithm ends.*

This procedure maximize the radius of the quadratically-invariant sphere contained in Ω_d, which belongs to the LRDA, by solving a generalized SOS problem in λ (Appendix A.2.4) and performing an exploration in ρ. Indeed, such a maximum-size sphere may appear at an intermediate point, as examples 4.4.1 and 4.4.2 in Section 4.4 will later show. Hence, no bisection or related idea is applicable in principle, as the radius provable with the above theorem sometimes increases when increasing the domain of interest and some others decreases, with local maxima and minima.

Better results might be obtained by increasing the degree of the Lyapunov function and/or Positivstellensatz multipliers, by adding more multipliers with cross products of the region constraints or by exploring regions Ω_d different from a symmetrically scaled Ω.

4.2.3 Domain of attraction expansion

The proposed approach and others mentioned in literature, obtain conservative results in the sense that the proven local domain of attraction is the largest level set of the Lyapunov function $V(x) < V_M$ contained in Ω_d. However, as previously mentioned on Section 4.1.1, there may be points outside $V(x) < V_M$ which belongs to $\mathcal{D}_\Omega$ too.

Results can be improved in the same region of interest Ω_d, using the obtained Lyapunov function $V(x)$, and without increasing computational complexity of LMI/SOS problems.

Usually, the largest level set of the Lyapunov function contained in the modelling region Ω_d defines the LRDA, because outside this level set a trajectory that leaves Ω_d is possible. However, it is not necessary to take into account all the possible trajectories, only such ones which will leave Ω by some *forbidden* regions of the boundary (Khalil, 2002, Chap. 8.2). The following lemma provides the optimization problem which ensures that any trajectory starting from points inside a region Θ will not touch these forbidden regions and, therefore, $\Theta \subset \mathcal{D}_\Omega$.

Lemma 4.4. *Given a fuzzy modelling region Ω_d defined as* (4.18)*, with the known sets B_i defining its boundary ($\partial\Omega_d$),*

$$B_i = \{g_i(x) = 0, g_j(x) \geq 0 \; j \neq i\} \quad i : 1, \ldots, n_g$$

and given a polynomial Lyapunov function $V(x)$ obtained with Theorem 4.1, or any other procedure, the largest region proved to belong to the local domain of attraction, using SOS techniques, is obtained by solving the following problem:

Maximize γ subject to

$$V(x) - \gamma - F_i(x) - G_i(x) \in \Sigma_x \quad i : 1, \ldots, n_g \tag{4.30}$$

for polynomials $F_i(x) \in \wp(-\dot{g}_i, g_1, \ldots, g_{n_g})$ and $G_i(x) \in \mathcal{I}(g_i)$.

The largest proven local domain of attraction Θ is:

$$\Theta = \{x \in \Omega_d : V(x) \leq \gamma\} = \{x : \min\left(\gamma - V(x), g_i(x)\right) > 0\} \tag{4.31}$$

Proof. Let us define the *allowed* regions in the frontier of Ω_d ($\partial\Omega_d$) as

$$R_i = \{x \in \partial\Omega_d : \dot{g}_i(x) > 0, g_i(x) = 0, g_j(x) \geq 0\} \quad \begin{matrix} i : 1, ..., n_g \\ j \neq i \end{matrix} \tag{4.32}$$

i.e., the boundaries where system trajectories point to inside Ω_d. Therefore, there is no need for $\Theta \subset \Omega_d$ if it does not intersect with any *forbidden* region, i.e., the zones of the frontier which do not belong to any R_i. Thus, the objective is to prove the maximum γ such that:

$$\{x : V(x) \leq \gamma, x \in B_i\} \Rightarrow \dot{g}_i(x) > 0 \quad i : 1, \ldots, n_g \tag{4.33}$$

With such γ, (4.31) defines an invariant set. Indeed, parts of $\partial\Theta$ are in $\partial\Omega_d$ and others in $V(x) = \gamma$. By (4.33), if one point x belonging to the frontier of Θ is included in $\partial\Omega_d$, then it does not abandon Ω_d because $\dot{g}_i(x) > 0$ and, if such point x is in the interior of Ω_d, condition of decreasing $V(x)$ (above sections) ensures it to continue belonging to Θ in the future.

Therefore, as (4.31) is an invariant set and only the origin belongs to the set $\{x : \dot{V}(x) = 0\}$, the region Θ is a subset of $\mathcal{D}_\Omega$.

In order to check (4.33), having a Lyapunov function $V(x)$ in Ω_d, it has to be checked that all the sets

$$c_i = \{V(x) \leq \gamma, \dot{g}_i \leq 0, g_i(x) = 0, g_j(x) \geq 0\} \quad \begin{matrix} i : 1, ..., n_g \\ j \neq i \end{matrix} \tag{4.34}$$

are empty. Then the problem is to maximize γ such that all c_i remain empty. It can be done, applying Theorem A.1, if the following equality is true

$$h^2 + (\gamma - V(x)) + F_i(x) + G_i(x) = 0 \tag{4.35}$$

for polynomials $(h, G_i(x)) \in \mathcal{R}_x$ and $F_i(x) \in \Sigma_x$, which is proved with a sign change, leading to condition (4.30). □

The new found DA estimate will be always bigger or equal than the one obtained with results stated in previous sections. See Example 4.4.2 in Section 4.4 for such a case.

Note. If Lemma 4.4 results in $\gamma \to \infty$, it means that the proven Θ is the whole region Ω_d, so the nonlinear system can be modelled in a bigger region obtaining, probably, a larger domain of attraction estimate.

4.3 Iterative DA estimate expansion

Given the above results, the objective of this section is presenting a methodology to expand the domain of attraction starting from a *previously computed* proven subset of it: any (possibly small) subset of the domain of attraction found with current LMI/SOS results (to be denoted as, say, B_1) can be used as a "*seed*" for an iterative algorithm that expands it. The methodology is discussed for both continuous-time and discrete-time cases.

This section shows that, once any seed set B_1 is available, there is no longer need of using actual Lyapunov functions, but only proving that there is a set B_2 such that trajectories starting in it fall into B_1, so B_2 belongs to the DA too. A SOS approach provides a numerical tool to obtain such a set and an iterative algorithm naturally ensues by using $B_2 \cup B_1$ as the new seed. If $B_1 \subset B_2$, SOS algorithms provide B_2 which is an estimate of the DA expressed in closed form as a polynomial.

The function defining the boundary of the resulting DA estimates in this section is free from the restriction of being decrescent and positive in its interior. That allows for improved estimates over previous results.

The results presented on this section have been published in the journal paper Pitarch, Sala and Ariño (2014b).

4.3.1 Auxiliary lemmas

Consider a compact set defined by o_q polynomial boundaries $\Theta = \{x : Q_l(x) \leq 1\}$ $l : 1, ..., o_q$, and an inner region $B = \{x \in \Theta : V(x) < 1\}$ containing the origin ($V(0) < 1$). The following definition will be later taken in the rest of the chapter as the best low-degree fit of Θ.

Definition 4.3. *Consider a decision-variable polynomial of predefined degree denoted by $R(x)$. The* ***best fitting region*** $\Theta_R = \{x \in \Theta : R(x) \leq 1\}$*, fulfilling $B \subset \Theta_R \subset \Theta$, is defined to be the solution of the following problem:*

Minimize τ subject to

$$1 + \tau \geq R(x) \; when \; x \in \Theta_m, \; m = 1, \ldots, o_q \tag{4.36}$$

$$R(x) \leq 1 \; when \; x \in B \tag{4.37}$$

$$R(0) = 0 \tag{4.38}$$

being Θ_m each one of the o_q portions of the frontier of Θ, defined as $\Theta_m = \{x : \; V(x) \geq 1, \; Q_m(x) = 1\}$.

In this way, Θ_R will be an inner approximation to Θ with a *single* polynomial restriction.

Note that condition (4.38) is needed in order to avoid the trivial solution $\tau = 0$, $R = 1$, requiring that at least one point has a value different from one[3].

Let us discuss an auxiliary result regarding lower-complexity SOS conditions for stability. As commented in Chapter 3, some developments (particularly in discrete-time) require a high-degree polynomial in both state and auxiliary membership variables $\sigma^2 = \mu$. As an alternative, a dummy variable ρ may be introduced jointly with the equality constraint $\rho = x_{k+1}$, i.e., $\rho - \sum_{i=1}^{r} \mu_i P_i = 0$. In this way, Lemma 3.3 may be reformulated, following a Positivstellensatz argumentation.

[3] There is no loss of generality in setting $R(0)$ to zero, as a straightforward argumentation with affine scalings shows (details omitted for brevity).

Lemma 4.5. *The system* (4.4) *is globally stable if there exist functions* $V(x)$ *and* $G_1(\rho, x)$ *such that:*

$$V(x) - V(\rho) - \epsilon(x) + G_1(\rho, x) > 0 \tag{4.39}$$

with[4] $G_1 \in \mathcal{I}(\rho - \sum_{i=1}^{r} \mu_i P_i)$ *arising from the equality constraint.*

Note that (4.39) is not yet a SOS problem (because of the nonlinear functions μ_i appearing in G_1); however, it is a fuzzy summation so well-known semidefinite relaxations based on Polya's theorem (Sala and Ariño, 2007a) may be applied.

For instance, if $G_1(\rho, x)$ were chosen as the simple expression

$$G_1(\rho, x) = \phi(x, \rho) \cdot (\rho - \sum_{i=1}^{r} \mu_i P_i)$$

being $\phi(x, \rho)$ a polynomial vector in $\mathcal{R}^n_{\rho,x}$, then (4.39) is a single-dimensional fuzzy summation whose positiveness for $\mu_i \in \Gamma$ is proved if the r SOS conditions below hold:

$$V(x) - V(\rho) - \epsilon(x) + \phi(x, \rho)(\rho - P_i(x)) \in \Sigma_{\rho,x} \quad i = 1, \dots, r \tag{4.40}$$

In fact, the above proposed structure of G_1 will be the actual choice in later examples.

4.3.2 Discrete-time DA estimation

Given a particular region B_1 which belongs to the DA of a nonlinear system (4.2), a larger estimate of the DA can be calculated following the next result.

Lemma 4.6. *Let* $B_1 = \{x \in \mathbb{R}^n : V_1(x) < 1\} \subset \mathcal{D}$ *be a (previously proven) bounded subset of the domain of attraction of* (4.2) *and let* N *be a horizon parameter (number of future samples) fixed* a priori*. Then, any region* B_2 *such that*

$$B_2 \subseteq \{x \in \mathbb{R}^n : V_2(x) < 1\}, \tag{4.41}$$

where[5] $V_2(x) = V_1(f^{[N]}(x))$*, belongs also to the domain of attraction of the system* (4.2).

[4] A slight abuse of notation is involved in the definition of the ideal as it is generated by an expression which is not a polynomial. In this context, the ideal will be considered to be the product of arbitrary polynomials –to be obtained by SOS optimization– by any product of the generating functions.

[5] Notation: $f^{[N]}(x) = (\underbrace{f \circ f \circ \ldots \circ f}_{N})(x)$.

Proof. Following system dynamics (4.2), the points which in N future samples will be inside B_1 are those defined by:

$$V_1(x_{k+N}) < 1 \equiv V_1(f^{[N]}(x_k)) < 1$$

So the region B_2 is a subset of the DA as any starting point in B_2 will enter the open set B_1 in a finite number of time steps. Hence, it will later reach the origin as $B_1 \subset \mathcal{D}$. □

Corollary 4.1. *If B_1 contains the origin, when $N \to \infty$, B_2 exactly coincides with the actual DA of the origin of the nonlinear discrete-time system.*

Proof. Indeed, no point reaching the origin can avoid entering B_1 in a finite number of time steps, as the origin is in its interior. □

Remark 4.1. Note that B_1 does not need to be a Lyapunov level set like the ones considered in classical results (which implicitly consider $N = 1$). In fact, there is no need of it being even an "invariant" set as understood in literature (Khalil, 2002).

4.3.2.1 Application to fuzzy polynomial systems

Despite of Lemma 4.6 gives an exact description of the N-step DA, unless f is linear, the result is a very high-degree expression if fuzzy polynomial models for system (4.2) are used (both in the state variables and in the membership functions). So, the results in the above lemma may be of little use if a reasonably simple approximation of the DA of a nonlinear system were needed for subsequent analysis or representation.

In order to obtain a simpler reliable representation for the DA, the following lemmas propose the use of fuzzy polynomial models in order to describe the nonlinear dynamics. Hence, inspired on the "best fitting region" of Definition 4.3, they obtain a user-defined low degree polynomial in order to characterize the LRDA.

The basic idea motivating the results below is obtaining a low-degree approximation of the 1-step DA $V(f(x)) < 1$ in Lemma 4.6 and, later, iterating such approximation.

Lemma 4.7. *Consider a known seed set $B_1 \subset \mathcal{D}_\Omega$ defined by a polynomial $V_1(x)$, $B_1 = \{x \in \Omega : V_1(x) < 1\}$ and a user-defined modelling region Ω defined by o_q restrictions $\Omega = \{x : Q_l(x) \leq 1\}$ $l : 1, ..., o_q$, such that it is compact. Then, the region $B_2 = \{x : Q_l(x) \leq 1, V_2(x) \leq 1\}$ belongs to $\mathcal{D}_\Omega$*

and $B_2 \supset B_1$, if a function $V_2(x)$ can be found solving the following problem:

Minimize τ subject to

$$V_2(0) = 0 \qquad (4.42)$$
$$V_2(x) - 1 - F_1(x,\rho) + G_1(x,\rho) > 0 \quad \forall x, \rho \qquad (4.43)$$
$$V_2(x) - 1 - F_2(x,\rho) + G_2(x,\rho) > 0 \quad \forall x, \rho \qquad (4.44)$$
$$1 - V_2(x) + \tau - F_3(x,\rho) + G_3(x,\rho) + G_4(x,\rho) > 0 \quad \forall x, \rho \qquad (4.45)$$
$$1 - V_2(x) - F_4(x) > 0 \quad \forall x \qquad (4.46)$$

Where

- $\tau > 0$,
- $F_1(x,\rho) \in \wp(V_1(\rho) - 1, 1 - Q_1(x), ..., 1 - Q_{o_q}(x))$,
- $F_2(x,\rho) \in \wp(Q_1(\rho) - 1, ..., Q_{o_q}(\rho) - 1, 1 - Q_1(x), ..., 1 - Q_{o_q}(x))$,
- $F_3(x,\rho) \in \wp(1 - Q_1(x), ..., 1 - Q_{o_q}(x))$,
- $F_4(x) \in \wp(1 - V_1(x), 1 - Q_1(x), ..., 1 - Q_{o_q}(x))$,
- $G_3(x,\rho) \in \mathcal{I}(V_1(\rho) - 1)$,
- $\{G_1(x,\rho),\ G_2(x,\rho),\ G_4(x,\rho)\} \in \mathcal{I}(\rho - \sum_{i=1}^{r} \mu_i P_i(x))$

Note that the same abuse of notation issue discussed in footnote 4 has been assumed.

Proof. By condition (4.43), the region $V_2 < 1$ will be an inner approximation (actually it has to fulfill the requirement only inside the modelling region Ω) to the region defined by $V_1(x_{k+1}) < 1$ (the points which in *one* sample will be inside B_1): the condition implies that $V_2(x)$ is greater than 1 when $V_1(x_{k+1}) \geq 1$ and $x \in \Omega$.

Condition (4.44) implies that $V_2(x)$ should be greater than one for those points $x \in \Omega$ such that $x_{k+1} \notin \Omega$. Jointly with (4.43) the condition discards the points $x \in \Omega$ for which $V_1(x_{k+1}) < 1$ but $x_{k+1} \notin \Omega$.

So conditions (4.43),(4.44) together mean that all points in $V_2(x) < 1$ will fulfill $V_1(x_{k+1}) \leq 1$ and $x_{k+1} \in \Omega$, i.e. $x_{k+1} \in B_1$. Hence the obtained level set can be used as B_2 in (4.41), for $N = 1$.

Figure 4.2, in which Ω is, for clarity, only defined by a circle $Q(x) < 1$, illustrates the different regions involved in the conditions: the pink region $V_2 < 1$ must not intersect green zones ($V_1(x_{k+1}) > 1$) and red ones ($Q(x_{k+1}) > 1$).

Lastly, conditions (4.42), (4.45) and (4.46) are the adaptation of the best-fit conditions (4.38), (4.36) and (4.37), respectively, to the setting now in consideration, in order to obtain the "optimal" V_2 according to Definition 4.3. □

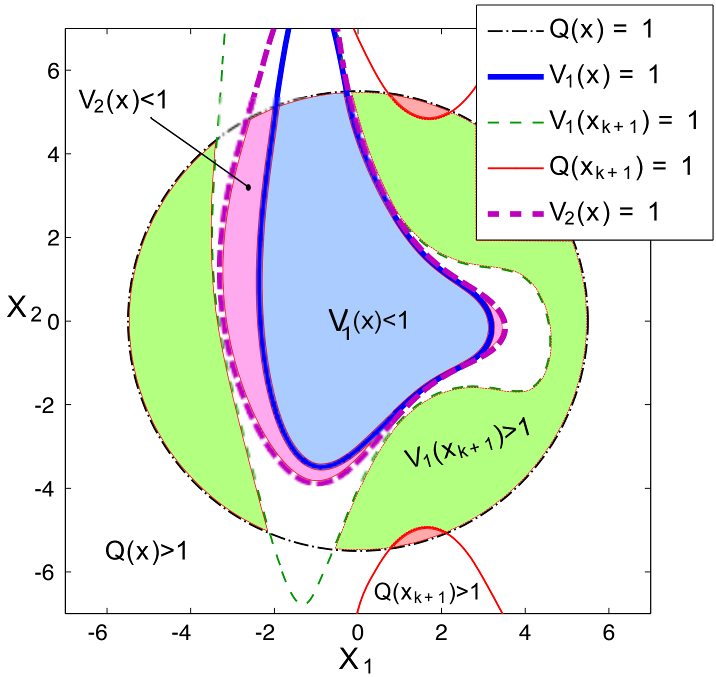


Figure 4.2: Example of regions involved in Lemma 4.7 (Blue: B_1, Pink+Blue: B_2, plus other relevant boundaries in the legend).

The optimization problem in the above Lemma cannot be solved via SOS techniques. The reason is that conditions must involve only polynomial terms in order to be able to use semidefinite programming. However, it can be converted in an straightforward way to a SOS problem if all $V_1(x), V_2(x), Q_1(x), \ldots, Q_{o_q}(x)$ belong to the polynomials $\mathcal{R}_x$ and semidefinite relaxations are suitably applied: the case is identical to the one when transforming (4.39) to (4.40) in a previous section (details omitted for brevity).

Remark 4.2. Note that the slack variables ρ in Lemma 4.7 are introduced instead of directly using $\sum_i \sigma_i^2 P_i$ ($\sigma^2 = \mu$), in order to reduce the degree of the conditions. Therefore, the computational complexity of the resulting semidefinite problem (Lemma 4.5) is lower.

Remark 4.3. Usually, B_1 will have been obtained with a shape-independent fuzzy technique in literature and this is why, in Lemma 4.7, the LRDA condition $B_1 \subset D_\Omega$ has been assumed. If B_1 had been obtained with a shape-dependent or other nonlinear stability analysis technique, then the resulting B_2 will be a larger set possibly including points of the DA outside the LRDA, so it would be a better solution: evidently, the larger the initial estimate B_1 is, the better the proposed methodology will work.

Remark 4.4. Note that, as in Lemma 4.6, B_1 does not need to be a Lyapunov level set fulfilling $V_1(x_{k+1}) - V_1(x_k) < 0$ in all its interior, even if the previous remark suggests it as a reasonable seed set. Note also that the resulting B_2 is also free from the above Lyapunov decrease condition. In fact, we don't even need to enforce neither $V_1 > 0$ nor $V_2 > 0$ inside the level set (with Positivstellensatz conditions). These are the reasons why the proposed methodology obtains better results than previous literature.

Iterative procedure

As a natural choice, using the obtained B_2 in Lemma 4.7 to define a new region B_1, a sequence of new functions and associated regions would be readily obtained by repeatedly applying Lemma 4.7. See Example 4.4.5 in Section 4.4.

Remark 4.5. There is also the possibility of remodelling while iterating, defining a new larger region $\Omega_2 \supset \Omega$ in order to obtain larger LRDA estimates, in particular when $\{x : V_2(x) \leq 1\} \not\subset \Omega$. Note that, in that case, conditions (4.44) must make reference to the *previous* modelling region Ω when setting up F_2, in order to fullfill Lemma 4.6. The other Positivstellensatz polynomials F_1, F_3 and F_4 must belong to the cone formed with the constraints associated to Ω_2. For further details, see Example 4.4.5 in Section 4.4.

4.3.3 DA estimation in continuous-time systems

The following theorem states conditions so that, given a particular region B_1 elsewhere proven to belong to the DA of (4.1), a larger one can be found via invariant set considerations.

Theorem 4.2. *Let* $\Theta = \{x : Q_l(x) \leq 1,\ l : 1, ..., o_q\}$ *be a compact user-defined region of interest, with* $Q_l(x)$ *differentiable. Let* $B_1 = \{x \in \Theta : V_1(x) < 1\} \subset \mathcal{D}$ *be a (previously proven) bounded subset of the domain of attraction of the origin of system* (4.1)*. If we can find a differentiable function* $V_2(x)$ *such that, given* $\epsilon > 0$*, the following conditions hold:*

$$V_2(x) \geq 1\ \forall\, x \in C_m \qquad m : 1, .., o_q \tag{4.47}$$

$$\dot{V_2}(x) < -\epsilon(x)\ when\ V_1(x) \geq 1, Q_l(x) \leq 1\ \forall l \tag{4.48}$$

where

$$C_m = \{x :\ Q_m = 1, \dot{Q}_m > 0,\ V_1 \geq 1,\ and\ Q_l \leq 1\ \forall l \neq m\}$$

Then, the interior of the region $B_2 = \{x : Q_l(x) \leq 1,\ V_2(x) \leq 1\} \cup B_1$ *belongs to* $\mathcal{D}$*.*

Proof. As $B_1 \subset \Theta$, we have $B_1 \subseteq B_2 \subseteq \Theta$. Condition (4.48) means that $\dot{V_2}(x)$ is strictly negative in $\Theta \backslash B_1 = \{x :\ x \in \Theta,\ x \notin B_1\}$.

We will now prove that all trajectories starting in x_0 in the interior of $B_2 \backslash B_1$ reach in finite time B_1.

Indeed, as $V_2(x_0) < 1$, and for all l, $Q_l(x_0) < 1$ then $V_2(x(t)) < 1$ for all $t \geq t_0$ while in $B_2 \backslash B_1$ by (4.48). Hence, it will never exit $B_2 \backslash B_1$ neither via the frontier $V_2(x) = 1$, evidently, nor via $Q_l(x) = 1$ because $V_2(x) \geq 1$ or $\dot{Q}_l(x) < 0$ in such points due to (4.47). So, the only way of a trajectory to exit $B_2 \backslash B_1$ will be entering B_1.

As the fact that the trajectory remains forever in $B_2 \backslash B_1$ is not possible we can conclude that the trajectory from the above x_0 will enter B_1 in finite time (see below).

Indeed, we have $B_2 \backslash B_1 \subset \Theta \backslash B_1$. Also, $\Theta \backslash B_1$ is compact and, by (4.48), $\dot{V_2}(x(t)) < -\epsilon$ when $x(t) \in \Theta \backslash B_1$. $V_2(x)$ will achieve a minimum α in $\Theta \backslash B_1$. Consider a trajectory such that $V_2(x(0)) \leq 1$ and $\dot{V_2}(x(t)) < \epsilon$ for all $t \geq 0$. In that case, for all $t > (1-\alpha)/\epsilon$ we would have $V_2(x(t)) < \alpha$, so such trajectory is not possible inside $\Theta \backslash B_1$: the state must have left $\Theta \backslash B_1$ (hence, $B_2 \backslash B_1$) in finite time. □

Note that some sets C_m may be empty so, in those cases, there is no need of checking condition (4.47).

Corollary 4.2. *If the condition*

$$V_2(x) \leq 1\ when\ V_1(x) \leq 1, Q_l(x) \leq 1\ \forall l \tag{4.49}$$

is also enforced, then $B_2 = \{x \in \Theta : \; V_2(x) \leq 1\}$*, and* $B_1 \subset B_2$*. So* B_2 *and* V_2 *can be used again for finding new points in the domain of attraction, replacing* V_1 *and* B_1 *with them.*

The advantage of the above corollary is that there is no need of considering the union of regions discussed in Theorem 4.2 when defining B_2, simplifying further computations. An iterative algorithm naturally ensues (see Section 4.3.3.2).

4.3.3.1 Application to fuzzy polynomial systems

In the following, fuzzy polynomial models (4.3) and restrictions will be used in the context of the above theorem to obtain LRDA estimates $\mathcal{D}_\Omega$ of the domain of attraction $\mathcal{D}$ of (4.1) in a modelling region Ω. In this way, SOS programming can be used. In order for the polynomial model to be valid, the condition $\Theta \subset \Omega$ must be enforced by a suitable definition of Q_l, being Θ the region of interest discussed in Theorem 4.2.

Remark 4.6. The "region of interest" Θ is introduced, instead of the full modelling region Ω, in order to reduce conservatism by eliminating the need of checking $\dot{V}_2 < 0$ in the whole Ω, which may be infeasible. Indeed, note that if there are equilibrium points in $\Theta \backslash B_1$ then (4.48) will not hold. A suitable choice for Θ will be later discussed.

Lemma 4.8. *Consider a known set* $B_1 \subset \mathcal{D}_\Omega$ *defined by a polynomial* $V_1(x)$*,* $B_1 = \{x \in \Omega : V_1(x) < 1\}$ *and a user-defined region* Θ *defined by* o_q *restrictions* $\Theta = \{x : Q_l(x) \leq 1\}$ $l : 1, ..., o_q$*, such that* $\Theta \subset \Omega$ *and it is compact. Then, the region* $B_2 = \{x : Q_l(x) \leq 1, V_2(x) \leq 1\}$ *belongs to* $\mathcal{D}_\Omega$ *and* $B_2 \supset B_1$*, if a continuous differentiable function* $V_2(x)$ *can be found solving the following SOS problem:*

Minimize τ subject to

$$V_2(0) = 0 \tag{4.50}$$

$$-(\frac{\partial V_2(x)}{\partial x}\rho + \epsilon) - F_1(x,\rho) + G_1(x,\rho) \in \Sigma_{x,\rho} \tag{4.51}$$

$$V_2(x) - 1 - F_{2m}(x,\rho) + G_{2m}(x,\rho) \in \Sigma_{x,\rho} \; m : 1, ..., o_q \tag{4.52}$$

$$1 - V_2(x) + \tau - F_{3m}(x) + G_{3m}(x) \in \Sigma_x \quad m : 1, ..., o_q \tag{4.53}$$

$$1 - V_2(x) - F_4(x) \in \Sigma_x \tag{4.54}$$

Where

- $\tau > 0$, $\epsilon > 0$,
- $F_1(x,\rho) \in \wp(V_1(x) - 1, 1 - Q_1(x), ..., 1 - Q_{o_q}(x))$,
- $F_{2m}(x,\rho) \in \wp(V_1(x) - 1, \frac{\partial Q_m(x)}{\partial x}\rho)$,
- $F_{3m}(x) \in \wp(V_1(x) - 1)$,
- $F_4(x) \in \wp(1 - V_1(x), 1 - Q_1(x), ..., 1 - Q_{o_q}(x))$,
- $G_{2m}(x,\rho) \in \mathcal{I}(Q_m(x) - 1, \rho - \sum_{i=1}^{r} \mu_i P_i(x))$,
- $G_{3m}(x) \in \mathcal{I}(Q_m(x) - 1)$,
- $G_1(x,\rho) \in \mathcal{I}(\rho - \sum_{i=1}^{r} \mu_i P_i(x))$.

Proof. Conditions (4.51) and (4.54) mean (4.48) and (4.49) respectively. As $\dot{Q}_m = \frac{\partial Q_m}{\partial x}\rho$, constraining $\rho = \sum_i \mu_i P_i(x)$ by "Positivstellensatz" multipliers, then condition (4.52) implies (4.47), also condition (4.53) implies (4.36), and condition (4.54) implies (4.37). □

Note that, inspired in Definition 4.3, minimization of τ above allows obtaining a region B_2 which best fits Θ subject to the additional constraint of belonging to $\mathcal{D}_\Omega$.

Remark 4.7. As in the discrete case, the above optimization problem doesn't involve polynomial finite conditions. So, in order to be able to use semidefinite programming, a recasting is needed by taking $(V_1(x), V_2(x), Q_1(x), ...,$ $Q_{o_q}(x)) \in \mathcal{R}_x$ and a finite number of terms from the cones and ideals. See, again, the transformation from (4.39) to (4.40) (details omitted for brevity).

The above lemma generalises particular cases in literature, as follows:

Corollary 4.3. *If $B_1 = \{0\}$ and all conditions of Lemma 4.8 are set with the particular choices $F_1(x,\rho) \in \wp(1 - Q_1(x), ..., 1 - Q_{o_q}(x))$, $F_{2m} = 0$, $F_{3m} = 0$, and* (4.54) *is omitted, V_2 is a Lyapunov function whose level set $\{x : V_2 \leq 1\}$ belongs to the DA of the origin, recovering classical local-stability results (Lemma 4.2).*

Proof. If $B_1 = \{0\}$ relaxing requirements of positiveness and decrescence inside $\{V_1 \leq 1\}$ should not be done because such V_1 does not exist. Hence, the term $V_1(x) - 1$ should be removed from the generator of the cones. Also, (4.54) which refers to conditions inside $B_1 \cap \Theta$ (i.e, the origin) is redundant with (4.50). The rest of conditions can then be interpreted as the usual ones on Lyapunov functions (locally in Θ). □

Corollary 4.4. *If conditions in Corollary 4.3 are solved getting V_2 and, later, only* (4.52) *is posed setting a new V_2 equal to an scaled version of the one just computed, then Lemma 4.4 is obtained.*

Indeed, Section 4.2.3 discusses only *a posteriori* scaling of Lyapunov functions.

4.3.3.2 Iterative procedure

Lemma 4.8 starts with a seed set $B_1 = \{x : V_1 < 1\}$ and a user-defined region Θ which, obviously, should intersect with the seed set (in most of practical cases, it will actually contain the seed set). The result is a new level set $\{x : V_2 < 1\}$ larger than B_1 such that its intersection with Θ belongs to the DA.

a) Progressive enlargement of the DA estimate: As a natural choice, if Θ were fixed, using the larger V_2 obtained with Lemma 4.8 to define a new seed region B_1, then the conditions of Lemma 8 are fulfilled and, thus, it can be applied again with the new seed. Hence, a sequence of new functions and associated regions would be readily obtained by repeatedly applying Lemma 4.8.

b) Choice of region of interest Θ: There are various posibilities for choosing a region Θ but:

- a large Θ might eventually lead to (4.51) being infeasible, *e.g.*, if Θ included more than one equilibrium point.
- a small Θ would lead to little improvement in the domain of attraction estimates and, also, the restrictions (4.53) and (4.54) would be hard to fulfill if V_2 were a low-degree polynomial and Θ and V_1 defined complicated shapes.

Furthermore, as iterations progress and the DA estimates grow larger (encompassing most of Θ), then constraining Θ to the initial “small” choice may not be a good option. This fact, jointly with the above issues arising in the choice of Θ motivate incorporating iterations in the size and shape of such region, as discussed below.

c) Proposal for modification of Θ: Although there might be alternative options, for instance, the new region of interest can be defined by a user-defined “zoom” factor $\upsilon \geq 1$ as:

$$\Theta = \{x \in \Omega : V_1(x) \leq \upsilon,\ \upsilon \in \mathbb{R}\} \tag{4.55}$$

The smaller υ is, the smaller the region $\Theta - B_1$ is, so condition $\dot{V}_2 < 0$ there becomes less restrictive.

If $V_1(x)$ were $\mathcal{C}^1$ differentiable, and enhanced proposal for the choice of Θ may be based on the evident fact that for any fixed time $\delta > 0$, the set $\{x_0 \in \mathbb{R}^n : V_1(x(\delta)) < 1, x(0) = x_0\}$ is included in the domain of attraction $\mathcal{D}$. Intuitively, from the first order Taylor series expansion of $V_1(x(t))$,

$$V_1(x(\delta)) \approx V_1(x_0) + \delta \frac{\partial V_1}{\partial x} \dot{x}(0)$$

the new region of interest in Corollary 4.8 can be, choosing $\delta > 0$:

$$\Theta = \left\{ x \in \Omega : V_1(x) + \delta \frac{\partial V_1(x)}{\partial x} \sum_{i=1}^{r} \mu_i P_i(x) \le \upsilon, \ x \in G \right\} \tag{4.56}$$

where $G \in \Omega$ is, in general case, a sphere limiting the search zone and ensuring compatness ($G \equiv \Omega$ if Ω is compact). The constant υ has the same meaning as in (4.55) and δ is a new user-defined constant.

The idea behind this option is to explore stability in "promising" directions: the region (4.56) will include the set of points x which "approximately" in δ seconds will be inside B_1.

Note that the accuracy of these steps is not very relevant because region Θ can actually be arbitrarily defined by the user in Theorem 4.2. Also, in order to be less conservative, the original nonlinear system may also be remodelled: indeed, for a given $\Theta \subset \Omega$, the closer the modelling region Ω is to Θ the less uncertain the fuzzy model will be.

To clarify the proposed methodology, the examples 4.4.3 and 4.4.4 in Section 4.4 use the following algorithm (particular case of Lemma 4.8):

Algorithm 4.2. *Starting from a known $B_1 = \{V_1(x) < 1\}$, $B_1 \in \mathcal{D}$ and $V_1(x) \in \mathcal{R}_x$, carry out the following steps:*

1. *Choose a starting combination of region increase parameters $\delta \ge 0$ (gradient) and $\upsilon \ge 1$ (zoom), defining a candidate region of interest*[6] (4.56).

[6] A slack variable $\rho = \sum_i \mu_i P_i$ will be introduced as in Lemma 4.5 in next steps. Note also that region (4.56) might be defined by a high-degree polynomial for $\delta \neq 0$, so it may have a strange shape and can take large values far from the origin. Hence, to avoid numerical problems, a low-degree best-fitting region (Definition 4.3) to (4.56) may also be obtained in this step for later use if needed. Details omitted for brevity.

2. *Find a new polynomial $V_2(x)$ solving the following SOS problem:*

   Minimize τ such that

$$V_2(0) = 0 \tag{4.57}$$

$$\begin{aligned} &-(\frac{\partial V_2}{\partial x}\rho + \epsilon) - \psi_{1i}(V_1 - 1) - \psi_{2i}(R - x^T x) \\ &- \psi_{3i}(\upsilon - V_1 - \delta\frac{\partial V_1}{\partial x}\rho) + \phi_1(\rho - P_i) \in \Sigma_{x,\rho} \quad i : 1, ..., r \end{aligned} \tag{4.58}$$

$$1 - V_2 - \psi_4(1 - V_1) - \psi_5(R - x^T x) \in \Sigma_x \tag{4.59}$$

$$\begin{aligned} &V_2 - 1 + \phi_{2i}(V_1 + \delta\frac{\partial V_1}{\partial x}\rho - \upsilon) - \psi_{6i}(R - x^T x) \\ &\qquad + \phi_3(\rho - P_i) \in \Sigma_{x,\rho} \quad i : 1, ..., r \end{aligned} \tag{4.60}$$

$$\begin{aligned} &V_2 - 1 - \psi_{7i}(\upsilon - V_1 + \delta\frac{\partial V_1}{\partial x}\rho) + \phi_{4i}(R - x^T x) \\ &\qquad + \phi_5(\rho - P_i) \in \Sigma_{x,\rho} \quad i : 1, ..., r \end{aligned} \tag{4.61}$$

$$\begin{aligned} &1 - V_2 + \tau + \phi_{6i}(V_1 + \delta\frac{\partial V_1}{\partial x}\rho - \upsilon) - \psi_{8i}(R - x^T x) \\ &\qquad + \phi_7(\rho - P_i) \in \Sigma_{x,\rho} \quad i : 1, ..., r \end{aligned} \tag{4.62}$$

   where $\epsilon > 0$, $\tau > 0$, R is a user-defined radius of a sphere belonging to Ω, $\psi_j \in \Sigma_x$, $\psi_{ji} \in \Sigma_{x,\rho}$, $\phi_k \in \mathcal{R}^n_{x,\rho}$ and $\phi_{ki} \in \mathcal{R}_{x,\rho}$.

3. *If the above problem is feasible, set $V_1(x) = V_2(x)$ and return to Step 1.*

4. *If problem in Step 2 is not feasible, then:*

   (a) *If $\upsilon > 1$, set $\upsilon = \max(1, \upsilon - \Delta_\upsilon)$ (Δ_υ user-defined step) and go back to Step 2.*

   (b) *If $\upsilon = 1$, reduce δ by Δ_δ (user-defined step) and go back to Step 2.*

   (c) *If $\upsilon = 1$ and $\delta \leq 0$ **stop** the algorithm, due to lack of progress. The finally proved DA estimate B_2 is obtained in closed-form by the current $V_1(x)$, computed in the last feasible iteration (i.e., after setting $V_1 = V_2$ in Step 3):*

$$B_2 = \{x : V_1(x) < 1,\ x_1^2 + x_2^2 < R^2\}$$

Note. Conditions on the above algorithm are a particularization of those in Lemma 4.8 as follows:

- (4.57), (4.58), (4.59) and (4.62) correspond to (4.50), (4.51), (4.54), and (4.53), respectively
- (4.60) and (4.61) are conditions (4.52) but forcing $\{x : V_2(x) = 1\}$ to be contained inside Θ to avoid the result of each iteration being an "intersection" (i.e., forcing B_1 in next iteration to be defined by only *one* polynomial inequality), setting $F_{2m} = 0$.

Remark 4.8. With condition (4.59), i.e., $V_2 \leq 1$ when $V_1 \leq 1$, Corollary 4.2 applies and the proved domain of attraction increases in each iteration. Note that improvements come from the fact that there is no need for either $V_1 > 0$, $\dot{V}_1 < 0$, $V_2 > 0$ or $\dot{V}_2 < 0$ in *all* the interior of the level sets, contrary to usual Lyapunov approaches.

4.4 Examples

This section presents several examples in order to show the effectiveness of the proposed methodologies. The evolution of the largest sphere, which is guaranteed to belong to the DA, is shown for different modelling region sizes. Also the final shape of the found DA estimate is shown for the used methodology in each example.

Example 4.4.1. Consider a 2-rules fuzzy polynomial model (4.3) with its corresponding membership functions given by:

$$\mu_1(x) = e^{-(x_1^2+x_2^2)}; \quad \mu_2(x) = 1 - \mu_1(x)$$

$$p_1(x) = \begin{pmatrix} -0.116x_1 + 0.015x_2^4 + 0.0603x_2 \\ 0.2x_1^2x_2 - 0.0603x_1 - 0.4711x_2 \end{pmatrix}$$

$$p_2(x) = \begin{pmatrix} -0.1106x_1 + 0.03x_2^4 - 0.2796x_2 \\ -0.1x_1^2x_2 + 5.8714x_1 - 0.4763x_2 \end{pmatrix}$$

The region Ω_d is defined as a square region in the state space given by:

$$\Omega_d = \left\{ x : \left| \begin{pmatrix} 0 & \frac{1}{\rho_d} \end{pmatrix} x \right| \leq 1, \left| \begin{pmatrix} \frac{1}{\rho_d} & 0 \end{pmatrix} x \right| \leq 1 \right\} \tag{4.63}$$

where d is the iteration number and ρ_d is the scale factor which changes in each iteration.

In this case, the maximum and minimum of the membership functions μ_i in Ω_d is easy to compute due to their monotonicity: those values are obtained in the center (max μ_1, min μ_2) and in the extreme vertex of the square (min μ_1, max μ_2), which will be denoted by ξ. Therefore, the v_{di} vertices are:

$$v_{11} = \mu_1^M = \mu_1(0), \quad v_{21} = \mu_1^m = \mu_1(\xi)$$

$$v_{12} = \mu_2^m = \mu_2(0), \quad v_{22} = \mu_2^M = \mu_2(\xi)$$

The procedures presented on Section 4.2 allow to determine the largest sphere around $x = 0$ for which local stability conditions hold. In this first example, quadratic stability ($V(x) = x^T Px$) has been chosen for simplicity. Higher-degree polynomial Lyapunov functions will be used on next examples.

Consider as a first iteration $\rho_1 = 0.01$, and increments of 0.01 for next iterations until infeasibility of the SOS problem is reached. The largest provable sphere with Algorithm 4.1 is obtained by performing an exploration in Ω_d, replacing conditions (4.29) by:

$$-\frac{dV}{dx}(\mu_1^M p_1 + \mu_2^m p_2) - \epsilon(x) + \psi_{11}(x_1^2 - \rho_d^2) + \psi_{12}(x_2^2 - \rho_d^2) \in \Sigma_x \quad (4.64)$$

$$-\frac{dV}{dx}(\mu_1^m p_1 + \mu_2^M p_2) - \epsilon(x) + \psi_{21}(x_1^2 - \rho_d^2) + \psi_{22}(x_2^2 - \rho_d^2) \in \Sigma_x \quad (4.65)$$

The ellipsoidal estimates and the largest sphere contained in each one of them are shown on Figure 4.3, for some exploration values ρ_d. Let us focus on how ill-conditioned ellipsoids prove a small "stable" area in comparison with the whole square Ω_d, in which the Lyapunov function is forced to decrease. This fact is even more stressed when looking at the found spherical regions: For instance, with a 0.6 side square Ω_d, the maximum sphere has a radius of 0.18, i.e., this area is only 7% of the modelling region.

Now, if some decay-rate performance α is to be proven for this system, the term

$$-2\alpha[x_1\ x_2]P[x_1\ x_2]^T$$

has to be added to the conditions (4.64) and (4.65). Then, starting with $\alpha = 0$ and increasing it progressively, SOS conditions become more restrictive so the maximum provable radius will be smaller. Figure 4.4 shows the radius evolution of the largest provable sphere with Ω_d for different decay rates. The top line correspond to simple stability ($\alpha = 0$).

As it can be seen, radius evolution is not always positive due to the fact that the larger the modelling region is, the more uncertainty may be added between

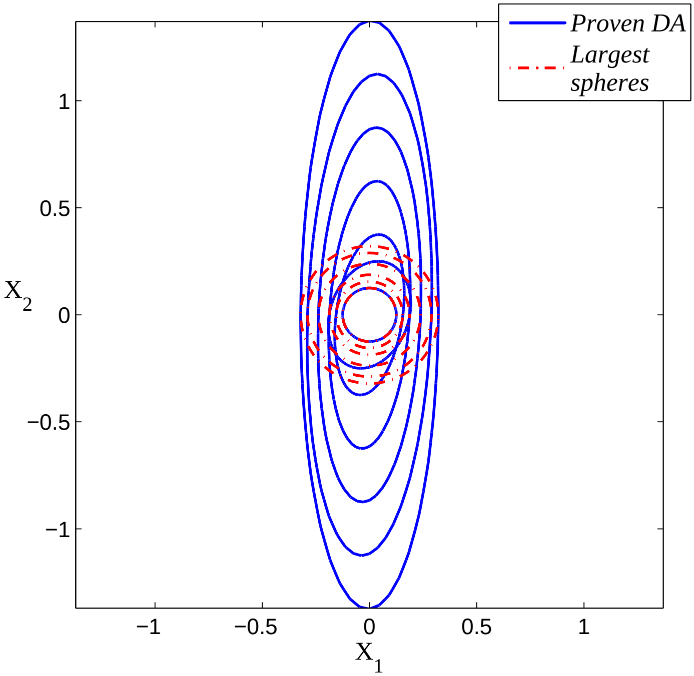


Figure 4.3: DA estimates for different regions Ω_d (squares) using quadratic Lyapunov functions.

the fuzzy vertex models. This fact is reflected progressively with a worse conditioning on the Lyapunov functions. However, after an initial degradation, no new nonlinearities are found when increasing the modelling region, so the radius increases monotonically until infeasibility. In conclusion, no monotonic properties can be proved on the radius evolution because it depends on the particular features of the analyzed system.

Example 4.4.2. Consider the following continuous-time nonlinear system:

$$\begin{aligned} \dot{x}_1 &= -3x_1 + 0.5x_2 \\ \dot{x}_2 &= x_2(-2 + 3\sin(x_1)) \end{aligned} \tag{4.66}$$

which has equilibrium points at $x = 0$, at $(x_1 = 0.7297 + 2k\pi, x_2 = 6x_1)$ and at $(x_1 = -0.7297 + (2k+1)\pi, x_2 = 6x_1)$ for $k \in \mathbb{Z}$.

The objective is to find a Lyapunov function in the region

$$\Omega = \{x_1, x_2 : 4^2 - x_1^2 \geq 0,\ 4^2 - x_2^2 \geq 0\} \tag{4.67}$$

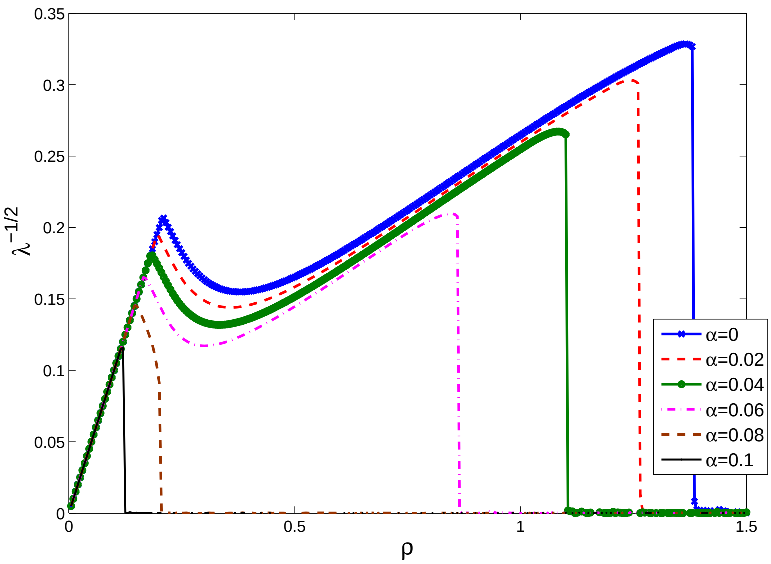


Figure 4.4: Radius evolution for different decay rates using quadratic Lyapunov functions (fuzzy polynomial model).

in which the nonlinearity $\sin(x)$ will be modelled by following the Taylor series approach presented on Section 3.1.1. The resulting fuzzy polynomial model (4.3) using Taylor decomposition until degree 3 is:

$$\mu_1(x) = \frac{\sin(x_1) - x_1 + 0.074325x_1^3}{-0.0923417x_1^3}; \quad \mu_2(x) = 1 - \mu_1(x)$$

$$p_1(x) = \begin{cases} -3x_1 + 0.5x_2 \\ -2x_2 + 3x_2(x_1 - 0.166667x_1^3) \end{cases}$$

$$p_2(x) = \begin{cases} -3x_1 + 0.5x_2 \\ -2x_2 + 3x_2(x_1 - 0.074325x_1^3) \end{cases}$$

Then, following the methodology described on Section 4.2 with Algorithm 4.2.3 and expressing the exploration region as in (4.63), the largest spheres are obtained by performing iterations with ρ_d. Note that the extreme values of the membership functions are computed in a similar way to the above example because they are monotonic with Ω_d.

First, if quadratic stability is analyzed, similar results to Example 4.4.1 are obtained. In addition, the orientation of the found ellipsoids changes with the region of study Ω_d. After that, the use of higher-degree Lyapunov functions has been analyzed in order to check the improvement on the obtained DA estimates. Figure 4.5 shows how with using 4^{th} and 5^{th} degree Lyapunov functions, larger spheres belonging to the DA are found than in the quadratic case. Indeed, since $\rho_d = 2.1$, conditioning of quadratic Lyapunov functions decrease quickly and they give smaller spheres.

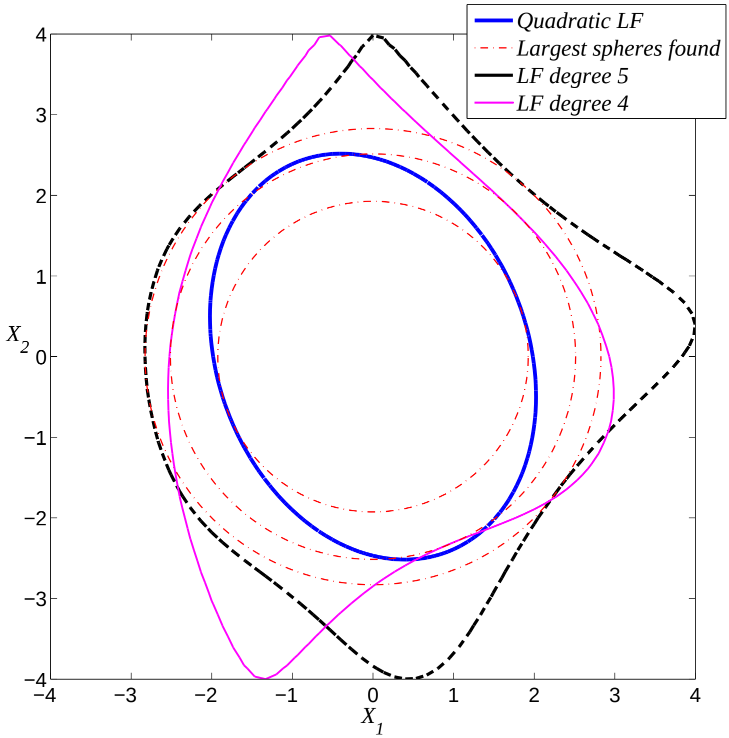


Figure 4.5: Better DA estimates found with different Lyapunov functions.

Once a Lyapunov function is obtained, for instance the 4^{th} order one

$$\begin{aligned}V(x) = 0.016x_1^4 + 0.00742x_1^3x_2 - 0.0133x_1^3 + 0.0147x_1^2x_2^2 \\ + 0.0246x_1^2x_2 + 0.0133x_1^2 - 0.00139x_1x_2^3 + 0.0374x_1x_2^2 \\ - 0.0272x_1x_2 - 0.0029x_2^4 - 0.0041x_2^3 + 0.133x_2^2,\end{aligned}$$

the "a posteriori" procedure explained on Section 4.2.3 can be applied. Thus, defining constraints (4.18) as $g_1 = \rho_d - x_2$, $g_2 = \rho_d - x_1$, $g_3 = -\rho_d - x_1$ and $g_4 = -\rho_d - x_2$, the borders of the squared region of study are:

$$B_1(x) = \{x_1 | -\rho_d \leq x_1 \leq \rho_d,\ x_2 = \rho_d\}$$
$$B_2(x) = \{x_2 | -\rho_d \leq x_2 \leq \rho_d,\ x_1 = \rho_d\}$$
$$B_3(x) = \{x_1 | -\rho_d \leq x_1 \leq \rho_d,\ x_2 = -\rho_d\}$$
$$B_4(x) = \{x_2 | -\rho_d \leq x_2 \leq \rho_d,\ x_1 = -\rho_d\}$$

With the zones of the frontier where system trajectories point to inside Ω_d, Lemma 4.4 can compute a larger level set of $V(x)$ until it touches some border zone in which $\dot{g}(x) < 0$. As a result of this procedure, Figure 4.6 shows; the numerically computed *forbbiden* regions of the border (in red) where system trajectories leave the modelling square, the previously proven DA using Algorithm 4.1 and the improving obtained with the expansion of Lemma 4.4.

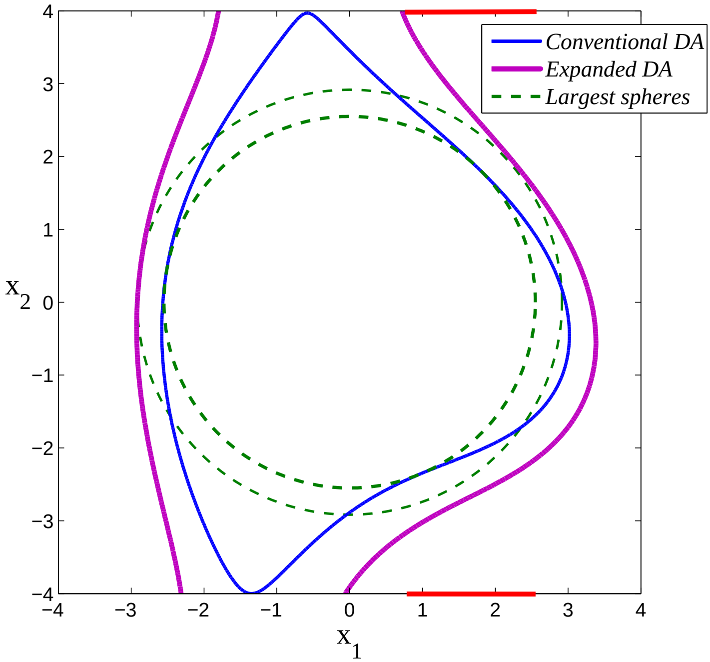


Figure 4.6: Conventional DA estimate and expanded one (Lemma 4.4) using 4^{th} order Lyapunov function. In red the zones of $\partial\Omega$ where $\dot{g}(x) < 0$.

As it can be seen, in this case, by means of this expansion procedure the stable spherical region around the origin is enlarged, without increasing the

Lyapunov-function order. For completeness, the LRDA surface corresponding to the fourth order Lyapunov function is depicted on Figure 4.7.

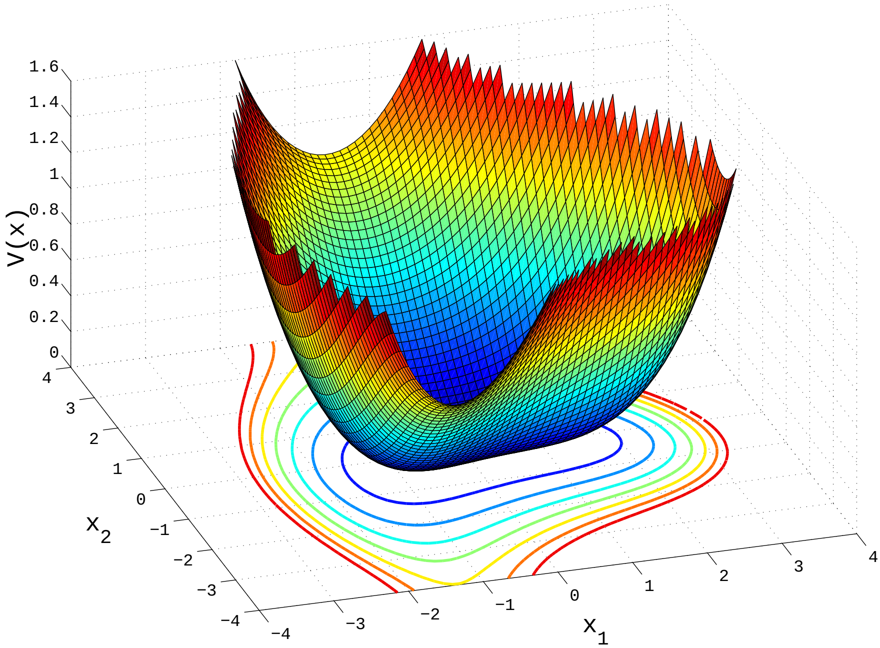


Figure 4.7: DA surface using a 4^{th} order Lyapunov function.

In order to have an idea of which amount of conservativeness is implied with this technique, on Figure 4.8 several system trajectories have been traced from different initial conditions. In this way, the real domain of attraction of the nonlinear system is estimated by simulation. Then the best proven LRDA obtained with the proposed methodology (Algorithm 4.1 + Lemma 4.4), using a 7^{th} degree Lyapunov function, is superposed. It can be observed that the largest sphere has been well estimated because, indeed, the Lyapunov function level set (which overflow the border of Ω due to expansion for Lemma 4.4) fits the real DA in the upper right quadrant, and the sphere is tangent to this level set.

Finally, on Figure 4.9, the evolution of the largest-found sphere contained in the Lyapunov level set is presented for different degrees on the Lyapunov functions. Also it compares the results before and after applying the expansion procedure of Section 4.2.3. It is shown that the higher degree of the Lya-

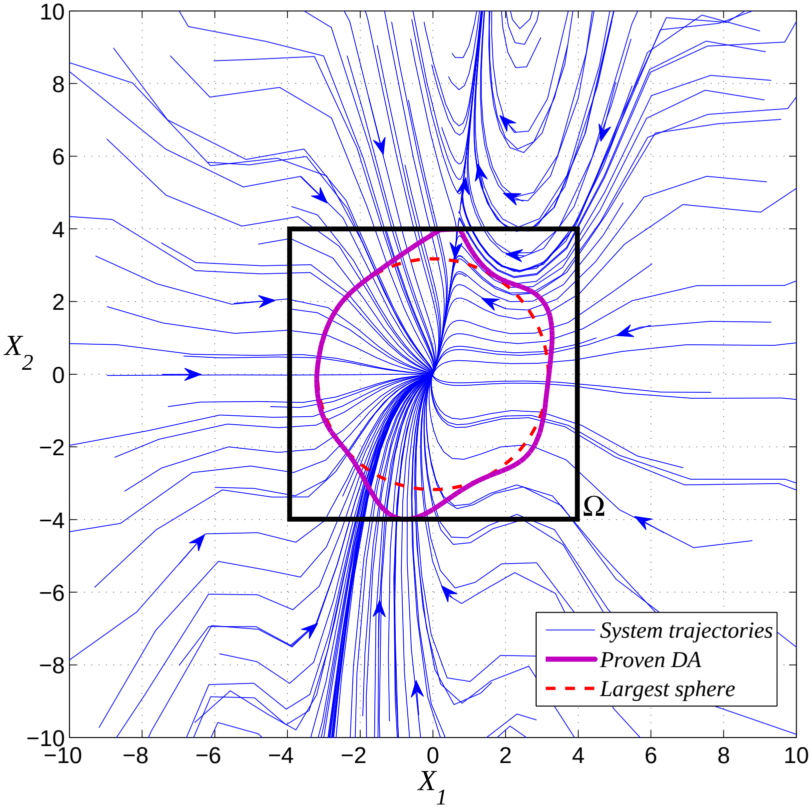


Figure 4.8: Phase plane of the nonlinear system jointly with the estimated LRDA using a 7^{th} order Lyapunov function.

punov function is, the lower the improvement with the expansion procedure is achieved. The reason is that, with higher degree, more flexibility is provided to the Lyapunov function in order to adapt the real DA. Indeed, on Figure 4.8 is slightly hard to realize that the maximum Lyapunov level set overflows the border $\partial\Omega$. This fact is also checked with green curves on Figure 4.9.

Computational cost. In order to give an idea of the required computational resources for the proposals in this chapter, in this particular example the resolution of one iteration with Algorithm 4.4 in an Intel® Core™2 Duo P8600 and 4 Gb of DDR3 RAM takes 94 sec. for a quadratic Lyapunov function, 100 sec. for degree 4 and 118 sec. for degree 6. MATLAB® 2008a with SDPT3 4.0 and SOSTOOLS 2.03 have been used for implementation.

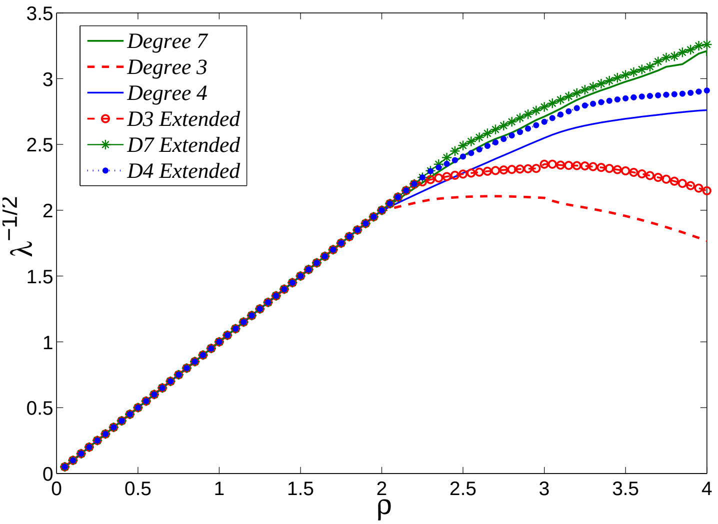


Figure 4.9: Evolution of the largest sphere centered on the origin with the scale factor of the modelling region.

Example 4.4.3. Non-fuzzy polynomial system.

Now, a simple example from Khalil (2002, Example 8.9) is provided in order to show the performance of the proposed methodology in Section 4.3 over standard level-set ones in the referred source.

Consider the polynomial system:

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2 \end{aligned} \tag{4.68}$$

For the above system, linearization shows that the origin is stable: there is a neighborhood of it belonging to its DA provable with a Lyapunov function $V(x) = 1.5x_1^2 - x_2x_1 + x_2^2$, see Khalil (2002) for details. However, phase plane simulation shows that it has an unstable limit cycle so the DA of the origin is limited by it.

The Lyapunov-based methodology proposed in Khalil (2002) obtains an initial estimate of the DA from a rough bounding of $\dot{V}$ given by $\{x : V(x) \leq 0.801\}$. Then, zooming out this region by performing a trial-and-error contour plotting, the above estimate is expanded to $\{x : V(x) \leq 2.25\}$.

Now, using the proposal in Section 4.3, the region $B_1 = \{x : V(x) \leq 2.25\}$ is used as the algorithm *seed* region. The initial step-size parameters are set to $\nu = 1.1$, $\delta = 0.2$ and a 4^{th} degree polynomial boundary V_2 is chosen. With $\Delta_\nu = 0.1$, $\Delta_\delta = 0.1$, Algorithm 4.2 runs for 9 iterations until it stops due to lack of progress. The largest region obtained with a 4^{th} degree polynomial boundary is $\{x : V_2(x) < 1\}$, where

$$V_2(x) = 0.18157x_1^2 - 0.58255x_1x_2 + 0.0058x_2^2 + 0.0327x_1^4$$

$$+0.15975x_1^3x_2 + 0.14346x_1^2x_2^2 - 0.0709x_1x_2^3 + 0.053x_2^4$$

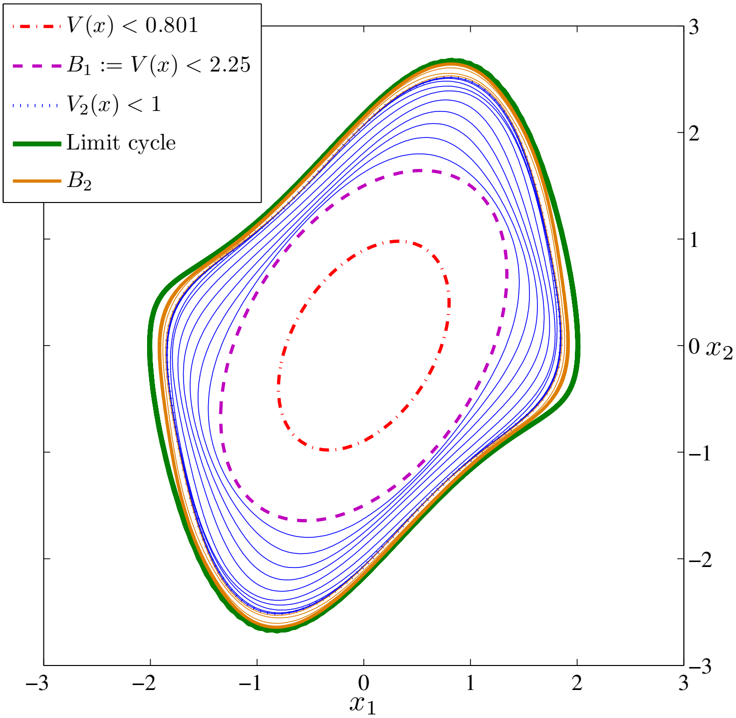


Figure 4.10: Domain of attraction evolution using 4^{th} order polynomial curves (blue) and 6^{th} order ones (brown). Seed set B_1 taken from Khalil (2002).

Then, four more iterations are executed by reducing starting algorithm parameters to $\nu = 1.02$, $\delta = 0.05$ and also setting a 6^{th} degree for the new polynomial boundaries V_2. Finally, the new DA estimate is, explicitly:

$$B_2 = \{x : -0.02023x_1^5x_2 - 0.401x_1x_2 + 0.595x_1^2 - 0.1633x_1^4$$

$$+0.3378x_2^2 - 0.0514x_2^4 + 0.0206x_1^6 + 0.055x_2^6 - 0.15867x_1^2x_2^2$$

$$+0.09x_1x_2^3 - 0.0208x_1x_2^5 + 0.182x_1^3x_2 - 0.0578x_1^3x_2^3$$
$$+0.049x_1^2x_2^4 + 0.0388x_1^4x_2^2 < 1\}$$

The improvement over estimates in Khalil (2002) can be checked on Figure 4.10. In fact, the obtained boundary of B_2 is pretty close to the actual limit cycle (see Figure 8.2 in the cited source, and green contour below for numerical simulation-based approximations to it) which is the *exact* shape of the DA for which a closed-form solution is, however, unavailable.

Example 4.4.4. Continuous-time non-polynomial system.

Consider the nonlinear system presented in Example 4.4.2.

The objective is to estimate the domain of attraction of the origin in a state-space modelling region Ω defined as a sphere of radius R_e centered in $x = 0$:

$$\Omega = \{(x_1, x_2) \,|\, x_1^2 + x_2^2 < R_e^2\}$$

For instance, for $R_e = 10$, we have the two equilibrium points inside Ω: $e_0 = (0, 0)$ and $e_1 = (0.7297, 4.378)$. Linearization shows that e_0 is a stable node (two negative real Jacobian eigenvalues), and e_1 is a saddle point (one stable and one unstable eigenvalues).

Taking into account the range $-10 \leq x_1 \leq 10$ and using the 5^{th} degree Taylor expansion of $\sin(x_1)$, there exists an exact fuzzy-polynomial representation in Ω such that $\sin(x_1) = \mu_1(x)p_1(x) + \mu_2(x)p_2(x)$, where:

$$p_1(x) = x_1 - \frac{1}{6}x_1^3 + 9.16 \cdot 10^{-3}x_1^5$$

$$p_2(x) = x_1 - \frac{1}{6}x_1^3 + 1.56 \cdot 10^{-3}x_1^5$$

which gives a two-vertices fuzzy polynomial model (4.3) with membership functions ($z \equiv x$):

$$\mu_1(x) = \frac{\sin(x_1) - p_2(x)}{7.6 \cdot 10^{-3}x_1^5}, \quad \mu_2(x) = 1 - \mu_1(x)$$

For other sizes of the modelling region Ω, resulting in different ranges of x_1, suitable vertex models may be obtained by the same Taylor-series methodology.

A starting region B_1 is obtained with well-known methodologies (Chesi, 2011; Papachristodoulou and Prajna, 2002): a search was made for a polynomial Lyapunov function $V_1(x)$ giving the maximum radius R_1 of a circle included in its level set $\{V_1(x) \leq 1\}$, and such that $\dot{V}_1$ decreases in a spherical modelling region around the origin of radius R_e.

As there is a saddle point e_1, whatever the choice for V_1 is, we will have $\dot{V}_1(e_1) = 0$. Forcefully, *any* Lyapunov function search from literature (for instance, Lemma 3.4, Theorem 4.1) will *not* be feasible for $R_e \geq \|e_1\| = 4.44$. So, to obtain a first seed set, R_e was set to 4.42 in the numerical implementations, corresponding to curve C_1 in Figure 4.11. In fact, because of the inherent conservatism from fuzzy modelling, the single saddle point in the original nonlinear system becomes a "strip" of possible equilibrium points (for different values of μ) in the fuzzy model.

The Lyapunov function is found by adapting Theorem 4.1 to solving the SOS problem of maximize R_i subject to

$$\begin{aligned} V - \epsilon x^T x + \psi_1(x^T x - R_e^2) &\in \Sigma_x \\ V - 1 - \psi_2(x^T x - R_e^2) &\in \Sigma_x \\ 1 - V + \psi_3(x^T x - R_i^2) &\in \Sigma_x \\ -(\frac{\partial V}{\partial x} p_i(x) + \epsilon x^T x) - \phi_i(R_e^2 - x^T x) &\in \Sigma_x \quad i = 1, 2 \end{aligned}$$

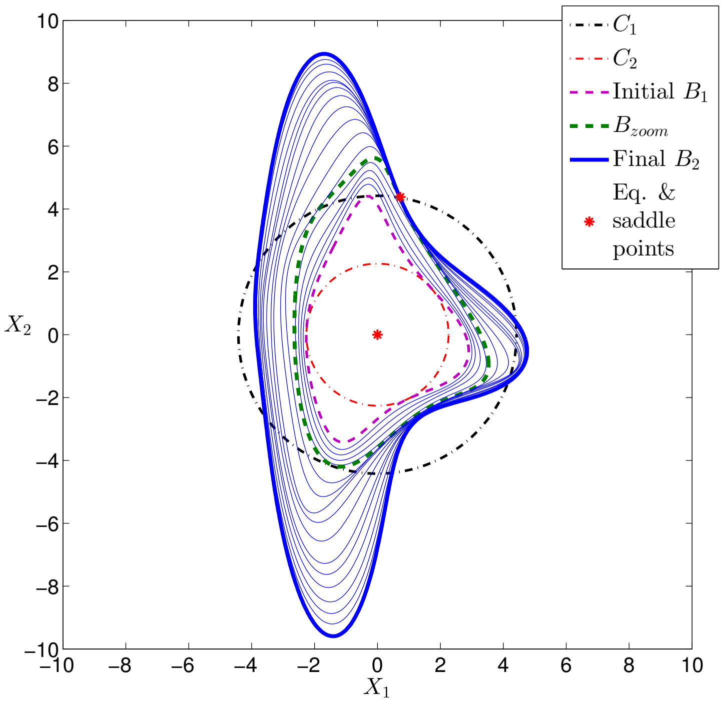
for $\epsilon = 0.001$, and multipliers $\{\phi_i,\ \psi_j\} \in \Sigma_x$. The Lyapunov function's degree has been set to 4. Obviously, higher degrees would yield better results, but the objective of the methodology is showing that improvements in DA estimation can be made *without increasing the polynomial degree*.

The largest circle proved to belong to the DA with this standard methodology is C_2, and the Lyapunov level set is limited by the dashed-purple curve "Initial B_1" in Figure 4.11.

The proved domain of attraction is then enlarged following Algorithm 4.2, looking for 4^{th} degree new polynomials $V_2(x)$. Figure 4.11 shows how the estimated domain of attraction increases from the Lyapunov-only solution, i.e.,"Initial B_1", as iterations progress. First, with a zoom factor $\upsilon = 1.2$ and $\delta = 0$, and $\Delta_\upsilon = 0.1$, Algorithm 4.2 works for five iterations reaching region labelled as B_{zoom} in the figure.

Using B_{zoom} as seed, restarting the algorithm with $\delta = 0.03$, $\Delta_\delta = 0.01$ and $\upsilon = 1$, the algorithm runs for 12 more iterations, and gives the best feasible DA proved (curve "Final B_2" in the Figure).

Although simulations show that the domain of attraction is quite larger, iterations find hard to obtain a better estimate using a *closed* 4th degree boundary. Indeed, each new candidate region has to be valid for the family of "all" systems between p_1 and p_2: however, the difference between the vertex poly-



C_1: Starting modelling region Ω with a single equilibrium (Eq) point inside (classical Lyapunov techniques used locally in C_1);

Initial B_1: Level set of the Lyapunov function proving C_2;

B_{zoom}: Last iteration with $\nu \neq 1$, $\delta = 0$;

Final B_2: Last iteration with $\nu = 1$, $\delta \neq 0$.

Figure 4.11: Domain of attraction evolution using 4^{th} order polynomial curves.

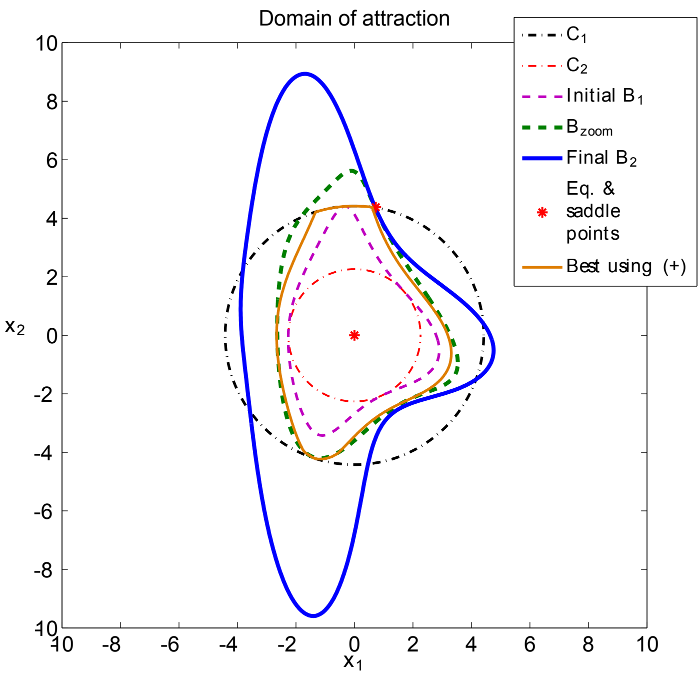
nomials grows larger as we depart further from the origin. Anyway, the obtained result “Final B_2” is substantially larger than the initial Lyapunov level set “Initial B_1” from usual methodologies in literature.

In summary, the largest set proved to belong to the LRDA $\mathcal{D}_\Omega$ is the set:

$$\begin{aligned} B_2 = \{x : &-0.2828x_1 - 0.1238x_2 - 0.1315x_1^2 - 0.0918x_2x_1 - 0.0468x_2^2 \\ &+ 0.0056x_1^3 + 0.0252x_1^2x_2 + 0.1111x_2^2x_1 + 0.0039x_2^3 + 0.0099x_1^4 \\ &+ 0.0123x_1^3x_2 + 0.0358x_1^2x_2^2 + 0.002x_1x_2^3 + 0.0017x_2^4 < 1\} \end{aligned}$$

Note. In general the proved DA with Lemma 4.8 is an intersection between a level set and the region of interest, i.e., $B_2 = \{x : V_2(x) < 1 \cap \Theta\}$. However, in this particular case, the intersection notation is not needed (in fact the possibility is intentionally not allowed enforcing $\{x : V_2(x) < 1\} \subset \Theta \subset \Omega$). The next example considers the more general case.

Note also that the techniques in Section 4.2.3 obtain a DA estimate larger than "Initial B_1" but smaller than $B_{zoom} \cap C_1$, much smaller than the one "Final B_2" obtained in this work. See Figure 4.12.



Curves C_1, C_2, Initial B_1, B_{zoom} and Final B_2 are as in Figure 4.11;
Best using $(+)$: Largest DA estimate using the expansion method in Section 4.2.3.

Figure 4.12: DA estimate comparison with proposed techniques.

Example 4.4.5. Discrete-time system.

Consider the following nonlinear system obtained by the Euler discretization of (4.66) at sample time $T = 0.1$ seconds:

$$\begin{aligned} x_{1k+1} &= 0.7x_{1k} + 0.05x_{2k} \\ x_{2k+1} &= x_{2k}(0.8 + 0.3\sin(x_{1k})) \end{aligned} \tag{4.69}$$

which has the same equilibrium as (4.66). However, due to the large sampling period in the Euler approximation, the domain of attraction may change, as discussed below. Also, for illustration, the degree of the fuzzy-polynomial approximation of $\sin(x_1)$ has been chosen differently.

The objective again is to estimate the domain of attraction of the origin in a state-space circular modelling region of radius R_e centered in $x = 0$. The discrete system has the same equilibrium points as the continuous-time one.

For instance, using the 3^{th} degree Taylor expansion of $\sin(x_{1k})$ computed in the range $|x_1| < 10$, there exists an exact fuzzy-polynomial representation in Ω such that $\sin(x_{1k}) = \mu_1(x_k)p_1(x_k) + \mu_2(x_k)p_2(x_k)$, where:

$$p_1(x_k) = x_{1k} - \frac{1}{6}x_{1k}^3$$

$$p_2(x_k) = x_{1k} - 0.01054x_{1k}^3$$

which gives a two-vertices fuzzy polynomial model (4.4) with membership functions ($z_k \equiv x_k$):

$$\mu_1(x_k) = \frac{\sin(x_{1k}) - p_2(x_k)}{-0.15612x_{1k}^3}, \quad \mu_2(x_k) = 1 - \mu_1(x_k)$$

A starting region B_1, is again obtained with well-known Lyapunov methodologies (Tanaka, Ohtake and Wang, 2008). The way is to search for a polynomial $V_1(x)$ which gives the maximum radius R_1 of a circle included in the region $\{x : V_1(x) < 1\}$ such that V_1 decreases in a circular region around the origin of radius R_e. Let us detail how initial V_1 was crafted in this example:

As in Example 2, whatever the choice for V_1 is, any Lyapunov function search from literature will not be feasible for $R_e \geq \|e_1\| = 4.44$, so R_e was set to 4.15 in the numerical implementations[7], hence Ω in the previous sections corresponds to curve C_1 in Figures 4.13 and 4.14.

The starting Lyapunov function may be found by two approaches:

1. Solving the SOS problem of maximising R_i subject to

$$\begin{aligned} V(x) - \epsilon x^T x + \psi_1(x^T x - R_e^2) &\in \Sigma_x \\ V(x) - 1 - \psi_2(x^T x - R_e^2) &\in \Sigma_x \\ 1 - V(x) + \psi_3(x^T x - R_i^2) &\in \Sigma_x \end{aligned}$$

[7] R_e cannot be increased without leading to an infeasible problem due to the intrinsic conservativeness issues of the fuzzy-polynomial approach (Sala and Ariño, 2009).

$$Z(\sigma)V(x) - V(\sum_i \sigma_i^2 p_i(x)) - Z(\sigma)\epsilon x^T x \\ - \phi_1 Z(\sigma)(R_e^2 - x^T x) \ \in \Sigma_{x,\sigma}$$

where $\epsilon = 0.001$, $Z(\sigma)$is used to make conditions homogeneous[8] in σ^2 , and $\{\phi_1 \ \psi_j\} \in \Sigma_x$ are Positivstellensatz multipliers.

The drawback with this approach is that the degree of the polynomial conditions above grows quickly with the Lyapunov function's degree (because computations involve products of σ^2 and powers of x).

2. If the idea of introducing slack variables ρ is applied (Lemma 4.5), the above problem can be expressed as maximizing R_i subject to:

$$\begin{aligned} V - \epsilon x^T x + \psi_1(x^T x - R_e^2) &\in \Sigma_x \\ V - 1 - \psi_2(x^T x - R_e^2) &\in \Sigma_x \\ 1 - V + \psi_3(x^T x - R_i^2) &\in \Sigma_x \\ V - V(\rho) - \epsilon x^T x - \phi_{1i}(R_e^2 - x^T x) + \phi_2(\rho - p_i(x)) &\in \Sigma_{x,\rho} \quad i : 1, 2 \end{aligned}$$

where $\epsilon = 0.001$ and multipliers $\phi_2 \in \Sigma^n_{x,\rho}$, $\phi_{1i} \in \Sigma_{x,\rho}$, $\psi_j \in \Sigma_x$.

In the example, the second approach has been used, and the Lyapunov function's degree has been set to 4. The largest circle proved to belong to the DA with this standard methodology is C_2, and the Lyapunov level set is limited by the dashed-blue curve Initial B_1 in figures 4.13 and 4.14.

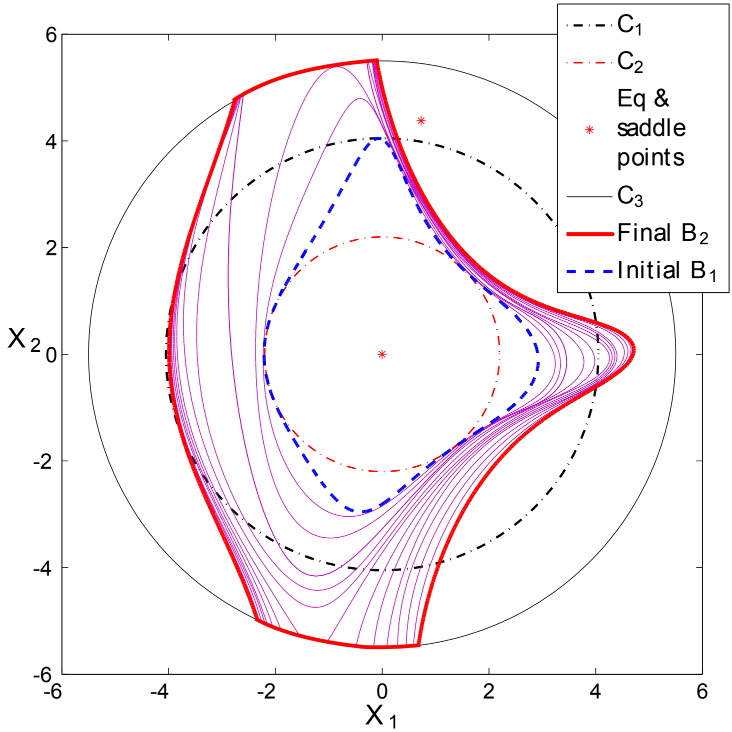
The proven domain of attraction is then enlarged following Lemma 4.7, as proposed in Section 4.3.2.1, iteratively searching for new polynomials $V_2(x)$ of 4^{th} degree. Two trials of the iterations with different modelling regions have been considered.

a) Circle of radius $R_e = 5.5$: Consider the user-defined spherical region (C_3 in Figure 4.13):

$$C_3 = \{x : x_1^2 + x_2^2 < 5.5^2\}$$

so $\Omega \equiv C_3$ in this case. Note, importantly, that it includes the saddle point so no Lyapunov function can be ever found to decrease in all C_3.

[8]The change $\mu \equiv \sigma^2$ is enforced. Also, suitable manipulations (multiplication by powers of $1 = \sum_i \sigma_i^2$) in the term $V(\sum_i \sigma_i^2 p_i(x))$ are implicitly assumed for homogenization.



C_1:	Starting modelling region close to largest circle with a single equilibrium (Eq) point inside;
C_2:	largest circle in DA proved with classical Lyapunov techniques over C_1;
C_3:	New modelling region, including the saddle point (always infeasible with previous literature results);
Initial B_1:	Level set of the classical Lyapunov function proving C_2;
Final B_2:	Last iteration of iterative algorithm in Section 4.3.2.1.

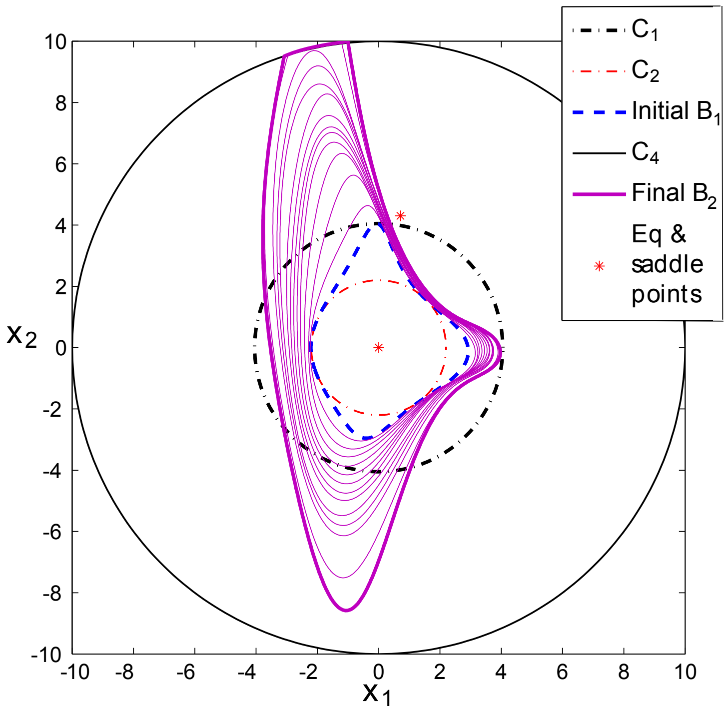
Figure 4.13: DA evolution using 4^{th} order polynomial curves and fixed R_e.

Figure 4.13 shows how the estimated domain of attraction increases from the Lyapunov-only solution "Initial B_1" as iterations progress.

The final estimation of the LRDA is given by:

$$B_2 = \{x : V_2(x) < 1,\ x_1^2 + x_2^2 < 5.5^2\}$$

with $V_2(x) = 0.003054x_1^4 - 0.00132x_1^3x_2 - 0.02021x_1^3 + 0.01636x_1^2x_2^2 - 0.001495x_1^2x_2 + 0.004x_1^2 + 0.00075x_1x_2^3 + 0.03096x_1x_2^2 + 0.02511x_1x_2 + 0.32495x_1 - 0.00034x_2^4 + 0.0025x_2^3 + 0.02942x_2^2 + 0.030556x_2$.



C_1: Starting modelling region close to largest circle with a single equilibrium (Eq) point inside (same as Fig. 4.13);

C_2: largest circle in DA proved with classical Lyapunov techniques over C_1 (same as Fig. 4.13);

C_4: Circular modeling region for $R_e = 10$;

Initial B_1: Lyapunov level set proving C_2 (same as Fig. 4.13);

Final B_2: DA estimate in last iteration.

Figure 4.14: DA evolution using 4^{th} order polynomial curves for increasingly larger modelling region radius.

b) Circle of radius $R_e = 10$: Note that, as iterations progress in the above case (a), the obtained sets approach the boundary of the modelling region C_3 (actually, they *cross* it). Hence, that suggest that larger regions might be obtained if the modelling region is expanded. This second case considers expanding a little the modelling region in each iteration until a final target $R_e = 10$ is reached (or the algorithm stops improving).

Figure 4.14 shows the final DA estimation. The new LRDA found ("Final

B_2" on the picture) is

$$B_2 = \{x : V_2(x) < 1,\ x_1^2 + x_2^2 < 10^2\}$$

being $V_2(x) = 0.00864x_1^4 - 0.00214x_1^3x_2 - 0.0454x_1^3 + 0.01313x_1^2x_2^2 + 0.00427x_1^2x_2 - 0.0042x_1^2 + 0.003525x_1x_2^3 + 0.0388x_1x_2^2 + 0.0394x_1x_2 + 0.4642x_1 + 0.0006x_2^4 + 0.00454x_2^3 + 0.01627x_2^2 - 0.05847x_2$.

4.5 Conclusions

In this chapter some local stability problems in nonlinear systems have been addressed within a polynomial framework. The objective was estimating as better as possible the real domain of attraction of a nonlinear system and expressing it in *closed form* by polynomial boundaries. Procedures have been presented for both continuous-time and discrete-time cases. Several improvements over existent literature have been proposed:

1. Expressing the membership functions as a polytopic expression in order to take into account its local information in a reduced region of study.
2. Trying to avoid ill-shaped solutions by means of forcing to obtain the largest prefixed-shape region which is proved to belong to the DA.
3. Expanding the actually proven DA by allowing the Lyapunov level set to not be contained in the modelling region.
4. Remove the classical Lyapunov stability constraints using an iterative methodology for expansion based on invariant-set considerations: starting from a "seed" region, the DA estimate is enlarged iteratively.

It has been checked that the largest scale factor of a prefixed-shape region, proven to belong to the DA, is a non-monotonic function with the modelling region size. Therefore, an iterative exploration is required to find the optimum.

The iterative methodology for expansion of the DA (Section 4.3) always improves the starting "seed" subset of the DA, or at least finds the same set. Nevertheless, the procedure may fail after several iterations (i.e. return infeasible solutions due to numerical problems) when there won't be enough freedom to find an improving solution. This fact will occur for instance when the previously proven subset of the DA is pretty close to other equilibrium points or limit cycles, because there might not be enough space for a low-degree polynomial boundary to fit the actual shape of the DA. In addition, the Taylor-series developments up to a prefixed degree are very close to the actual nonlinear system around the equilibrium point, but the polynomial vertex models separate abruptly far from the origin, thus increasing the size of the fuzzy sector and its associated conservativeness.

In those cases, the procedure can progress further by increasing the degree of the decision-variable polynomials (boundaries and/or Positivstellesatz multipliers) and/or taking higher-degree terms in the Taylor-series developments in the modelling phase, at the prize of increasing computational effort, or perhaps, increasing/reducing the size of the modelling region Ω in order to give more freedom (if the last successful estimate is too much close to Ω) /reducing conservativeness associated to the fuzzy modelling.

Chapter 5

Inescapable-set Estimation with Nonvanishing Disturbances

You step onto the road, and if you don't keep your feet, there's no knowing where you might be swept off to.

John Ronald Reuel Tolkien,
The Lord of the Rings

ABSTRACT: Fuzzy TS and polynomial models are local and, obviously, a larger modelling region yields lower performance bounds. In systems subject to disturbances, determining the optimal size of the modelling region is needed in order to avoid small ones (state escapes from them) or overly large ones (suboptimal performance). This chapter addresses such problem and, furthermore, suggest a multicriteria fuzzy control-design strategy in order to ensure disturbance-rejection performance without leaving a particular modelling region. The presented results allow extending the concept of Lyapunov level sets used in stability analysis to the case where disturbances are present.

The above chapter has addressed the stability analysis problem of obtaining well-shaped domain of attraction estimates for nonlinear systems. Nevertheless, the case of stability analysis in presence of disturbances is still disregarded. Normally disturbance rejection is addressed by $\mathcal{H}_\infty/\mathcal{H}_2$ optimum control designs, which are wide-applied with fuzzy TS systems. However, those optimum designs do not ensure stability of the original nonlinear system for any particular disturbance (if a disturbance pushes the system out of the modelling region, system's behavior is unknown) or if initial conditions are not zero.

The preliminary work Salcedo, Martínez and García-Nieto (2008) analyzes those above presented problems of optimum disturbance-rejection designs for TS systems and the PDC control law case, concluding that those methods cannot be used in applications with bounded persistent disturbances. Indeed, estimating stable regions under disturbances is still an open problem. In addition, it also proposes an $\mathcal{L}_1$ control-design method that assumes an a priori bound on the disturbance level, which allows the output $\mathcal{L}_1$ norm to be minimized.

This chapter presents a generalization of such preliminary work to the fuzzy polynomial framework and analyzes the system's behavior with nonvanishing disturbances ("bounded power" during a certain integral time), computing an estimate of the smallest inescapable set. The polynomial control synthesis problem is modified by taking into account the validity of the obtained results in disturbance-rejection control designs. The developments presented in this chapter were preliminary results available at the time this thesis was defended, published in the conference paper Pitarch, Sala, Bedate and Ariño (2013). Nevertheless, they have been further extended afterward in the journal paper Pitarch, Sala and Ariño (2015) for pure stability analysis, and more deeply studied in a general disturbance-invariant control framework in Sala and Pitarch (2016).

The structure of this chapter is as follows: preliminaries and main objectives are stated on next section, a precise disturbance analysis, defining the initial, transitory and final regions for non-vanishing disturbances, is presented on Section 5.2 jointly with the main results provided for the studied disturbance properties, Section 5.3 addresses the control design, an academic example is provided on Section 5.4 and finally Section 5.5 concludes the chapter.

5.1 Problem statement

Consider the nonlinear continuous-time system:

$$\begin{aligned} \dot{x} &= A(x)x + B(x)u + Ew \\ y &= C(x) + Du \end{aligned} \tag{5.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^b$ is the control input, $w \in \mathbb{R}^c$ is a disturbance vector and $A(x)$, $B(x)$, $C(x)$ are matrices of appropriate dimension containing nonlinear terms.

If there exist a known control action u, system (5.1) reduces to:

$$\begin{aligned} \dot{x} &= f(x) + Ew \\ y &= c(x) \end{aligned} \tag{5.2}$$

where $f(x)$ and $c(x)$ are, in general case, vectors of nonlinear functions of the state.

Consider a modelling region Ω defined by $g_i(x)$ polynomial boundaries as follows:

$$\Omega = \{x : g_1(x) > 0, \ldots, g_k(x) > 0\} \tag{5.3}$$

Then, by using the polynomial sector nonlinearity approach (Section 3.1.1), a fuzzy-polynomial model for (5.1) can be obtained in the form

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{r} \mu_i(x) \left(A_i(x) z(x) + B_i(x) u \right) + Ew \\ y &= \sum_{i=1}^{r} \mu_i(x) C_i(x) + Du \end{aligned} \tag{5.4}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^m$ is the output, $u \in \mathbb{R}^b$ is the control action, $w \in \mathbb{R}^c$ is the disturbance input, $z(x) \in \mathcal{R}_x^l$ is a vector of polynomials of x, $(A_i(x), B_i(x)) \in \mathcal{R}_x^n$ and $C_i(x) \in \mathcal{R}_x^m$ are polynomial vertex models, $\mu_i(x)$ are their corresponding nonlinear membership functions and r is the number of fuzzy rules.

In a similar way, if there exist a known control input u, system (5.2) can be equivalently expressed in Ω as the fuzzy-polynomial model in the form:

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{r} \mu_i(x) p_i(x) + Ew \\ y &= \sum_{i=1}^{r} \mu_i(x) c_i(x) \end{aligned} \tag{5.5}$$

The objectives of this chapter are:

1. Generalizing the existent preliminary work to the fuzzy polynomial framework and including a more general class of disturbances (“bounded power” during a certain integral time).

2. Analyzing system's behavior with non-vanishing disturbances acting and computing an estimate of the inescapable set given particular disturbance features.

3. Proposing a control design methodology which, starting inside a certain initial set, ensures that the transient trajectories do not exceed the fuzzy-modelling region.

5.2 Invariant sets under nonvanishing disturbances

Now, a stability analysis is presented for a class of non-vanishing disturbances whose main feature is to give a limited power during a certain time period.

Theorem 5.1. *Assume, for the system* (5.2)*, that the disturbance fulfills*

$$\int_t^{t+T} w(t)^T w(t) \leq \beta \quad \forall t \geq 0 \tag{5.6}$$

for a known time-horizon T*, and that there exist a nonnegative function* $V(x)$ *fulfilling:*

$$\dot{V}(t) + \alpha V(t) - w(t)^T w(t) \leq 0 \tag{5.7}$$

Then,

$$\lim_{t\to\infty} V(t) \leq \frac{\beta}{1 - e^{-\alpha T}} \tag{5.8}$$

and, during the transient from an initial condition $V(0)$*,* V *can be bounded as:*

$$V(t) \leq \max(V(0) + \beta, \frac{\beta}{1 - e^{-\alpha T}}) \quad \forall t \geq 0 \tag{5.9}$$

Proof. Consider the linear first-order differential equation

$$\dot{V}(t) + \alpha V(t) - w(t)^T w(t) = -g(t)$$

where some $g(t) \geq 0$ exists by (5.7). Its solution, with initial condition $V(0)$ is:

$$V(t) = e^{-\alpha t} V(0) + \int_0^t (w(\tau)^T w(\tau) - g(\tau)) e^{-\alpha(t-\tau)} \, d\tau$$

As g is nonnegative, this can be rewritten, considering the largest N such that $NT \leq t$ as:

$$V(t) \leq e^{-\alpha t}V(0) + \int_0^T w(\tau)^T w(\tau) e^{-\alpha(t-\tau)}\, d\tau + \int_T^{2T} w(\tau)^T w(\tau) e^{-\alpha(t-\tau)}\, d\tau + \cdots + \int_{NT}^t w(\tau)^T w(\tau) e^{-\alpha(t-\tau)}\, d\tau \quad (5.10)$$

so that, after some straightforward manipulations, we have the bound (5.10) explained better:

$$V(t) \leq e^{-\alpha t}V(0) + (1 + e^{-\alpha T} + \cdots + e^{-\alpha NT})\beta \quad (5.11)$$

And, as $t \geq NT$

$$V(t) \leq e^{-\alpha NT}V(0) + (1 + e^{-\alpha T} + \cdots + e^{-\alpha NT})\beta \quad (5.12)$$

Note that the right-hand side bound, call it ψ, as N progresses is the output of the first-order equation:

$$\psi_{N+1} = e^{-\alpha T}\psi_N + \beta$$

Hence it is monotonic. If the initial value ψ_0 is larger than the steady-state one $\frac{\beta}{1-e^{-\alpha T}}$, then the bound will be $\psi_0 = V(0) + \beta$; otherwise it will be $\psi = \frac{\beta}{1-e^{-\alpha T}}$. □

Note that, from the proof, the switching point will be

$$V(0) + \beta = \frac{\beta}{1 - e^{-\alpha T}}$$

which yields to:

$$V(0) = \frac{\beta}{1 - e^{-\alpha T}} - \beta = \frac{e^{-\alpha T}\beta}{1 - e^{-\alpha T}}$$

Particular cases:

- Norm-bounded disturbances: if $\|w\|_2 \leq \beta$, the above theorem, letting $T \to \infty$, results in

$$V(t) \leq V(0) + \beta$$

- Peak-bounded disturbances: if $w^T w \leq \gamma$, the above theorem, letting $\beta = \gamma T$ and $T \to 0$, results in

$$V(t) \leq \max(V(0), \frac{\gamma}{\alpha}) \qquad \lim_{t \to \infty} V(t) \leq \frac{\gamma}{\alpha}$$

Given an initial set $\Theta_i = \{V \leq V(0)\}$, the above result defines two important types of sets (see Figure 5.1):

- a "*transient inescapable set*" Θ_t from (5.9) in which the state is guaranteed to never leave
- a "*stationary inescapable set*" Θ_f from (5.8), $\Theta_f \subset \Theta_t$, where the states will ultimately lie (and not escape).

Some optimization problems in such sets can be stated, see next section.

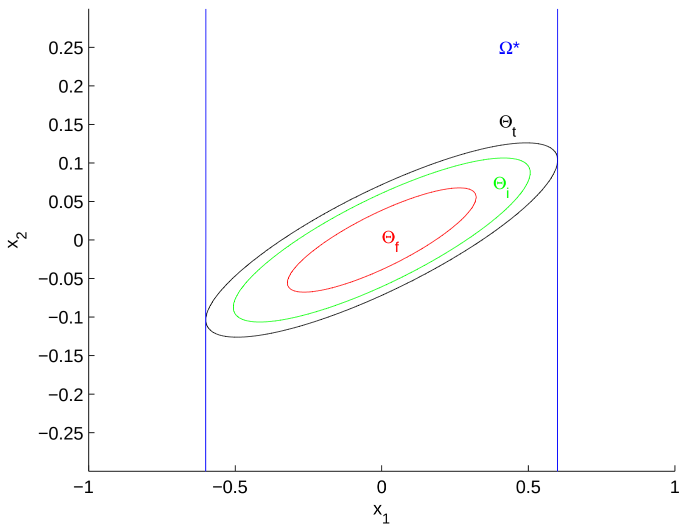


Figure 5.1: "Initial", "final" and "transitory" regions.

5.2.1 Application to local fuzzy models

A TS or fuzzy-polynomial model computed using sector-nonlinearity approach, is only valid locally in a modelling region Ω. Therefore, the above sets Θ_i, Θ_t should not leave Ω. Hence, Theorem 5.1 may be stated, for local models, as follows.

Theorem 5.2. *Given an initial set Θ_i, if there exist a Lyapunov function $V(x)$ fulfilling*

$$\Theta_i \subset \{x : V(x) \leq V_0\} \tag{5.13}$$

$$\{x : V(x) \leq V_0 + \beta\} \subset \Omega \tag{5.14}$$

$$\{x : V(x) \leq \frac{\beta}{1 - e^{-\alpha T}}\} \subset \Omega \tag{5.15}$$

$$\dot{V}(x) + \alpha V(x) - w^T w \leq 0 \quad \forall x \in \Omega \tag{5.16}$$

then the system does not leave Ω if initial conditions are in Θ_i and, ultimately, reaches the level set $\Theta_f = \{x : V \leq \frac{\beta}{1-e^{-\alpha T}}\}$.

Proof. Condition (5.16) makes the final reachable set to be $\Theta_f = \{x : V \leq \frac{\beta}{1-e^{-\alpha T}}\}$ by Theorem 5.1. Condition (5.13) ensures that the Lyapunov level set V_0 includes the region of initial conditions Θ_i. Condition (5.14) ensures the region $\{x : V(x) \leq V_0 + \beta\}$ to be inside the modelling region and condition (5.15) ensures the region $\{x : V \leq \frac{\beta}{1-e^{-\alpha T}}\}$ to be also inside the modelling region. Then, conditions (5.14) and (5.15) together mean (5.9). Moreover, transient region Θ_t is always inside Ω, where the fuzzy model is valid to represent (5.1). □

Remark 5.1. Note that, for global models, conditions (5.14)–(5.15) regarding Ω should be dismissed and condition (5.16) must hold for all $x \in \mathbb{R}^n$. Also, for systems initially in equilibrium, condition (5.13) should be disregarded too.

Remark 5.2. If $\Theta_i = 0$, the region of interest is given by an output equation $x^T C^T C x \leq \gamma^2$ and the disturbance is bounded by $w^T w \leq \beta$, the inescapable-ellipsoid results in Salcedo, Martínez and García-Nieto (2008) are obtained.

The above conditions are nonlinear in α. However, the set-inclusion conditions and the Lyapunov decrease can, in some interesting cases, be cast as convex programming conditions. Hence, fixing α, such problems can be solved via convex programming; the optimal solution in some sense would, however, require some bisection or exploration strategies over α between convex programming iterations (see details in Seiler and Balas (2010)).

Let us state some interesting problems to which Theorem 5.2 applies. Consider the fuzzy polynomial model (5.5), initial conditions $x(0) = 0$ and a modelling region Ω defined by polynomial boundaries, in a similar way to (4.18).

Corollary 5.1 (Bounding reachable set from origin)**.** *If there exists a polynomial $V(x)$ fulfilling*

$$V(x) - \epsilon(x) \in \Sigma_x \tag{5.17}$$

$$V(x) - \frac{\beta}{1-e^{-\alpha T}} + s_i(x)g_i(x) \in \Sigma_x \; i:1,\ldots,k \tag{5.18}$$

$$-\frac{dV(x)}{dx}(p_j(x)+Ew) - \alpha V(x) + w^T w - \sum_{i=1}^{k} s_{ji}(x)g_i(x) \in \Sigma_x \; j:1,\ldots,r \tag{5.19}$$

where $\epsilon(x)$ is a radially-unbounded positive polynomial and $(s_i(x), s_{ji}(x)) \in \Sigma_x$ are Positivstellensatz multipliers, then system (5.2) *does not leave Ω if initial conditions are in $\Theta_i = \{x : V \leq \frac{e^{-\alpha T}\beta}{1-e^{-\alpha T}}\}$, and it does not leave either $\Theta_f = \{x : V \leq \frac{\beta}{1-e^{-\alpha T}}\}$.*

Proof. Conditions (5.19) mean (5.16) and, jointly with (5.17), make $V(x)$ be a Lyapunov function of system (5.2) locally in Ω by Theorem 5.2 and Theorem A.1. Conditions (5.18) mean (5.15). If initial conditions $\Theta_i \subset \{x : V \leq \frac{e^{-\alpha T}\beta}{1-e^{-\alpha T}}\}$, then $V_0 \leq \frac{\beta}{1-e^{-\alpha T}} - \beta$ and condition (5.15) will be always more restrictive than (5.14) in Theorem 5.2. Therefore, conditions (5.13) and (5.14) can be disregarded here. □

This allows solving:

- Maximum disturbance level β tolerated in a given set Ω starting from the origin (actually, a small set around the origin whose size depends on β).
- Minimum reachable set Θ_f for a given disturbance level: starting from the linearized model on the origin, an iterative methodology of computing $\Theta_f^{[k]}$ and remodelling for $\Omega^{[k+1]} = \Theta_f^{[k]}$ could be enforced until reach $\Theta_f^{[k]} \subset \Omega^{[k]}$ or infeasibility.

Particularization: TS quadratic case. Considering $V = x^T P x$, a TS fuzzy model (i.e., $p_i(x) = A_i x$) and Ω a sphere of radius $\sqrt{\rho_\Omega}$, the above corollary reduces to the following LMI problem:

$$P > \epsilon_1 I \tag{5.20}$$

$$P > (\frac{\beta}{1-e^{-\alpha T}})\rho_\Omega^{-1} I \tag{5.21}$$

$$\begin{pmatrix} PA_i + A_i^T P + \alpha P & PE \\ E^T P & -I \end{pmatrix} + \epsilon_2 iI < 0 \quad i:1,\ldots,r \tag{5.22}$$

Where $\epsilon_1 > 0$ and $\epsilon_{2i} > 0$ are user defined tolerances to ensure strict positivity (or negativity) of matrices.

Note that maximising β for a fixed α is a convex problem, as well as minimising ρ_Ω.

Consider now a given set of initial conditions defined also by a set of polynomial boundaries:

$$\Theta_i = \{x : h_1(x) > 0, \ldots, h_m(x) > 0\} \tag{5.23}$$

If initial conditions $\Theta_i \not\subset \{x : V \leq \frac{e^{-\alpha T}\beta}{1-e^{-\alpha T}}\}$, computed from Corollary 5.1, the following result is a direct application of Theorem 5.2 to the fuzzy polynomial case.

Corollary 5.2 (Bounding the inescapable set from prescribed initial-condition set Θ_i)**.** *If there exists a polynomial $V(x)$ fulfilling*

$$V(x) - \epsilon(x) \in \Sigma_x \tag{5.24}$$

$$V_0 - V(x) - \sum_{i=1}^{m} s_{1i}(x)h_i(x) \in \Sigma_x \tag{5.25}$$

$$V(x) - V_0 + \beta + s_{2i}(x)g_i(x) \in \Sigma_x \; i : 1, \ldots, k \tag{5.26}$$

$$V(x) - \frac{\beta}{1 - e^{-\alpha T}} + s_{3i}(x)g_i(x) \in \Sigma_x \; i : 1, \ldots, k \tag{5.27}$$

$$-\frac{dV(x)}{dx}(p_j(x) + Ew) - \alpha V(x) + w^T w - \sum_{i=1}^{k} s_{4ji}(x)g_i(x) \in \Sigma_x \; j : 1, \ldots, r \tag{5.28}$$

where $\epsilon(x)$ is a radially-unbounded positive polynomial and $(s_{1i}(x), s_{2i}(x), s_{3i}(x), s_{4ji}(x)) \in \Sigma_x$ are Positivstellensatz multipliers, then system (5.2) *does not leave Ω and the state will remain inside $\Theta_f = \{x : V \leq \frac{\beta}{1-e^{-\alpha T}}\}$ after enough time.*

Proof. Conditions (5.24) and (5.28) make $V(x)$ be a Lyapunov function for system (5.2) locally in Ω by Theorem A.1, as in Corollary 5.1. Then conditions (5.25), (5.26) and (5.27) mean (5.13), (5.14) and (5.15) respectively. □

Problems that can be solved:

- Maximize β for fixed Ω and Θ_i. Maximum allowed disturbance power to hold trajectories inside the modelling region.

- Maximize size of Θ_i for a fixed β. Largest set of initial conditions ensuring that a particular region is not abandoned.
- Minimize the size of Ω for fixed β and Θ_i. Minimum modelling region ensuring the fuzzy model to be valid, starting from a set of initial conditions, with a particular disturbance acting.

Remark 5.3. First problem requires only exploration in α, however second and third problems are totally nonconvex and need exploration in α and Θ_i, Ω respectively.

Particularization: TS quadratic case. Denoting $V'(x) = V(x)/V_0 = x^T P x$, considering a TS model and Ω, Θ_i spheres of radius $\sqrt{\rho_\Omega}$ and $\sqrt{\rho_0}$ respectively, we have the following GEVP ($\gamma = V_0^{-1}$):

$$P > \epsilon_1 I \tag{5.29}$$

$$P < \rho_0^{-1} I \tag{5.30}$$

$$P > (1+\gamma\beta)\rho_\Omega^{-1} I \tag{5.31}$$

$$P > (\frac{\gamma\beta}{1-e^{-\alpha T}})\rho_\Omega^{-1} I \tag{5.32}$$

$$\begin{pmatrix} PA_i + A_i^T P + \alpha P & PE \\ E^T P & -\gamma I \end{pmatrix} + \epsilon_{2i} I < 0 \quad i : 1, \ldots, r \tag{5.33}$$

Where $\epsilon_1 > 0$ and $\epsilon_{2i} > 0$ are user defined tolerances to ensure strict positivity (or negativity) of matrices.

In the quadratic case, problems of;

- maximize β for fixed ρ_0 and ρ_Ω,
- maximize the radius ρ_0 for fixed β and ρ_Ω,

can be solved only by exploring in α. The problem of minimize ρ_Ω for fixed β and ρ_0 is still not an LMI problem.

Remark 5.4. As the objective of the chapter is discussing the inescapable-set problem in the polynomial framework, detailed comparative analysis with other proposals in TS LMI literature is omitted. Related TS results can be consulted in the already cited work Salcedo, Martínez and García-Nieto (2008) as well as in Li, Xu and Li (2013), stemming from the basic LMI analysis for linear cases in Boyd, Ghaoui, Feron and Balakrishnan (1994). Furthermore, some of these works discuss stabilization. A proposal for $\mathcal{L}_1$ controller design in the polynomial case, derived from the above results in stability analysis, is discussed in next section.

5.3 Inescapable-set issues in stabilization

This section makes use of the invariant-set analysis under nonvanishing disturbances previously carried out in the above section. The objective now is designing a controller which fulfills some performance or stability condition while a bounded-power disturbance is acting.

Define a candidate Lyapunov function as

$$V(x) = z(x)^T P(\tilde{x}) z(x) \tag{5.34}$$

where $P^{-1}(\tilde{x}) \in \mathcal{R}_{\tilde{x}}^{l\times l}$ and $\tilde{x} \in \mathbb{R}^b$ are the state variables which corresponding row in $B(x)$ is equal to 0, i.e., do not depend on the control input (Prajna, Papachristodoulou and Wu, 2004b).

Consider now the fuzzy-polynomial model (5.4) modelled in (5.3), with a PDC state-feedback control law

$$u = -\sum_{i=1}^{r} \mu_i(x) K_i(x) z(x) \tag{5.35}$$

where $K_i(x) = M_i(x)P(\tilde{x}) \in \mathcal{R}_x^{b\times l}$ are the controller-gain matrices to be designed. If initial conditions $x(0) = 0$ and the modelling region (5.3) can be expressed in the form

$$\begin{aligned}\Omega = \{x : \sigma_1 - z(x)^T N_1(x)^T N_1(x) z(x) > 0, \ldots,\\ \sigma_k - z(x)^T N_k(x)^T N_k(x) z(x) > 0\end{aligned} \tag{5.36}$$

then, by performing the standard technique of change of variables $v = Pz, X = P^{-1}$ (Bernussou, Peres and Geromel, 1989), a closed-loop reachable set computation can be carried out.

Corollary 5.3 (Closed-loop inescapable set). *If matrices $X(\tilde{x})$, $M_i(x)$ can be found, with α fixed, fulfilling:*

$$v^T \left(X(\tilde{x}) - \epsilon I\right) v \in \Sigma_{x,v} \tag{5.37}$$

$$\phi^T \begin{pmatrix} X(\tilde{x}) + s_i(x) g_i(x) I & X(\tilde{x}) N_i(x)^T \\ N_i(x) X(\tilde{x}) & \frac{\sigma_i(1-e^{-\alpha T})}{\beta} I \end{pmatrix} \phi \in \Sigma_{x,\phi} i : 1, \ldots, k \tag{5.38}$$

$$- 2v^T R(x) \left(A_i(x)X(\tilde{x}) - B_i(x)M_i(x)\right) v - 2v^T R(x)Ew + v^T \frac{\partial X(\tilde{x})}{\partial x}(A_i(x)z(x) + Ew + \alpha X(\tilde{x}))v + w^T w - \sum_{c=1}^{k} s_{ic}(x,v)g_c(x) \in \Sigma_{x,v} \quad i : 1, \ldots, r \quad (5.39)$$

$$- v^T R(x) \left(A_i(x)X(\tilde{x}) - B_i(x)M_j(x) + A_j(x)X(\tilde{x}) - B_j(x)M_i(x)\right) v - 2v^T R(x)Ew + v^T \frac{\partial X(\tilde{x})}{\partial x} \left(A_i(x)z(x) + Ew + \alpha X(\tilde{x})\right) v + w^T w - \sum_{c=1}^{k} s_{ijc}(x,v)g_c(x) \in \Sigma_{x,v} \quad \begin{matrix} i : 1, \ldots, r \\ r \geq j > i \end{matrix} \quad (5.40)$$

where $\epsilon > 0$ *acts as a tolerance,* $R(x) = \frac{\partial z(x)}{\partial x} \in \mathcal{R}_x^{m \times n}$ *and* $s_i(x) \in \Sigma_x$, $(s_{ic}(x,v), s_{ijc}(x,v)) \in \Sigma_{x,v}$ *are Positivstellensatz multipliers, then controller* (5.35) *makes system* (5.1) *to not leave* Ω *if initial conditions are in* $\{x : V \leq \frac{e^{-\alpha T}\beta}{1-e^{-\alpha T}}\}$ *and, furthermore, it does not leave* $\{x : V \leq \frac{\beta}{1-e^{-\alpha T}}\}$*. Controller gains can be obtained by* $K_j(x) = M_j(x)X(\tilde{x})^{-1}$.

Proof. If initial conditions $\Theta_i \subset \{x : V \leq \frac{e^{-\alpha T}\beta}{1-e^{-\alpha T}}\}$, then conditions (5.13) and (5.14) can be disregarded as in Corollary 5.1. Conditions (5.39) and (5.40) mean (5.16) after carrying out some operations with the change of variable $v = Pz, X = P^{-1}$ and the evident fact of

$$P(\tilde{x})X(\tilde{x}) = I,$$

$$\frac{dP(\tilde{x})}{dt}X(\tilde{x}) + P(\tilde{x})\frac{dX(\tilde{x})}{dt} = 0.$$

So, jointly with (5.37), they make $V(x)$ to be a Lyapunov function for system (5.1), with controller (5.35), locally in Ω by Theorem 5.2 and Theorem A.1. The use of $X(\tilde{x})$ instead of $X(x)$ allows conditions (5.39)-(5.40) to be convex due to the fact that term $v^T \frac{\partial X(\tilde{x})}{\partial x}(B_i(x)K_j(x)z)v = 0$ in $\dot{V}(x)$. Condition (5.15) fulfilled locally in Ω is, for each i:

$$z^T P(\tilde{x})z - \frac{\beta}{1 - e^{-\alpha T}} + s_i(x)^* g_i(x) \in \Sigma_x \quad (5.41)$$

with $g_i = \sigma_i - z^T N_i^T(x) N_i(x) z$. Then, if Ω can be expressed as (5.36), then the Positivstellensatz multiplier can be selected as:

$$s_i(x)^* = s_i^{**}(x) + \frac{\beta}{\sigma_i(1 - e^{-\alpha T})} \in \Sigma_x$$

Also, without loss of generality, we can express the multiplier as:

$$s_i^{**}(x) = z^T P(\tilde{x}) s_i(x) P(\tilde{x}) z$$

Then, substituting all in (5.41), we have

$$z^T (P(\tilde{x}) + P(\tilde{x}) s_i(x) g_i(x) I P(\tilde{x}) - N_i(x)^T \frac{\beta}{\sigma_i(1 - e^{-\alpha T})} I N_i(x)) z \in \Sigma_x \tag{5.42}$$

which, doing the change of variable $v = Pz$ and using Schur complement, leads to (5.38). □

Remark 5.5. Note that conditions (5.40) may be relaxed via dimensionality expansion or via artificial decision variables by using Polya's theorem (Sala and Ariño, 2007b), at the prize of increasing computational effort. Note also that Corollary 5.3 does not necessarily involve more complex constraints than the ones proposed in literature (Ichihara, 2008; Tanaka, Ohtake and Wang, 2009b), from a computational point of view. Nevertheless, the problem may become intractable due to the inclusion of w, v independent variables, matrix size growing by Schur complement, the chosen degree of Positivstellesatz multipliers and, of course, system's order and chosen degree of the Lyapunov function (term $v^T \frac{\partial X(\tilde{x})}{\partial x}(A_i(x)z(x) + Ew + \alpha X(\tilde{x}))v$ on conditions (5.39)-(5.40)).

Discussion

Once the conditions for inescapability have been set up on controller-design basic framework, the issue is to decide which objective to choose in order to optimize. Holding the problem to be convex (with α fixed a priori), we may either:

- Minimize the modelling region in order to compute the smallest final inescapable set Θ_f (exploration in Ω required).
- Maximize the disturbance power β which is proved to hold system trajectories inside the modelling region ($\Theta_t = \Theta_f \subset \Omega$).

Or in a multicriteria setting, jointly with the following common objectives;

- optimizing the $\mathcal{L}_2 \to \mathcal{L}_2$ induced norm,
- optimizing an integral quadratic measure of the disturbance-free transient (guaranteed-cost control),

we may optimize a weight on the three criteria or set bounds in one or two of them.

In summary, the above suggests simultaneously setting up conditions (5.37)-(5.40) jointly with:

$$\dot{V} + \zeta_1^T \zeta_1 - \gamma_1 w^T w \leq 0 \tag{5.43}$$

$$\dot{V} + \gamma_2 \zeta_2^T \zeta_2 \leq 0 \tag{5.44}$$

where ρ_Ω is a size parameter of a set of variables whose maximum deviation must be bounded, γ_1 and ζ_1 are disturbance-rejection $\mathcal{H}_\infty$ performance and variables, respectively, and γ_2 and ζ_2 are guaranteed-cost $\mathcal{H}_2$ performance and variables, respectively.

Note. Limitations in control action can also be included. For instance the ones proposed in Tanaka and Wang (2001), i.e., on the region $\{V \leq \frac{\beta}{1-e^{-\alpha T}}\} \to \|u\| \leq \eta^2$.

From engineering insight, different performance compromises might be wished using a multi-model approach (setting different fuzzy models for different state space regions):

- Far from the origin, it is suggested to optimize guaranteed-cost performance, possibly subject to a constraint stating that the final inescapable set Θ_f should be reasonably smaller than Ω (for instance in order to facilitate the switch to other controller with better performance).
- Close to the origin: disregard transient guaranteed-cost performance and consider $\mathcal{H}_\infty$ performance and size of inescapable sets (an approximation to $\mathcal{L}_1$ control) to concentrate on disturbance rejection.

5.4 Example

On the following, an academic example is presented in order to remark the importance of the problem addressed in Section 5.3.

Example 5.4.1. Consider the following first-order nonlinear system:

$$\begin{aligned} \dot{x} &= -4.7x - \sin(x) + 0.7u + 1.6w \\ y &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 2 \end{pmatrix} u \end{aligned} \tag{5.45}$$

where x, u and w are as defined in (5.1).

Following the polynomial fuzzy-modelling methodology in Section 3.1.1, a two-rules polynomial-fuzzy model (5.4) of $\sin(x)$ can be computed:

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{2} \mu_i(x) A_i(x) x + 0.7u + 1.6w \\ y &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 2 \end{pmatrix} u \end{aligned} \tag{5.46}$$

For instance, we are interested in getting a fuzzy model of (5.45) valid between the range $\Omega = -4 \leq x \leq 4$, which gives the following two polynomial vertex models with their corresponding membership functions:

$$A_1(x) = -5.7 + 0.16667x^2;$$
$$A_2(x) = -5.7 + 0.074325x^2;$$
$$\mu_1 = \frac{\sin(x) - x + 0.074325x^3}{-0.0923417x^3}; \quad \mu_2 = 1 - \mu_1$$

Once this model is available, disturbance-rejection controllers can be designed. For instance, a classical $\mathcal{H}_\infty$ non-fuzzy polynomial optimal controller

$$u = -K(x)x \tag{5.47}$$

can be designed fulfilling

$$\dot{V} + y^T y - \gamma w^T w < 0 \tag{5.48}$$

which, using a quadratic Lyapunov function, is the direct extension of Scherer and Weiland (2004) to polynomial framework. This can be done by solving the following SOS problem:

minimize γ subject to

$$X > 0, \quad \gamma > 0 \tag{5.49}$$

$$v^T \begin{bmatrix} \psi_i(x) & X & 2M(x) & 1.6 \\ X & 1 & 0 & 0 \\ 2M(x) & 0 & 1 & 0 \\ 1.6 & 0 & 0 & \gamma \end{bmatrix} v \in \Sigma_{x,v} \; i : 1, 2 \tag{5.50}$$

where

$$\psi_i(x) = -2(A_i(x)X - 0.7M(x)) - \tau_i(x)(4^2 - x^2),$$

$\tau_i(x) \in \Sigma_x$ are Positivestellesatz multipliers in order to make (5.48) hold only locally in Ω and the polynomial controller gain[1] can be obtained by $K(x) = M(x)X^{-1}$.

By setting up a maximum degree of two for $K(x)$, the solution found is:

$$\gamma = 0.524, \quad K(x) = 0.0186242 + 0.0024x^2 \tag{5.51}$$

Disturbance analysis

Usually, the controller design ends here in the major part of literature results, without taking care about the *validity* of the found solution in front of a particular disturbance. Let us study for instance, what happens if the process (5.45) under control (5.51) is at $x = 0$ and a disturbance w with the following features appears

$$\beta = 21, \quad T = 0.75$$

where β is the maximum disturbance power available to be spent each T seconds. Then, the closed-loop system can be analyzed by using (5.46) and Corollary 5.1. Effectively, starting from the linearized model computed on the origin

$$\dot{x} = -5.713x + 1.6w \tag{5.52}$$

Corollary 5.1 gives the minimal inescapable region $\Theta_f^{[0]}$, where the above considered disturbance may push the linearized system (5.52). Thus, by exploring between $0.01 \leq \alpha \leq 4.5$, the obtained region has a radius of 2.68 (maximum radius of the Lyapunov level set).

Once this range is known, computation is repeated using the fuzzy system (5.46) modelled in $\Omega^{[1]} = \Theta_f^{[0]}$, obtaining a new and probably larger range (indeed, the TS model in the larger zone, obtains an inescapable region of radius 3.15); iterations on this procedure give rise to progressive larger inescapable regions. The procedure continues up to get convergence in the found $\Theta_f^{[k+1]} \subset \Omega^{[k]}$ or stops if any iteration gives an infeasible solution.

[1] Note that (5.50) is required to be SOS on independent variables v, which are product of the usual change of variable required in order to present the problem in LMI form.

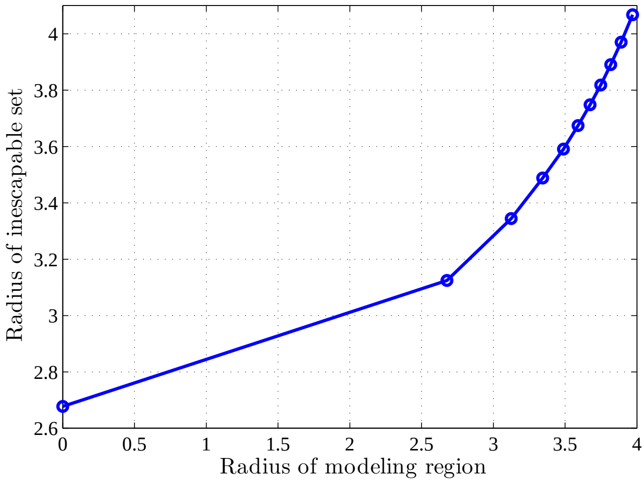


Figure 5.2: Evolution of the proved inescapable region with modelling region for system (5.45).

The evolution of the analysis procedure is shown on Figure 5.2. In this case, in the worst case of the proposed disturbance, all the trajectories of system (5.45) are not guaranteed to remain inside the fuzzy modelling region, so, as outside Ω the fuzzy-polynomial model is not valid, *stability* (understood as bounded output) of (5.45) under this disturbance *is not guaranteed* using controller (5.51).

Multicriteria design

If stability has to be ensured for a disturbance with the known bound $\beta = 21$ (and associated $T = 0.75$), the $\mathcal{H}_\infty$ controller design must include shape constraints in order to keep the inescapable region inside the TS modelling region. This can be done by solving the SOS problem proposed in Corollary 5.3 jointly with constraints (5.49) and (5.50).

Then, minimizing γ by carrying out an exploration in the decay rate α, the obtained solution is

$$\gamma = 4.2679, \quad K(x) = 65.9675 - 0.4481x^2 \tag{5.53}$$

with an optimal target decay $\alpha = 22$ (with $\alpha = 1 \to \gamma = 4.3294$ and with $\alpha = 200 \to \gamma = 4.5018$). As it is expected, the new controller is more powerful in order to counteract the disturbance, however the minimum $\mathcal{H}_\infty$ achieved bound γ is bigger. The advantage in this case is that stability is guaranteed in front of that expectable disturbance.

5.5 Conclusions

In this chapter, the BIBO stability problem of nonlinear systems under period-bounded nonvanishing disturbances has been addressed. Existent inescapable-set estimation approaches for TS systems have been extended to the fuzzy polynomial framework.

A general methodology of analysis is presented in order to check whether a closed-loop system (designed with classical approaches) ensures that trajectories do not escape the fuzzy modelling region (guaranteed stability) when confronting a disturbance with known characteristics.

In addition, fuzzy-polynomial design constraints are developed in order to enable a multicriteria controller design by combining them with other classical requirements. However, the assumptions of initial conditions being zero and the special structure required for the modelling region make the problem more restrictive, therefore loosing generality.

Future work will be pointed in addressing the more general controller-design problem when initial conditions are not zero and also in introducing a switching-controller methodology for each of the found inescapable/reachable sets.

Chapter 6

Local Fuzzy Polynomial Observers

If there was an observer on Mars, they would probably be amazed that we have survived this long.

Noam Chomsky

ABSTRACT: This chapter proposes a fuzzy-polynomial observer synthesis for nonlinear systems based on SOS techniques. With the designed observers, the estimation error is proven to converge asymptotically to zero, fulfilling some decay-rate performance. Dealing with the presence of external disturbances and measurement noise is also introduced on the design phase. Design conditions are relaxed by introducing the information of some operating regions of the state space and estimation error.

Many problems in decision making, monitoring, fault detection, and control require the knowledge of state variables and time-varying parameters that are not directly measured by sensors. In such situations, observers (or estimators) can be employed, using the measured input and output signals along with a dynamic model of the system, in order to estimate the unknown states or parameters. An essential requirement in designing an observer is guaranteeing the convergence of the estimates to the true values, or at least to a small neighborhood around them. However, for nonlinear or time-varying systems the design and tuning of an observer is generally complicated.

Polynomial fuzzy-modelling methodologies for nonlinear systems which were discussed previously on Chapter 3.3, in particular the Taylor series approach (Section 3.1.1), allow to design polynomial observer laws in order to estimate the *unmeasurable* process states or parameters.

The available sum-of-squares tools and software allow doing observer design for polynomial systems by means of convex optimization. However, observers for polynomial systems are not very developed yet and existent approaches which deal with disturbance rejection, are still in preliminary phases. See details on them in Section 3.3.2.

In this chapter, the main objective is extending the existent observer-design methodology done for polynomial systems to a more general class of nonlinear ones, by employing fuzzy polynomial models and also using fuzzy-observer gains. Furthermore, this chapter presents a more complete disturbance-rejection analysis by considering a more general class of process disturbance (and measurement noise) than the existent in previous polynomial-observer literature.

Developments are presented for both continuous-time and discrete-time design cases, either with domain of attraction guarantee or robustness versus performance multicriteria objective. In addition, some additional constraints are introduced on the design phase in order to avoid undesirable behaviors in practice, i.e., remove fast or oscillating dynamics from the set of feasible solutions, because it can excite some unmodelled dynamics, leading to unpredictable responses on the physical process.

This chapter is organized as follows: next section discusses the actual literature results related to observer design for nonlinear systems using fuzzy modelling techniques, Section 6.2 addresses a polynomial observer design which ensures some domains of attraction under vanishing disturbances, the observer design under bounded nonvanishing disturbances is presented and discussed on Section 6.3, an academic example is included on Section 6.4 in order to show the effectiveness of the proposed results and, finally, Section 6.5 summarizes the conclusions of this work.

6.1 Preliminaries

Nowadays research in observer design using fuzzy methodologies and Lyapunov theory is very focused on nonlinear systems for which a systematic modelling methodology (sector nonlinearity) can be used. The more extended proposals in literature are based on Takagi-Sugeno models, which are a time-varying convex combination of linear models (Tanaka and Wang, 2001, Chap. 4),(Howell and Hedrick, 2002; Koenig, 2006; Guerra, Kerkeni, Lauber and Vermeiren, 2012). As it was pointed previously on Chapter 2, the resulting fuzzy observer laws have linear consequents and the computed observer gains are constant matrices.

Chapter 3.3 in this thesis presented the existent literature results on observer design for polynomial systems, which has been addressed recently since sum-of-squares tools and software are available. In this case, the observer gains are allowed to be polynomial in both estimated state and measurements (Ichihara, 2009a). The treatment of process disturbances has also been addressed within the observer design phases in a preliminary way, assuming that disturbances fulfill some kind of bounds.

More recently, the idea has been extended to a more general class of nonlinear systems using fuzzy polynomial models, allowing to design both observer and controller simultaneously under some restrictive assumptions, in which the separation principle holds (Tanaka, Ohtake, Seo, Tanaka and Wang, 2012).

6.1.1 Problem statement

Following Lyapunov methodologies, the above-mentioned proposals in literature are able to design fuzzy polynomial observers for nonlinear systems. However those proposals have some drawbacks or limitations which are highlighted on the following.

The work of Ichihara (2009a) proposes a “local” fuzzy-polynomial observer design methodology for continuous-time systems which combine disturbance rejection and invariant-set guarantees. This is done by relaxing Lyapunov stability conditions, introducing information of regions in the state space and in the estimation error by means of Positivstellensatz (Theorem A.1). The main inconveniences are:

- The considered disturbances are vanishing. They must be bounded by an integral bound $\sqrt{\int_0^\infty w^T w\, dt} = \|w\|^2_{\mathcal{L}_2} \leq \beta$. This is a restrictive assumption because the major part of disturbances are non-vanishing in real world.
- The methodology is only addressed in a continuous-time case so possible measurement noise, usually present in a discrete-time implementa-

tion, is not taken into account, i.e., there is no sensor dynamics with associated process-disturbance consideration (Dufour and Bertrand, 1994; Walter and Pronzato, 1997).

- The approach is only valid for polynomial systems, a particular class of nonlinear ones.
- There is no decay-rate guarantee. Moreover, the presented SOS inequalities are not proven to be strictly positive (zero is sum of squares). Therefore, when disturbances are not acting the error is not ensured to decrease until zero but only to remain in its proven invariant set.

The work in Tanaka, Ohtake, Seo, Tanaka and Wang (2012) extends the classical continuous-time fuzzy TS observer design to the fuzzy polynomial case. Also, if some assumptions are made about the fuzzy-modelling premise variables, the paper proposes a methodology for designing together the observer and controller. The drawbacks of this work are the following:

- No local information is added into Lyapunov conditions. Therefore the obtained designs, if found feasible, are global, which is a good point. However, the SOS problem will be usually infeasible because high-degree polynomials take large values far away from the origin and it is difficult to obtain sum-of-squares certificates, even for small values of their negative coefficients. For example, if the highest degree of the polynomial to be checked is not even, positiveness cannot be proved for all $x \in \mathbb{R}^n$.
- There is no process disturbance or measurement noise (in discrete-time) consideration, neither decay-rate guarantee.

Given the above preliminary works and its limitations, the main objective of this chapter is presenting and developing an unified observer design for fuzzy polynomial systems including the advantages of both existent approaches. In particular, the achieved objectives are listed below:

1. Extending the existent theory in observer design for polynomial systems with vanishing disturbances to a more general class of nonlinear systems and disturbances using fuzzy polynomial models.
2. Developing theory for multiobjective $\mathcal{H}_\infty$ plus decay-rate observers for fuzzy polynomial systems, relaxing conservativeness by adding operating-region information.

3. Extending the above design methodologies to the discrete-time case and adding extra constraints, regarding the fastest components of the dynamics, in order to avoid undesirable behaviors in practice, for instance due to Euler discretization.

On the following, preliminary definitions and notation are presented for the rest of the chapter.

Consider a continuous-time nonlinear system with a known control input, modelled as a fuzzy-polynomial system in a region of the state space Ω:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} \mu_i(z(t))\,(p_i(x(t)) + E_i(x(t))w(t)) \\ y(t) &= C(x(t)) + R\eta(t) \end{aligned} \tag{6.1}$$

Or an equivalent discrete-time model at sample time T_s ($t = k \cdot T_s$):

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{r} \mu_i(z_k)\,(\mathcal{P}_i(x_k) + \mathcal{E}_i(x_k)w_k) \\ y_k &= C(x_k) + R\eta_k \end{aligned} \tag{6.2}$$

Where z are the premise variables (might be states or not), r denotes the number of fuzzy rules (usually a power of 2), $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^m$ are the measurments, the input $w \in \mathbb{R}^c$ is considered to be an unmeasurable process disturbance and $\eta \in \mathbb{R}^d$ is the measurement disturbance or noise.

Denote now the estimated states of a nonlinear system by $\hat{x}$ and the estimation error by $e = x - \hat{x}$.

Definition 6.1. *Given a fuzzy polynomial system* (6.1)*, a fuzzy-polynomial continuous-time observer for it is defined to be the dynamic system with equations:*

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(z(t))\,(p_i(\hat{x}(t)) + L_i(\hat{x}(t), y(t))(y(t) - C(\hat{x}(t))) \tag{6.3}$$

In a similar way, a fuzzy-polynomial discrete-time observer for a system (6.2) *is defined by*

$$x_{k+1} = \sum_{i=1}^{r} \mu_i(z_k)\,(\mathcal{P}_i(\hat{x}_k) + L_i(\hat{x}_k, y_k)(y_k - C(\hat{x}_k))) \tag{6.4}$$

where the observer gains L_i may depend polynomially on sensor measurements and estimated states.

6.2 Observer design under vanishing disturbances

This section extends the results of Ichihara (2009a) to fuzzy polynomial systems. The results and procedures presented in this section have been published in the conference papers; Sala, Pitarch, Bernal, Jaadari and Guerra (2011) for the continuous-time case, and Pitarch and Sala (2012) for the discrete-time one. In the following, only the discrete-time design is presented and the continuous-time one is omitted for brevity, because developments are very similar. See the cited references for more details.

Consider a fuzzy polynomial model (6.2) of a nonlinear system without measurement noise ($\eta = 0$) and a fuzzy polynomial observer (6.4) for that system.

The error dynamics $e = x - \hat{x}$ will be given by:

$$e_{k+1} = \sum_{i=1}^{r} \mu_i(z_k)(\mathcal{P}_i(x_k) - \mathcal{P}_i(\hat{x}_k) - L_i(y_k, \hat{x}_k)(C(x_k) - C(\hat{x}_k)) + \mathcal{E}_i(x_k)w_k) \tag{6.5}$$

The input w is considered to be an unmeasurable disturbance, bounded by an integral bound $\sum_{k=0}^{\infty} w_k^T w_k = ||w||_2^2 \leq \beta$ or an instantaneous bound $w_k^T w_k \leq \beta^2$. On the following the notation $z_k \equiv z(k)$ will be used for simplicity.

A candidate Lyapunov function may be defined as $V(e) = e^T Q e$. Then, its discrete increment $\Delta V = V(e_{k+1}) - V(e_k)$ can be written as:

$$\Delta V = \sum_{i=1}^{r} \mu_i(z_k)(*)^T Q(\mathcal{P}_i(x_k) - \mathcal{P}_i(\hat{x}_k) + \mathcal{E}_i(x_k)w_k - L_i(y_k, \hat{x}_k)(C(x_k) - C(\hat{x}_k))) - e_k^T Q e_k \tag{6.6}$$

Computing inescapable set starting from a low error

Ensuring the condition

$$\Delta V(x_k, \hat{x}_k, w_k) - w_k^T w_k < 0 \tag{6.7}$$

ensures

$$V_k < V_0 + \sum_{k=0}^{\infty} w_k^T w_k$$

for all $k \in \mathbb{R}$. The bound $|w_k|_{\mathcal{L}_2}^2 \leq \beta$ will be assumed available.

The inequality (6.7) will be required to hold for all x_k, $\hat{x}_k$ in a particular region defined by the sets:

$$\chi_S = \{x_k \in \mathbb{R}^n | G_s(x_k) > 0\}$$

$$\chi_E = \{e_k \in \mathbb{R}^n | G_e(e_k) > 0\}$$

where χ_S is assumed known (a bounded region of state space where the "true" system state lies), and χ_E is an user-defined region fulfilling:

- $\chi_S \subset \chi_E$ because $e(0) = x(0)$ if the observer starts in $\hat{x}(0) = 0$.
- $\chi_S \oplus \chi_E \subset \Omega$ is required to ensure $\hat{x}_k \in \Omega$
- χ_E must be reasonably small in order to limit the maximum error in the transient (and to limit the required size of Ω).

Obviously, in order to use SOS conditions, it is assumed that $G_s(x_k)$, $G_e(e_k)$ are defined by polynomials. For instance:

$$G_s(x_k) = 1 - x_k^T S_X x_k, \quad G_e(e_k) = 1 - e_k^T S_E e_k$$

with $S_X \succ 0$ and $S_E \succ 0$.

Then, if the initial error (i.e., initial state) is small enough so that $V_0 \leq \alpha$, then

$$V_k = e_k^T Q e_k \leq \alpha + \beta \quad \forall\, k \tag{6.8}$$

Obviously, the increment of actual error caused by β will be small if Q is big. This suggests maximizing a parameter γ subject to $Q > \gamma I$ and (6.7).

Once the above optimal Q has been obtained, then, the maximum value ϕ so that the ellipsoid $x_k^T Q x_k \leq \phi \subset \chi_E$ can be computed and, subsequently $\alpha = \phi - \beta$ will give a guaranteed initial state ellipsoid $x_k^T Q x_k \leq \alpha$ (equal to initial error as $\hat{x}_0 = 0$ by assumption) so that the error will be guaranteed to not leave its maximum allowed region χ_E, see Figure 6.1.

Lemma 6.1. *An observer* (6.4) *for system* (6.2) *which minimizes the effect of disturbances, i.e., maximizes the set of initial conditions* $\{e : V(e) \leq \alpha\}$ *which holds the estimation error within the allowed region* χ_E*, is obtained by solving the following SOS optimization problem:*

Maximize γ *subject to*

$$x^T(Q - \gamma I)x \quad \in \quad \Sigma_x \tag{6.9}$$

$$\psi^T \begin{bmatrix} F1_i & F2_i^T \\ F2_i & Q \end{bmatrix} \psi \quad \in \quad \Sigma_{x,\hat{x},w,\psi} \quad i : 1, ..., r \tag{6.10}$$

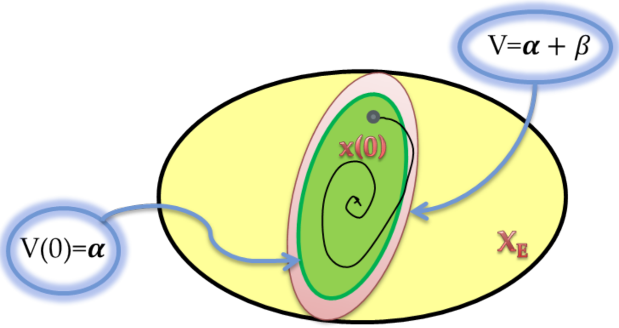


Figure 6.1: Example of regions involved in Lemma 6.1.

with

$$
\begin{aligned}
F1_i &= e^T(Q - \epsilon I)e + w^T w - \phi_i(x, \hat{x}) & (6.11)\\
F2_i &= Q(\mathcal{P}_i(x) - \mathcal{P}_i(\hat{x}) + \mathcal{E}_i(x)w - H_i(y, \hat{x})(C(x) - C(\hat{x})) & (6.12)
\end{aligned}
$$

where $\epsilon > 0$ acts as a tolerance in order to ensure that the error tends to zero in absence of disturbances and $\phi_i(x, \hat{x}) \in \wp(G_s(x), G_e(e))$. The discrete observer gains can be obtained by $L_i(y_k, \hat{z}_k) = Q^{-1}H_j(y_k, \hat{z}_k)$.

Proof. Condition (6.9) means $V(e) > 0$ and, at same time, computes the smallest ellipsoid by maximizing the eigenvalues of Q.

Condition (6.10) means that (6.7) holds inside the considered state-space regions of interest:

$$
\begin{aligned}
&V(e_{k+1}) - V(e_k) - w_k^T w_k + \phi(x_k, \hat{x}_k) = \\
&\qquad e_{k+1}^T Q Q^{-1} Q e_{k+1} - e_k^T Q e_k - w_k^T w_k + \phi(x_k, \hat{x}_k) < 0 \quad (6.13)
\end{aligned}
$$

which applying Schur complement, using Proposition A.1 and fuzzy polynomial models, leads to (6.10). □

Remark 6.1. The *fuzzy* gain in Lemma 6.1 is only useful if the premise variables z are "known or measurable", such as $\mu_i(y)$. If not, the use of fuzzy gains is useless because of the double summation $\sum \mu_i(x)\mu_j(\hat{x})$, for which any L_j must hold for all i vertex models. In this case, the choices to proceed with the design can be either a single observer gain $L(y, \hat{x})$ or bounding the unknown term

$$
\left\| \sum_{i=1}^{r} (\mu(x) - \mu(\hat{x})) p_i(x) \right\| \leq \rho \|e\| \quad (6.14)
$$

locally in Ω, in a similar way to what is done in Ichalal, Marx, Ragot and Maquin (2010); Lendek, Berna, Guzmán-Giménez, Sala and García (2011).

Decay rate. Deriving the developments in Section 5.2 now for discrete-time systems, if inequality (6.7) is changed to

$$V_{k+1} - \delta V_k - w_k^T w_k < 0 \tag{6.15}$$

and the following term is renamed as

$$V_{k+1} - \delta V_k = \mathcal{G}_k$$

then:

$$V_k = \delta^k V_0 + \sum_{i=0}^{k-1} \delta^i \mathcal{G}_{k-i-1} \tag{6.16}$$

Hence, as by (6.15), $|\mathcal{G}_k| \leq w_k^T w_k$, it implies:

$$V_k < \delta^k V_0 + \sum_{i=0}^{k-1} \delta^i w_{k-i-1}^T w_{k-i-1}$$

So considering bounded vanishing disturbances $\sum_{i=0}^{\infty} w_i^T w_i \leq \beta$:

$$\sum_{i=0}^{k-1} \delta^i w_{k-i-1}^T w_{k-i-1} \leq \beta, \quad 0 \leq \delta \leq 1 \tag{6.17}$$

allows bounding V_k by:

$$V_k \leq \delta^k V_0 + \beta \tag{6.18}$$

Furthermore, considering non-vanishing disturbances:

$$\sum_{i=0}^{N-1} \delta^i w_{k-i-1}^T w_{k-i-1} \leq \beta, \quad 0 \leq \delta \leq 1 \tag{6.19}$$

then we have

$$\sum_{i=0}^{\infty} \delta^i w_i^T w_i = \sum_{j=0}^{\infty} \sum_{i=0}^{N-1} \delta^{i+Nj} w_{i-1+Nj}^T w_{i-1+Nj}$$

Hence, as by (6.15), $|\mathcal{G}_k| \leq w_k^T w_k$, it implies after some straightforward manipulations:

$$V_k < \delta^k V_0 + \frac{1}{1-\delta^N} \beta$$

Hence, replacing (6.11) by

$$F1_i = e^T(\delta Q - \epsilon I)e + w^T w - \phi_i(x, \hat{x})$$

on conditions (6.10) allows giving the required decay-rate result.

6.3 $\mathcal{H}_\infty$ polynomial observer design

This section addresses the fuzzy-polynomial observer design for nonlinear systems when there exist nonvanishing disturbances but there is no information about their maximum power or energy bound. In this case, a trade-off between $\mathcal{H}_\infty$ norm bounds and speed of convergence performance is taken into account in the design process: $\mathcal{H}_\infty$ is a worst-case measure (of disturbance/model error rejection) so its solutions are conservative in practice whereas decay-rate optimization produces unacceptable noisy estimates. The theoretical results presented in this section have been used for the experimental applications (Part III of the thesis) and appear published in the journal paper Pitarch and Sala (2014).

Both continuous-time and discrete-time design methodologies can be pursued in order to obtain an observer with the objective of achieving acceptable behaviour in practice. Nevertheless, let us first present the stability and $\mathcal{H}_\infty$ performance analysis of a discrete-time observer.

6.3.1 Stability and performance analysis

Let us first consider how to prove discrete-time stability and performance of a previously-designed observer (i.e., the assumption that L_i does not contain decision variables will be preliminarily stated) at sample time T_s, as follows.

Consider a fuzzy polynomial model (6.2) of a nonlinear system and a fuzzy polynomial observer (6.4) for that system.

In order to carry out theoretical stability analysis, the observer error $e_k = x_k - \hat{x}_k$ follows the equation:

$$e_{k+1} = \sum_{i=1}^{r} \mu_i(z) \left(\mathcal{P}_i(x_k) - \mathcal{P}_i(\hat{x}_k) - L_i(\hat{x}_k, y_k)(C(x_k) + R\eta_k - C(\hat{x}_k))\right) + \mathcal{E}_i w_k \quad (6.20)$$

which, of course, in the linear case (constant $\mathcal{P}, C, L, \mathcal{E}$, i.e., $\mathcal{P}_i(x_k) - \mathcal{P}_i(\hat{x}_k) = \mathcal{P}(x_k - \hat{x}_k) = Ae_k$) reduces to the well-known:

$$e_{k+1} = (A - LC)e_k - LR\eta_k + Ew_k$$

Less conservative conclusions can be obtained if the observer error ($e = x - \hat{x}$ where $\hat{x}$ is the estimated estate) is assumed to lie in a compact region Ω_e. Indeed, the well-known fact that linear systems are globally stable does not hold in the polynomial case; hence, many times the problems are infeasible if state and error regions (Ω_x and Ω_e, respectively) are not introduced.

In the developments here, the error and the state are assumed to be always contained inside regions

$$\Omega_e = \{x, \hat{x} : G_e(x, \hat{x}) \geq 0\}, \quad \Omega_x = \{x : G_x(x) \geq 0\} \tag{6.21}$$

where $G_e(x, \hat{x}), G_x(x)$ are defined by polynomials or intersections of them.

This assumption is reasonable when the initial estimated state is $\hat{x}(0) = 0$ (usual observer start-up condition) and the initial state is inside some level-set of a Lyapunov function (to be later computed) : in that case, $e(0) = x(0)$ and stability will make the Lyapunov function decrease (Ichihara, 2009a; Khalil, 2002).

Note. On the following developments, the premise variables z for the membership functions are assumed to be known or measurable. If not, some extra considerations have to be done. See Remark 6.1 on Section 6.2.

From the above considerations, on the following we will assume that the state and estimation error do not leave some Lyapunov level sets, so that, in turn, we can assert $e \in \Omega_e$ and $x \in \Omega_x$ being Ω_e and Ω_x described by polynomial boundaries as in (6.21). If the disturbances push the system out of such regions, the stability or performance is not guaranteed because the fuzzy model is not valid outside Ω_x. See such an example on Section 5.4. Also, the following notation will be intentionally used as a shorthand:

- $\bar{\mathcal{P}}_i(x_k, \hat{x}_k)$ stands as $\mathcal{P}_i(x_k) - \mathcal{P}_i(\hat{x}_k)$,
- likewise $\bar{C}(x_k, \hat{x}_k)$ stands as $C(x_k) - C(\hat{x}_k)$.
- Subscript notation of a sample k is omitted for simplicity, i.e., e stands as e_k where appropriate.

Consider now a quadratic candidate Lyapunov function on the error in the form

$$V(e) = e^T Q e \tag{6.22}$$

where Q is a constant symmetric positive-definite matrix.

The following result states sufficient conditions for the above observer to be locally stable.

Theorem 6.1. *Given a discrete-time polynomial observer* (6.4)*, a $\mathcal{L}_2^2$ gain attenuation bound $\gamma > 0$ on the estimation error for system* (6.2) *against the worst-case disturbance, locally in regions on error Ω_e and state Ω_x, is obtained by solving the following SOS optimization problem:*

Minimize γ under the constraints

$$x^T Q x - \epsilon(x) \in \Sigma_x \tag{6.23}$$

$$\begin{aligned} &- (*)^T Q \left(\bar{\mathcal{P}}_i(x,\hat{x}) - L_i(\hat{x},y)\bar{C}(x,\hat{x}) + \mathcal{E}_i w - L_i(\hat{x},y)R\eta\right) \\ &+ e^T(Q - D^T D)e + \gamma W^T W - s_{1i}(x,\hat{x}) \in \Sigma_{x,\hat{x},w,\eta} \quad i = 1,\ldots,r \end{aligned} \tag{6.24}$$

Also, a decay rate $\alpha > 0$ is proved if (6.24) *is replaced by:*

$$\begin{aligned} &\alpha e^T Q e - (\bar{\mathcal{P}}_i(x,\hat{x})^T - \bar{C}(x,\hat{x})^T L_i(\hat{x},C(x))^T) Q (\bar{\mathcal{P}}_i(x,\hat{x}) \\ &\qquad - L_i(\hat{x},C(x))\bar{C}(x,\hat{x})) - s_{2i}(x,\hat{x}) \in \Sigma_{x,\hat{x}} \quad i = 1,\ldots,r \end{aligned} \tag{6.25}$$

where $W = \begin{bmatrix} w \\ \eta \end{bmatrix}$ is the disturbance vector, $(s_{1i}, s_{2i}) \in \wp(G_x(x), G_e(x,\hat{x}))$ are Positivstellensatz terms, $\epsilon(x)$ is an arbitrary radially unbounded positive polynomial and D is a constant matrix which stands as an user-defined scaling weight, in order to choose a particular combination of errors to minimize. If the above problem renders feasible, then:

1. *If initial condition $e_0 = 0$, $\sqrt{\gamma}$ is an upper bound on the (weighted) $\mathcal{L}_2^2$ gain, which is the nonlinear interpretation to the linear $\mathcal{H}_\infty$ norm (see Section 2.4.2), i.e.:*

$$\sup_{0<||w||_2<\infty} \frac{||De||_2^2}{||W||_2^2} < \gamma \tag{6.26}$$

2. *If the initial condition $e_0 \neq 0$, the initial estimation error decays exponentially with rate α in absence of disturbances, i.e.:*

$$V(e_k) \leq \alpha^k V(e_0)$$

Proof. By following classical Lyapunov asymptotic stability conditions Khalil (2002):

- Condition (6.23) makes $Q \succ 0$. Therefore $V = e^T Q e > 0$ when $e \neq 0$.
- Conditions (6.24) mean $V(e_{k+1}) - V(e_k) + e_k^T D^T D e_k - \gamma W_k^T W_k < 0 \forall \mu_i$ when $x \in \Omega_x$ and $e \in \Omega_e$ by the local information added with s_{1i} multipliers following Theorem A.1. So, in absence of disturbances W, V is a Lyapunov function of system (6.2). Furthermore, if initial conditions are $x_0 = 0$ and adding from time zero till time k (and letting later $k \to \infty$) the above discrete-time inequality results in:

$$V(e_k) + e_k^T D^T D e_k - \gamma W_k^T W_k < 0 \ \Rightarrow \ ||De||_2^2 < \gamma ||W||_2^2$$

- Conditions (6.25) mean $V(e_{k+1}) - \alpha V(e_k) < 0 \forall \mu_i$ when $x \in \Omega_x$ and $e \in \Omega_e$, again with s_{2i} following Theorem A.1. Parameter α is the desired discrete decay rate. Then V is also a valid Lyapunov function ensuring exponential stability of system (6.2). Note that $L_i(\hat{x}, y)$ has been replaced by $L_i(\hat{x}, C(x))$ because decay-rate conditions consider a noise-free dynamics in nonzero initial conditions, so $y = C(x) + R\eta$ applies for the particular case $\eta = 0$.

□

Using this result, both the $\mathcal{H}_\infty$ disturbance/model error rejection gain and decay performance of a given discrete-time fuzzy-polynomial observer for a nonlinear system can be checked. However, if the observer gains $L_i(\hat{x}, y)$ have to be found, Theorem 6.1 cannot be used because the product of decision variables (Q, L) leads to a nonconvex optimization problem. This will be addressed next: the following subsections consider observer design, both in pure discrete-time and in continuous-time.

6.3.2 Direct discrete-time design

On the following, a discrete-time polynomial observer design methodology is presented.

Theorem 6.2. *Given a decay rate parameter $\alpha > 0$, a polynomial observer* (6.4) *which gives a $\mathcal{L}_2^2$ gain attenuation bound γ on the estimation error for system* (6.2) *against the worst-case disturbance, over regions on the error and* *the state* (6.21)*, is obtained by solving the following SOS problem for all* i *:* $1, \ldots, r$*:*

Minimize γ *subject to*

$$x^T Q x - \epsilon(x) \in \Sigma_x \tag{6.27}$$

$$\rho^T \begin{bmatrix} e^T(Q - D^T D)e + \gamma W^T W - \Psi_{1i}(x, \hat{x}) & (*)^T \\ Q\bar{\mathcal{P}}_i(x, \hat{x}) - H_i(\hat{x}, y)\bar{C}(x, \hat{x}) & \\ +Q\mathcal{E}_i w - H_i(\hat{x}, y)R\eta & Q \end{bmatrix} \rho \in \Sigma_{x,\hat{x},w,\eta,\rho} \tag{6.28}$$

$$\rho^T \begin{bmatrix} \alpha e^T Q e - \Psi_{2i}(x, \hat{x}) & (*)^T \\ Q\bar{\mathcal{P}}_i(x, \hat{x}) - H_i(\hat{x}, C(x))\bar{C}(x, \hat{x}) & Q \end{bmatrix} \rho \in \Sigma_{x,\hat{x},\rho} \tag{6.29}$$

$$2e^T Q\bar{\mathcal{P}}_i(x, \hat{x}) - 2e^T H_i(\hat{x}, C(x))\bar{C}(x, \hat{x}) - \Psi_{3i}(x, \hat{x}) \in \Sigma_{x,\hat{x}} \tag{6.30}$$

Where $\Psi_{ji} \in \wp(G_x(x), G_e(x, \hat{x}))$*,* $j = 1, 2, 3$*,* $\gamma > 0$*,* $W = \begin{bmatrix} w \\ \eta \end{bmatrix}$*,* $\epsilon(x)$ *and* D *are the same as in Theorem 6.1. The polynomial observer gains can be computed as* $L_i(\hat{x}, y) = Q^{-1} H_i(\hat{x}, y)$*.*

Proof. Condition (6.27) ensures positive-definiteness of the Lyapunov function, which in the linear LMI case amounts to (6.34).

Under no disturbances, the basic decay-rate discrete condition $\Delta V = V_{k+1} - \alpha V_k < 0$ can be expressed as:

$$-\Delta V = \sum_{i=1}^{r} \mu_i(z) e^T \left(\alpha Q - (*)^T Q Q^{-1} Q \left(\bar{\mathcal{P}}_i(x, \hat{x}) \right. \right. \\ \left. \left. - L_i(\hat{x}, C(x))\bar{C}(x, \hat{x}) \right) \right) e > 0 \tag{6.31}$$

Then, by using the convex-sum property and adding local information following Theorem A.1 in order to make the positiveness condition of (6.31) hold only locally in the required regions, it leads to

$$e^T \left(\alpha Q - (*)^T Q Q^{-1} Q \left(\bar{\mathcal{P}}_i(x, \hat{x}) - L_i(\hat{x}, C(x))\bar{C}(x, \hat{x}) \right) \right) e \\ - \Psi_{2i}(x, \hat{x}) > 0 \quad i : 1, \ldots, r \tag{6.32}$$

This is, by Schur complement, equivalent to the convex SOS conditions

$$\rho^T \begin{bmatrix} \alpha Q - \Psi_{2i}(x,\hat{x}) & (*)^T \\ Q\bar{\mathcal{P}}_i(x,\hat{x}) - H_i(\hat{x}, C(x))\bar{C}(x,\hat{x}) & Q \end{bmatrix} \rho \in \Sigma_{x,\hat{x},\rho} \quad i: 1,\ldots,r \tag{6.33}$$

which can be checked for SOS with Proposition A.1. In this way, (6.29) is obtained.

In the disturbance case, conditions (6.28) are obtained by a similar procedure with the dissipation inequality $V_{k+1} - V_k + e_k^T D^T D e_k - \gamma^2(w^T w + \eta^T \eta) < 0$ plus information about locality in multipliers in Ψ_{1i} (details omitted for brevity). Note that $L_i(\hat{x}, C(x))$ is used in decay-rate conditions but $L_i(\hat{x}, y)$ in dissipation ones, see proof of Theorem 6.1.

Finally, the meaning of conditions (6.30) in the linear case, would be forcing the real part of the poles to be non-negative (pole-region placement results in Boyd, Ghaoui, Feron and Balakrishnan (1994)) , which is actually (6.37). See Remark 6.2. □

Remark 6.2. Conditions (6.30) are stated in order to avoid high-frequency oscilations with period $2T_s$ in the error dynamics. This is done via the extensions to SOS of standard LMI pole-region constraints, avoiding what in a linear case could be understood as discrete poles with negative real part, see (6.37) below. Although they may be considered as "optional", including them is reasonable in practice both for having an acceptable transient and to avoid exciting high-frequency dinamics in the underlying physical system: in this way the result will be more tolerant to errors (such as those due to Euler discretization and unmodelled dynamics).

With some manipulations, the following result in TS observer literature can be easily obtained (details omitted for brevity) as a particular case.

Corollary 6.1 (Tanaka and Wang (2001))**.** *With a TS model*

$$x_{k+1} = \sum_{i=1}^{r} \mu_i(z) A_i x_k + E w_k, \quad y_k = C x_k + R\eta_k$$

and fuzzy observer gains L_i, the above result reduces (by reordering and extracting variables (x, e, w, η) from matrices) to the well-know LMI optimization problem proving global $\mathcal{H}_\infty$ bound and decay rate:

Minimize γ subject to:

$$Q \succ 0 \tag{6.34}$$

$$\begin{bmatrix} Q - D^T D & A_i^T Q - C^T H_i^T & 0 & 0 \\ QA_i - H_i C & Q & QE & -H_i R \\ 0 & E^T Q & \gamma I & 0 \\ 0 & -R^T H_i^T & 0 & \gamma I \end{bmatrix} \succ 0 \tag{6.35}$$

$$\begin{bmatrix} \alpha Q & A_i^T Q - C^T H_i^T \\ QA_i - H_i C & Q \end{bmatrix} \succ 0 \tag{6.36}$$

$$A_i^T Q - C^T H_i^T + QA_i - H_i C \succ 0 \tag{6.37}$$

where, $\gamma > 0$, $\alpha > 0$ and the observer gains can be obtained as $L_i = Q^{-1} H_i$.

Again, conditions (6.37) state that poles must lie in the right-half plane (of course, inside the circle of radius α from (6.36)).

6.3.3 Continuous-time based design

The following result presents a continuous-time fuzzy-polynomial observer (6.3) design methodology. Consider the fuzzy-polynomial continuous-time system (6.1). The notation $\bar{p}_i(x, \hat{x}) = p_i(x) - p_i(\hat{x})$ and $\bar{C}(x, \hat{x}) = C(x) - C(\hat{x})$ will be used; also, explicit notation of time t is omitted for simplicity, i.e., e stands for $e(t)$, etc.

Theorem 6.3. *Given a decay rate parameter $\alpha > 0$, a polynomial observer* (6.3) *which gives a $\mathcal{L}_2^2$ gain attenuation bound γ on the estimation error for system* (6.1) *against the worst-case disturbance, over regions on the error and the state* (6.21)*, is obtained by solving the following SOS problem for all i : $1, \ldots, r$:*

Minimize γ subject to:

$$x^T Q x - \epsilon(x) \in \Sigma_x \tag{6.38}$$

$$\begin{aligned} &- 2e^T \left(Q\bar{p}_i(x, \hat{x}) - H_i(\hat{x}, y)\bar{C}(x, \hat{x}) + QE_i w - H_i(\hat{x}, y)R\eta \right) \\ &\qquad - e^T D^T D e + \gamma W^T W - s_{1i}(x, \hat{x}) \in \Sigma_{x,\hat{x},w,\eta} \end{aligned} \tag{6.39}$$

$$-2e^T (Q\bar{p}_i(x, \hat{x}) - H_i(\hat{x}, C(x))\bar{C}(x, \hat{x})) - 2\alpha e^T Q e - s_{2i}(x, \hat{x}) \in \Sigma_{x,\hat{x}} \tag{6.40}$$

$$2e^T(Q\bar{p}_i(x,\hat{x}) - H_i(\hat{x}, C(x))\bar{C}(x,\hat{x})) + \frac{2}{T_s}e^T Q e - s_{3i}(x,\hat{x}) \in \Sigma_{x,\hat{x}} \quad (6.41)$$

Where $\gamma > 0$, $W = \begin{bmatrix} w \\ \eta \end{bmatrix}$, $(s_{1i}, s_{2i}, s_{3i}) \in \wp(G_x(x), G_e(x,\hat{x}))$ *and* $\epsilon(x)$, D *are the same as in Theorem 6.1. The polynomial observer gain can be computed as* $L_i(\hat{x}, y) = Q^{-1}H_i(\hat{x}, y)$ *such that its Euler-discretized implementation results in:*

$$\hat{x}_{k+1} = \sum_{i=1}^{r} \mu_i \left(T_s p_i(\hat{x}_k) + \hat{x}_k + T_s L_i(\hat{x}_k, y_k)(y_k - \hat{y}_k)\right) \quad (6.42)$$

Proof omitted for brevity, see Appendix B.4 for details.

Remark 6.3. Conditions (6.41) are stated in order to avoid state dynamics faster than a decay $\frac{1}{T_s}$ for robustness considerations regarding unmodelled dynamics and discrete-time implementation via Euler integration (the additional conditions (6.41) exclude many (but not all) of the solutions which would render (6.42) unstable).

Indeed, further commenting on the above remark, conditions (6.41) and (6.30) are actually the same if, by Euler discretization method, $\bar{\mathcal{P}}_i(x_k, \hat{x}_k) - L_i(y_k, \hat{x}_k)\bar{C}(x_k, \hat{x}_k) = Ie(t) + T_s(\bar{p}_i(x(t)) - L_i(y(t), x(t))\bar{C}(x(t), \hat{x}(t)))$. Indeed, replacing it on conditions (6.30), setting $\Psi_{3i} = 0$, it leads to:

$$2e^T Q e + 2T_s e^T Q(\bar{p}_i - L_i\bar{C}) \geq 0 \rightarrow 2e^T Q\bar{p}_i - 2e^T H_i\bar{C}) \geq -\frac{2}{T_s}e^T Q e$$

which is (6.41) locally in Ω_x, Ω_e.

Computational cost. Note that the theorems presented in this chapter are evaluated *off-line*, so when computational requirements are discussed, they refer to memory and CPU of the workstation with MATLAB®code carrying out sum-of-squares optimizations. Such optimizations are *not needed in on-line operation*, but only a very simple direct evaluation of (6.4) once the coefficients of L are fixed. Such on-line evaluations can be easily encoded on a few lines of code on any low-range industrial control computer.

By following Theorem 6.2, a discrete polynomial observer can be directly designed and implemented. However, the need of applying the Schur complement (which increases matrix size), makes the size and complexity of the associated SDP optimization problem grow with respect to the continuous-time design with Theorem 6.3. Note that computational complexity of SOS programs grows quickly with the system order and multiplier degree. Indeed, the number of monomials of a polynomial of degree d in n variables is $(n+d-1)!/(d!(n-1)!)$. For instance, in the application presented in Chapter 8 we have 21 variables amounting to 6 states, 6 estimated states and 9 process and measurement noises. Degree 4 polynomials on them result in 10626 monomials. A higher-degree model forcing checking positiveness of degree 6 polynomials in them would jump to 230230 monomials.

Note also that, in general, if a polynomial model has $p(x)$ of degree d, the derivative of the Lyapunov function $V(x) = x^T Qx$ is of degree $d+1$. However, the increment of $V(x_k)$ in the discrete case has the term $\mathcal{P}^T(x)Q\mathcal{P}(x)$ which is of degree $2d$.

In summary, the difference in needed computing resources in the discrete design versus the continuous one, possibly grows significantly as problem complexity increases.

6.4 Example

This section presents an academic example in order to show the effectiveness of the proposed methodologies. Both, the computed fuzzy-polynomial observer gains and the estimation error trajectories are shown for the performed simulations.

In addition, the observer design procedures addressed on the above sections have been also applied to experimental platforms, see Chapter 8 and 9.

Example 6.4.1. Consider a two-rule TS fuzzy polynomial model (6.1) of a nonlinear system with membership functions and its corresponding polynomial functions $p_i(x)$ and $E(x)$ given by:

$$\mu_1 = 1 - e^{-y^2}; \quad \mu_2 = 1 - \mu_1; \quad E(x) = \begin{bmatrix} 0 \\ 0.5x_1x_2 \end{bmatrix}; \tag{6.43}$$

$$p_1(x) = \begin{bmatrix} 1.2x_2 \\ -x_1 + 1.2x_2 - x_1^2x_2 \end{bmatrix};$$

$$p_2(x) = \begin{bmatrix} 0.98x_2 \\ -x_1 + 0.98x_2 - x_1^2x_2 \end{bmatrix};$$

and the output equation is $y = x_1^3 + 0.5x_2$.

The Euler discretization of the above fuzzy polynomial model at sampling period $T = 0.01$s, with its corresponding polynomial functions $\mathcal{P}_i(x_k)$ and $\mathcal{E}(x_k)$ is given by:

$$\mu_1 = 1 - e^{-(0.1y_k)^2}; \quad \mu_2 = 1 - \mu_1; \quad \mathcal{E}(x_k) = \begin{bmatrix} 0 \\ 0.005x_{1k}x_{2k} \end{bmatrix};$$

$$\mathcal{P}_1(x_k) = \begin{bmatrix} x_{1k} + 0.012x_{2k} \\ -0.01x_{1k} + 1.012x_{2k} - 0.01x_{1k}^2 x_{2k} \end{bmatrix};$$

$$\mathcal{P}_2(x_k) = \begin{bmatrix} x_{1k} + 0.0098x_{2k} \\ -0.01x_{1k} + 1.0098x_{2k} - 0.01x_{1k}^2 x_{2k} \end{bmatrix};$$

Note. The following results are only approximate as the Euler discretization of a continuous-time system is not exact; they are valid for small enough sampling period, i.e., the larger the sampling period is the more inaccurate the found solution is, but not because of the proposed SOS methodology.

The region of study χ_S, and χ_E for the states, estimated states and error are specified by the matrices:

$$S_X = S_E = 10^{-2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e., a sphere of radius 10 is assumed to be enough to contain the trajectories of the state and estimated state over the course of simulations. And the highest degree of the fuzzy-polynomial observer $L(y_k, \hat{x}_k)$ is fixed at 2.

With those assumptions, we are also in the case of measurable premises so that an actual fuzzy observer can be sought.

One solution found for the problem is:

$$\gamma = 0.0269; \; Q = \begin{bmatrix} 0.37536 & 0.005358 \\ 0.005358 & 0.054277 \end{bmatrix};$$

and the fuzzy observer is:

$$L = \mu_1 L_1 + \mu_2 L_2 = \mu_1 \begin{pmatrix} L_{11} \\ L_{12} \end{pmatrix} + \mu_2 \begin{pmatrix} L_{21} \\ L_{22} \end{pmatrix}$$

being the computed polynomial observer gains L_1, L_2 given by:

$$L_1(y_k, \hat{x}_k) = 10^{-3}\Bigg(\begin{bmatrix} 20.11 \\ 31.55 \end{bmatrix} + \begin{bmatrix} -0.0338 \\ -0.0957 \end{bmatrix}\hat{x}_{1k}^2 + \begin{bmatrix} -0.004 \\ 0.0069 \end{bmatrix}\hat{x}_{2k}^2$$
$$+ \begin{bmatrix} -0.0017 \\ -0.0066 \end{bmatrix} y_k\hat{x}_{1k} + \begin{bmatrix} 0 \\ -0.002 \end{bmatrix} y_k\hat{x}_{2k}\begin{bmatrix} 0.0001 \\ -0.0023 \end{bmatrix}\hat{x}_{1k}\hat{x}_{2k}\Bigg)$$

$$L_2(y_k, \hat{x}_k) = 10^{-3}\Bigg(\begin{bmatrix} 16.345 \\ 27.95 \end{bmatrix} + \begin{bmatrix} -0.0212 \\ -0.07 \end{bmatrix}\hat{x}_{1k}^2 + \begin{bmatrix} -0.0054 \\ -0.00625 \end{bmatrix}\hat{x}_{2k}^2$$
$$+ \begin{bmatrix} -0.0008 \\ -0.0035 \end{bmatrix} y_k\hat{x}_{1k} + \begin{bmatrix} 0.00002 \\ -0.002 \end{bmatrix} y_k\hat{x}_{2k}\begin{bmatrix} 0.00027 \\ 0.0007 \end{bmatrix}\hat{x}_{1k}\hat{x}_{2k}\Bigg)$$

With this fuzzy discrete observer the results show that $\phi = \alpha + \beta = 5.42$, so we have proved that starting with any initial conditions inside $V_0 < \alpha = 4.22$ the error dynamics is stable in the sense that it will not abandon the interest zone χ_E.
Once this observer is in place, the decay rate can be proved to be $\delta = 0.9995$, so it is actually proved that under vanishing disturbances, the error will reach zero.

For simulation, consider two different scenarios; first starting from initial conditions $x(0) = [-2\ 7]$ and considering that there is a disturbance $w_k = 0.4775\,0.9^k$ applied at time $t = 0.3$ seconds, and then another scenario starting from $x(0) = [-1.5\ \ -7.8]$ and considering a disturbance $w_k = \sqrt{1.2}$ only if $k = 1$. As it can be seen easily, in the two cases the disturbances verify $\sum_{k=0}^{\infty} w_k^T w_k = 1.2$, so $\beta = 1.2$ can be set up as the square integral disturbance bound.

On next figures we can see the trajectory of the states and estimated states corresponding to the first scenario.

In figures 6.2 and 6.3 the states x_1 and x_2 are reached in about one second by the estimated ones instead of the disturbance. It can be checked too in Figure 6.4 where trajectories of the system and the observed system are shown.

In figure 6.5 the evolution of the Lyapunov function of the error is represented for the two scenarios considered. It can be checked that the Lyapunov function decreases except when the disturbance is applied but, instead of this, finally it tends to zero.

Finally Figure 6.6 shows the trajectory of the errors starting from the initial conditions (inside the ellipsoid $V(e) < \alpha$). The two trajectories do not abandon the region $V(e) < \alpha + \beta$, therefore they do not abandon the allowed region χ_E. In addition, trajectories go to zero when disturbances have lost energy.

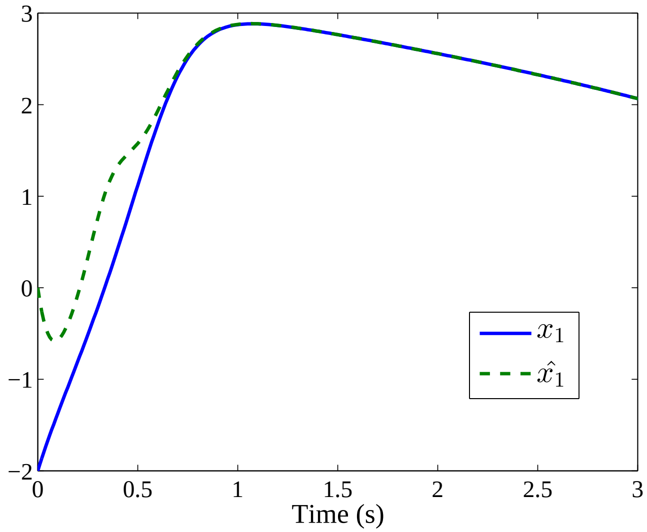


Figure 6.2: Evolution of x_1 and $\hat{x_1}$

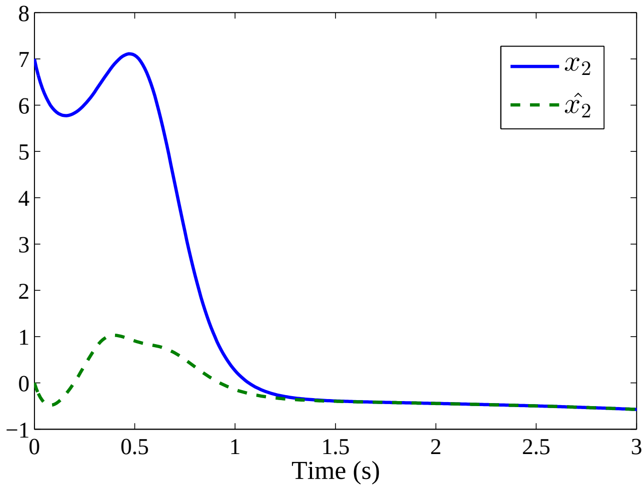


Figure 6.3: Evolution of x_2 and $\hat{x_2}$

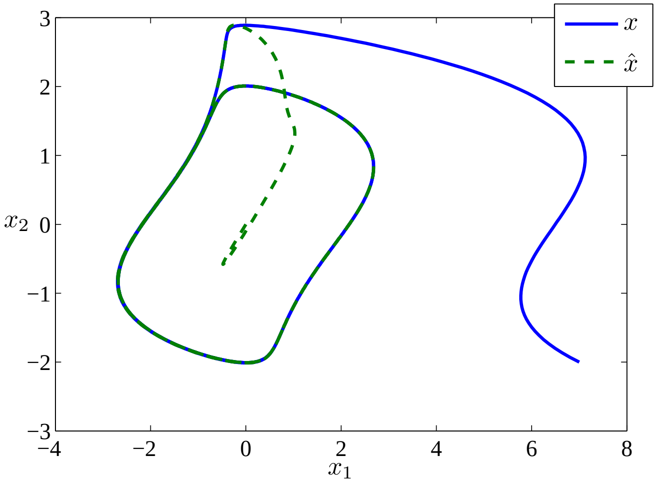


Figure 6.4: Trajectory of x and $\hat{x}$

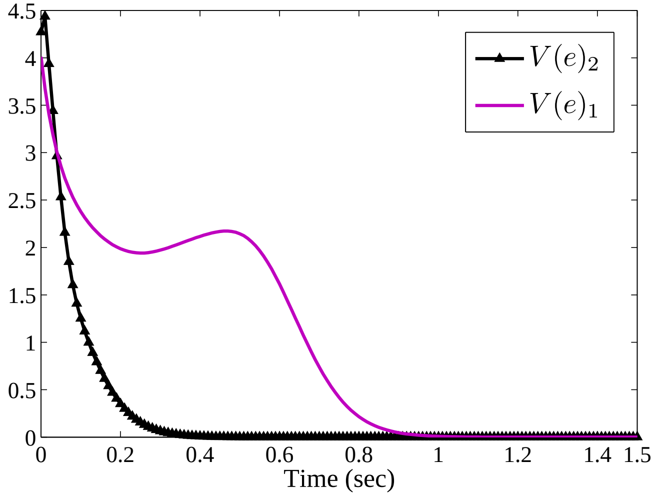


Figure 6.5: Lyapunov function V(e)

6.5 Conclusions

In this chapter, stable fuzzy-polynomial observers for nonlinear systems have been designed under bounded (vanishing and nonvanishing) disturbances. The

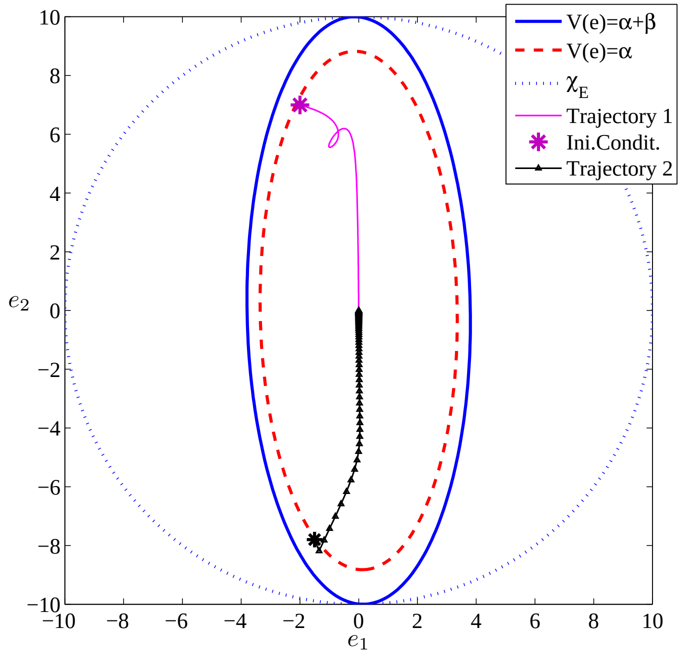


Figure 6.6: Error trajectory

result are observer gains which can be polynomial on the output and the estimated states. The developments have been presented in discrete-time for inescapable-set computation with vanishing disturbances, being the continuous-time ones very similar, and discussed in both continuous-time and discrete-time cases for the $\mathcal{H}_\infty$ versus decay-rate multicriteria design.

If premise variables are measurable or some bounds can be assumed on them, then fuzzy observer gains can be designed. If premises are unknown, a single polynomial observer gain is always an option. Takagi-Sugeno observer designs (and, of course, linearized time-invariant ones) can be considered as particular cases of the proposed methodology.

Chapter 7

Discrete-time Control Synthesis with Input Saturation

Immense power is acquired by assuring yourself in your secret reveries that you were born to control things.

Christian Grey

ABSTRACT: This chapter presents a discrete-time control design methodology using a Lyapunov function with dependence on present and past states. This proposed approach is used to bypass the usual difficulty of expressing the problem in a convex way, holding the extra decision variables given by polynomial Lyapunov function approaches. Also polynomial controllers are allowed to depend on both present and past states. Furthermore, in the particular case of knowing the saturation limits of the control action, information about the relationship between the present and past states can be introduced via Positivstellensatz multipliers. Sum-of-squares (SOS) techniques and available SDP software are used in order to find the controller.

Stability analysis and control design for polynomial systems has received attention in recent literature, both in continuous-time (Ichihara, 2009b; Prajna, Papachristodoulou and Wu, 2004b; Hancock and Papachristodoulou, 2013; Chesi, 2011) and discrete-time settings (Tanaka, Ohtake and Wang, 2008; Xu, Xie and Wang, 2007). The basic framework uses Sum of Squares (SOS) techniques and Positivstellensatz argumentations in order to prove local stability /stabilization.

Furthermore, many smooth nonlinear systems can be also transformed to polynomial ones, either by Taylor-series approximation (Section 3.1.1) or by a change of variable and state augmentation (Section 3.1.2). Therefore, the polynomial approach may be used to obtain alternative nonlinear control solutions to others in literature (Slotine, 1991; Khalil, 2002) based on a systematic modelling and convex-programming approach. The approach, however, involves some conservative choices in order to get a reasonable computational cost: finite degree of Lyapunov functions, finite degree and number of KKT-like multipliers associated to algebraic constraints (Jarvis-Wloszek, Feeley, Tan, Sun and Packard, 2005). Moreover, if the controller has to be designed via Lyapunov theory, the polynomial discrete-time design case usually leads to a nonconvex problem which has to be solved via iterative LMI algorithms (Appendix A.1.4.2) or similar. In order to avoid this problem, Prajna, Papachristodoulou and Wu (2004b); Tanaka, Ohtake and Wang (2008) proposed restricting the dependence of the Lyapunov function to the states which are not affected directly by the control action. This outperforms the classical quadratic case control design but it is still quite restricted.

In this work, the stabilization problem for polynomial systems with input bounds is addressed in a convex way using the whole state information. The main idea is introducing delayed states in the Lyapunov function which breaks up some bilinear terms and also provides the state-feedback controller with extra degrees of freedom (polynomial depending on the present and past values). The use of Lyapunov functions with dependence on delayed scheduling parameters has been successfully applied in the Takagi-Sugeno LMI framework (Lendek, Guerra and Lauber, 2012; Guerra, Kerkeni, Lauber and Vermeiren, 2012). In the discrete-time case here considered, due to the construction of the matrices involved, there is no need of Krasovskii-like terms in Lyapunov functions, as other developments need (Gassara, Hajjaji, Kchaou and Chaabane, 2014).

In this chapter, the idea is applied to include a full delayed state in polynomial systems. Information about the relationship between present and past state values is introduced by specifying bounds in the control action and Positivstellensatz multipliers. The approach improves over existent ones in literature, if restricted to convex optimization setups. The recent work Valmorbida, Tarbouriech and Garcia (2013) proposes a similar development addressing the polynomial control synthesis for discrete-time systems under actuator saturation. However, their proposal leads to a nonconvex bilinear matrix inequality problem, which needs to be solved iteratively without guarantees of global optimality, as widely known (Fukuda and Kojima, 2001). Intentionally, analysis and comparison with BMI approaches has been left out of the scope of this paper because the BMI results depend on the initial conditions and iteration step sizes. The objective of the chapter is, hence, lifting conservativeness in polynomial control by using delayed-state Lyapunov functions and saturation bounds while keeping the resulting SOS conditions convex. The main result of this chapter has been published in the journal paper Pitarch, Sala, Lauber and Guerra (2016).

The chapter is organized as follows: next section states the notation followed in the rest of the chapter as well as summarizes the existent drawbacks related to the current issue and presents the problem statement, Section 7.2 presents a convex control design methodology by delayed polynomial Lyapunov functions, Section 7.3 shows an academic example demonstrating the effectiveness of the proposed approach and, finally, a conclusions section closes the chapter.

7.1 Preliminaries

The existent approaches in convex literature addressing the polynomial controller design for nonlinear systems are usually very conservative (see Section 3.3). This is due to the loss of information made when applying some changes of variables in a similar way to the Takagi-Sugeno LMI case (Section 2.3.1). The main reason for making those changes of variables is breaking up some bilinear terms which appear as products of decision variables present in the polynomial Lyapunov function $V(x)$ and the polynomial control law $K(x)$.

Moreover, those changes of variables prevent the direct extension of the domain of attraction results developed in Chapter 4. Indeed, local relaxations of Lyapunov stability conditions with Positivstellensatz loose their powerful meaning in the synthesis phase. This is because, after a change of variable, local information on the new variables is unknown or should be introduced via non-convex terms.

Let us first introduce some notation about sum-of-squares results to be used throughout the chapter. The $N \times N$ SOS polynomial matrices in variables x (see Proposition A.2) will be denoted by Σ_x^N. Evidently, SOS polynomial matrices are positive semi-definite matrices for all values of x.

Denote by $\mathcal{I}^N(f_1,\dots,f_t)$ the set of $N \times N$ matrices whose elements belong to the ideal $\mathcal{I}(f_1,\dots,f_t)$. Denote also by $\wp^N(f_1,\dots,f_t)$ the subset of $\mathcal{I}^N(f_1,\dots,f_t)$ formed by the cone of matrices which are positive semi-definite for any values of the argument variables.

Using the above notation, the cited work recasts the so-called Positivstellensatz theorem (Section A.2.2) in order to assert local non-negativeness of polynomials in a region with polynomial boundaries. The lemma below generalizes the concept to local positive semi-definiteness of polynomial matrices:

Lemma 7.1. *The polynomial matrix $P(x) \in \mathcal{R}_x^{N\times N}$ is positive semi-definite in a region $\Omega = \{x : g_i(x) > 0, h_j(x) = 0, i : 1,\dots,r, j : 1,\dots,v\}$ if there exist polynomial matrices $G(x) \in \wp^N(g_1,\dots,g_r), H(x) \in \mathcal{I}^N(h_1,\dots,h_v)$ which verify:*

$$P(x) - G(x) + H(x) \in \Sigma_x^N \tag{7.1}$$

Proof. Multiplying (7.1) by auxiliary variables $\nu \in \mathbb{R}^N$ on the left and right, it results in a polynomial sum-of-squares condition $\nu^T(P(x)-G(x)+H(x))\nu \in \Sigma_{x,\nu}$ by Proposition A.1, so that, if it holds, $\nu^T P(x)\nu$ is nonnegative in Ω as required. □

Note that computational checking of (7.1) can be done with the LMI's deriving from Proposition A.2. For instance, one choice of matrices above for computations may be

$$G(x) = \sum_{i=1}^{r} S_i(x)g_i(x) \quad H(x) = \sum_{j=1}^{v} Z_j(x)h_j(x)$$

where S_i are SOS matrices and Z_j are arbitrary ones, both with unknown coefficients. S and Z can be full matrices or, for instance, only diagonal ones depending on the available computing resources.

7.1.1 Problem statement

Up to the author's knowledge, the general problem of finding a Lyapunov function and a controller gain together has not been posed in convex form. In fact, recent literature still employs the assumptions of Prajna, Papachristodoulou and Wu (2004b) in order to put the problem in convex form. See, for instance, the work with piecewise polynomial Lyapunov functions in Lam, Narimani, Li and Liu (2013). Let us recall now those well-known stability results for discrete-time polynomial systems:

Consider a polynomial discrete-time system in the form

$$x_{k+1} = A(x_k)z(x_k) + B(x_k)u_k \tag{7.2}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^b$ are the state and control input vector at time instant k respectively, $A(x_k) \in \mathcal{R}_{x_k}^{n\times l}$, $B(x_k) \in \mathcal{R}_{x_k}^{n\times b}$ and $z(x_k) \in \mathcal{R}_{x_k}^{l}$. On the sequel, shorthand $z_k = z(x_k)$ will be used for brevity.

Define a Lyapunov function candidate $V : \mathbb{R}^n \to \mathbb{R}$ as

$$V(x_k) = z_k^T Q^{-1}(x_k) z_k \tag{7.3}$$

where $Q(x_k) \in \mathcal{R}_{x_k}^{l\times l}$ is a polynomial matrix in the states. Consider a state feedback controller in the form

$$u_k = -K(x_k)z_k \tag{7.4}$$

where $K(x_k) = M(x_k)Q^{-1}(x_k)$ is the feedback gain and $M(x_k) \in \mathcal{R}_{x_k}^{c\times l}$. According to Lyapunov theory, the controller (7.4) stabilizes the system (7.2) if conditions

$$V(0) = 0 \tag{7.5}$$

$$V(x_k) > 0, \; \forall x_k \in \mathcal{D}, x_k \neq 0 \tag{7.6}$$

$$\Delta V = V(x_{k+1} - V(x_k)) \leq 0, \{x_{k+1}, x_k\} \in \mathcal{D} \tag{7.7}$$

are satisfied (Khalil, 2002). Further if the inequality (7.7) is strict for $x_k \in \mathcal{D}\backslash\{0\}$, then the system is asymptotically stable. Moreover, if $\mathcal{D} = \mathbb{R}^n$, stability is global.

In the controller synthesis problem (i.e., the controller (7.4) has to be found simultaneously with the Lyapunov function (7.3)), some conservative assumptions are addressed in literature (Xu, Xie and Wang, 2007; Tanaka, Ohtake and Wang, 2008) in order to put the problem in a convex way:

- If Q is constant and $z_k = x_k$, the controller synthesis problem becomes convex by Schur complement, resulting in finding Q and coefficients of polynomials in $M(x_k)$ such that, for an arbitrary $\epsilon > 0$:

$$\begin{bmatrix} Q & (*)^T \\ A(x_k)Q - B(x_k)M(x_k) & Q \end{bmatrix} - \epsilon I \in \Sigma_{x_k}^{2l}$$

- Following the idea introduced for continuous-time in the work Prajna, Papachristodoulou and Wu (2004b), use a Lyapunov function defined by $Q(\tilde{x}_k)$, where $\tilde{x}_k = Ex_k \in \mathbb{R}^L$, being E a constant matrix fulfilling[1] $EB(x_k) = 0$. If z_k can be expressed as

$$z_k = T(\tilde{x}_k)x_k \tag{7.8}$$

with $T(\tilde{x}_k) \in \mathcal{R}^{l\times n}_{\tilde{x}_k}$, the problem is still convex.

If the above problems render infeasible, local stability conditions can be posed based on modifying conditions (7.6) and (7.7) in order to make them hold locally in a region of interest $\Omega \subset \mathbb{R}^n$. Lemma 7.1 enables checking such conditions with SOS programming (sufficient conditions). For instance, the local stability results in (Xu, Xie and Wang, 2007) can be adapted to the notation here as follows:

Corollary 7.1. *If polynomial matrices $G(x_k), H(x_k)$ as defined in Lemma 7.1 can be found fulfilling*

$$\begin{bmatrix} Q & (*)^T \\ A(x_k)Q - B(x_k)M(x_k) & Q \end{bmatrix} - \epsilon I - G(x_k) + H(x_k) \in \Sigma^{2l}_{x_k} \tag{7.9}$$

with $\epsilon > 0$, then $\Delta V(x_k)$ is locally negative in a region of the state space Ω except at the origin.

When conditions (7.5), (7.6) and (7.7) hold for all $x \in \Omega$, the system is said to be *locally stable* in Ω, implying that all level sets $\{x : V(x) \leq \gamma\} \subset \Omega$ are invariant (Section 3.2.2).

Summarizing, using the nullifier E and setting $V = z^T Q^{-1}(Ex_k)z$ to ignore the bilinearity is what previous literature does. In this chapter, the efforts are directed to partially overcome such issue (with, of course, some conservatism), so this is the core development. Inspired on the Lyapunov functions with dependence on delayed scheduling parameters in Guerra, Kerkeni, Lauber and Vermeiren (2012); Lendek, Guerra and Lauber (2012), a full delayed-state polynomial Lyapunov function is used to reduce the conservativeness of the above results, as discussed on next section.

Note that, although introducing delays in the Lyapunov function or in the system's dynamics do not help to improve in the general control case[2], this

[1] A particular case (Tanaka, Ohtake and Wang, 2008) is choosing E to be a row-selector matrix extracting the state variables whose corresponding row of $B(x)$ is zero (i.e. $\tilde{x}$ are states that do not directly depend on the control input).

[2] For non-delayed systems, if a delayed Lyapunov function proving stability exists, there will exist an undelayed Lyapunov proving it too. However, the issue at stake is how to obtain it via convex conditions. Converse Lyapunov does not ensure that there is a convex characterization of the sought non-delayed Lyapunov function.

proposal can obtain better results in the particular case of knowing the saturation limits of the control action.

The specific objective of this chapter is designing a discrete-time polynomial control law in a *convex* way, ensuring some saturation bounds on the control action. Furthermore, by introducing available information about the relationship between present and past-state values will allow reducing the conservativeness with existent methodologies.

7.2 Synthesis via delayed-state Lyapunov functions

Consider a *delayed*-rational candidate Lyapunov function $V(x_k, x_{k-1})$ in the form

$$V(x_k, x_{k-1}) = z_k^T Q^{-1}(\tilde{x}_k, x_{k-1}) z_k \tag{7.10}$$

and a state-feedback control law which can depend on present and past states

$$u_k = -K(x_k, x_{k-1}) z_k \tag{7.11}$$

where, $K(x_k, x_{k-1}) = M(x_k, x_{k-1}) Q^{-1}(\tilde{x}_k, x_{k-1})$ being $Q(\tilde{x}_k, x_{k-1}) \in \mathcal{R}^{l\times l}_{x_k, x_{k-1}}$ and $M(x_k, x_{k-1}) \in \mathcal{R}^{b\times l}_{x_k, x_{k-1}}$. It will be assumed that there exists a constant matrix $E \in \mathbb{R}^{L\times n}$ such that z_k can be expressed as (7.8) and another constant matrix $E^\perp$ such that $E^T E^\perp = 0$ and the rows of E and $E^\perp$ form a basis of $\mathbb{R}^n$. Obviously, by definition, the columns of B belong to the row space of $E^\perp$.

Consider a region Ω of the augmented state space:

$$\begin{aligned} \Omega_0 &= \{x : z(x)^T U z(x) \le R^2\} &(7.12)\\ \Omega &= \{x_k, x_{k-1} : x_k \in \Omega_0, x_{k-1} \in \Omega_0\} &(7.13) \end{aligned}$$

and a second region Φ, $\Phi \subset \Omega$, where initial conditions are supposed to lie in, described as

$$\Phi = \{x_0, x_{-1} : \max(z_0^T Y z_0, z_{-1}^T Y z_{-1}) \le \beta^2\} \tag{7.14}$$

where U and Y are constant user-defined matrices with suitable dimension.

Consider also that each individual control input has known saturation bounds

$$|e_j u_k| \le \sigma_j, \quad \sigma_j \in \mathbb{R}, \quad j : 1, \dots, b \tag{7.15}$$

where e_j is the standard canonical row vector in $\mathbb{R}^b$ whose *j*-th component is one and the rest are zero. Hence, a set of vectors $\bar{u}_i$, $i : 1, \dots, 2^b$ can be constructed such that the control action u belongs to its convex hull.

Theorem 7.1. *Assume* $\{x_0, x_{-1}\} \in \Phi$*. Then, system* (7.2) *with the control law* (7.11) *is locally stable in region* (7.13)*, satisfies the control input saturation* (7.15) *and* Φ *belongs to the domain of attraction of the origin if the following SOS problem is feasible for all* $i : 1, \ldots, 2^b$ *and* $j : 1, \ldots, b$*:*

$$\begin{bmatrix} Q(Ex_k, x_{k-1}) & (*)^T \\ \Psi(x_k, x_{k-1}) & Q(E \cdot A(x_k)z_k, x_k) \end{bmatrix} - \epsilon I - \Upsilon_{1i} \in \Sigma^{2l}_{x_k, x_{k-1}} \tag{7.16}$$

$$(Q(Ex_k, x_{k-1}) - \epsilon I - W_1 - \mathcal{H}_1 \in \Sigma^{l}_{x_k, x_{k-1}} \tag{7.17}$$

$$\begin{bmatrix} Q(Ex_k, x_{k-1}) & (*)^T \\ e_j M(x_k, x_{k-1}) & \sigma_j^2 \end{bmatrix} - \Upsilon_{2i} \in \Sigma^{l+1}_{x_k, x_{k-1}} \tag{7.18}$$

$$R^2 U^{-1} - Q(Ex_k, x_{k-1}) - W_2 - \mathcal{H}_2 \in \Sigma^{l}_{x_k, x_{k-1}} \tag{7.19}$$

$$\begin{bmatrix} \beta^{-2} z_k^T Y z_k & z_k^T \\ z_k & Q(Ex_k, x_{k-1}) \end{bmatrix} - \Upsilon_{3i} \in \Sigma^{l+1}_{x_k, x_{k-1}} \tag{7.20}$$

where

$$\Psi(x_k, x_{k-1}) = T(E \cdot A(x_k))(A(x_k)Q(x_{k-1}) - B(x_k)M(x_k, x_{k-1})),$$

$$\Upsilon_{di} = S_{di} + \mathcal{H}_{di} + \sum_{c=1}^{n-L} H_{dc}\phi_{ci}, \quad d : 1, 2, 3 \tag{7.21}$$

being $\epsilon > 0$ *and:*

$$\begin{aligned}
&\tilde{\phi}_c = e_c(Ex_k - E \cdot A(x_{k-1})z_{k-1}), \\
&\phi_{ci} = e_c(E^{\perp}x_k - E^{\perp}A(x_{k-1})z_{k-1} - E^{\perp}B(x_{k-1})\bar{u}_i), \\
&W_1 \in \wp^l(R^2 - z_k^T U z_k, R^2 - z_{k-1}^T U z_{k-1}), \\
&W_2 \in \wp^l(R^2 - z_{k-1}^T U z_{k-1}), \\
&S_{1i} \in \wp^{2l}(R^2 - z_k^T U z_k, R^2 - z_{k-1}^T U z_{k-1}), \\
&S_{2i} \in \wp^{l+1}(R^2 - z_k^T U z_k, R^2 - z_{k-1}^T U z_{k-1}), \\
&S_{3i} \in \wp^{l+1}(\beta^2 - z_{k-1}^T Y z_{k-1}), \\
&\mathcal{H}_1 \in \mathcal{I}^l(\tilde{\phi}_1, \ldots, \tilde{\phi}_L), \\
&\mathcal{H}_2 \in \mathcal{I}^l(R^1 - z_k^T U z_k, \tilde{\phi}_1, \ldots, \tilde{\phi}_L), \\
&\mathcal{H}_{1i} \in \mathcal{I}^{2l}(\tilde{\phi}_1, \ldots, \tilde{\phi}_L), \\
&\{\mathcal{H}_{2i}, \mathcal{H}_{3i}\} \in \mathcal{I}^{l+1}(\tilde{\phi}_1, \ldots, \tilde{\phi}_L), \\
&H_{1c} \in \mathcal{R}^{(2l)\times(2l)}_{x_k, x_{k-1}}, \ \{H_{2c}, H_{3c}\} \in \mathcal{R}^{(l+1)\times(l+1)}_{x_k, x_{k-1}}.
\end{aligned}$$

Proof. Using the candidate Lyapunov function (7.10), stability condition $\Delta V = V_{k+1} - Vk < 0$ now becomes:

$$\Delta V = z_{k+1}^T Q^{-1}(\tilde{x}_{k+1}, x_k) z_{k+1} - z_k^T Q^{-1}(\tilde{x}_k, x_{k-1}) z_k < 0$$

Substituting z_{k+1} by its value

$$z_{k+1} = T(E \cdot A(x_k))(A(x_k) + B(x_k)K(x_k, x_{k-1}))z_k,$$

performing the well-known change of variable

$$\rho = Q^{-1}(\tilde{x}_k, x_{k-1}) z_k \tag{7.22}$$

and applying Schur complement, it leads to:

$$\nu^T \begin{bmatrix} Q(\tilde{x}_k, x_{k-1}) & (*)^T \\ \Psi(x_k, x_{k-1}) & Q(\tilde{x}_{k+1}, x_k) \end{bmatrix} \nu \geq 0$$

The relationship between present and past states is:

$$\begin{array}{c} E^{\perp}(x_k - A(x_{k-1})z_{k-1} - B(x_{k-1})u_{k-1}) = 0, \\ E(x_k - A(x_{k-1})z_{k-1}) = 0. \end{array} \tag{7.23}$$

This information can be introduced in the SOS constraints with terms $\mathcal{H}$ formed by the ideals associated to the above equalities. However, in order to avoid introducing new variables u in the SOS program, equalities in (7.23) depending on $E^{\perp}$ must be introduced with arbitrary multiplier matrices H_{dc}, conforming the rightmost summation in the definition of Υ_{di} in (7.21), but keeping linearity in ϕ_{ci}. In fact, to actually get (7.21), a last step is needed: as the resulting expressions are affine in u_{k-1}, they will hold if they do in all the vertices given by vectors $\bar{u}_i$, from convexity arguments. Note that multipliers H_{dc} must be shared between all vertices.

Now, positive semi-definite matrix multipliers W, S are provided in order to add information about Ω in SOS conditions so that they need to hold only locally (note that multipliers S_{di} can actually be different for different $\bar{u}_i$). After these steps, (7.16) and (7.17) are obtained, so (7.10) is a valid Lyapunov function locally in Ω by Lemma 3.2 and Theorem A.1.

Define now Θ as the Lyapunov level set:

$$\Theta = \{x_k, x_{k-1} : V(x_k, x_{k-1}) \leq 1\}$$

Conditions (7.18) ensure that u does not take values larger than the saturation bounds σ inside the region $\Pi = \Theta \cap \Omega$. They are obtained from the inequality

$$z_k^T Q^{-1}(\tilde{x}_k, x_{k-1}) z_k - z_k^T e_j^T K(x_k, x_{k-1})^T \sigma_j^{-1} I(*) \geq 0$$

in a similar way to the quadratic case (Boyd, Ghaoui, Feron and Balakrishnan, 1994) for Θ, but relaxed with local information on Ω and system dynamics analogous to the above discussed multipliers[3].

As a last step in the proof, as locality conditions only hold in Ω, we need to ensure that there exists an invariant subset of Ω containing the initial set Φ.

Let us assume $V(x_k, x_{k-1}) \geq 1 \forall x_k \in \partial\Omega_0, x_{k-1} \in \Omega_0$ which is enforced by (7.19) as later shown. Let us prove that $\Pi = \Theta \cap \Omega$ is invariant. Indeed, the points $x_k \in \partial\Omega_0$ and $x_{k-1} \in \Omega_0$ are outside Π, so the trajectories will never leave Π through that part of $\partial\Omega$.

If $x_k \in \Omega_0, x_{k-1} \in \partial\Omega_0, V(x_k, x_{k-1}) \leq 1$ then $x_{k+1} \in \Omega_0, x_k \in \Omega_0$ and $V(x_{k+1}, x_k) < 1$. Indeed, $V(x_{k+1}, x_k) < 1$ from (7.16); then expression (7.19), from the above paragraph, discards the option of x_{k+1} leaving Ω_0. Hence, if $\{x_k, x_{k-1}\} \in \Pi$, we have $\{x_{k+1}, x_k\} \in \Pi$.

To enforce $V(x_k, x_{k-1}) \geq 1 \forall x_k \in \partial\Omega_0, x_{k-1} \in \Omega_0$, similar issues to those arising in (7.18) discussed in footnote 3 apply. Thus, resorting to similar argumentations gives (7.19).

The last set of SOS constraints must ensure the initial condition set $\Phi \subset \Pi$. As $\Phi \subset \Omega$ by assumption, $\Phi \subset \Theta$ has to be ensured, too. It can be proved by enforcing $V(x_k, x_{k-1}) \leq 1 \forall \{x_k, x_{k-1}\} \in \Phi$. A sufficient condition for this to hold is

$$\frac{1}{\beta^2} z_k^T Y z_k - z_k^T Q^{-1}(\tilde{x}_k, x_{k-1}) z_k \geq 0$$

enforced locally in Ω by (7.20), after applying Schur complement and Positivstellensatz. Details omitted for brevity.

So, $\{x_0, x_{-1}\} \in \Phi \subset \Pi \subset \Omega$, invariance of Π has been ensured by SOS constraints and $\Pi \subset \Theta$ ensures the control action bounds (7.15) are met, so multipliers arising from (7.23) are valid.

Now, the proven invariant set in the augmented space is *not* a Lyapunov function level set: the level set Θ can actually extend outside the local-stability region Ω, removing conservativeness. So, the discrete-time analog to La-Salle

[3] Actually, we should prove $\sigma_j - z e_j^T K^T K e_j z > 0$ via multipliers in the cone $(1 - z^T Q^{-1} z)$ and the rest of constraints defining (7.13) and system dynamics (7.23). However, the need of the change of variable (7.22) forces the use of some constant (S-procedure like) multipliers because relationship between ρ and x is lost (details omitted for brevity).

invariance theorem needs to be invoked: the system will converge to the largest invariant set in $\Delta V = 0$, and only the origin verifies the zero-increment condition (details omitted for brevity).

□

Remark 7.1. With $Q(\tilde{x}_k)$, $z_k = T(\tilde{x}_k)x_k$ and $u = -K(x_k)z_k$, Theorem 7.1 reduces to cases in Xu, Xie and Wang (2007); Tanaka, Ohtake and Wang (2008). A more general version encompassing the "natural" case $V = z_k^T Q(x_k)$ z_k may be crafted by letting $Q(x_k, x_{k-1})$. In that case, the SOS problems would involve variables $\{x_k, x_{k-1}, x_{k+1}\}$. However, in order to keep convexity, new multipliers analog to (7.21) are needed, with additional ϕ_c and ϕ_{ci} now referring to the relationship between x_k and x_{k+1}. Details are omitted for brevity.

Remark 7.2. Presence of x_{k-1} in Q instead of only $Q(\tilde{x}_k)$ (or $Q(x_k)$, remark above), allows controller $M(x_k, x_{k-1})$ to take into account present and past information, so it provides more degrees of freedom to find a solution. Note also that, even if, of course, an undelayed controller $u(x_k)$ achieving the same performance will likely exist, maybe it cannot be obtained with convex SOS conditions.

In this approach, the bilinearity has been resolved by conceiving a full-rank matrix $[E\ E^{\perp}]$ and an implicit change of coordinates, so that:

(a) In the nullspace of B, we can add an arbitrary multiplier because the control action and the matrix K do not appear. Also, the Lyapunov function can depend on Ex_k due to the nullification of B. So, no conservatism from the "delay" trick is induced in this subspace.

(b) In the image space of B, in order to avoid decision variables in "K", the actual control variable must be kept. Then, as $H(u - Kz)$ would be bilinear (due to the product of multiplier H and controller K decision variables), saturation constraints on u should be added either by Positivstellensatz conditions or, as we chose, by convex-hull argumentations. This may be conservative (we are only considering bounds on u independent of decision variables, instead of $u = Kz$) but allows for more general Lyapunov functions and controllers which effectively achieved improved results. See Example 7.3.1.

Remark 7.3. In discrete-time, Lyapunov-Krasovskii (LK) functions are actually a particular case of generic Lyapunov functions of an augmented finite-dimensional realization (well known, for instance in Hetel, Daafouz and Iung

(2008); Gonzalez, Sala, Garcia and Albertos (2013)). From the realization $\psi_k = (x_k\; x_{k-1})^T$ so

$$\psi_{k+1} = \begin{bmatrix} A - BK & 0 \\ I & 0 \end{bmatrix} \begin{bmatrix} z_k \\ z_{k-1} \end{bmatrix}$$

it can be proved that, using a candidate "full" Lyapunov function

$$Q = \begin{bmatrix} P_{11} & P_{12} \\ (*) & P_{22} \end{bmatrix}$$

all blocks except P_{11} must forcefully be zero in the linear case, and also if changes of variable leading to (7.17) are enforced. So, this is the motivation on why (7.10) is taken as a LK candidate (equivalent Lyapunov function of the augmented system) instead of other more complex constructions which would not be useful with the proposed changes of variables and developments.

7.2.1 Extension to fuzzy-polynomial systems

The above design methodology can be extended to a more general class of nonlinear systems. In particular, the ones which can be expressed as a fuzzy-polynomial model in the form

$$x_{k+1} = \sum_{i=1}^{r} \mu_i(x_k) A_i(x_k) x_k + B(x_k) u_k \tag{7.24}$$

where x, u are as in (7.2) and μ are the membership functions of the fuzzy model. The problem is addressed, in a preliminary way, in the conference paper Pitarch and Sala (2013b).

Consider now a particularization of (7.10) as a candidate Lyapunov function

$$V(x_k, x_{k-1}) = x_k^T P^{-1}(x_{k-1}) x_k \tag{7.25}$$

and a state-feedback control law (7.11) such that $K(x_k, x_{k-1}) = M(x_k, x_{k-1})$ $P^{-1}(x_{k-1})$, being $P(x_{k-1}) \in \mathcal{R}^{l\times l}_{x_{k-1}}$.

Consider also the region of the augmented state

$$\Omega_0 = \{x : x^T U x \leq R^2\} \tag{7.26}$$

$$\Omega = \{x_k, x_{k-1} : x_k \in \Omega_0, x_{k-1} \in \Omega_0\} \tag{7.27}$$

and the region (7.14) of initial conditions. Suppose that each individual control action fulfill the saturation bounds (7.15).

Corollary 7.2. *Assume* $\{x_0, x_{-1}\} \in \Phi$. *Then, system* (7.24) *with the controller* (7.11) *is locally stable in the region* (7.27), *satisfies the control input saturation* (7.15) *and* Φ *belong to the domain of attraction of the origin if the following SOS problem is feasible for all* $i : 1, \ldots, r$, $j : 1, \ldots, b$, $h : 1, \ldots, 2^b$*:*

$$\begin{bmatrix} Q(x_{k-1}) & (*)^T \\ \Psi_i(x_k, x_{k-1}) & Q(x_k) \end{bmatrix} - \epsilon I - \Upsilon_{1i} \in \Sigma_{x_k, x_{k-1}^{2l}} \tag{7.28}$$

$$x_{k-1}) - \epsilon I - W_1 - \mathcal{H}_1 \in \Sigma^l_{x_{k-1}} \tag{7.29}$$

$$\begin{bmatrix} Q(x_{k-1}) & (*)^T \\ e_j M(x_k, x_{k-1}) & \sigma_j^2 \end{bmatrix} - \Upsilon_{2i} \in \Sigma^{l+1}_{x_k, x_{k-1}} \tag{7.30}$$

$$R^2 U^{-1} - Q(x_{k-1}) - W_2 - \mathcal{H}_2 \in \Sigma^l_{x_{k-1}} \tag{7.31}$$

$$\begin{bmatrix} \beta^{-2} x_k^T Y x_k & x_k^T \\ x_k & Q(x_{k-1}) \end{bmatrix} - \Upsilon_{3i} \in \Sigma^{l+1}_{x_k, x_{k-1}} \tag{7.32}$$

where

$$\Psi_i(x_k, x_{k-1}) = A_i(x_k) Q(x_{k-1}) - B(x_k) M(x_k, x_{k-1})),$$

$$\Upsilon_{dh} = S_{dih} + \sum_{c=1}^{n} H_{dc} \phi_{ich}, \quad d : 1, 2, 3 \tag{7.33}$$

being $\epsilon > 0$ *and:*

$$\begin{aligned}
&\phi_{ich} = e_c(x_k - A_i(x_{k-1}) x_{k-1} - B(x_{k-1}) \bar{u}_h), \\
&\{W_1, W_2\} \in \wp^l(R^2 - x_{k-1}^T U x_{k-1}), \\
&S_{1ih} \in \wp^{2l}(R^2 - x_k^T U x_k, R^2 - x_{k-1}^T U x_{k-1}^T), \\
&S_{2ih} \in \wp^{l+1}(R^2 - x_k^T U x_k, R^2 - x_{k-1}^T U x_{k-1}^T), \\
&S_{3ih} \in \wp^{l+1}(\beta^2 - x_{k-1}^T Y x_{k-1}), \\
&H_{1c} \in \mathcal{R}^{(2l)\times(2l)}_{x_k, x_{k-1}}, \ \{H_{2c}, H_{3c}\} \in \mathcal{R}^{(l+1)\times(l+1)}_{x_k, x_{k-1}}.
\end{aligned}$$

Proof. The proof is easy to derive by following the argumentations of the one on Theorem 7.1 and the convex-sum property of the fuzzy-vertex models A_i. Details omitted for brevity. □

Indeed, Corollary 7.2 is a simplified version of Theorem 7.1 in which terms depending on $\tilde{x}$ have been removed from the Lyapunov function matrix Q in order to keep linearity on the system's dynamics $\sum_i \mu_i A_i$ and, therefore, being able to express the Lyapunov stability constraints with the convex hull formed by all the vertex models.

Remark 7.4. Note that fuzzy controller gains $M_i = K_i P^{-1}$ may be taken into account in Corollary 7.2, as well as fuzzy B_i if required (using Polya's relaxations for conditions on which $B_i M_j$ appear). However those options have not been stated in the corollary for clarity reasons and the fact that computational requirements increase a lot with the amount of new vertex-model conditions.

7.3 Example

Next, an academic example is provided to show the effectiveness of the presented synthesis methodology.

Example 7.3.1. Consider the following polynomial system:

$$x_{k+1} = \begin{bmatrix} -0.7 & 0.05 \\ 0.3x_{2_k}(1 - 0.166x_{1_k}^2) & 0.8 \end{bmatrix} x_k + \begin{bmatrix} -0.02 \\ 0.05x_{1_k} \end{bmatrix} u_k \qquad (7.34)$$

The goal will be to obtain the largest possible region of initial conditions Φ, with a predefined shape, for a fixed degree in the Lyapunov function and multipliers. Note that, given the model, $E = 0$, $E^{\perp} = I$, $z_k = x_k$ is the only option.

Conditions to find a global controller with quadratic $V(x_k)$, i.e., constant Q, are infeasible. Note that setting $Q(\tilde{x}_k)$ is not a viable option, as $E = 0$.

Now define a spherical state-space region of interest Ω and a spherical region of initial conditions Φ as:

$$\Omega = \max(x_{1_k}^2 + x_{2_k}^2, x_{1_{k-1}}^2 + x_{2_{k-1}}^2 \leq 4.5^2$$

$$\Phi = \max(x_{1_k}^2 + x_{2_k}^2, x_{1_{k-1}}^2 + x_{2_{k-1}}^2 \leq \beta^2$$

The objective will be maximizing the size parameter β for fixed control action bounds σ, $|u_k| \leq \sigma$. The maximum degree for $M(x_k, x_{k-1})$ and $Q(x_{k-1})$ is set to two. The parametrizations of Positivstellensatz terms are:

$$\begin{aligned}
W_1 &= \psi_1(4.5^2 - x_{1_k}^2 - x_{2_k}^2) + \varsigma_1(4.5^2 - x_{1_{k-1}}^2 - x_{2_{k-1}}^2),\\
W_2 &= \psi_2(4.5^2 - x_{1_{k-1}}^2 - x_{2_{k-1}}^2),\\
S_{di} &= \psi_{di}(4.5^2 - x_{1_k}^2 - x_{2_k}^2) + \varsigma_{di}(4.5^2 - x_{1_{k-1}}^2 - x_{2_{k-1}}^2),\ d:1,2\\
S_{3i} &= \psi_{3i}(\beta^2 - x_{1_{k-1}}^2 - x_{2_{k-1}}^2),\ \mathcal{H}_1 = \mathcal{H}_2 = 0
\end{aligned}$$

where $\{\psi, \varsigma\}$ are diagonal matrices of appropiate dimension whose entries belong to $\Sigma_{x_k, x_{k-1}}$.

Constant and delayed Lyapunov functions are compared. The largest β obtained until infeasibility with the different approaches is shown in Table 7.1. Strict feasible solutions are found by software SOSOPT using the image representation of the SOS problem (Balas, Packard, Seiler and Topcu, 2012; Seiler, Zheng and Balas, 2013).

β	$\sigma = \infty$	$\sigma = 6.3$	$\sigma = 1.05$
Q, M	Inf	Inf	Inf
$Q, M(x_k)$	1.273	1.272	0.937
$Q(x_{k-1}), M(x_k, x_{k-1})$	1.275	1.383	1.162

* $Inf \equiv infeasible$

Table 7.1: Comparison of different approaches

The table shows that a linear controller cannot be proved to stabilize the system in region Ω. A polynomial controller $M(x_k)$ using Xu, Xie and Wang (2007) never needs a bound larger than $\sigma \cong 6.3$ (maximum β is approximately the same as in the non-saturation case in second column).

The last row shows that improvement with respect to Xu, Xie and Wang (2007) has been achieved (8.73% increase of β with $\sigma = 6.3$ and 24% with a 6 times lower bound $\sigma = 1.05$). Analyzing the results, it is shown that, without saturation constraints, there is not enough information between past and present states, so there is no improvement over prior literature results. On the other hand, if saturation is too restrictive, the percent improvement over previous work is high, but the proved region remains small.

Computational cost. In order to give a rough idea about the amount of computing resources required for designing a controller with the proposed approach in this paper, Table 7.2 shows the amount of RAM memory, the time spent in the parsing phase and the time employed by the solver to obtain a solution for each of the considered approaches with $\mu = 6.3$.

	Problem Size	RAM	Parser Time	Solver Time
Q, M	774×201	10 Mb	1.19 s	0.96 s
$Q, M(x_k)$	784×201	10 Mb	1.08 s	0.76 s
$Q(x_{k-1})$ $M(x_k, x_{k-1})$	21041×3320	230 Mb	25.84 s	65.26 s

Table 7.2: Approximate computational resources with the different approaches

The code was executed in an Intel® Core™2 Duo CPU P8600 2.4GHz, 4 Gb DDR3 RAM machine running MATLAB® R2011b with SOSOPT 2.01 and SeDuMi 1.3.

7.4 Conclusions

This chapter develops a convex stabilization design for polynomial systems, which reduces some sources of conservatism in previous literature results. An extension from the classical polynomial Lyapunov function is given based on including delayed states and knowledge about limits on the control input. Furthermore, it is possible to improve solutions by including information about the present and past states in the proposed control law. The percentage improvement in performance with respect to prior results increases as input bounds get smaller. The input bound can be actually considered as a design parameter, with a maximum value given by the physical saturation limits. The methodology has been translated to fuzzy-polynomial systems in a preliminary way.

Future work will be focused on developing refinements for the synthesis design procedure in order to extend the delayed Lyapunov function technique to a more general class of nonlinear systems.

Part III

Experimental Applications

Chapter 8

State Estimation in a 3DoF Electromechanical Platform

No amount of experimentation can ever prove me right; a single experiment can prove me wrong.

Albert Einstein

ABSTRACT: This chapter discusses how the theoretical developments on polynomial observer design, presented previously in this thesis on Chapter 6, apply to an experimental bench. The objective is estimating the unknown (or difficult to measure) speeds in a fixed quadrotor which will be further required for control tasks on such system. This chapter tries to verify if the fuzzy-polynomial methodologies are advantageous over the classical ones from both points of view; modelling accuracy plus complexity and obtained performance.

Mechatronic systems are usually nonlinear because of trigonometric and Coriolis-like terms in their differential equation models. Some of these expressions are polynomials or can be approximated by polynomial ones (or, ultimately, by linear ones with larger approximation errors). Those models can be expressed as time-varying polynomial dynamic systems (Section 3.1). Indeed, there exist an amount of applications on where the nonlinear processes are mainly defined by polynomial nonlinearities (Zhao and Wang, 2008; Peñarrocha, Dolz, Aparicio and Sanchis, 2014). On those cases, the fuzzy-polynomial modelling and control framework addressed on this thesis can be helpful to improve over linear or fuzzy TS classical techniques.

The design of state observers for nonlinear systems using TS (or quasi-LPV) models has been actively considered recently in practical control applications (Lendek, Berna, Guzmán-Giménez, Sala and García, 2011; Lendek, Guerra, Babuška and De Schutter, 2010; Bouarar, Guelton and Manamanni, 2013). The proposed polynomial tools are a generalization of them (quasi-LPV are a particular case of fuzzy polynomial), allowing to obtain better results due to the lower mismodelling possible with polynomial equations.

The objective of this chapter is designing a discrete-time polynomial observer for a three degrees of freedom (3DoF) fixed quadrotor and validating it with experimental data. Experiments are presented to show that fuzzy polynomial modelling and SOS tools can be effectively applied in actual applications. In order to demonstrate the results, this chapter considers the application of the fuzzy polynomial techniques, previously developed in Section 6.3, to angular speed estimation on a quadrotor with noisy sensors. A detailed comparative analysis between fuzzy TS and non-fuzzy (linearised) techniques is reported too.

Furthermore, in order to get sensible solutions, the chapter discusses a practical design which proposes a tradeoff between tracking speed versus $\mathcal{H}_\infty$ worst-case attenuation in a multicriteria setting. The concrete goals are:

1. Developing multiobjective $\mathcal{H}_\infty$ plus decay-rate observers for fuzzy polynomial systems, relaxing conservativeness by adding local information.

2. Adding extra constraints regarding the fastest components of the dynamics, in order to avoid undesirable behaviors in practice due to approximate discretization.

3. Proposing a practical methodology to choose from the Pareto-front solutions.

4. Comparing the achieved results with other techniques in literature (with theoretical guarantees absent in linearised designs for nonlinear systems), and validating the results on an experimental benchmark.

The experimental results presented in this chapter have been published in the journal paper Pitarch and Sala (2014).

This chapter is structured as follows: Section 8.1 presents the experimental platform (quadrotor) and its approximate polynomial model, Section 8.2

presents the considered observer design cases (fuzzy polynomial, TS and LTI) for comparison, Section 8.3 discusses the many compromises required to craft a successful engineering application, Section 8.4 presents a comparative analysis between linear, TS and polynomial observers, both theoretically and experimentally. Finally, Section 8.5 provides the conclusions of this experiment.

8.1 Description and modelling of the test setup

The experimental platform chosen to evaluate the performance of the designed observers is a three degrees of freedom (3DoF) system provided by Quanser$^{\text{TM}}$ (Quanser, 2011). The bench platform, shown in Fig. 8.1, consists in a quadrotor mounted on a 3DoF pivot joint, such that the body can freely move in roll, pitch and yaw. The data acquisition tasks and the implementation of required control algorithms were carried out in a PC running Linux-RT on top of an Ubuntu 12.04 installation. The communications between the quadrotor platform and the PC were made with a PMC I/O board. The sensors of the platform are encoders that measure the position of the three orientation-axes of the quadrotor ϕ, θ and ψ. The control inputs are the voltages V_1, V_2, V_3 and V_4 applied to each of the 4 propellers of the quadrotor. High-resolution encoders are available to estimate the "true" speeds, but intentionally, a random noise has been added to them in order to evaluate the behavior with much lower-quality sensors. Indeed, the objective of the experiment is testing the differences between linear and polynomial approaches in far-from-ideal cases: as expected from common sense, if sensors were "excellent" then any optimal observer design would, basically, disregard the (more uncertain) model equations.

The author is grateful to Ph.D. students A. Berna, J. Guzmán and associate professor P.J. García for their laboratory data acquisition work.

An approximate non-linear model of the 3DoF platform is presented in the following equations giving accelerations in roll, pitch and yaw coordinates, as given in Bouabdallah (2007):

$$\begin{aligned}
\ddot{\phi} &= \frac{J_r\dot{\theta}}{I_{xx}}u_g + \frac{I_{yy}-I_{zz}}{I_{xx}}\dot{\theta}\dot{\psi} + u_1 \\
\ddot{\theta} &= \frac{J_r\dot{\phi}}{I_{xx}}u_g + \frac{I_{zz}-I_{xx}}{I_{yy}}\dot{\psi}\dot{\phi} + u_2 \\
\ddot{\psi} &= \frac{I_{xx}-I_{yy}}{I_{zz}}\dot{\theta}\dot{\phi} + u_3
\end{aligned} \tag{8.1}$$

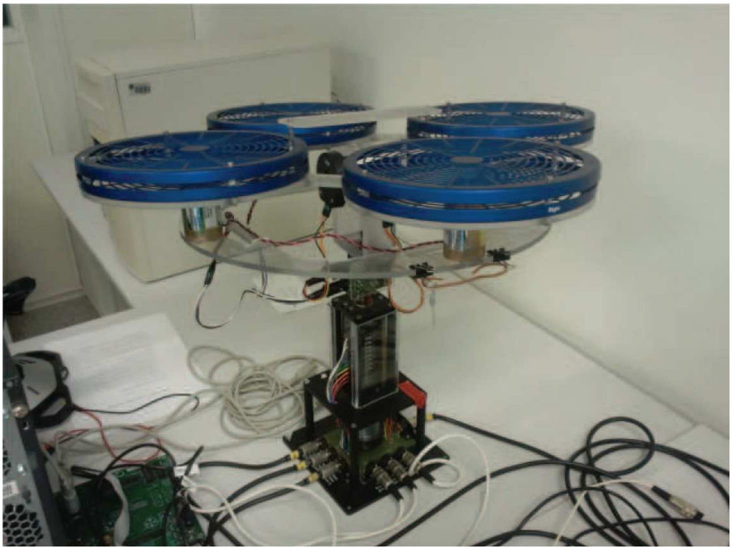


Figure 8.1: The Quanser quadrotor 3DoF system.

where the gyroscopic effects in the roll and pitch dynamics contain the term

$$u_g(V_1, \ldots, V_4) = K_v(V_1 + V_3 - V_2 - V_4) \tag{8.2}$$

which is the sum of the applied (known) voltages. Furthermore, each acceleration input (u_1, u_2, u_3) from the nonlinear propellers' actuation, depends on the applied voltages as follows [1]:

$$\begin{aligned} u_1 &= \frac{blK_v^2(V_2^2 - V_4^2)}{I_{xx}} \\ u_2 &= \frac{blK_v^2(V_3^2 - V_1^2)}{I_{yy}} \\ u_3 &= \frac{dK_v^2(V_1^2 - V_2^2 + V_3^2 - V_4^2)}{I_{zz}} \end{aligned} \tag{8.3}$$

The symbols used and their values, where applicable, are given in Table 8.1 (extracted from Quanser (2011)). The terms $\frac{J_r\dot{\theta}}{I_{xx}}K_v(V_1 + V_3 - V_2 - V_4)$ and $\frac{J_r\dot{\phi}}{I_{xx}}K_v(-V_1 - V_3 + V_2 + V_4)$ denote gyroscopic effects. The terms $\frac{I_{yy}-I_{zz}}{I_{xx}}\dot{\theta}\dot{\psi}$, $\frac{I_{zz}-I_{xx}}{I_{yy}}\dot{\psi}\dot{\phi}$, $\frac{I_{xx}-I_{yy}}{I_{zz}}\dot{\theta}\dot{\phi}$, denote Coriolis effects.

[1]The square is shorthand for $\text{sign}(V_i)V_i^2$ to account for upwards and downwards thrust.

Symbol	Meaning	Type	Unit
ϕ	Roll angle	Measured	rad
$\dot{\phi}$	Roll angular velocity	Estimated	rad/s
θ	Pitch angle	Measured	rad
$\dot{\theta}$	Pitch angular velocity	Estimated	rad/s
ψ	Yaw angle	Measured	rad
$\dot{\psi}$	Yaw angular velocity	Estimated	rad/s
V_i	Voltage to propeller i	Known input	V
K_v	Transformation constant	54.945	rad s/V
J_r	Rotators inertia	$6 \cdot 10^{-5}$	kgm^2
I_{xx}	Inertia X-axis	0.0552	kgm^2
I_{yy}	Inertia Y-axis	0.0552	kgm^2
I_{zz}	Inertia Z-axis	0.1104	kgm^2
b	Thrust coefficient	$3.9351 \cdot 10^{-6}$	N/Volt
d	Drag coefficient	$1.1925 \cdot 10^{-7}$	Nm/Volt
l	Distance pivot-motor	0.1969	m
m	Mass	2.85	kg
g	Gravity acceleration	9.81	$\mathrm{m/s}^2$
T_s	Sampling time	0.005	s

Table 8.1: Quadrotor variables and parameters

8.1.1 Fuzzy modelling

Fuzzy-polynomial model. The bounds on the term u_g can be computed based on the bounds of the voltage input, and they are a lower bound $\underline{u_g} = 4K_v V_{\min}$ and an upper one $\overline{u_g} = 4K_v V_{\max}$. Following well-known procedures (Rugh and Shamma, 2000), if we define the weighting functions

$$\mu_1(V_1, \ldots, V_4) = \frac{\overline{u_g} - u_g(V_1, \ldots, V_4)}{\overline{u_g} - \underline{u_g}} \tag{8.4}$$

$$\mu_2(V_1, \ldots, V_4) = 1 - \mu_1(V_1, \ldots, V_4) \tag{8.5}$$

then, the term u_g is expressed as $u_g = \mu_1 \underline{u_g} + \mu_2 \overline{u_g}$. Note that the arguments to μ_i and u_g are omitted for brevity.

With the above input voltage bounds and taking $x = (\phi, \dot{\phi}, \theta, \dot{\theta}, \psi, \dot{\psi})$ as states, denoting as premise variables $z(t) = (V_1, \ldots V_4)$, the quadrotor model (8.1) can be expressed at sample time $T_s = 5$ ms (by Euler-discretization method) in the form:

$$
\begin{aligned}
x_{k+1} &= \sum_{i=1}^{2} \mu_i(z_k)\mathcal{P}_i(x_k) + \mathcal{B}(u_k) + T_s E w_k \\
y_k &= C(x_k) + R\nu_k
\end{aligned} \tag{8.6}
$$

where x_k, z_k denote the samples $x(kT_s)$, $z(kT_s)$, model matrices are given by

$$
\begin{aligned}
\mathcal{P}_1(x_k) &= \begin{bmatrix} x_{1_k} + 0.005x_{2_k} \\ x_{2_k} + 0.012x_{4_k} - 0.005x_{4_k}x_{6_k} \\ x_{3_k} + 0.005x_{4_k} \\ x_{4_k} - 0.012x_{2_k} + 0.005x_{2_k}x_{6_k} \\ x_{5_k} + 0.005x_{6_k} \\ x_{6_k} \end{bmatrix}, \\
\mathcal{P}_2(x) &= \begin{bmatrix} x_{1_k} + 0.005x_{2_k} \\ x_{2_k} - 0.012x_{4_k} - 0.005x_{4_k}x_{6_k} \\ x_{3_k} + 0.005x_{4_k} \\ x_{4_k} + 0.012x_{2_k} + 0.005x_{2_k}x_{6_k} \\ x_{5_k} + 0.005x_{6_k} \\ x_{6_k} \end{bmatrix},
\end{aligned}
$$

$$
\mathcal{B}(u_k) = \begin{bmatrix} 0 \\ 0.005u_{1_k} \\ 0 \\ 0.005u_{2_k} \\ 0 \\ 0.005u_{3_k} \end{bmatrix}, \; C(x_k) = \begin{bmatrix} x_{1_k} \\ x_{3_k} \\ x_{5_k} \end{bmatrix},
$$

$x \in \mathbb{R}^6$ is the state, $u \in \mathbb{R}^3$ is the control input, $y \in \mathbb{R}^3$ is the output and $w \in \mathbb{R}^6$, $\nu \in \mathbb{R}^3$ are *unknown* process disturbance inputs and measurement noise respectively. Matrices E and R are scaling matrices for expected disturbance and noise powers respectively, later specified. In this way, we may assume that the norm of the vector $[w^T \; \nu^T]^T$ is 1 in later designs without loss of generality.

The speeds of the quadrotor $\eta = (\dot\phi, \dot\theta, \dot\psi) = (x_2, x_4, x_6)$ will be assumed to lie in the operating region:

$$
\Omega_x = \{x_2, x_4, x_6 : \eta^T\eta \le R_{max}^2\}, \tag{8.7}
$$

with $R_{max}^2 = 3\left(\frac{\pi}{2}\right)^2$. The above bound include a hypercube centered in the origin spanning the interval $[-\pi/2, \pi/2]$ on each coordinate.

Takagi-Sugeno model. In order to compare the experiment with existent results in TS fuzzy literature, the quadrotor dynamics (8.1) was also modeled as a TS system (linear consequents):

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{8} \mu_i(z_k)\mathcal{A}_i x_k + \mathcal{B}u_k + T_s E w_k \\ y_k &= C x_k + R\nu_k \end{aligned} \tag{8.8}$$

The model had 8 vertices, because the Coriolis terms (products of angular speeds), say for instance $\dot{\phi}\dot{\theta}$, had to be modelled as a TS model interpolating between vertices $\mathbf{p}\dot{\theta}$ and $\mathbf{q}\dot{\theta}$ where $\mathbf{p}$ and $\mathbf{q}$ are constants denoting *a priori* chosen speed bounds $\dot{\phi}_{max}$ and $\dot{\phi}_{min}$, respectively; the resulting membership function depends on $\dot{\phi}$. Details on the model and procedure can be consulted in Lendek, Berna, Guzmán-Giménez, Sala and García (2011).

Linear model. In order to compare with standard linear time-invariant techniques, the linearized model of the quadrotor around $x = 0$, discretized by Euler method (as the nonlinear model) for $T_s = 5$ ms, is:

$$\begin{aligned} x_{k+1} &= \mathcal{A}x_k + \mathcal{B}u_k + T_s E w_k \\ y_k &= C x_k + R\nu_k \end{aligned} \tag{8.9}$$

where:

$$\mathcal{A} = \begin{bmatrix} 1 & 0.005 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.005 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.005 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0.005 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.005 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.005 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

8.2 Observer design

This section discusses the observer design for the particular case of the quadrotor discrete-time model. First, by following the design methodologies presented in Chapter 6, and given the above presented fuzzy models of the quadrotor, a discrete fuzzy-polynomial observer (6.4) will be addressed. Afterwards, TS and linear observers will be addressed too in order to compare with the fuzzy-polynomial one.

Fuzzy polynomial observer. In order to carry out observer design, the observer error $e_k = x_k - \hat{x}_k$ follows the equation (6.20). The observer speed error ($e_s = \eta - \hat{\eta}$ where $\hat{\eta}$ are the estimated speeds) is assumed to lie inside the same sphere as the actual speed, see (8.7), i.e.:

$$\Omega_e = \{e_s^T e_s \leq R_{max}^2\}. \tag{8.10}$$

From the above considerations, on the following we will assume that the state and estimation error do not leave expected operation regions $e \in \Omega_e$ and $x \in \Omega_x$, being Ω_e and Ω_x described in (8.10) and (8.7) respectively.

Then, by considering a common quadratic candidate Lyapunov function on the error in the form $V(e) = e^T Q e$ and the above particular operating regions for the quadrotor, a fuzzy polynomial observer (6.4) can be designed by Theorem 6.2.

Remark 8.1. In the particular fuzzy-polynomial model of the quadrotor (8.6), as the arguments of μ_i are measurable (z are the input voltages), on-line implementation of (6.4) can be carried out with very low computational requirements.

Fuzzy TS observer. With some manipulations, existent results in fuzzy TS observer literature can be obtained as a particular case in a straightforward way. Note, however, that in the plant in consideration in this application, TS modelling has an important side effect: the observer problem gets converted to one with *unmeasurable premises*, as the memberships must be evaluated with speed estimates $\hat{\eta}$ instead of the actual ones η. This did not occur in the polynomial case, and it will lead to an important performance decrease as discussed in the experimental results (Section 8.4) and will require theoretical refinements, addressed next.

With the TS model (8.8), the observer error dynamics is given by

$$e_{k+1} = \sum_{i=1}^{8} \mu_i(\hat{x}_k)((\mathcal{A}_i - L_i C)e_k - L_i R \nu_k) + T_s E w_k + \sum_{i=1}^{8} (\mu_i(z_k) - \mu_i(\hat{z}_k))\mathcal{A}_i x_k \quad (8.11)$$

and the observer-model mismatch must fulfill a Lipschitz-like bound

$$\left\| \sum_{i=1}^{8} (\mu_i(z) - \mu_i(\hat{z}))\mathcal{A}_i x \right\| \le \sigma \|e\| \quad (8.12)$$

in order to proceed further and set up LMI's (Lendek, Berna, Guzmán-Giménez, Sala and García, 2011; Lendek, Guerra, Babuška and De Schutter, 2010). The bound σ depends on the shape of the memberships (actually, bounds in $\|\frac{\partial \mu}{\partial x}\|$), the model $\mathcal{A}_i$ and the modelling region Ω_x. For this quadrotor case, the taken bound is $\sigma = 0.003$. See the above cited references for details about how to compute it.

Then, if an analogue development to Theorem 6.2 is carried out for TS models (adding, however, the Lipschitz-bound construction), the SOS results, as a particular case, get converted to the classical LMIs (basically, similar to the ones reported in Lendek, Berna, Guzmán-Giménez, Sala and García (2011); Lendek, Guerra, Babuška and De Schutter (2010)) given in the following corollary (details omitted for brevity):

Corollary 8.1. *If the quadrotor's TS model* (8.8) *is used and non-polynomial fuzzy observer gains L_i are to be designed, the observer is stable if the following LMI problem is feasible for $\alpha > 0$ fixed and $i = 1, \ldots, 8$:*

Minimize γ such that

$$Q \succ 0 \quad (8.13)$$

$$\begin{bmatrix} Q - D^T D - \tau_1 \sigma^2 I & (*) & 0 & 0 & 0 \\ Q\mathcal{A}_i - H_i C & Q & (*) & (*) & Q \\ 0 & T_s E^T Q & \gamma I & 0 & 0 \\ 0 & -R^T H_i^T & 0 & \gamma I & 0 \\ 0 & Q & 0 & 0 & \tau_1 I \end{bmatrix} \succ 0 \quad (8.14)$$

$$\begin{bmatrix} \exp(-2\alpha T_s)Q - \tau_2 \sigma^2 I & (*) & 0 \\ Q\mathcal{A}_i - H_i C & Q & Q \\ 0 & Q & \tau_2 I \end{bmatrix} \succ 0 \quad (8.15)$$

$$\mathcal{A}_i^T Q - C^T H_i^T + Q\mathcal{A}_i - H_i C \succ 0 \tag{8.16}$$

where, $\tau_1 > 0$ and $\tau_2 > 0$ are Lagrange multipliers, $\gamma > 0$, and the fuzzy observer gains can be obtained by $L_i = Q^{-1}H_i$.

LTI observer design. Furthermore, if the quadrotor's linearized model (8.9) is used ($\mathcal{A}_i = \mathcal{A}$), the well-known LTI observer L design (with the same performance criteria) trivially results from setting $\sigma = 0$ in the above corollary, which also leads to Corollary 6.1.

8.3 Design compromises in practice

There are some issues to be discussed in order to obtain acceptable responses in practice from the above observer design techniques, addressed next.

8.3.1 Disturbance rejection vs. decay trade-off

In many practical cases, such as the application here discussed, there is a trade-off between different relevant aspects:

- *Performance*, i.e., trying to maximize decay-rate parameter α in (6.29) for fast convergence from nonzero initial conditions. This point is very important from the tracking point of view problem.
- *Worst-case* disturbance rejection, i.e., *robustness*, as $\mathcal{H}_\infty$ bounds can be understood as robustness margins to unstructured time-varying uncertainty via the small-gain theorem (Khalil, 2002).
- *Actual-case* disturbance rejection (i.e., actual mean error performance with the random noises, operating ranges and modelling errors of a test experiment).

Note that $\mathcal{H}_\infty$ disturbance rejection design is minimizing the effect of the "worst-case" disturbance (which can be either an external disturbance or a modeling error). However, it does not directly minimize the effect of "actual" disturbances (particularly, zero mean *random* noise).

From the above considerations, a better $\mathcal{H}_\infty$ attenuation bound (in the design phase) may result in, for instance, a larger accumulated integral squared error index (ISE) in a particular experiment (compared to an alternative design with theoretically worse $\mathcal{H}_\infty$ bound).

Moreover, in order to minimize the effect of the worst-case disturbance, the obtained gains may be very small so achieved decay performance is very bad. On the other hand, decay-rate optimization does not take into account the affordable amount of risk if the worst-case disturbance appears. Also, desired decay-rate performance is very related to the amount of random noise in sensor readings.

Figures 8.2a and 8.2b illustrate the above-discussed extremal cases[2]. For instance, with a noisy sensor, the optimal $\mathcal{H}_\infty$ estimated speeds are smooth (noise is very-well filtered) but with an unacceptable error from tracking point of view (Figure 8.2a). If a fast decay rate is required, the estimated speeds are also useless because there is a high amount of accumulated error due to noise spikes (Figure 8.2b).

Therefore, as there is a trade-off, there is no single "optimal" observer design, but a collection of optimal ones (the multicriteria Pareto front (Marler and Arora, 2004; Miettinen, 1999)). For instance, the Pareto front can be built by; (a) providing the fastest decay rate for a given robustness bound, (b) conversely, the better robustness bound for a given decay rate or (c) a weighted combination of both following some importance criteria.

Hence, it's a choice of the end-user where to lie in the tradeoff: in practice both extreme designs may be useless (Figure 8.2) whereas an intermediate one (Figure 8.3) will be satisfactory[3] for the "actual case" performance.

8.3.2 Proposed methodology

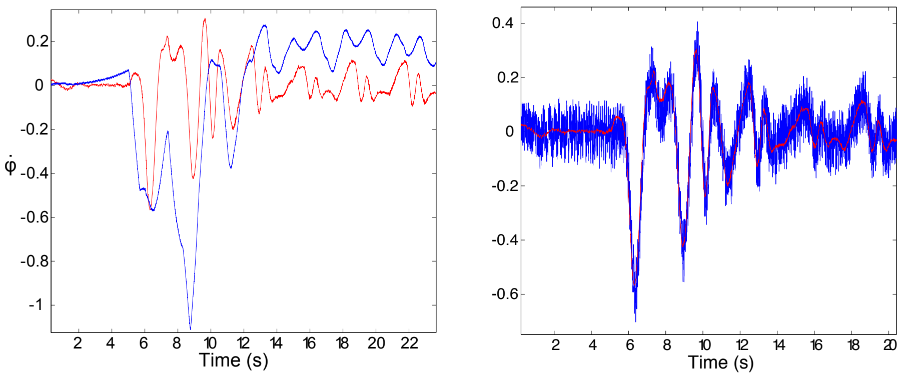
The above design compromises suggest the following methodology in order to select, in practice, a particular observer:

I. As the actual disturbances, model errors and the to be tracked signal bandwidth in practical operation are not known with precision at design time, obtain a whole Pareto-front for multicriteria $\mathcal{H}_\infty$ plus decay-rate settings.

II. Carry out a representative experiment and collect input-output data.

[2]Data come from the actual experimental platform, but the discussion in this section applies to more general settings.

[3]The referred figures have been obtained with the experiments and datasets discussed in Section 8.4.



(a) Slow decay (lowest $\mathcal{H}_\infty$ bound) (b) Spiky estimation (fast decay)

Figure 8.2: Useless speed estimates in the Pareto front. Red line: actual speed, blue line: estimated speed.

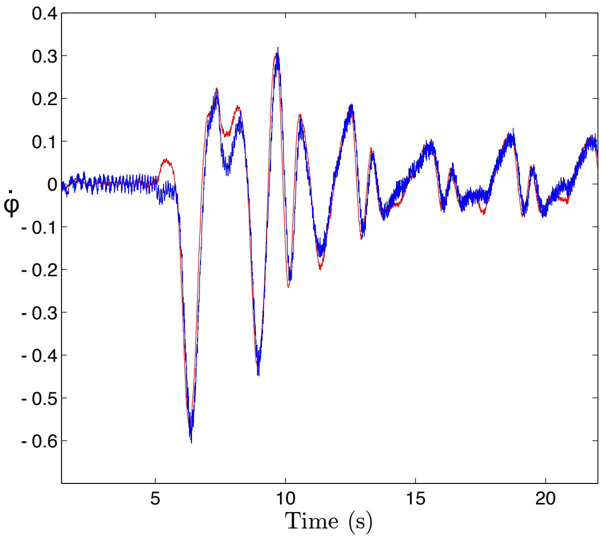


Figure 8.3: Acceptable speed estimates (same sensor data as Figure 8.2).

III. Test the different observers in the Pareto front over the same data (single experiment from step II).

IV. Evaluate achieved performance (see details on the particular setting for the 3DoF system in Section 8.4.2.2).

V. Choose the design which has achieved the best experimental performance with the “actual case” disturbances and modelling errors.

In this way, the “practically” optimal solution is selected from a set of multiple “theoretically” optimal ones.

8.3.3 Choice of disturbance size parameters

Even if the presented results are correct in theory, there is no clear rule on how to choose observer design parameters E, R, D. Indeed, the main idea is that the resulting time response of the observer error will depend on:

- The proportion between process noise/model error E and sensor noise/-modelling error R.
- The accuracy of the model (expecting polynomial models to be more accurate than plain linearized ones).
- The relative proportions between the elements of D: giving more weight to some position or speed errors trades off a larger error for the less weighted variables.
- The decay-rate parameter discussed above.

For simplicity, from the above considerations, it is suggested to choose simple structures for the design matrices (such as diagonal), and tune actual performance of the observer via the decay-rate parameter instead of devoting too much time in modifying such matrices.

8.4 Experimental results

The above results have been experimentally tested on the platform described in Section 8.1.

The objectives of the experiment are; (a) showing that, with actual experimental data, reasonable solutions regarding the performance/robustness are achieved with the proposed polynomial design and (b) showing that the use of polynomial techniques dominate the non-polynomial ones based on fuzzy TS models or on linearised models.

The expected disturbance, modelling errors and noise sizes on this application, jointly with considerations discussed in Section 8.3, made us select the following scaling matrices in order to run the optimization problem:

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0.02 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.02 \end{bmatrix} \quad R = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}$$

Indeed, by definition process disturbances on the speed dynamic equations do not exist. The constant weighting matrix D in (6.26) is chosen as follows

$$D = diag(0.001, 0.1, 0.001, 0.1, 0.001, 0.1);$$

in order to take into account speed estimation errors as the objective of the experiments.

A Pareto front with different decay rates, obtaining different suboptimal $\mathcal{H}_\infty$ norm bounds, is made for different observer designs (linear, TS and polynomial), being $[y_k - C\hat{x}_k]$ the output estimation error at sample k.

Note that the decay-rate parameter is used to tune the aggressiveness of the observer instead of the manipulation of the disturbance-size design weights[4].

The compared observer design strategies were:

- **[LM-D]**: $\hat{x}_{k+1} = \mathcal{A}\hat{x}_k + \mathcal{B}u_k - L[y_k - C\hat{x}_k]$, classical linear observer design with linearized model (Corollary 6.1 with $\mathcal{A}_i = \mathcal{A}$, $\mu = 0$). Without theoretical guarantees in practice (only for infinitesimally small disturbances and initial conditions on state and estimated state).

- **[LM-A]**: $\hat{x}_{k+1} = \sum_{i=1}^{2} \mu_i \mathcal{A}_i(\hat{x}_k) + \mathcal{B}(u_k) - L[y_k - C(\hat{x}_k)]$, analyzes the above LM-D obtained linear gain with the actual fuzzy polynomial model (Theorem 6.2 with fixed $L_i(y_k, \hat{x}_k) = L$). This yields *a posteriori* guaranteed performance in all operating region.

- **[TS-8GD]**: $\hat{x}_{k+1} = \sum_{i=1}^{8} \mu_i(\mathcal{A}_i\hat{x}_k - L_i[y_k - C\hat{x}_k]) + \mathcal{B}u_k$, optimal LMI design with fuzzy Takagi-Sugeno model (Corollary 6.1) and 8 fuzzy observer gains L_i. Guaranteed *a priori* performance in all operating region.

[4] Actually, quite a few other values for those matrices were tested, but they are not reported because results are very similar to the ones presented (once the suitable Pareto-front decay-rate tuning is accounted for).

- **[TS-1GD]**: $\hat{x}_{k+1} = \sum_{i=1}^{8} \mu_i(\mathcal{A}_i\hat{x}_k - L[y_k - C\hat{x}_k]) + \mathcal{B}u_k$, optimal LMI design with fuzzy TS model (Corollary 6.1) and *one* observer gain $L_i = L$. Guaranteed *a priori* performance in all operating region.

- **[TS-1GA]**: $\hat{x}_{k+1} = \sum_{i=1}^{2} \mu_i\mathcal{A}_i(\hat{x}_k) + \mathcal{B}(u_k) - L[y_k - C(\hat{x}_k)]$, analyzes the above obtained single-gain observer with the fuzzy polynomial model (Theorem 6.2 with fixed $L_i(y_k, \hat{x}_k) = L$). *A posteriori* guaranteed performance in the operating region.

- **[SOS-2G]**: $\hat{x}_{k+1} = \sum_{i=1}^{2} \mu_i(\mathcal{P}_i(\hat{x}_k) - L_i[y_k - C(\hat{x}_k)]) + \mathcal{B}(u_k)$, optimal SOS design with fuzzy polynomial model (Theorem 6.2), and 2 non-polynomial[5] observer gains. Guaranteed *a priori* performance in all operating region.

Computational cost. In the off-line gain computation, with the chosen quadrotor case, Table 8.2 shows the demanded computational resources on a machine with Windows® XP, Intel Pentium®III at 640 MHz and 512 Mb of RAM. Only the two more demanding alternatives (TS-8GD and SOS-2G with degree of multipliers s, q equal to 2) are shown for brevity. The results were obtained using MATLAB® 6.5 R13, YALMIP R20110318 and SeDuMi 1.21.

	Problem size	RAM	YALMIP time	Solver time
SOS-2G	10262×1180	140 Mb	31.48 s	20.05 s
TS-8GD	11596×168	4 Mb	2.17 s	13.02 s

Table 8.2: Approximate computational resources for design.

8.4.1 Pareto-front results

The results for the six different strategies are shown in Table 8.3. The corresponding Pareto fronts for each design strategy are depicted in Figure 8.4a and their corresponding guaranteed performance on the whole operating region are depicted in Figure 8.4b. Basically, looking at the latter figure, the

[5] Polynomial techniques presented in Section 8.2 allow us to design polynomial observer gains $L_i(y, \hat{x})$. However, in this quadrotor particular case the use of polynomial terms in observer gains didn't seem to obtain better results than those with constant gains L_i, above reported.

α	**1.5**	**6**	**10**	**14**	**18**	**22**
LM-D	1.628	6.45	11.06	15.89	20.9	26.09
TS-8GD	13.94	30.64	50.23	73.95	102.87	138.65
TS-1GD	14.89	32.63	52.87	77.35	107.25	144.46
SOS-2G	5.986	9.915	14.36	19.18	24.26	29.57
LM-A	22.37	39.44	31.34	30.64	33.07	36.93
TS-1GA	9.708	22.3	35.3	49.71	65.65	83.21

Table 8.3: Set of optimal solutions in the Pareto-front sense with different approaches (rows 1-4). Rows 5-6: *a posteriori* analysis results of designs in rows 1 and 3.

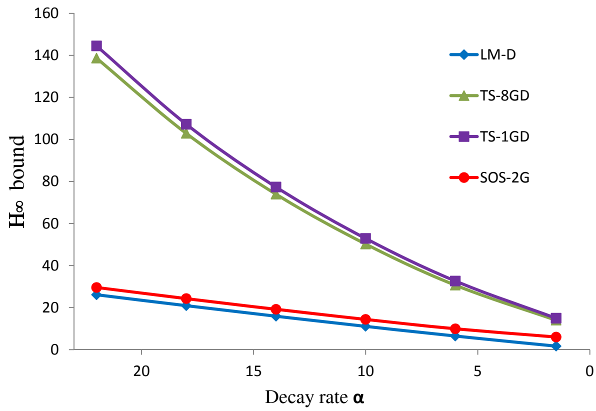
overall conclusion is that the fuzzy-polynomial approach dominates the linear and TS alternatives in both performance criteria.

For better clarity, let us discuss in detail the meaning of all the data in the referred table and figures:

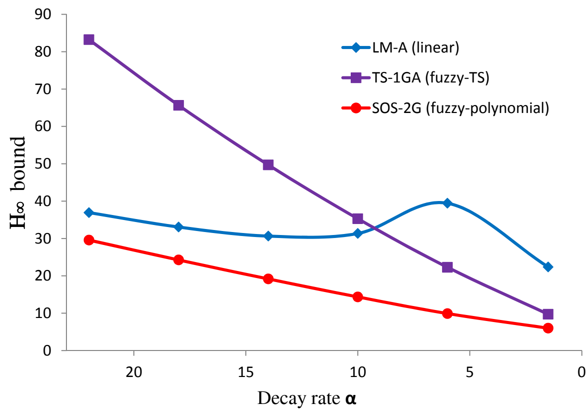
- LM-D performances (row 1) assume that the process is linear, valid when the operation region is infinitesimal. Hence, they are meaningless, overly optimistic. LM-A (row 5) evaluates the design (a posteriori) in the non-infinitesimal region of study with polynomial models, and results are more meaningful (markedly *worse*, as expected). The blue line in Figure 8.4a depicts the meaningless linear performance, whereas the same line in Fig. 8.4b depicts the results in row 5.

- Performances obtained with the TS model and 8 fuzzy gains (TS-8GD, row 2) dominate those obtained with single gain (TS-1GD, row 3) in the Pareto sense, as intuitively expected (the table data appear in Figure 8.4a with green and purple lines, respectively). However the improvement of using fuzzy observer gains is not very significant, so the choice of using a single gain appears to be justified in the present application: a posteriori evaluation is carried out only for TS-1GD as TS-8GD gives very similar results.

- Valid TS designs (TS-8GD, TS-1GD) are very conservative due to the need of fulfilling the Lipschitz condition (8.12). This can be noted by comparing them with the fuzzy polynomial design results (SOS-2G, row 4 in Table 8.3 and orange lines in Figure 8.4), even in an *a posteriori*



(a) Pareto front for each observer design strategy



(b) Theoretically guaranteed performance for the *linear*, *fuzzy-TS*, and *fuzzy-polynomial* designs on the whole operating region.

Figure 8.4: Representation of results in Table 8.3.

analysis (TS-1GA). Indeed, the purple TS-1GA line in Figure 8.4b is dominated by the orange SOS-2G one.

- A posteriori performance of TS designs outperform linear ones for low decay requirements, as comparing rows 5 and 6 in Table 8.3 and its corresponding (blue and purple) curves in Figure 8.4b illustrates. However, for high decays, the need of ensuring Lyapunov constraints with the Lipschitz bound (8.12) seems to make the TS design very conservative.

- Using the SOS-2G design (row 4), the worst-case disturbance attenuation is always better than the obtained with all other LMI strategies which ensure *guaranteed* performance for the same decays (i.e., rows 2 and 3), also better than the polynomial analysis of LM and TS-1G in rows 5 and 6. Hence, the theoretical solutions of the SOS strategy dominate linear and TS ones (see Figure 8.4b), as intuitively expected.

Based on the above considerations, the best chosen design will be the fuzzy polynomial model and SOS observer design approach (Table 8.3, SOS-2G), as clearly shown in Figure 8.4b. The theoretical results must, however, be confronted to actual experimental performance, as done next.

8.4.2 Evaluation of final design

Considering the above compromises, this section presents the final chosen observer striking a good performance/robustness balance for this application and evaluates it on the experimental platform.

8.4.2.1 Data generation.

With the objective of validating the SOS proposed approach, the system has been subjected to an excitation achieving large enough angular speeds for the nonlinear terms to be significant. Hence, a sinusoidal excitation was introduced in ψ from second 5 till 60 and a reference in θ and ϕ changes every 30 seconds from 10 to -10 degrees to an underlying low-gain stabilizing PI controller, providing excitation in these degrees of freedom. The collected input-output data appear in Figures 8.5 and 8.6.

The initial conditions were close to the linearization point, and in the first 5 seconds no input excitation has been applied. The set of data has been obtained during a device maneuver using a direct low-noise encoder. This data confirms that the system states satisfy the bounds from Section 8.1.

The objective is to check performance of observer designs in a demanding environment (otherwise, everything works perfectly if sensors are very good). In order to do that, the encoder signals have been intentionally corrupted with random noise (variance 0.001 rad) plus a chirp sinusoidal signal (amplitude $8 \cdot 10^{-4}$ rad and frequency varying between 0.01 and 10 Hz in cycles of 5 sec) simulating a deterministic decalibration.

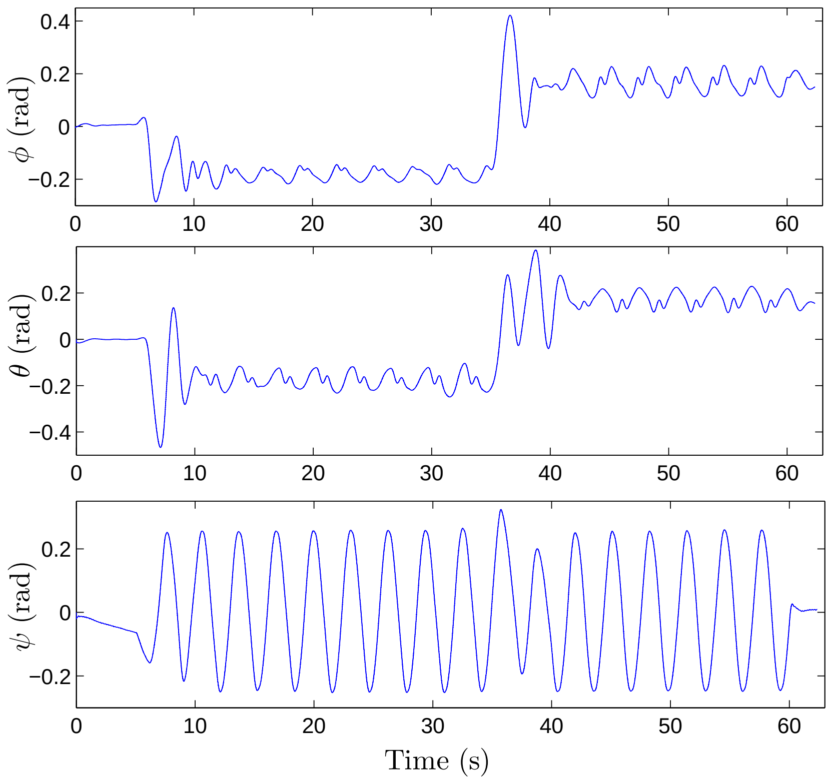


Figure 8.5: Measurements

8.4.2.2 Evaluation of observer performance

Given the Pareto-front designs in Table 8.3, the problem now is selecting one of them as the chosen design for the application. Let us follow the methodology in Section 8.3. In order to evaluate observer performance, a "precise" state trajectory is required. With that trajectory, actual observer performance will be evaluated in terms of integral square error (ISE).

The direct low-noise encoder measurements have been used for that task. The position estimates are precise, but, as there is no direct access to the speed state variables, a noncausal zero-phase filter (`filtfilt` in MATLAB® with $0.5/(1-0.5z^{-1})$ in forward and reverse time, plus further noncausal numerical differentiation $(z-z^{-1})/(2T_s)$ in the speed coordinates) has been used to com-

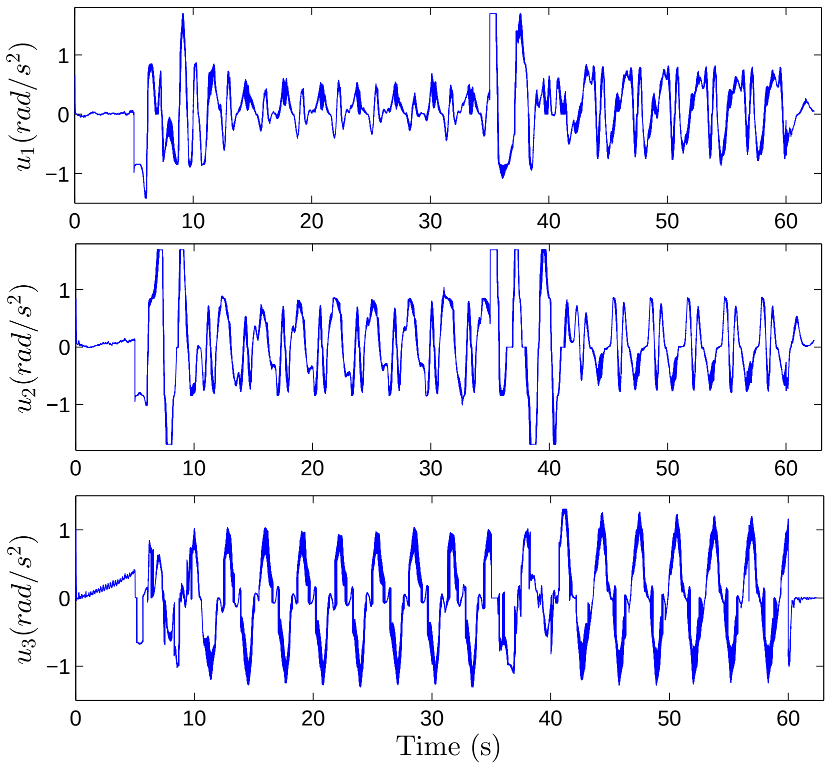


Figure 8.6: Excitations

pute (Merry, de Molengraft and Steinbuch, 2010) (off-line) a target "actual" value of speeds from *clean* position measurements (data in Figure 8.5). The resulting data have been assumed to be the "true" speeds to which observers should converge (note that observers are under the constraints of dealing with noise and being causal).

Evaluating the observer ISE, we finally selected the options with decay rate of $\alpha = 18$: it offered a good enough response from tracking point of view and a reduced ISE compared to other Pareto-front candidate solutions.

8.4.2.3 Adopted solution

Let us compare the LM-A option (designed with linearised model) which, in theory, gives guaranteed performance ($\alpha = 18, \mathcal{H}_\infty = 33.07$), with the fuzzy

TS single-gain design TS-1GA ($\alpha = 18, \mathcal{H}_\infty = 65.65$) and the polynomial SOS-2G design ($\alpha = 18, \mathcal{H}_\infty = 24.264$). Note that, thanks to the polynomial techniques, the SOS designs are *more robust* than the linear or TS ones from a theoretical point of view (lower $\mathcal{H}_\infty$ bound for the same decay rate). The selected observer designs are tested and compared on the quadrotor. The obtained gains are:

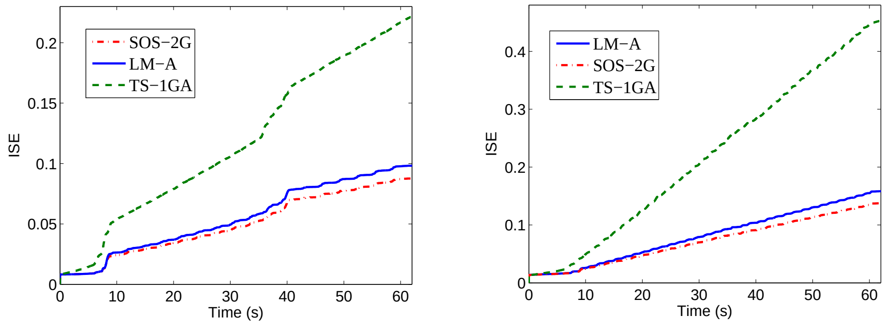
$$L_{LM-A} = \begin{bmatrix} 0.713 & 0 & 0 \\ 11.253 & 0 & 0 \\ 0 & 0.713 & 0 \\ 0 & 11.253 & 0 \\ 0 & 0 & 0.713 \\ 0 & 0 & 11.253 \end{bmatrix}$$

$$L_{TS-1GA} = \begin{bmatrix} 1.163 & 0 & 0 \\ 52.013 & 0 & 0 \\ 0 & 1.163 & 0 \\ 0 & 52.013 & 0 \\ 0 & 0 & 1.191 \\ 0 & 0 & 53.61 \end{bmatrix}$$

$$\begin{aligned} L(1)_{SOS-2G} &= \begin{bmatrix} 0.73 & -0.062 & 0 \\ 12.51 & -0.889 & 0 \\ 0.062 & 0.73 & 0 \\ 0.889 & 12.51 & 0 \\ 0 & 0 & 0.82 \\ 0 & 0 & 14.238 \end{bmatrix} \\ L(2)_{SOS-2G} &= \begin{bmatrix} 0.73 & 0.062 & 0 \\ 12.51 & 0.889 & 0 \\ -0.062 & 0.73 & 0 \\ -0.889 & 12.51 & 0 \\ 0 & 0 & 0.82 \\ 0 & 0 & 14.238 \end{bmatrix} \end{aligned}$$

The following evaluates how the theoretical advantage translates into experimental behaviour.

The accumulated integral square errors (ISE) are shown in Figure 8.7. The LM-A and TS-1GA observers have the same decay rate performance as the optimal SOS-2G. However, the improvement of the SOS polynomial design with respect to the alternatives is confirmed in this experiment (12.5% less ISE than LM-A and, roughly, less than half than TS-1GA). Note that the non-measurable premise TS-1GA setup ($\sigma = 0.003$) results in an overly conservative setup whose theoretical performance and experimental ISE are far higher than the other setups[6].



(a) Pitch speed ISE (b) Yaw speed ISE

Figure 8.7: Experimental ISE computation

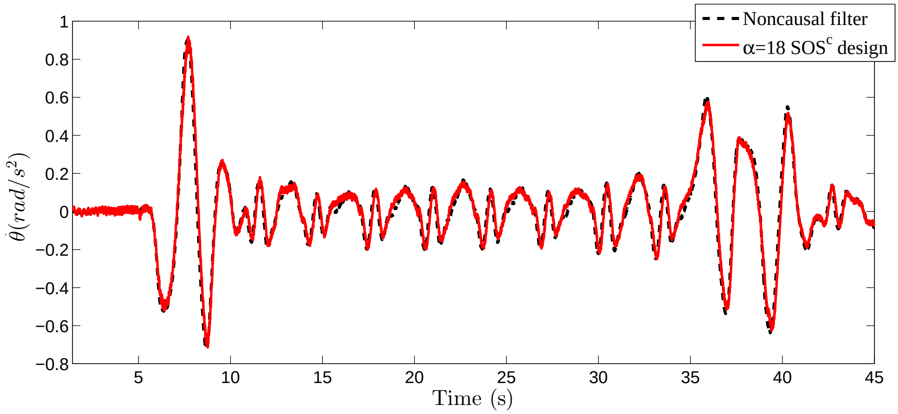
In summary, the results show that:

- The polynomial methodology clearly outperforms the TS design with unmeasurable premises TS-1GA.
- The polynomial setup obtains the best theoretically guaranteed worst-case $\mathcal{H}_\infty$ bound (both in the *a priori* formal design and in the *a posteriori* stability analysis step), see Table 8.3.
- In actual-case performance the linear-only and polynomial observer setups achieve similar performance, beating the TS observer[7]. Note also that this experiment is likely not the "worst case" situation.

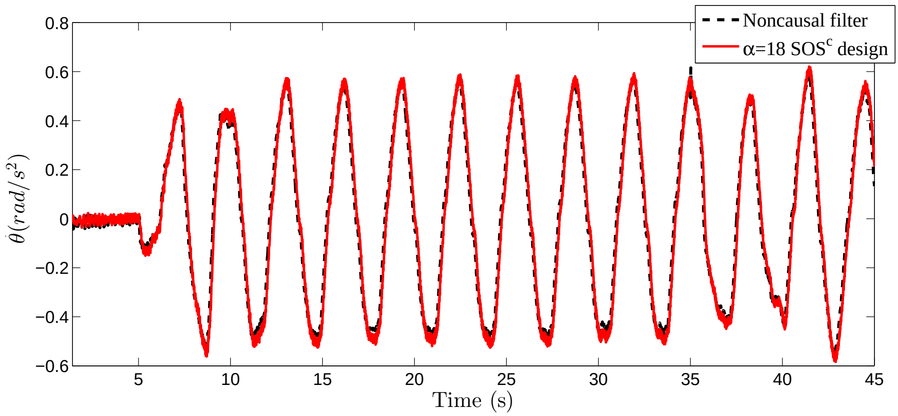
After the above confirmation that fuzzy polynomial techniques outperform TS and linear results, a detail of the time response with the finally chosen solution is shown in Figure 8.8, where roll dynamics has been omitted because

[6]Note, however, that TS-1GA and the SOS-2G are able to ensure validity of the design *a priori*: the underlying assumptions in the linear case do not actually hold in the experimental setting and its validity must be proved *a posteriori* (LM-A), as discussed before, so linearised theoretical and actual performance are a matter of "pure chance".

[7]In Lendek et al. (2011) the TS $\sigma = 0.003$ observer outperformed the linear observer; however, that was due to the fact that the actual implementation of the linear observer used the linearized model both in the "prediction" step $x_{k+1} = Ax_k + Bu_k$ and in the correction one $L[y_k - C\hat{x}_k]$. In the work presented here, the full nonlinear model $x_{k+1} = f(x_k, u_k)$ has been used in the prediction step, avoiding the unnecessary introduction of such linearization error. This produces, a much accurate linear observer performance than in Lendek et al. (2011).



(a) Pitch estimated speed



(b) Yaw estimated speed

Figure 8.8: Speed estimation with causal observer versus noncausal filter.

there exists symmetry with pitch and, therefore, results are similar. The temporal evolution shows that the SOS-2G observer design can follow the real states without a high noise amplitude.

8.5 Conclusions

In this chapter, a discrete-time polynomial observer design for state estimation of an electromechanical plant (fixed quadrotor) with fuzzy polynomial model is presented. The design setting is multiobjective: both decay rate and $\mathcal{H}_\infty$

conditions are used. Fuzzy TS designs (and, of course, linearized time invariant ones) can be considered as particular cases of the proposed methodology.

The obtained observer has been implemented and checked in an experimental quadrotor platform, comparing with the fuzzy TS and classical linear designs. The results clearly show that a mixed $\mathcal{H}_\infty$/decay design is needed, in order to achieve observers which are both fast and have reasonable worst-case attenuation guarantees. Furthermore, it is shown that linear and TS approaches have worse theoretical performance guarantees in a polynomial system. Obviously, the possible improvements of polynomial approaches need a model accurate enough so that the linearization errors (which polynomials avoid) contribute significantly to the overall uncertainty.

In actual experimental data, polynomial-based designs achieved a similar behaviour to the linear ones when close to the linearization point and a 12% lower integral square error when operating in high-speed trajectories away from it. However, the amount of conservatism introduced in the TS design with non-measurable premises makes its performance decrease, being even worse than that obtained by the linear observer. Importantly, the proposed techniques allow for theoretical *guarantee* of performance, which is not the case in the naive extrapolation of linearised designs far from the equilibrium point.

Chapter 9

Parameter Estimation in the Air Path of a TCSI Engine

Aerodynamics is for losers who can't build good engines.

Enzo Ferrari

ABSTRACT: This chapter deals with the nonlinear dynamics of the air system in a turbocharged spark ignited (TCSI) engine and proposes a preliminary strategy based on fuzzy polynomial techniques. The main goal is applying some theoretical developments of this thesis in order to design an observer for parameter estimation, which are unknown a priori and very important for further control strategies. The adopted framework handles easily the high nonlinearities and facilitates considerably the stability analysis of the whole turbocharged air system. The proposed design strategy considers, not only estimation error stability, but also a trade-off between speed convergence performance and disturbance rejection.

Nowadays, environmental constraints and customer's high demands in terms of efficiency and performance make vehicle design and manufacturing a big challenge. Combined technology of turbocharging with downsizing in IC engines is fully exploited only with an efficient air path management system. In this context, the knowledge of the dynamics in the manifold (mass flow and pressure) is essential to improve control tasks, which translates into better engine efficiency/performance and, therefore, meeting the new demand requirements.

Modelling of the air path and control in such kind of applications for IC engines have been recently considered using fuzzy TS approaches (Khiar, Lauber, Floquet, Colin, Guerra and Chamaillard, 2007; Khiar, Lauber, Guerra, Floquet, Colin and Chamaillard, 2008; Leroy, Chauvin and Petit, 2009). This chapter proposes an alternative way to do it by using fuzzy polynomial techniques.

The bench platform is a turbocharged spark ignition engine (TCSI) mounted in actual cars from the manufacturer Renault. The goal of the experiments presented in this chapter is showing the usefulness of fuzzy polynomial techniques to deal with the nonlinear modelling and observer design for the air path. Moreover, dealing with unknown parameters as new state variables is easier by polynomial techniques: indeed products of parameters and states are just polynomials. This fact allows reducing complexity in the modelling and further design phases, in comparison to classical TS designs.

In a similar way to the application presented in Chapter 8, a combined $\mathcal{H}_\infty$ attenuation and decay-rate performance is used at the design phase in order to obtain a successful practical solution. The concrete goals are:

1. Designing a fuzzy polynomial observer in order to estimate the air mass flow and pressure in the manifold plus the unknown coefficients corresponding to the loss of charge in air dynamics.

2. Checking the designed observer with the experimental engine in different working regimes and discussing if the proposed on-line coefficient estimation can improve observation and further control tasks.

The approach presented on this chapter is still in a preliminary stage and further refinements will be addressed on future work. Therefore, the results presented here are still not published.

The chapter layout is as follows: Section 9.1 describes the test TCSI engine, Section 9.1 deals with the fuzzy polynomial modelling of the air path dynamics, Section 9.3 discusses the user-defined parameters and presents the proposed design to obtain a suitable observer in practice, the experimental tests carried out with the IC engine are presented on Section 9.4, including a discussion on the obtained results and, finally, Section 9.5 provides the conclusions of the work.

9.1 System description

In turbocharged engines, the exhaust gases provide recovered energy to compress the fresh air in the intake manifold and thus increasing the mass-flow rate entering the cylinders. A schematic of a 4-cylinders TCSI engine is depicted in Figure 9.1.

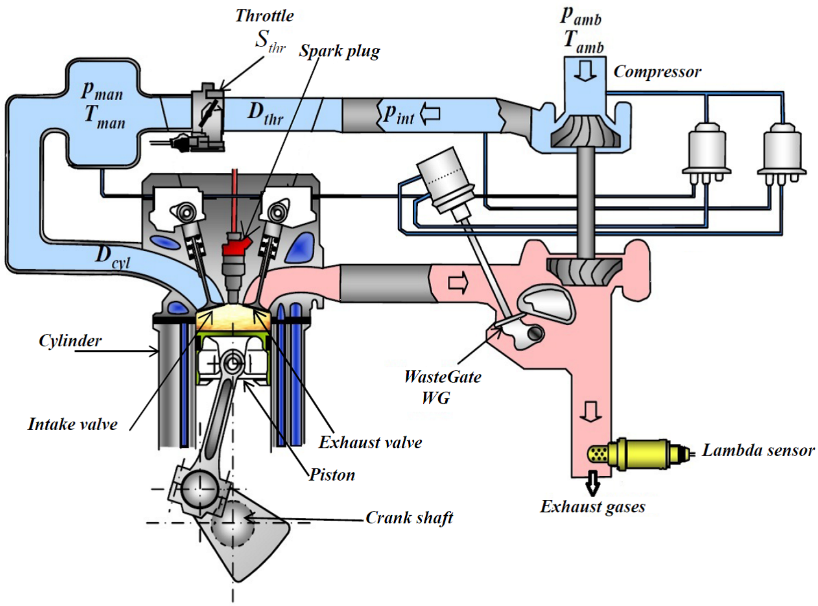


Figure 9.1: Schematic of a turbocharged gasoline engine.

The engine must operate at stoichiometric ratio due to pollution constraints. This leads to a tight connection between the torque response and the intake manifold pressure (P_{man}). The goal of the air path management is controlling this pressure. At high load, since pumping losses are minimized (throttle is fully open), the manifold pressure is similar to the boost one coming from the turbocharger ($P_{int} \approx P_{man}$) and it is controlled by the wastegate. However, at low load the throttle is activated to control the intake manifold pressure. This strategy results in a switching controller requirement.

There exist many recent studies focused on the control issue of the turbocharged air system. However, most of them propose linearizations around nominal points (Santillo and Karnik, 2013) which present some unavoidable drawbacks like achieving performance/robustness throughout the wide operating range and spending a lot of effort in tunning for each operating point. To overcome those drawbacks, fuzzy strategies provide a simpler way to analyze and design controllers. Furthermore, all variables needed for the controller must be measured or estimated.

On the following, the model of the air flow is presented for later use in observer design. Only the main important equations governing the air system behaviour are recalled. For more details the reader is referred to Eriksson, Nielsen, Brugård, Bergström, Pettersson and Andersson (2002); Eriksson (2007).

The pressure dynamics in the intake manifold (P_{man} in Pa) is derived from the ideal gas relationship (Heywood, 1988):

$$\dot{P}_{man} = \frac{R_0 T_{man}}{V_{man}}(D_{thr} - D_{cyl}) \tag{9.1}$$

Where $V_{man} = 8 \cdot 10^{-4}$ m^3 is the manifold volume, $R_0 = 287.058$ J/Kg°K is the perfect gas constant and D_{thr}, D_{cyl} are the air flow through the throttle and into the cylinders in Kg/s respectively. The manifold pressure P_{man} and temperature $T_{man} \approx 303$ °K are measured.

The mass flow rate across the throttle (D_{thr}) is modeled based on the following adiabatic orifice flow (Heywood, 1988):

$$D_{thr} = \beta_1 \frac{P_{int}}{\sqrt{R_0 T_{atm}}} f\left(\frac{P_{man}}{P_{int}}\right) g(\phi_{thr}) \tag{9.2}$$

Where ϕ_{thr} is the throttle butterfly valve-opening angle in radians ($0 \div \pi/2$), P_{int} is the boost pressure provided by the turbocharger in ($1.2 \div 1.8 \cdot 10^5$ Pa) and the temperature before the throttle is supposed to be close to the ambient one ($T_{amb} = 293°K$). The nonlinear function $g(\phi_{thr})$, jointly with the loss-of-charge coefficient β_1, expresses the geometric flow characteristics for the throttle. The differential pressure function $f\left(\frac{P_{man}}{P_{int}}\right)$ is defined by:

$$f\left(\frac{P_{man}}{P_{int}}\right) = \begin{cases} \sqrt{\frac{2\gamma}{\gamma-1}} r^{\frac{1}{\gamma}} \sqrt{1 - r^{\frac{\gamma-1}{\gamma}}} & if\ r > \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \\ \sqrt{\gamma}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} & if\ r \leq \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \end{cases} \tag{9.3}$$

Where $\gamma = 1.4$ is the adiabatic constant and $r = \frac{P_{man}}{P_{int}}$ is the pressure ratio between the boost and the manifold. The geometric flow characteristics across the throttle valve are modelled as a nonlinear function of the valve lift (characterized by the maximum surface opening), the intake valve open timing, the maximum lift and the intake valve opening duration:

$$g(\phi_{thr}) = a_1(1 - cos(a_2\phi_{thr} + a_3)) + a4 \tag{9.4}$$

The coefficients $a_1 = 0.0464$, $a_2 = 1.3793$, $a_3 = 1.6702 \cdot 10^{-6}$ and $a_4 = 0.0052$ are problem data obtained by experimental least squares (LS) identification with data taken at sample-time $T_s = 10$ ms, collected in separate experiments for several engine-speed operating regimes. For details on the modelling and identification procedure see Khiar (2007).

The mass-flow rate into the cylinders is modelled as an average nonlinear function depending on the engine speed N_e and the manifold pressure:

$$D_{cyl} = \beta_2 \frac{V_d}{120 R_0 T_{man}} (s_1 P_{man} + s_2) N_e \tag{9.5}$$

Where β_2 is a parameter expressing the loss of charge throughout the exhaust manifold, $V_d = 6 \cdot 10^{-4}$ m^3 is the total engine cylinder capacity, N_e is the engine speed regime (1500 $\div$ 4000 rpm) and coefficients $s_1 = 0.4643$, $s_2 = -15259$ are again obtained by a previous LS identification from sampled data collected in separate experiments.

The idea is to design an observer in order to estimate both loss-of-charge coefficients β_1 and β_2. This observer will be based on the measurement of the air-mass flow provided by a hot wire anemometer and on the measurement of the manifold pressure provided by a piezoelectric sensor located after the throttle.

The dynamics of the air-mass flow sensor is considered to be first order

$$\tau_1 \dot{D}_{thr_m} = D_{thr} - D_{thr_m} \tag{9.6}$$

where D_{thr_m} is the sensor state, D_{thr} is the actual air mass flow through the throttle and $\tau_1 = 40$ ms is the time constant of the sensor.

The dynamics of the pressure sensor is also considered first order:

$$\tau_2 \dot{P}_{man_m} = P_{man} - P_{man_m} \tag{9.7}$$

where P_{man_m} is the sensor state and $\tau_2 = 20$ ms is the time constant of the piezoelectric sensor.

Finally, putting together all dynamics, the computed nonlinear system for the air path is

$$\begin{aligned} \dot{x}_1 \;=\; & 25 x_2 u_1 f\left(\frac{x_3}{u_1}\right) \big(1.2725 \cdot 10^{-4}(1 - (\cos(1.3793 u_2 + \\ & \qquad 1.6702 \cdot 10^{-6}))) + 1.9313 \cdot 10^{-5}\big) - 25 x_1 \end{aligned}$$

$$\begin{aligned}
\dot{x}_2 &= 0 \\
\dot{x}_3 &= x_2 u_1 f\left(\frac{x_3}{u_1}\right)\left(1.2725\cdot 10^{-4}(1-(\cos(1.3793u_2+1.6702\cdot 10^{-6})))\right. \\
&\qquad \left. +1.9313\cdot 10^{-5}\right) - x_5(0.0029x_3 - 95.366)u_3 \\
\dot{x}_4 &= 50(x_3 - x_4) \\
\dot{x}_5 &= 0
\end{aligned}$$

where $x = (x_1, x_2, x_3, x_4, x_5) = (D_{thr_m}, \beta_1, P_{man}, P_{man_m}, \beta_2)$ is the considered state vector, $u = (u_1, u_2, u_3) = (P_{int}, \phi_{thr}, N_e)$ are measured input signals and the parameters β_1 and β_2 have been considered constant (zero dynamics).

9.2 Fuzzy polynomial modelling of the air path system

From the above continuous-time system, an Euler-discretized nonlinear system can be stated at sample time $T_s = 0.01$:

$$\begin{aligned}
x_{1_{k+1}} &= 0.75x_{1_k} + 0.25x_{2_k}u_{1_k} f\left(\frac{x_{3_k}}{u_{1_k}}\right)\left(1.2725\cdot 10^{-4}(1-(\cos(1.3793u_2\right. \\
&\qquad \left. +1.6702\cdot 10^{-6}))) + 1.9313\cdot 10^{-5}\right) \\
x_{2_{k+1}} &= x_{2_k} \\
x_{3_{k+1}} &= x_{3_k} + 0.01(x_{2_k}u_{1_k} f\left(\frac{x_{3_k}}{u_{1_k}}\right)(1.2725\cdot 10^{-4}(1-(\cos(1.3793u_{2_k}+ \\
&\qquad 1.6702\cdot 10^{-6}))) + 1.9313\cdot 10^{-5}) - x_{5_k}(0.0029x_{3_k} - 95.366)u_{3_k}) \\
x_{4_{k+1}} &= 0.5(x_{4_k} + x_{3_k}) \\
x_{5_{k+1}} &= x_{5_k}
\end{aligned} \tag{9.8}$$

There exists previous literature using TS fuzzy models in order to deal with a similar observer problem for the above air-path dynamics (see Kerkeni, Lauber and Guerra (2010)). However, results in the cited reference were developed for an atmospheric engine, i.e., without turbocharger ($P_{int} = P_{atm}$), and only the lower constant part of equation (9.3) was considered. Here, polynomial techniques allow to deal with the on-line parameter estimation without the requirement of a high number of fuzzy rules. This is due to the fact that the products of the parameters with states are just polynomials, which do not need further conservative modelling in opposition to TS modelling.

Given the nonlinear model (9.8), a four-rules fuzzy polynomial model (6.2) can be computed as follows:

- Computing the maximum and minimum of the nonlinear term $\Psi = u_1 f\left(\frac{x_3}{u_1}\right) g(u_2)$, which can be done because the nonlinear functions f, g are bounded and also the turbocharger pressure is between an operative range:

$$\overline{\Psi} = \overline{P}_{int}\overline{f}\left(\frac{x_3}{u_1}\right)\overline{g}(u_2), \quad \underline{\Psi} = \underline{P}_{int}\underline{f}\left(\frac{x_3}{u_1}\right)\underline{g}(u_2),$$

with $\overline{P}_{int} = 1.8{\cdot}10^5$, $\underline{P}_{int} = 1.2{\cdot}10^5$, $\overline{f}\left(\frac{x_3}{u_1}\right) = 0.647$, $\underline{f}\left(\frac{x_3}{u_1}\right) = 0.05$, $\overline{g}(u_2) = 2.1797 \cdot 10^{-4}$ and $\underline{g}(u_2) = 1.9313 \cdot 10^{-5}$ as extreme values[1].

- Bounding the engine speed u_3 between its expected operation range $\overline{N}_e = 4000, \underline{N}_e = 1050$.

Then, with those bounds, the computed polynomial vertex models are:

$$\mathcal{P}_1(x_k) = \begin{bmatrix} 0.75x_{1_k} + 2.3157x_{2_k} \\ x_{2_k} \\ x_{3_k} - 0.11608x_{3_k}x_{5_k} + 0.68584x_{2_k} + 0.037648x_{5_k} \\ 0.5x_{4_k} + 0.5x_{3_k} \\ x_{5_k} \end{bmatrix}$$

$$\mathcal{P}_2(x_k) = \begin{bmatrix} 0.75x_{1_k} + 0.009989x_{2_k} \\ x_{2_k} \\ x_{3_k} - 0.11608x_{3_k}x_{5_k} + 2.0429 \cdot 10^{-5}x_{2_k} + 0.037648x_{5_k} \\ 0.5x_{4_k} + 0.5x_{3_k} \\ x_{5_k} \end{bmatrix}$$

$$\mathcal{P}_3(x_k) = \begin{bmatrix} 0.75x_{1_k} + 2.3157x_{2_k} \\ x_{2_k} \\ x_{3_k} - 0.030471x_{3_k}x_{5_k} + 0.68584x_{2_k} + 0.0098825x_{5_k} \\ 0.5x_{4_k} + 0.5x_{3_k} \\ x_{5_k} \end{bmatrix}$$

[1]Note that, from the air-path dynamics considered in Section 9.1, the value $\underline{f}(x_3)$ should be zero. However, in that extreme situation ($P_{man} = P_{int}$), the parameter β_1 becomes unobservable. As the objective of this application is to give a parameter estimation during normal operation, $\underline{f}(x_3)$ has been set to a minimum nonzero value ($r = 0.9985$ instead of $r = 1$)

$$\mathcal{P}_4(x_k) = \begin{bmatrix} 0.75x_{1_k} + 0.009989x_{2_k} \\ x_{2_k} \\ x_{3_k} - 0.030471x_{3_k}x_{5_k} + 2.0429 \cdot 10^{-5}x_{2_k} + 0.0098825x_{5_k} \\ 0.5x_{4_k} + 0.5x_{3_k} \\ x_{5_k} \end{bmatrix}$$

in which the manifold pressure x_3 and its dynamics have been rescaled to be in Atm instead of Pa.

The corresponding membership functions $\mu_i(x, u)$ are constructed by:

$$\mu_1(x_3, u) = M_1(\Psi)N_1(u_3) \qquad \mu_2(x_3, u) = M_1(\Psi)N_2(u_3)$$
$$\mu_3(x_3, u) = M_2(\Psi)N_1(u_3) \qquad \mu_4(x_3, u) = M_2(\Psi)N_2(u_3)$$

with

$$M_1(\Psi) = \begin{cases} \frac{u_1 g(u_2) - \underline{P}_{int}\underline{g}(u_2)}{\overline{P}_{int}\overline{g}(u_2) - \underline{P}_{int}\underline{g}(u_2)} & if\ r \leq 0.5283 \\ \frac{\Psi(x_3, u_1, u_2) - \underline{\Psi}}{\overline{\Psi} - \underline{\Psi}} & if\ 0.5283 < r < 1 \end{cases} \qquad M_2(\Psi) = 1 - M_1(\Psi)$$

$$N_1(u_3) = \frac{u_3 - \underline{N}_e}{\overline{N}_e - \underline{N}_e} \qquad N_2(u_3) = 1 - N_1(u_3)$$

9.3 Observer design

The objective of the observer design will be estimating β_1, β_2 in order to see which values are needed to obtain a good estimate of the manifold pressure and air flow from its measurements. The on-line parameter estimation should improve the prediction step, adapting the model to deal with the unmodelled dynamics. Thus, it also will give us an idea about the lack of accuracy made by using a simple nonlinear model of the air path.

In order to carry out polynomial observer design, the observer error $e_k = x_k - \hat{x}_k$ follows the equation (6.20). The manifold pressure P_{man} and the parameter β_2 (the ones which enter polynomially in the fuzzy model) and their estimated values are assumed to lie between their respective expected ranges:

$$\begin{aligned} \Omega_x &= \{0.34 \leq x_3 \leq 1.8,\ 0.1 \leq x_5 \leq 1.1\} \\ \Omega_{\hat{x}} &= \{0.34 \leq \hat{x}_3 \leq 1.8,\ 0.1 \leq \hat{x}_5 \leq 1.1\} \end{aligned} \tag{9.9}$$

Choice of disturbance-size parameters. External disturbances, process miss-modeling and sensor noises are taken into account in the fuzzy polynomial model (6.2) by variables w, ν, weighted with suitable power matrices $\mathcal{E}, R$.

The process disturbances in $\dot{x}_1$ and $\dot{x}_3$ are mainly introduced by the measurement noise present in the boost-pressure sensor (P_{int}). In order to estimate those weights, the 3σ level (being σ the standard deviation) of the noise distribution has been computed from a set of measured data taken in steady state. The obtained value is $\sigma = 6.6735 \cdot 10^3$. Then, the worst-case disturbance-weighting values for that 3σ level are computed as follows:

$$\begin{aligned} E_1 &= \frac{T_s}{\tau_1}\left(\frac{3\sigma}{\sqrt{R_0 T_{atm}}}\overline{f}(x_{3_k})\overline{g}(u_{2_k})\right) = 0.0026 \\ E_3 &= \frac{T_s R_0 T_{man}}{V_{man}}\left(\frac{3\sigma}{\sqrt{R_0 T_{atm}}}\overline{f}(x_{3_k})\overline{g}(u_{2_k})\right) = 0.0763 \end{aligned}$$

The process disturbance on pressure sensor dynamics is set arbitrarily to $E_4 = 0.001$ because it has been considered to be much smaller than the rest of engine process noise inputs. Finally, the weighting matrix $\mathcal{E}$ is given by:

$$\mathcal{E} = \begin{bmatrix} E_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_3 & 0 & 0 \\ 0 & 0 & 0 & E_4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Sensor power noises for the manifold pressure and air flow have been estimated from the experimental data. Again the standard deviation has been computed for both measurements but in this case the 6σ level has been computed in order to give the worst-case values for the $\mathcal{H}_\infty$ design. Thus, the values of the weighting matrix R are:

$$R = \begin{bmatrix} 0.0016 & 0 \\ 0 & 0.441 \end{bmatrix}$$

Note. In this case, the membership functions depend on the "unknown" state P_{man}, so the observer-model mismatch should fulfill a Lipschitz-like bound (6.14). Note however that the manifold pressure is measured by the sensor, so the difference between the state P_{man} and the measurement should be small. Therefore, as the inclusion of a Lipschitz constraint makes the design problem too conservative, the sensor measurement has been used as premise for the memberships instead of P_{man}.

Setting a discrete decay-rate of $\alpha = 0.99994$ (Section 6.3.1), the equivalent of a continuous-time decay $\alpha_c = 0.003$ ($\alpha = e^{-2T_s\alpha_c}$), an acceptable enough robust/fast compromise is achieved on this application. For details see a discussion about design compromises in practice on Section 8.3. The constant weighting matrix D in (6.26) is chosen as:

$$D = diag(2, 0.001, 0.01, 0.01, 0.001);$$

Those values have been set by taking into account the different units of each state. Thus, multiplying by 200 the estimation of the air flow $e_1 = x_1 - \hat{x}_1$, which is in Kg/s, gives comparable values[2] to the estimation error of the pressure $e_3 = x_3 - \hat{x}_3$, $e_4 = x_4 - \hat{x}_4$, in Atm. In addition, the weight for estimation error on the loss-of-charge coefficients is set to 0.001, because the objective is finding some values for the coefficients which make the estimation of the physical states as good as possible. Therefore, e_1, e_3, e_4 must contribute more than e_2, e_5 to the performance index.

Then, by considering the above particular operating regions (9.9) and process disturbance and measurement noise weights, a fuzzy polynomial observer (6.4) can be designed by Theorem 6.2. The computed fuzzy observer gains are:

$$L_1 = \begin{bmatrix} 1.3764 & 0 \\ 0.2728 & 0 \\ 0.6348 & 0.9847 \\ 0.3957 & 1.0805 \\ 0.0795 & -0.0599 \end{bmatrix} \quad L_2 = \begin{bmatrix} 0.7575 & -0.0001 \\ 0.2726 & 0 \\ 0.4519 & 0.9854 \\ 0.3984 & 1.0812 \\ 0.0794 & -0.0599 \end{bmatrix}$$
$$L_3 = \begin{bmatrix} 1.3763 & -0.0001 \\ 0.2729 & 0 \\ 0.6134 & 1.0637 \\ 0.3603 & 0.985 \\ 0.0807 & -0.0643 \end{bmatrix} \quad L_4 = \begin{bmatrix} 0.7575 & 0 \\ 0.2726 & 0 \\ 0.4316 & 1.056 \\ 0.345 & 0.9788 \\ 0.0804 & -0.064 \end{bmatrix} \tag{9.10}$$

which give a theoretical $\mathcal{H}_\infty$ disturbance-rejection bound $\sqrt{\gamma} = 0.049$. The

[2] Note that the $\mathcal{H}_\infty$ design is minimizing γ such that $e^T D^T De \leq \gamma W^T W$ (Section 6.3.1) so, if $e_1 << e_i$, $i : 2, \ldots, 5$, then minimizing such γ barely contributes to minimize e_1.

found Lyapunov function $V(e) = e^T Qe$ proving those results is:

$$Q = \begin{bmatrix} 5.9114 & -6.9281 & -0.0052 & 0.0054 & -0.0006 \\ -6.9281 & 25.5387 & 0 & -0.0001 & -0.0111 \\ -0.0052 & 0 & 0.0171 & -0.0164 & 0.002 \\ 0.0054 & -0.0001 & -0.0164 & 0.0228 & 0.0001 \\ -0.0006 & -0.0111 & 0.0020 & 0.0001 & 0.034 \end{bmatrix}$$

Table (9.1) shows the demanded computational resources for the design phase. Feasible solutions were obtained with software SOSOPT (Balas, Packard, Seiler and Topcu, 2012; Seiler, Zheng and Balas, 2013) plus SeDuMi 1.3, executed in an Intel® Core™2 Duo CPU P8600 2.4GHz, 4 Gb DDR3 RAM machine running Windows 7 (64 Bit) and MATLAB R2011b.

Problem size	RAM	Parser time	Solver time
21007×3437	860 Mb	186.12 s	41.76 s

Table 9.1: Approximate computational resources for design.

9.4 Experimental evaluation

This section evaluates the computed observer on the experimental platform and gives a discussion on the obtained results. The test bench is a 600 cc gasoline engine developed by Renault for using in hybrid vehicles and develops a maximum power of 70 CV.

Data generation. The objective of the experiment is obtaining a representative set of data in which the engine nonlinearities were significant enough. In this way, a controlled electrical drive is coupled to the engine crankshaft in order to produce external load to the engine.

The test starts at steady state: low-regime speed, no external load. At the first second no input excitation has been applied to the throttle. Then, the IC engine has been excited with a low-amplitude reference throttle signal changing randomly each 3 seconds during 35 seconds. Afterwards, the throttle reference signal was applied with a higher amplitude until the end of the experiment, set in 100 seconds. Meanwhile, the engine speed was limited manually to several values with the objective of forcing the engine to produce (or receive) torque, simulating road slopes. Figure 9.2 shows the collected input signals to

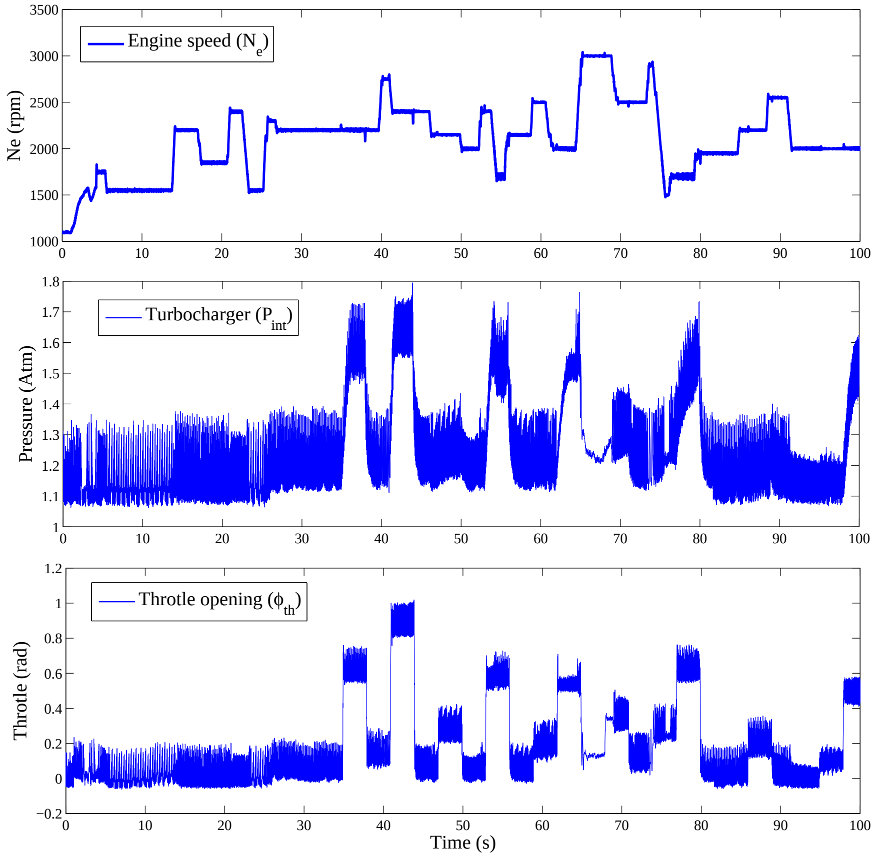


Figure 9.2: Collected input data.

the observer: the throttle opening given by a potentiometer, the pressure given by the turbocharger and the engine speed measured by an encoder.

Experimental results. Once fuzzy observer gains (9.10) are on place, the following evaluates how the theoretical observer design translates into experimental behaviour. In the work presented here, the full nonlinear model (9.8) has been used in the prediction step, avoiding the unnecessary introduction of error with the use of membership functions μ depending on the pressure measurement instead of the state P_{man}.

In order to obtain acceptable estimates of the physical states, the observer had to change the coefficients β_1 and β_2 following the temporal evolution shown in Figure 9.3.

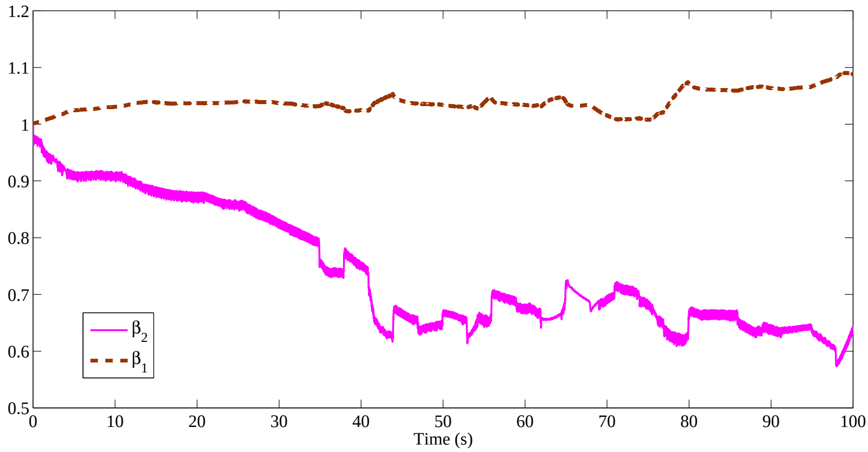


Figure 9.3: Evolution of the estimated coefficients during the experiment.

A detail of the time response with the computed observer is shown in Figures 9.4 and 9.5. The temporal evolution shows that the fuzzy polynomial observer gives an estimate of the manifold air flow and pressure close to the measurements (indeed, pretty close in the manifold pressure). Note also that estimated air flow achieve a larger amplitude than the measurement. This fact was quite expectable because the actual engine dynamics is very fast (the four-stroke cycle in each cylinder is not considered on (9.5), but only the mean) and sensors act as low-pass filters, specially the hot-wire anemometer with a time constant of 0.04 seconds.

Looking at the obtained responses, three main conclusions can be stated:

1. After the first seconds, when no excitation input is applied to the engine, the designed observer is able to estimate physical states and coefficients with a fast decay in normal operation.

2. The considered region of study (9.9) for observer design is fulfilled during the experiment. Note that, even being $\beta_1 > 1$, the observer holds stability, as the theoretical design guarantees.

3. The coefficients were supposed to be almost constant and lie between 0 and 1. However, the observer estimates $\beta_1 \geq 1$ in order to fit D_{thr_m}. This suggest that the considered model for the air path is not precise enough.

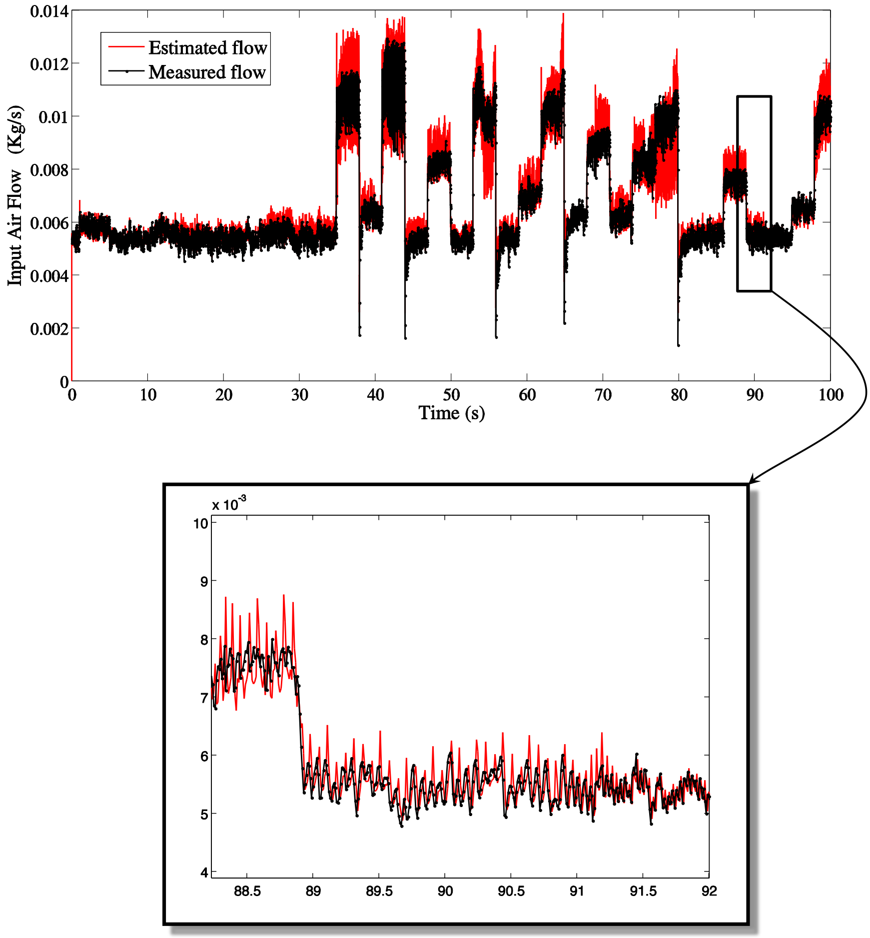


Figure 9.4: Evolution of the air flow through the throttle given by sensor (black) and the observer (red).

9.5 Conclusions

In this chapter, the application of fuzzy polynomial techniques has been demonstrated to be valid for dealing with state and parameter estimation. The preliminary experimental results confirm a good observer response.

In comparison with existent TS observer literature for this application, the

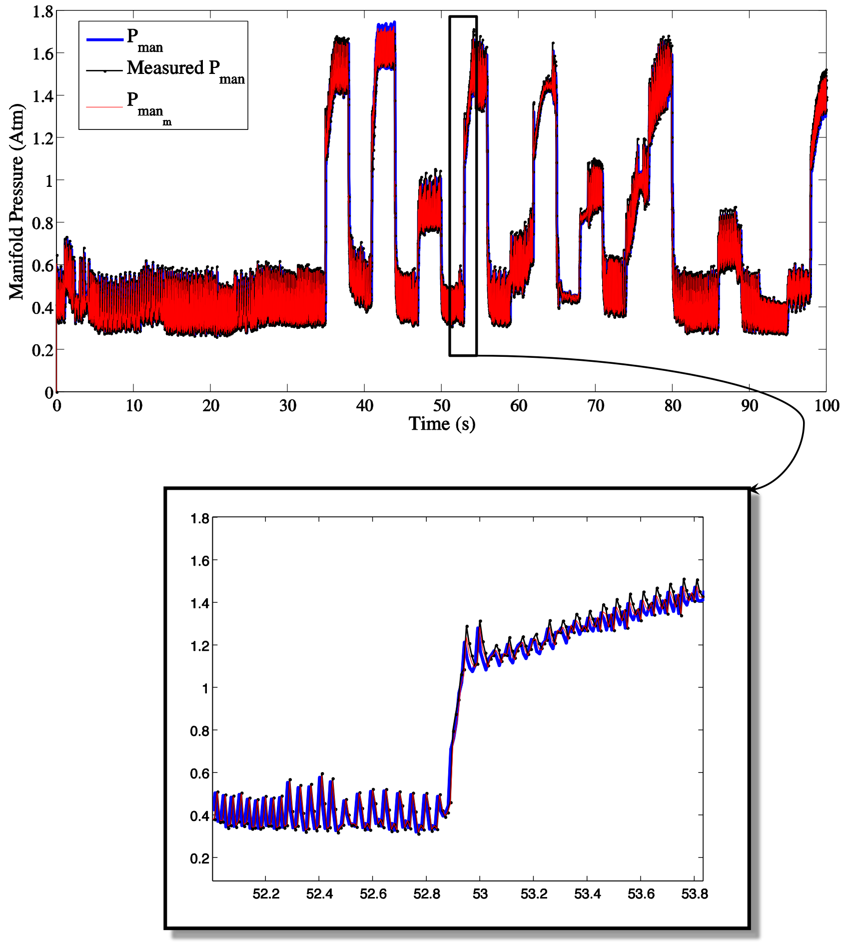


Figure 9.5: Evolution of the manifold pressure given by sensor (black) and the observer: P_{man_m} (red), P_{man} (blue).

fuzzy polynomial methodology allows designing an observer with less considerable number of fuzzy rules. The advantage is getting benefit from the fact that products of states with parameters are just polynomials and can be introduced directly to SOS programing packages.

Less conservative results can be achieved by approximating the nonlinear part of (9.3) by a polynomial (or a set of them) instead of two constants. However, required computational resources increase considerably as Positivstellensatz multipliers need to fit the same degree as the Lyapunov function discrete increment.

The experimental results show that the observer had to change the loss-of-charge coefficients from their initial values (considerably in the case of β_2), in order to better adapt the estimates of pressure and air flow in each operating regime. This suggests that the on-line knowledge of those coefficients may improve control performance. However, although the designed observer achieved an acceptable response in practice, the obtained evolution of the coefficient estimation suggests a deeper study in the nonlinear modelling phase: further refinements on the air flow dynamics, testing in different conditions, identifying possible patterns and evaluating the need or not of using such observer for future control tasks. Such issues will be addressed in future work, as this application is still in preliminary phases.

Conclusions of the thesis

This thesis presented ideas and methodologies in order to improve, or partially overcome, some existent drawbacks in present fuzzy polynomial literature for analysis and control of nonlinear systems. The main contributions of this work where summarized on the introduction and a particular conclusion section closes each chapter. Here, general conclusions and future-work lines are discussed.

- **The iterative methodology for improving the domain of attraction estimation**, presented on Chapter 4, allows a better fitting of the real shape of the nonlinear system's DA. Moreover, it combines nicely with other DA estimation techniques by being able to use such results as initial "seed" estimates for the algorithm. The procedure has its own limitations: a polynomial curve of prefixed degree cannot keep obtaining better estimates forever, i.e., improvements diminish as iterations progress. In addition, limited numerical accuracy of computers may cause the accumulation of numerical errors in each new iteration of the algorithm, forcing finally the stop criterion due to numerical problems. In addition, it has limitations derived from the Taylor-series fuzzy modelling: the polynomial vertex models separate fast with the distance to the origin, depending on the disregarded high-degree terms of the Taylor-series developments.

- **The stability analysis under disturbances** has been addressed in Chapter 5 with a systematic methodology even for persistent disturbances, with bounded power in every time period T. The analysis allows computing minimum reachable sets (final regions) and also transitory or inescapable regions, given an initial region and performing an exploration in a decay-rate parameter. However the advantages of the methodology vanish when initial condition regions are large: the problem is nonconvex and requires both exploration in the decay rate and in a region-size parameter.

The stabilization under nonvanishing disturbances poses further caveats, because only the case when initial conditions are assumed zero is computationally tractable and even that requires additional assumptions on the modelling-region constraints: the polynomial boundaries need to be of even degree and must be expressed as products of a vector of monomials $z(x)$, a matrix $N(x)$ and its corresponding transpose, in order to apply Schur complement. This, obviously, reduces generality on the obtained results.

- **Fuzzy-polynomial observers for nonlinear systems with bounded disturbances** have been designed in Chapter 6 for both inescapable-set and $\mathcal{H}_\infty$ settings, making a link between the different preliminary approaches in previous polynomial observer literature. In addition, the implementation issues of such observers in discrete time has been considered since the design phase: additional constrains have been introduced to the SOS $\mathcal{H}_\infty$ optimization problems in order to ensure acceptable behaviours in practice, both in pure discrete-time design such as discretized continuous-time one. The presented approach, however, requires further study of the different settings in Chapter 5 and the computational complexity/cost.

- **Polynomial controller synthesis for nonlinear systems** leads in general to a nonconvex problem, which only can be cast as a convex one by making some conservative assumptions and changes of variable. After those changes, the local information about regions in the state space is only introduced partially and desired domains of attraction are difficult to ensure. Thus, the advantages of having more degrees of freedom by using polynomial controller gains and polynomial Lyapunov functions become weak when putting the problem in convex form. In Chapter 7 a methodology for polynomial systems was presented in discrete time in order to remove some of this conservativeness by using full-delayed state Lyapunov functions and introducing existent information about past and present states, provided by the input saturations. The methodology was proven to overcome existent literature at the price of more computationally effort. However, although the idea can be extended to a more general class of nonlinear systems by using fuzzy polynomial modelling, more conservative assumptions are required, which make the proposal lose a bit of its initial power. Further work is required to remove those conservative assumptions and also to develop similar ideas for the continuous-time case.

- **Two experimental application of fuzzy-polynomial observer design** demonstrated the applicability of the polynomial techniques in practice with successful results. Fuzzy-polynomial methodologies arise as a good alternative to Takagi-Sugeno ones for some real applications: the approaches presented in this thesis overcame classical TS ones in the tested benchmarks, both by reducing the required number of fuzzy rules (Chapter 9) and by obtaining better performance bounds (Chapter 8).

The above review of the obtained results presents some open issues which need a further study and can lead to new research lines.

There exist many works in literature, apart from this thesis, dealing with the stability analysis and domain of attraction estimation. However, the analysis in presence of disturbances is still very scarcely developed for nonlinear systems. Chapter 5 has focused the problem and may serve as preliminary guidelines for a future thesis in the topic. Generalizing the results in Chapter 4 to the disturbance case and exploring its connections to other BMI proposals for DA estimation might prove interesting.

Furthermore, computation of inescapable or disturbance-invariant sets in observer design is also a very important problem in fuzzy systems: disturbances must not push the system out of the modelling region Ω and ensuring it with guaranteed regions is still an open problem, as Chapters 5 and 6 discuss. The combination of observer and controller design (output feedback) for fuzzy polynomial systems with the above inescapability and DA issues may be also an interesting problem, although it seems that only resorting to BMI-like results might be an option.

Or course, further testing of the developed techniques in the presented applications (as well as in other possible situations) is needed. In practical problems, the choice of design parameters, computational cost, and performance evaluation useful to the customer are important issues for which this thesis only provides preliminary discussions.

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And thus, due to much reading and little sleeping, his brain dried in such a way that he lost his sense.

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Appendices

Appendix A

Semidefinite and SOS Programming

Semidefinite programming is a branch of convex optimization which is encharged of minimizing a linear objective function over the intersection of the cone formed by positive semidefinite matrices with an affine space. In particular, in the automatic control framework, SDP is used in order to solve Linear Matrix Inequalities (LMI). Indeed, the SDP problems are a special case of conic programming and can be efficiently solved by numerical methods and software (Boyd, Ghaoui, Feron and Balakrishnan, 1994).

A sum-of-squares problem is an optimization problem with a linear cost and a particular type of constraint on the decision variables. At the end, these constraints are polynomials with decision-variable coefficients, which must fulfill the sum-of-squares property. If the polynomial constraints are affine in decision variables, the SOS optimization problem can be recasted as a semidefinite-programming one (Lasserre, 2001).

A.1 Linear Matrix Inequalities

Definition A.1. *Denote by $\mathbb{S}^n$ the space formed by all the $n \times n$ real symmetric matrices. A symmetric matrix A is* ***positive semidefinite*** *if all its eigenvalues are not negative and is denoted as $A \succeq 0$. In a similar way, $A \succ 0$ means A is* ***positive definite****, where all its eigenvalues are strictly positive and the opposite for negative semidefinite an definite.*

Definition A.2. *A* ***Linear Matrix Inequality*** *(LMI) is an expression in the form*

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0 \tag{A.1}$$

where $x \in \mathbb{R}^m$ is the decision variable and the symmetric matrices $F_i = F_i^T \in \mathbb{R}^{n\times n}$, $i = 0, \ldots, m$, are given (Boyd, Ghaoui, Feron and Balakrishnan, 1994, cap. 2).

The LMI (A.1) seems to be in a special form. However, it can represent a wide variety of convex constraints in x. In particular, the Lyapunov stability constraints (common in control theory) where the variables are matrices can be cast as LMI's. For example:

$$A^T P + PA \prec 0$$

where $A \in \mathbb{R}^{n\times n}$ is a given matrix and $P = P^T$ is the decision variable of the LMI. Of course, the above inequality can be expressed in the form (A.1) as follows; if $P_1, ..., P_m$ is a basis of $n\times n$ symmetric matrices ($m = n(n+1)/2$), then $F_0 = 0$, $F_i = -A^T P_i - P_i A$ are taken.

There exist two kinds of LMI problems:

1. **Feasibility problem**. This problem only requires the existence of, at least, one solution satisfying the set of constraints. This problem is widely used in the stability analysis and also in the synthesis problem, when only a controller stabilizing the system (or fulfilling some predefined performance criteria) is required.

2. **Optimization problem**. In this case, not only a feasible solution is to be found but, in addition, such solution must optimize some objective function over the set of feasible solutions. Obviously, the optimization problem is an implicit feasibility problem.

On the following, important definitions an properties in LMI framework are summarized.

A.1.1 Schur complement

Some nonconvex matrix-inequality problems can be recast as LMI using the well-known Schur complement result.

By Schur complement, the set of nonlinear matrix inequalities

$$R(x) > 0,\ Q(x) - S(x)^T R(x)^{-1} S(x) > 0 \tag{A.2}$$

where $Q(x) = Q(x)^T$, $R(x) = R(x)^T$, and $S(x)$ are affine in x, is equivalent to the following LMI:

$$M(x) = \begin{bmatrix} Q(x) & S(x)^T \\ S(x) & R(x) \end{bmatrix} > 0 \tag{A.3}$$

A.1.2 S-procedure

The constraint of requiring positiveness (or negativeness) for some quadratic function when other quadratic ones are, is very often in control problems. This constraint can be expressed as LMI, sometimes in a conservative way, by using the S-procedure detailed below:

If there exist p quadratic functions in the variable $\zeta \in \mathbb{R}^n$ in the form

$$F_i(\zeta) = \zeta^T T_i \zeta + 2u_i^T \zeta + vi, \quad i = 0, \dots, p,$$

where $T_i = T_i^T$, the following condition is considered:

$$F_0(\zeta) \geq 0 \,\forall\, \zeta \mid F_i(\zeta) \geq 0, \quad i = 1, \dots, p, \tag{A.4}$$

Obviously, if there exist $\tau_1 \geq 0, \dots, \tau_p \geq 0$ such that for all ζ

$$F_0(\zeta) - \sum_{i=1}^{p} \tau_i F_i(\zeta) \geq 0, \tag{A.5}$$

is fulfilled, then (A.4) holds.

In the particular case of having quadratic forms and strict inequalities, the procedure results as follows:

Given $T_0, \dots, T_p \in \mathbb{R}^{n \times n}$ symmetric matrices and considering the following condition

$$\zeta^T T_0 \zeta > 0 \,\forall\, \zeta \neq 0 \mid \zeta^T T_i \zeta \geq 0, \quad i = 1, ..., p. \tag{A.6}$$

If there exist $\tau_1 \geq 0, ..., \tau_p \geq 0$ such that

$$T_0 - \sum_{i=1}^{p} \tau_i T_i > 0, \tag{A.7}$$

then (A.6) holds too.

A.1.3 Finsler's lemma

Let $x \in \mathbb{R}^n$, $Q \in \mathbb{S}^n$ and $B \in \mathbb{R}^{m\times n}$ such that $rank(B) < n$. Then, the following statements are equivalent:

1. $x^T Q x < 0, \forall Bx = 0, x \neq 0$
2. $B^{\perp T} Q B^{\perp} \prec 0$
3. $\exists \mu \in \mathbb{R} : Q - \mu B^T B \prec 0$
4. $\exists X \in \mathbb{R}^{n\times m} : Q + XB + B^T X^T \prec 0$

A.1.4 Nonconvex matrix inequality problems

In general, when the problem won't be convex neither be recast as a convex one, obtaining solutions require global optimization methods (for instance, genetic algorithms). Those methods usually lead to a complex and slow process without a complete guarantee of reaching the optimum. However there exist some particular cases which are still computationally tractable by iterative LMI methods (ILMI), i.e., iterating by changing decision variables under some improvement criteria and solving a convex subproblem in each iteration.

A.1.4.1 Generalized Eigenvalue Problem (GEVP)

A GEVP tries to minimize the largest generalized eigenvalue of a pair of matrices which depend on a variable in an affine way, subject to a LMI-type constraint. The general form of a GEVP is:

$$\begin{array}{ll} minimize & \lambda \\ subject\ to: & \\ & \lambda B(x) - A(x) > 0, \quad B(x) > 0, \quad C(x) > 0 \end{array} \tag{A.8}$$

where A, B and C are symmetric matrices which are affine functions in x. This can be expressed in the following way:

$$\begin{array}{lll} minimize & \lambda_{max}(A(x), B(x)) & \\ subject\ to: & & \\ & B(x) > 0, & C(x) > 0 \end{array} \tag{A.9}$$

where $\lambda_{max}(X, Y)$ is the largest generalized eigenvalue of $\lambda Y - X$ with $Y > 0$, i.e., the largest eigenvalue of the matrix $Y^{-1/2} X Y^{-1/2}$. Therefore, a

GEVP is a quasiconvex optimization problem (Seiler and Balas, 2010; Boyd, Ghaoui, Feron and Balakrishnan, 1994) because the constraint is convex but the objective $\lambda_{max}(X, Y)$ is not.

The minimum objective in those kind of problems can be obtained by solving ILMI methods like bisection search o similar. When the matrices are all diagonal and $A(x)$, $B(x)$ are scalar, this problem reduces to a general linear-fractional programming one. Furthermore, many quasiconvex nonlinear functions can be represented as GEVP's with appropriate A, B and C (Boyd, Ghaoui, Feron and Balakrishnan, 1994).

A.1.4.2 Bilinear Matrix Inequalities (BMI)

Another particular case of nonlinear matrix inequalities are the bilinear (second order) ones whose general expression is:

$$F_{00} + \sum_{i=1}^{p_x} x_i F_{i0} + \sum_{i=1}^{p_y} y_i G_{i0} + \sum_{i=1}^{p_x} \sum_{j=1}^{p_y} x_i y_j H_{i0} < 0 \tag{A.10}$$

This kind of problem can still be addressed (without guarantees of reaching the optimum) by ILMI methods. The most used methods in literature are; [1] the V-K iterative procedure, based on fixing a decision variable V, solving for an suboptimal K and then fixing that value and solving for V (Banjerdpongchai, 1997), and [2] the Cone Complementary Linearization (CCL), based on adding N additional constraints $X_i X_j = I$, where (X_i, X_j) are some matrix decision variables, and minimizing iteratively the cost $J = \sum_{p=1}^{N} Trace(X_i X_j)$ until obtaining a feasible X fulfilling the original constraints (El Ghaoui, Oustry and AitRami, 1997).

A.1.5 Numerical resolution

The standard problems presented in the above section can be solved efficiently in polynomial time. By solving the problem we understand: determining if the problem is feasible or not and, in case it will be, computing a feasible point with the objective function that exceeds the global minimum only a prefixed precision.

There exist several software packages which are exclusive for those convex minimization problems. For instance, some programming languages like YALMIP (Löfberg, 2004) or CVX can be used with several convex minimization solvers, say SeDuMi (Self-Dual Minimization) (Sturm, 1999) or SDPT3

(Semidefinite Programming Tool) (Toh, Todd and Tütüncü, 1998). On the following, a simple example is shown in order to introduce the use of those tools for solving optimization problems. The code has been programmed in MATLAB using YALMIP plus SeDuMi.

Example A.1.1. Check global stability, in the Lyapunov sense, of the following linear system

$$\dot{x} = Ax = \begin{bmatrix} 0 & 1 & 0 \\ 9.8 & 0 & 1 \\ 0 & -10 & -10 \end{bmatrix} x$$

is a LMI problem which is stated as:

$$\begin{array}{ll} Find & P \succeq 0 \\ subject\ to: & \\ & \dot{V}(x) < 0, \quad where\ \dot{V}(x) = x^T P x\ (Lyapunov\ function) \end{array}$$

This can be implemented in MATLAB by the following code:

```
A=[0,1,0;9.8,0,1;0,-10,-10];
P=sdpvar(3); % Lyapunov matrix
lmis=[P>eye(3)]; % Constraint P>0
lmis=[lmis,A'*P+P*A<0]; %Constraint $d(V)/dt<0
solvesdp(lmis)
% Solutions
Lyap=double(P) % Matrix proving stability
```

A.2 Sum Of Squares

The general SOS problem is checking the non-negativity of a polynomial $f(x)$: $\mathbb{R}^n \to \mathbb{R}$, defined by powers of x and its associated coefficients. The idea is replacing the non-negativity by the sufficient condition of being SOS polynomials and searching for such decomposition. The basic ideas of the approach and main results are now summarized.

A.2.1 Positive polynomials: an outline

Sum Of Squares polynomials. The key idea in the SOS approach is trying to find an expression of a polynomial as the sum of squares of simpler polynomials.

Definition A.3. *The set of* ***Sum Of Squares*** *(SOS) polynomials in the variables "z", denoted as Σ_z, is the set defined by*

$$\Sigma_z = \left\{ p \in \mathcal{R}_z | p = \sum_{i=1}^{t} f_i^2,\ f_i \in \mathcal{R}_z \right\} \tag{A.11}$$

with $t \in \mathbb{Z}^+$.

It is very related (in fact, equivalent) to finding some matrix elements in order to make such matrix positive semidefinite. Indeed, an even-degree polynomial $p(z)$ is SOS if and only if there exist a vector of monomials $m(z)$ and a constant positive-definite matrix H such that $p(z) = m(z)^T Hm(z)$; in this way, SOS problems can be solved via SDP tools searching for such an H (see Section A.1). If the SDP problem is feasible, we will say that $p(z)$ is SOS. See Section A.2.4 for details.

Example A.2.1. For instance, existent SOS software (see Section A.2.5) find that polynomial

$$p(z) = z_1^4 - 4z_1^3 z_2 + 2z_1^3 + 4z_1^2 z_2^2 - 12 z_1^2 z_2 + z_1^2 + 16 z_1 z_2^2 - 8 z_1 z_2 + 16 z_2^2$$

can be written as:

$$p(z) = \left[\begin{array}{cccc} z_1 & z_2 & z_1 z_2 & z_1^2 \end{array}\right] \begin{pmatrix} 1 & -4 & -2 & 1 \\ -4 & 16 & 8 & -4 \\ -2 & 8 & 4 & -2 \\ 1 & -4 & -2 & 1 \end{pmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_1 z_2 \\ z_1^2 \end{bmatrix}$$

and, as the matrix is positive definite, the Cholesky factor gives the SOS decomposition $p(z) = (z_1 - 4z_2 - 2z_1 z_2 + z_1^2)^2$, so $p(z) \in \Sigma_z$. The reader is referred to Section A.2.6 for MATLAB® code details.

Evidently, all SOS polynomials are non-negative, but the converse is not true. There exist positive polynomials which are not sum of squares (Reznick, 2000; Blekherman, 2006). In general there exist only three combinations of degree d and number of variables n for which SOS polynomials are equivalent to the set of positive ones: $n = 2$, $d = 2$ and $n = 3$ with $d = 4$.

Unfortunately, polynomials resulting from Lyapunov stability conditions are not SOS in general. However, most of them are positive definite on a small enough range of their variables. Thus, local positivity of polynomials can be proven by using the well-known *Positivstellensatz* theorem, recalled next.

A.2.2 Positivstellensatz

This section recalls a central theorem from real algebraic geometry, the Positivstellensatz.

Definition A.4. *First, three main concepts are defined:*

- *Given polynomials $\{g_1, ..., g_t\} \in \mathcal{R}_z$, the* ***Multiplicative Monoid*** *is the finite set of all products involving g_j 's including 1 (empty product) and it will be denoted by $\mathcal{M}(g_1, ..., g_t)$.*

- *Given polynomials $\{f_1, ..., f_r\} \in \mathcal{R}_z$, the* ***Cone*** *is the set generated by the f_i 's in the form:*

$$\mathcal{P}(f_1, ..., f_r) := \left\{ s_0 + \sum_{i=1}^{l} s_i b_i | l \in \mathbb{Z}^+,\ s_i \in \Sigma_z, b_i \in \mathcal{M}(f_1, ..., f_r) \right\}.$$

  If the polynomials $s_i \in \Sigma_z$ and $f_i \in \mathcal{R}_z$, then $f_i^2 s_i \in \Sigma_z$. This allows expressing a cone of $\{f_1, ..., f_r\}$ as a sum of 2^r terms.

- *Given polynomials $\{h_1, ..., h_u\} \in \mathcal{R}_z$, the* ***Ideal*** *is the set generated by h_k 's in the form:*

$$\mathcal{I}(h_1, ..., h_u) := \left\{ \sum_{k=1}^{u} h_k p_k | p_k \in \mathcal{R}_z \right\}.$$

With those geometric definitions, the "Positivstellensatz" theorems is stated.

Theorem A.1 (Stengle (1974))**.** *Given polynomials $\{f_1, ..., f_r\}$, $\{g_1, ..., g_t\}$ and $\{h_1, ..., h_u\} \in \mathcal{R}_z$, the following statements are equivalent:*

1. *The set*
$$\left\{ z \in \mathbb{R}^n \;\middle|\; \begin{array}{l} f_1(z) \geq 0, ..., f_r(z) \geq 0 \\ g_1(z) \neq 0, ..., g_t(z) \neq 0 \\ h_1(z) = 0, ..., h_u(z) = 0 \end{array} \right\}$$
   is the empty set.

2. *There exist polynomials $f \in \mathcal{P}(f_1, ..., f_r),\ g \in \mathcal{M}(g_1, ..., g_t),\ h \in \mathcal{I}(h_1, ..., h_u)$ such that:*
$$f + g^2 + h = 0$$

This theorem is a powerful result which generalizes many well-known results like the S-procedure (Section A.1.2) and Finsler's lemma (Section A.1.3) to the polynomial case, by carefully choosing some SOS and arbitrary polynomial multipliers (Jarvis-Wloszek, Feeley, Tan, Sun and Packard, 2005, Pags. 4-7). In this way, many control problems within the polynomial framework can be addressed by combining the SOS optimization tools with the Positivstellensatz theorem. For instance, consider a region Ω defined by polynomial boundaries as follows:

$$\Omega = \{z : g_1(z) > 0, \ldots, g_{m_g}(z) > 0, h_1(z) = 0, \ldots, h_{m_h}(z) = 0\} \quad \text{(A.12)}$$

Corollary A.1. *If SOS polynomials $s_i(z) \in \Sigma_z$ and arbitrary ones $r_j(z) \in \mathcal{R}_z$ can be found fulfilling:*

$$p(z) - \epsilon(z) - \sum_{i=1}^{m_g} s_i(z)g_i(z) + \sum_{j=1}^{m_h} r_j(z)h_j(z) \in \Sigma_z \quad \text{(A.13)}$$

then $p(z)$ is locally greater or equal than $\epsilon(z)$ in the region Ω.

Proof. Indeed, note that, in the region Ω, the term $\sum_i s_i(z)g_i(z)$ is positive and $\sum_j r_j(z)h_i(z)$ is zero, so $p(z) - \epsilon(z) \geq \sum_i s_i(z)g_i(z) \geq 0$ for all $z \in \Omega$. □

The polynomials $(s_i(z), r_j(z))$ are denoted as *Positivstellensatz multipliers*, analogous to Lagrange and KKT ones in constrained optimization (Bertsekas, 1999).

The above corollary is a simplified version of the original Positivstellensatz result, in which less conservative expressions can be stated by setting higher degree multipliers (s_i, r_i), products of $p(z)$ with new multipliers or by adding more terms involving products of the $(p(z), g_i(z), h_j(z))$ belonging to the respective cone and ideal. However, more complex statements are avoided in practice because some of them lead to nonconvex problems and also the computational complexity increases considerably (Jarvis-Wloszek, Feeley, Tan, Sun and Packard, 2005).

Example A.2.2. The polynomial $p(x) = 1 - x^3$ is non-negative in the interval region of interest $\Omega = -0.5 \leq x \leq 0.5$. Indeed, let us first describe $\Omega = \{x : 0.25 - x^2 > 0\}$. If we execute the code included in Section A.2.6, we obtain that the quadratic multiplier $s(x) = 1.4076x^2$ fulfills

$$1 - x^3 - s(x)(0.25 - x^2) = (-0.61313x^2 + 1)^2 + \\ (-0.53472x^2 + 0.93508x)^2 + (0.86357x^2)^2 \in \Sigma_x \quad \text{(A.14)}$$

so we proved $1 - x^3 > 0 \; \forall x \in \Omega$.

A.2.3 SOS matrices

The propositions below will allow to use SOS programming to check positive-definiteness of matrices whose elements are polynomials.

Proposition A.1 (Prajna et al. (2004b))**.** *Let $L(x)$ be an $N \times N$ symmetric polynomial matrix of degree $2d$ in $x \in \mathbb{R}^n$. Furthermore, let $z(x)$ be a column vector whose entries are all monomials in x with degree no greater than d, and consider the following conditions:*

1. $L(x) \geq 0 \; \forall x \in \mathbb{R}^n$
2. *$v^T L(x) v$ is SOS, where $v \in \mathbb{R}^n$*
3. *There exists a positive semidefinite matrix Q such that*

$$v^T L(x) v = \left(v \otimes z(x)\right)^T Q \left(v \otimes z(x)\right)$$

where $\otimes$ denotes the Kronecker product.

Then (1) $\Leftarrow$ (2) *and* (2) $\Leftrightarrow$ (3)*.*

The proof of this proposition is based on the Cholesky decomposition (Higham, 1990) and the eigenvalue decomposition (Parrilo, 2000). In this way, the classical linear matrix inequality framework (positive-definiteness of matrices with linear expressions as elements (Boyd, Ghaoui, Feron and Balakrishnan, 1994)) is extended to the polynomial case.

Example A.2.3. According to the above result, testing if there exist coefficients a, b, such that the matrix

$$L(x) = \begin{pmatrix} 1 - x^2 & 0.1x + b \\ 0.1x + b & 3 - 0.1x^3 + ax^2 \end{pmatrix}$$

is positive semi-definite for all x such that $\{x \in \mathbb{R} : x^2 \leq 0.9\}$, can be done using Corollary A.1 by proving that the polynomial $H(x, v)$ below is SOS in the (augmented) set of variables (x, v):

$$H(x,v) = (1-x^2)v_1^2 + 2(0.1x+b)v_1v_2 + (3-0.1x^3+ax^2)v_2^2 + s(x,v)(0.9-x^2)$$

In this way, by setting a multiplier with arbitrary user-defined structure, say

$$s(x,v) = m_1 v_2^2 x^2 + m_2 v_1 v_2 x^2 + m_3 v_1 v_2 x + m_4 v_2^2 + m_5 v_1^2 x + m_6 v_1^2,$$

we must search for feasible a, b, m_1, ..., m_6 such that both $H(x,v)$ and $s(x,v)$ are SOS.

The obtained solution for this problem using SOSOPT is:

$$\begin{gathered} a = 0.5819, \quad b = -0.00030273, \\ m_1 = 1.1185,\ m_2 = 0,\ m_3 = 0.03929, \\ m_4 = 1.1366,\ m_5 = 0,\ m_6 = 1.0524 \end{gathered}$$

See next section for details on the solving procedure.

Moreover, the more terms and higher degree in $s(x,v)$ are, the less conservative the test is, at the expense of higher computational cost.

The above proposition increases the complexity due to the introduction of the auxiliary variables v. It is well-known that there exist better ways to deal with polynomial SOS matrices, from a computationally point of view:

Proposition A.2 (Scherer and Hol (2006))**.** *Let $F(x)$ be an $N \times N$ symmetric polynomial matrix of degree $2d$ in $x \in \mathbb{R}^n$. $F(x)$ is a SOS polynomial matrix if and only if there exist a constant matrix $Q \succeq 0$ satisfying*

$$F(x) = (I \otimes z(x))^T Q (I \otimes z(x)) \quad \forall x \in \mathbb{R}^n \tag{A.15}$$

with $z(x(t))$ being a column vector whose entries are all monomials in $x(t)$ with degree no greater than d.

Nevertheless, both results are mathematically equivalent.

A.2.4 SOS problems

On the following, basic formulation of the main SOS optimization problems is stated.

A.2.4.1 SOS Feasibility problem

The problem of feasibility is finding u such that

$$p_i(x,c) \quad \in \quad \Sigma_x,\ i = 1, 2, ..., N \tag{A.16}$$

where $c \in \mathbb{R}^m$ are *decision variables* (the "unknown" polynomial coefficients) and the polynomials p_i are given as part of the problem data and they are affine in c, i.e., they are of the form:

$$p_i(x,c) := p_{i0}(x) + p_{i1}(x)c_1 + \cdots + p_{im}(x)c_m$$

In this way, a solution belonging to the set of feasible polynomials fulfilling the constraints is found.

A.2.4.2 SOS Optimization problem

A SOS optimization problem is:

Minimize the linear objective function $d^T c$ subject to

$$p_i(x,c) \quad \in \quad \Sigma_x, \; i = 1, 2, ..., N \tag{A.17}$$

where c and p_i are the same as above and d is a weighting vector given as problem data.

Obviously the same problem can be also stated in terms of equality-only constraints, by introducing some extra SOS polynomials as dummy variables (Prajna, Papachristodoulou, Seiler and Parrilo, 2004a).

A.2.4.3 Generalized SOS problem

Similarly to GEVP's (Appendix A.1.4), SOS problems can be extended to allow one decision variable to enter bilinearly in the polynomial constraints. A *generalized SOS problem* is an optimization of the form:

Minimize λ subject to:

$$\lambda s_i(x,c) - p_i(x,c) \quad \in \quad \Sigma_x, \; i : 1, \ldots, N \tag{A.18}$$

$$s_i(x,c) \quad \in \quad \Sigma_x, \; i : 1, \ldots, N \tag{A.19}$$

$$q_i(x,c) \quad \in \quad \Sigma_x, \; i : 1, \ldots, M \tag{A.20}$$

where $\lambda \in \mathbb{R}$ and $c \in \mathbb{R}^m$ are decision variables. The polynomials p_i, s_i and q_i are given data and are affine in c. The optimization cost is linear in the decision variables and the constraints $s_i(x,c) \in \Sigma_x$ and $q_i(x,c) \in \Sigma_x$ are standard SOS constraints.

Note that this is not a convex SOS problem because the constraints $\lambda s_i(x,c)$ $-p_i(x,c) \in \Sigma_x$ are bilinear in the decision variables λ and c. However, the generalized SOS program is quasiconvex. The proof is omitted for brevity, see Seiler and Balas (2010). In consequence, the global minimum of a generalized SOS program can be computed via bisection on λ. Each step of the bisection involves holding λ fixed and solving for c such that satisfies the SOS constraints. This can be converted to a *feasibility* problem at each step of the bisection.

A.2.5 Numerical resolution

There exist software available to perform the conversion from SOS programs to SDP's. For instance, SOSOPT (Balas, Packard, Seiler and Topcu, 2012), SOSTOOLS (Prajna, Papachristodoulou, Seiler and Parrilo, 2004a), and YALMIP (Löfberg, 2009) are freely available MATLAB toolboxes for solving SOS optimizations. These packages allow the user to specify polynomial constraints using a symbolic or polynomial toolbox. The toolboxes convert the SOS optimization into an SDP which is solved with a freely available solver (say SeDuMi or SDPT3). Finally, these toolboxes convert the SDP solution back to a polynomial solution (Fig. A.1).

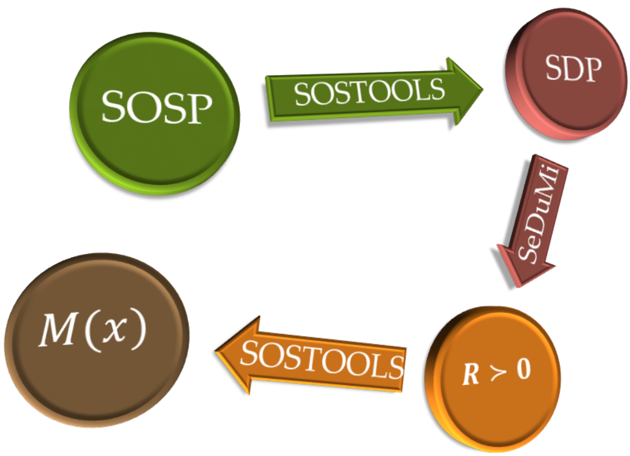

Figure A.1: Process of solving a SOS problem.

The following states the general steps in order to define and solve a SOS problem:

1. Initialize the problem.

2. Declare the problem variables: independent and decision variables plus polynomials on them.

3. Define sum-of-squares and polynomial constraints.

4. Specify the objective function (only for optimization problems).

5. Call the solver.

6. Obtain the solutions.

The main drawback is that size of resulting SDP grows very fast in both, the number of variables and degrees of the polynomials in the SOS optimization. While various techniques can be used to exploit problem structure (Gatermann and Parrilo, 2004; Seiler, Zheng and Balas, 2013), this computational growth is a generic trend in SOS optimizations. This roughly limits SOS methods to nonlinear analysis problems with at most 8-10 states and polynomial models with degree of at most 4-5, at least with up-to-date common computers.

Strictly-feasible solutions. Some complex SOS optimization problems may give solutions with numerical problems, meaning than the SDP solver has reached some minimum tolerances and accuracy on the computation is not totally guaranteed. In those cases, the problem has been solved by adding an additional constraint in order to force the solver to return a strict-feasible solution (certificate of positivity). This was done by following the techniques in Löfberg (2009), where *optimization* problems are reformulated as *feasibility* ones: in this case, if the optimization results in a optimal norm-bound γ^*, a strictly-feasible solution is obtained by posing a feasibility problem with the same constraints plus $\gamma < 1.001\gamma^*$ (see the cited reference for details).

The above procedure is very related to solving the so-called "generalized SOS problems" by bisection. Indeed all bisection iterations in such optimizations are just feasibility problems.

A.2.6 Code of examples

On the following, the MATLAB code for solving some simple examples presented in this thesis is shown:

Example YALMIPSOS:

Check if a polynomial $p(x, y, z)$ is SOS within the region defined by $\{-1 < x < 1, -1 < y < 1\}$.

```
%Define problem variables and the polynomial p:

x = sdpvar(1,1);
y = sdpvar(1,1);
z = sdpvar(1,1);
p = 12+y^2-2*x^3*y+2*y*z^2+x^6-2*x^3*z^2+z^4+x^2*y^2;

%Constraints:

g = [1-x;1+x;1-y;1+y];

%Define four parametric polynomials to act as Psatz
multipliers, quadratic for instance:

[s1,c1] = polynomial([x y],2);
[s2,c2] = polynomial([x y],2);
[s3,c3] = polynomial([x y],2);
[s4,c4] = polynomial([x y],2);

%Apply Positivstellensatz:

F=sos(p-[s1 s2 s3 s4]*g);
F=[F,sos(s1),sos(s2),sos(s3),sos(s4)];

%Call the solver with the necessary data:

options=sdpsettings('sos.model',2);
[sol,v,Q]=solvesos(F,[c1;c2;c3;c4],options);

%Obtaining solutions: V is the found SOS
decomposition and S1 is the first found
multiplier with its C1 coefficients.

S1=sdisplay(s1)
```

```
C1=double(c1)
monomials=sdisplay(v{1})
V=sdisplay(v{ 1}'*Q{1}*v{1}))}
```

Example SOSTOOLS:

Find a Lyapunov function for the following polynomial system:

$$\dot{x} = \begin{bmatrix} -x_1^3 - x_1 x_3^2 \\ -x_2 - x_1^2 x_2 \\ -x_3 + 3x_1^2 x_3 - \frac{3x_3}{(x_3^2+1)} \end{bmatrix}$$

```
% Define problem variables

pvar x1 x2 x3;
vars = [x1; x2; x3];

% Define the system dx/dt = f

f=[-x1^3-x1*x3^2;-x2-x1^2*x2;-x3+3*x1^2*x3
   -3*x3/(x3^2+1)];

% Initialize the SOS program:

prog = sosprogram(vars);

% Define the Lyapunov function structure
V(x1^2,x2^2,x3^2):

[prog,V] = sospolyvar(prog,[x1^2; x2^2; x3^2]);

% Define SOS constraints:

% Constr 1 : V(x)-epsilon(x1^2+x2^2+x3^2)>=0
prog = sosineq(prog,V-(x1^2+x2^2+x3^2));

% Constr 2: -dV/dx*(x3^2+1)*f>=0
expr=-(diff(V,x1)*f(1)+diff(V,x2)*f(2)
      +diff(V,x3)*f(3))*(x3^2+1);
```

```
prog = sosineq(prog,expr);

% Call the solver:
prog = sossolve(prog);

% Show the found Lyapunov function:
SOLV = sosgetsol(prog,V)
```

Code of Example A.2.1 using SOSTOOLS.

```
pvar z1 z2   %Independent variable definition

p=z1^4-4*z1^3*z2+2*z1^3+4*z1^2*z2^2-12*z1^2*z2+z1^2
+16*z1*z2^2+16*z2^2-8*z1*z2; %Defines the polynomial

[H,m]=findsos(p) %Calls solver to check if p is SOS
H = 16.0000   -4.0000    8.0000   -4.0000
    -4.0000    1.0000   -2.0000    1.0000
     8.0000   -2.0000    4.0000   -2.0000
    -4.0000    1.0000   -2.0000    1.0000
m = [   z2]   % SeDuMi finds the definite positive
    [   z1] % matrix H and the vector of monomials
    [z1*z2] % m is shown
    [ z1^2]
```

Code of Example A.2.2 using YALMIPSOS.

```
sdpvar x  %Define independent variables

%Create the Psatz multiplier x^2

[s,coef,monom]=polynomial(x,2,2);

%Define SOS constraints

constr=[sos(1-x^3-s*(0.25-x^2)),sos(s)];

obj=[];  %No objective to minimize
```

```
sol=solvesos(constr,obj,[],coef) %Solve SOS problem

%Display solution (the found Psatz multiplier)

disp('S='), sdisplay(double(coef)*monom)
S = '1.4076*x^2'
```

Code of Example 3.2.2 using YALMIPSOS.

```
%Define independent variables

sdpvar x1 x2 x3 sg1 sg2

% Defining system derivatives

dx1=-(sg1^2+sg2^2)*(x1^3)-(sg1^2+0.05*sg2^2)*x1*x3^2;
dx2=-sg1^2*x2-2*sg2^2*x2-(sg1^2+sg2^2)*x1^2*x2;
dx3=(sg1^2+sg2^2)*(3*x1^2*x3-4*x3);

% Tolerance epsilon and Lyapunov function

eps1=0.01*(x1^4+x2^4+x3^4);
eps2=0.01*(x1^4+x2^4+x3^4)*(sg1^2+sg2^2);
[V,coef,monom]=polynomial([x1,x2,x3],4,2);

% Define SOS constraints

dV=jacobian(V,x1)*dx1+jacobian(V,x2)*dx2
   +jacobian(V,x3)*dx3;
Constr=[sos(V-eps1),sos(-dV-eps2)];

obj=[];  %Minimization objective is not relevant

%Solve SOS problem

sol=solvesos(Constr,obj,[],coef)

%Display solution
```

```
disp('V='),sdisplay(clean(double(coef)'*monom,1e-7))
```

Code of Example 3.2.3 using SOSTOOLS.

```
pvar x1 x2 sg1 sg2 %Define independent variables

%Fuzzy polynomial model

dx1=(-3*x1+0.5*x2)*(sg1^2+sg2^2);
dx2=-2*x2*(sg1^2+sg2^2)+3*sg2^2*x1*x2;

alp=0.272; %Decay rate to prove

%Initialize SOS optimization program

SOSP=sosprogram([x1,x2,sg1,sg2]);

%Create a vector with the 2-degree monomials of x
m=monomials([x1,x2],2);
%Same but all the 2-degree monomials of [x,sg]
z=monomials([x1,x2,sg1,sg2],[2]);

%Define Lyapunov function and Psatz multipliers

[SOSP,V]=sospolyvar(SOSP,m);
[SOSP,s1]=sossosvar(SOSP,z); %s1 is defined SOS
[SOSP,s2]=sossosvar(SOSP,z); %s2 is defined SOS

%-dV/dx with decay rate

Q=-diff(V,x1)*dx1-diff(V,x2)*dx2
  -2*alp*V*(sg1^2+sg2^2);

%Define SOS constraints

SOSP=sosineq(SOSP,V-0.01*(x1^2+x2^2));
SOSP=sosineq(SOSP,Q-s1*(pi^2-x1^2)-s2*(pi^2-x2^2));

%Solve SOS problem and obtaining solutions
```

```
SOSP=sossolve(SOSP);
v=sosgetsol(SOSP,V)
```

Appendix B

Proofs

B.1 Proof of Proposition 2.1

Proof. Suppose a quadratic candidate Lyapunov function $V(x) = x^T P x$ for system (2.5). By condition (2.27), $V(x) > 0 \forall x \neq 0$.

Then consider the following inequality for all t:

$$\dot{V}(x(t)) + y^T(t)y(t) - \gamma u^T(t)u(t) \leq 0 \tag{B.1}$$

By integrating (B.1) from 0 to T and assuming that $x(0) = 0$, we obtain

$$V(x(T)) + \int_0^T (y^T(t)y(t) - \gamma u^T(t)u(t))dt \leq 0.$$

Since $V(x(T)) \geq 0$ by (2.27), ensuring (B.1) implies $\frac{\|y\|_2}{\|u\|_2} \leq \sqrt{\gamma}$.

Now, developing (B.1) (notation of time t is omitted for clarity) we have

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(z)\mu_j(z)x^T(A_i^T P + PA_i + C_i^T C_j)x$$
$$+ x^T PB_i u + u^T B^T Px - \gamma u^T u =$$
$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j[x^T u^T]\begin{bmatrix} A_i^T P + PA_i + C_i^T C_j & PB_i \\ B_i^T P & -\gamma I \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} < 0$$

which by Schur complement and the convex sum property of μ_i leads to the r LMI conditions (2.28). □

B.2 Proof of Lemma 3.1

Proof. Denote the Taylor approximations as:

$$f_n(x) := \sum_{i=0}^{n-1} \frac{f^{[i]}(0)}{(i)!} x^i \tag{B.2}$$

For later developments, denoting $f_0(x) = 0$, denote

$$T_n(x) := \frac{f(x) - f_n(x)}{x^n} = \frac{f^{[n]}(\psi(x))}{n!} \tag{B.3}$$

so the Taylor remainder is $f(x) - f_n(x) = T_n(x)x^n$.

In the region of interest Ω, $T_n(x)$ is bounded because $f^{[n]}(x)$ is continuous in Ω, as assumed in the lemma.

Denoting:

$$\psi_1 := \sup_{x \in \Omega} T_n(x), \quad \psi_2 := \inf_{x \in \Omega} T_n(x), \tag{B.4}$$

we may write:

$$T_n(x) = (\mu(x)\psi_1 + (1 - \mu(x))\psi_2)$$

with:

$$\mu(x) := \frac{T_n(x) - \psi_2}{\psi_1 - \psi_2} \tag{B.5}$$

Hence, as $f(x) = f_n(x) + T_n(x)x^n$, it can be expressed as:

$$f(x) = f_n(x) + (\mu(x)\psi_1 + (1 - \mu(x))\psi_2)x^n \tag{B.6}$$

so the polynomial consequent p_1 in (3.4) is given by $p_1(x) = f_n(x) + \psi_1 x^n$, and $p_2(x) = f_n(x) + \psi_2 x^n$. □

B.3 Proof of Theorem 3.1

Proof. Conditions (3.38) and (3.39) together mean (3.21) after carrying out some operations with the change of variable $\rho = P(\tilde{x})z(x), X(\tilde{x}) = P(\tilde{x})^{-1}$ and the evident fact of

$$P(\tilde{x})X(\tilde{x}) = I, \quad \frac{dP(\tilde{x})}{dx}X(\tilde{x}) + P(\tilde{x})\frac{dX(\tilde{x})}{dx} = 0.$$

So, jointly with (3.37), they make $V(x)$ to be a Lyapunov function for system (3.33), with controller (3.35), locally in Ω by Lemma 3.2 and Lemma A.1. The use of $X(\tilde{x})$ instead of $X(x)$ allows conditions (3.38)-(3.39) to be convex due to the fact that term $v^T \frac{dX(\tilde{x})}{dx}(B_i(x)K_j(x)z(x))v = 0$ in $\dot{V}(x)$. □

B.4 Proof of Theorem 6.3

Proof. Let's do the following change based on Euler discretization

$$\begin{aligned} e_{k+1} = &\sum_{i=1}^{2} \mu_i (\bar{\mathcal{P}}_i(x_k, \hat{x}_k) - L_i(\hat{x}_k, y_k)\bar{C}(x_k, \hat{x}_k) + \mathcal{E}_i w_k - L_i(\hat{x}_k, y_k) R\eta_k) \\ \cong &\sum_{i=1}^{2} \mu_i (e(t) + T_s(\bar{p}_i(x(t), \hat{x}(t)) - L_i(\hat{x}(t), y(t))\bar{C}(x(t), \hat{x}(t))) \\ &+ T_s E_i w(t) - T_s L_i(\hat{x}(t), y(t)) R\eta(t)) \end{aligned} \tag{B.7}$$

where T_s is the considered sample time.

In order to prove Lyapunov $\mathcal{H}_\infty$ stability, the following condition has to be fulfilled:

$$\begin{aligned} &\dot{V}(t) + e(t)^T D^T D e(t) - \gamma W(t)^T W(t) \cong \\ &\frac{V_{k+1} - V_k}{Ts} + e(t)^T D^T D e(t) - \gamma W(t)^T W(t) < 0 \end{aligned} \tag{B.8}$$

Then, using a quadratic candidate Lyapunov function and substituting (B.7) into (B.8), it leads to

$$\begin{aligned} &\sum_{i=1}^{2} \mu_i \left((*)^T Q \left(T_s(\bar{p}_i(x, \hat{x}) - L_i(\hat{x}, y)\bar{C}(x, \hat{x}) + E_i w - L_i(\hat{x}, y) R\eta) + e \right) \right) \\ &- e^T Q e + T_s(e^T D^T D e - \gamma W^T W) = 2T_s e^T Q \sum_{i=1}^{2} \mu_i \left(\bar{p}_i(x, \hat{x}) - \right. \\ &\left. L_i(\hat{x}, y)\bar{C}(x, \hat{x}) + E_i w - L_i(\hat{x}, y) R\eta \right) + e^T D^T D e - \gamma W^T W \big) + \\ &T_s^2 \sum_{i=1}^{2} \mu_i \left((*)^T Q \left(\bar{p}_i(x, \hat{x}) - L_i(\hat{x}, y)\bar{C}(x, \hat{x}) + E_i w - L_i(\hat{x}, y) R\eta \right) \right) < 0 \end{aligned} \tag{B.9}$$

This requires again a Schur complement on Q^{-1} (in the term which multiplies T_s^2) in order to obtain a convex constraint, equivalent to (6.28). However, if T_s is small enough, the nonconvex term on T_s^2 can be disregarded without making a large error. Then, if the convex-sum property is used and local information is added to (B.8) by using Positivstellensatz multipliers s_{1i} following Theorem A.1, constraints (6.39) are obtained.

If continuous decay-rate condition is added, a similar development can be done by substituting

$$e_{k+1} = \sum_{i=1}^{2} \mu_i(\bar{\mathcal{P}}_i(x_k, \hat{x}_k) - L_i(\hat{x}_k, C(x_k))\bar{C}(x_k, \hat{x}_k)) \cong$$

$$e(t) + \sum_{i=1}^{2} \mu_i((\bar{p}_i(x(t), \hat{x}(t)) - L_i(\hat{x}(t), C(x(t)))\bar{C}(x(t), \hat{x}(t))))T_s \quad \text{(B.10)}$$

into

$$\dot{V}(t) + 2\alpha e(t)^T Q e(t) \cong \frac{V_{k+1} - V_k}{T_s} + 2\alpha e(t)^T Q e(t) < 0$$

and again disregarding term on T_s^2 plus adding locality with Theorem A.1, it leads to conditions (6.40) (details omitted for brevity).

Finally conditions (6.41) say $\frac{dV}{dt} \geq -\frac{2}{T_s}V$ locally in Ω_x, Ω_e by adding again multipliers s_{3i}. Its meaning, in the linear case, would be forcing the real part of the state poles to be greater than $-\frac{1}{T_s}$ (Boyd, Ghaoui, Feron and Balakrishnan, 1994). □