# ŁOJASIEWICZ EXPONENT OF FAMILIES OF IDEALS, REES MIXED MULTIPLICITIES AND NEWTON FILTRATIONS

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ABSTRACT. We give an expression for the Łojasiewicz exponent of a wide class of n-tuples of ideals $(I_1, \dots, I_n)$ in $\mathcal{O}_n$ using the information given by a fixed Newton filtration. In order to obtain this expression we consider a reformulation of Łojasiewicz exponents in terms of Rees mixed multiplicities. As a consequence, we obtain a wide class of semi-weighted homogeneous functions $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ for which the Łojasiewicz of its gradient map $\nabla f$ attains the maximum possible value.

## 1. Introduction

Let $\mathcal{O}_n$ be the ring of complex analytic function germs $f : (\mathbb{C}^n, 0) \to \mathbb{C}$. S. Łojasiewicz proved in [19] (as a consequence of a more general result of functional analysis) that if $I$ is an ideal of $\mathcal{O}_n$ of finite colength and $g_1, \dots, g_s$ is a generating system of $I$, then there exists a real number $\alpha > 0$ for which there exist a constant $C > 0$ and an open neighbourhood $U$ of 0 in $\mathbb{C}^n$ such that

$$\|x\|^\alpha \leqslant C \sup_i |g_i(x)|, \tag{1}$$

for all $x \in U$. The infimum of such $\alpha$ is called the *Łojasiewicz exponent of* $I$ and is denoted by $\mathcal{L}_0(I)$. If $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ denotes a complex analytic map germ such that $g^{-1}(0) = \{0\}$, then the *Łojasiewicz exponent of* $g$ is defined as $\mathcal{L}_0(g) = \mathcal{L}_0(I)$, where $I$ denotes the ideal of $\mathcal{O}_n$ generated by the component functions of $g$. If $f \in \mathcal{O}_n$ has an isolated singularity at the origin, then the Łojasiewicz exponent of the gradient map $\nabla f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is particularly known in singularity theory, by virtue of the result of Teissier [30, p. 280] stating that the degree of $C^0$-determinacy of $f$ is equal to $[\mathcal{L}_0(\nabla f)] + 1$, where $[a]$ stands for the integer part of a given $a \in \mathbb{R}$. It is known that $\mathcal{L}_0(\nabla f)$ is an analytical invariant of $f$ but it is still unknown if $\mathcal{L}_0(\nabla f)$ is a topological invariant of $f$ (see [16] and [30]).

Let $\overline{\nu}_I : \mathcal{O}_n \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$ be the asymptotic Samuel function of $I$ (see [17] or [14, p. 139]). By a result of Nagata [21] the range of $\overline{\nu}_I$ is a subset of $\mathbb{Q}_{\geqslant 0} \cup \{+\infty\}$. If $J$ is any ideal of $\mathcal{O}_n$, let us define $\overline{\nu}_I(J) = \min\{\overline{\nu}_I(h_1), \dots, \overline{\nu}_I(h_r)\}$, where $h_1, \dots, h_r$ denotes any

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generating system of $J$. We will denote by $m_n$, or simply by $m$, the maximal ideal of $\mathcal{O}_n$. Lejeune and Teissier proved in [17] the following fundamental facts: $\mathcal{L}_0(I) = \frac{1}{\overline{\nu}_I(m)}$ (therefore $\mathcal{L}_0(I)$ is a rational number), relation (1) holds for $\alpha = \mathcal{L}_0(I)$, for some constant $C > 0$ and some open neighbourhood $U$ of $0 \in \mathbb{C}^n$, and $\mathcal{L}_0(I)$ is expressed as

$$\mathcal{L}_0(I) = \min\left\{\frac{p}{q} : p, q \in \mathbb{Z}_{\geqslant 1},\ m^p \subseteq \overline{I^q}\right\}, \tag{2}$$

where $\overline{J}$ denotes the integral closure of a given ideal $J$ of $\mathcal{O}_n$. The above expression was one of the motivations that lead the first author to introduce in [5] the notion of Łojasiewicz exponent of a set of ideals (see Definition 2.6). By substituting $m$ by a proper ideal $J$ of $\mathcal{O}_n$ in (2) we obtain what is known as the *Łojasiewicz exponent of $I$ with respect to $J$* (see relations (10) and (11)).

The effective computation of the Łojasiewicz exponent $\mathcal{L}_0(I)$ of a given ideal $I$ of $\mathcal{O}_n$ is a non-trivial problem, since it is intimately related with the determination of the integral closure of $I$. The authors applied in [6] the explicit construction of a log-resolution of $I$ to show an effective method to compute $\mathcal{L}_0(I)$. Newton polyhedra have proven to be a powerful tool in the estimation, and determination in some cases, of Łojasiewicz exponents, as can be seen in [2], [10], [18] and [22].

Let $w = (w_1, \dots, w_n) \in \mathbb{Z}^n_{\geqslant 1}$, we say that a monomial $x_1^{k_1} \cdots x_n^{k_n}$ has *$w$-degree $d$* when $w_1k_1 + \cdots + w_nk_n = d$. A polynomial function $f : \mathbb{C}^n \to \mathbb{C}$ is said to be *weighted homogeneous of degree $d$ with respect to $w$* when $f$ is written as a sum of monomials of $w$-degree $d$. A function $h \in \mathcal{O}_n$ is termed *semi-weighted* homogeneous of degree $d$ with respect to $w$ when $h$ is expressed as a sum $h = h_1 + h_2$, where $h_1$ is weighted homogeneous of degree $d$ with respect to $w$, $h_1$ has an isolated singularity at the origin and $h_2$ is a sum of monomials of $w$-degree greater that $d$.

The motivation of our work is the article [16] of Krasiński-Oleksik-Płoski, whose main result is a formula for the Łojasiewicz exponent $\mathcal{L}_0(\nabla f)$ of any weighted homogeneous function $f : \mathbb{C}^3 \to \mathbb{C}$ in terms of the weights and the degree of $f$. More precisely, if $f \in \mathcal{O}_3$ is weighted homogeneous with respect to $(w_1, w_2, w_3)$ of degree $d$ and $w_0 = \min\{w_1, w_2, w_3\}$ then it is proven in [16] that

$$\mathcal{L}_0(\nabla f) = \min\left\{\frac{d - w_0}{w_0},\ \prod_{i=1}^{3}\left(\frac{d}{w_i} - 1\right)\right\}. \tag{3}$$

We remark that when $d \geqslant 2w_i$, for all $i = 1, 2, 3$, then $\mathcal{L}_0(\nabla f) = \frac{d-w_0}{w_0}$. As a consequence of (3) we have that if $f : \mathbb{C}^3 \to \mathbb{C}$ is a weighted homogeneous function with respect to $(w_1, w_2, w_3)$, then $\mathcal{L}_0(\nabla f)$ is a topological invariant of $f$, by the results of Saeki [27] and Yau [32].

Let us fix a vector of weights $w = (w_1, \dots, w_n) \in \mathbb{Z}^n_{\geqslant 1}$ and let $w_0 = \min_i w_i$. Let $f \in \mathcal{O}_n$ be a semi-weighted homogeneous function of degree $d$ with respect to $w$. It is well-known that

$$\mathcal{L}_0(\nabla f) \leqslant \frac{d - w_0}{w_0}. \tag{4}$$

If $d < 2w_i$, for some $i \in \{1, \dots, n\}$, then it is easy to find examples where inequality (4) is strict. Assuming $d \geqslant 2w_i$, for all $i = 1, \dots, n$, then it is reasonable to conjecture that equality holds in (4).

In [7] we considered the problem of finding a sufficient condition on $f$ for equality in (4). We addressed this problem in the framework of Łojasiewicz exponents of sets of $n$ ideals in $\mathcal{O}_n$ (in the sense of [5]) and weighted homogeneous filtrations. Thus, we introduced in [7] the concept of sets of ideals admitting a *$w$-matching* (see Definition 4.1). The application of this notion to gradient maps lead to determine a wide class of functions for which equality holds in (4). In particular, we found that this equality is true for every semi-weighted homogeneous function $f \in \mathcal{O}_n$ of degree $d$ with respect to $w$ such that $w_i$ divides $d$, for all $i = 1, \dots, n$ (see [7, Corollary 4.16]).

In this article we show an extension of the main result of [7] to Newton filtrations in general (see Theorem 3.11). This extension projects to new results about the Łojasiewicz exponent of the gradient of semi-weighted homogeneous functions. In this direction, we emphasize that Corollary 4.12 (which is an immediate application of Corollary 4.11) shows a quite wide class of functions $f \in \mathcal{O}_n$ for which $\mathcal{L}_0(\nabla f)$ attains the maximum possible value, that is, such that equality holds in (4) (see also [16, Proposition 2]). The techniques that we will apply in this article come from multiplicity theory in local rings. More precisely, we use the notion of mixed multiplicities of a family of ideals of finite colength and its generalization to suitable families of ideals called Rees mixed multiplicities (see [4]).

Let us consider a Newton polyhedron $\Gamma_+$ in $\mathbb{R}^n_+$. The key ingredient in our approach to Łojasiewicz exponents in this article is the notion of $\Gamma_+$-linked pairs $(I; J_1, \dots, J_n)$, where $I, J_1, \dots, J_n$ are ideals of $\mathcal{O}_n$ (see Definition 3.10). This notion is expressed via the non-degeneracy condition explored in [8].

The article is organized as follows. In Section 2 we recall the basic definitions and previous results (mainly from [4] and [5]) that lead to the definition of Łojasiewicz exponent of a set of ideals. For the sake of completeness we also introduce in Section 2 some auxiliary results needed in the proof of the main result. In Section 3 we show the main result of the article (see Theorem 3.11) and discuss some examples. In Section 4 we particularize the techniques developed in Section 3 to weighted homogeneous filtrations and, as said before, we derive new results about the Łojasiewicz exponent of gradient maps.

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## 2. The Łojasiewicz exponent of a set of ideals

Let $(R, m)$ be a Noetherian local ring of dimension $n$ and let $I$ be an ideal of $R$ of finite colength (also called an $m$-primary ideal). Then we denote by $e(I)$ the Samuel multiplicity of $I$ (see [12], [14, §11] or [29] for the definition and basic properties of this notion). We recall that $e(I) = \ell(R/I)$ if $I$ admits a generating system formed by $n$ elements.

If $I_1, \dots, I_n$ are ideals of $R$ of finite colength, then we denote by $e(I_1, \dots, I_n)$ the mixed multiplicity of $I_1, \dots, I_n$ in the sense of Risler and Teissier [29] (see also [14, §17] or [31]).

**Definition 2.1.** [4] Let $(R, m)$ be a Noetherian local ring of dimension $n$. Let $I_1, \dots, I_n$ be ideals of $R$. Then we define the *Rees mixed multiplicity of* $I_1, \dots, I_n$ *as*

$$\sigma(I_1, \dots, I_n) = \max_{r \in \mathbb{Z}_+} e(I_1 + m^r, \dots, I_n + m^r), \tag{5}$$

when the number on the right hand side is finite. If the set of integers $\{e(I_1+m^r, \dots, I_n+m^r) : r \in \mathbb{Z}_+\}$ is non-bounded then we set $\sigma(I_1, \dots, I_n) = \infty$.

We remark that if $I_i$ is an ideal of $R$ of finite colength, for all $i = 1, \dots, n$, then $\sigma(I_1, \dots, I_n) = e(I_1, \dots, I_n)$. Moreover, if $I_1 = \cdots = I_n = I$, for some ideal $I$ of $R$ of finite colength, then $e(I_1, \dots, I_n) = e(I)$.

Let us suppose that the residue field $k = R/m$ is infinite. Let $I_1, \dots, I_n$ be ideals of $R$ and let $\{a_{i1}, \dots, a_{is_i}\}$ be a generating system of $I_i$, where $s_i \geqslant 1$, for $i = 1, \dots, n$. Let $s = s_1 + \cdots + s_n$. We say that a property holds for *sufficiently general* elements of $I_1 \oplus \cdots \oplus I_n$ if there exists a non-empty Zariski-open set $U$ in $k^s$ such that the said property holds for all elements $(g_1, \dots, g_n) \in I_1 \oplus \cdots \oplus I_n$ for which $g_i = \sum_j u_{ij} a_{ij}$, $i = 1, \dots, n$, where $(u_{11}, \dots, u_{1s_1}, \dots, u_{n1}, \dots, u_{ns_n}) \in U$.

The next proposition characterizes the finiteness of $\sigma(I_1, \dots, I_n)$.

**Proposition 2.2.** [4, p. 393] *Let $I_1, \dots, I_n$ be ideals of a Noetherian local ring $(R, m)$ such that the residue field $k = R/m$ is infinite. Then $\sigma(I_1, \dots, I_n) < \infty$ if and only if there exist elements $g_i \in I_i$, for $i = 1, \dots, n$, such that $\langle g_1, \dots, g_n \rangle$ has finite colength. In this case, we have that $\sigma(I_1, \dots, I_n) = e(g_1, \dots, g_n)$ for sufficiently general elements $(g_1, \dots, g_n) \in I_1 \oplus \cdots \oplus I_n$.*

If $I$ and $J$ are ideals of finite colength of $R$ such that $J \subseteq I$ then it is well-known that $e(J) \geqslant e(I)$ (see for instance [29, p. 292]). The following result extends this inequality to Rees mixed multiplicities.

**Lemma 2.3.** [5, p. 392] *Let $(R, m)$ be a Noetherian local ring of dimension $n \geqslant 1$. Let $J_1, \dots, J_n$ be ideals of $R$ such that $\sigma(J_1, \dots, J_n) < \infty$. Let $I_1, \dots, I_n$ be ideals of $R$ for which $J_i \subseteq I_i$, for all $i = 1, \dots, n$. Then $\sigma(I_1, \dots, I_n) < \infty$ and*

$$\sigma(J_1, \dots, J_n) \geqslant \sigma(I_1, \dots, I_n).$$

Let us recall some basic definitions. We will denote by $\mathbb{R}_+$ the set of non-negative real numbers. We also set $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$. Let us fix a coordinate system $x_1, \dots, x_n$ in $\mathbb{C}^n$. If $k = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$, we will denote the monomial $x_1^{k_1} \cdots x_n^{k_n}$ by $x^k$. Let $A \subseteq \mathbb{Z}_+^n$, then the *Newton polyhedron determined by* $A$, denoted by $\Gamma_+(A)$, is the convex hull of the set

$\{k+v : k \in A, v \in \mathbb{R}^n_+\}$. A subset $\Gamma_+ \subseteq \mathbb{R}^n_+$ is called a *Newton polyhedron* when $\Gamma_+ = \Gamma_+(A)$, for some $A \subseteq \mathbb{Z}^n_+$. A Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n_+$ is termed *convenient* when $\Gamma_+$ meets the $x_i$-axis in a point different from the origin, for all $i = 1, \dots, n$.

If $h \in \mathcal{O}_n$ and $h = \sum_k a_k x^k$ denotes the Taylor expansion of $h$ around the origin, then the *support of* $h$ is the set $\mathrm{supp}(h) = \{k \in \mathbb{Z}^n_+ : a_k \neq 0\}$. If $h \neq 0$, then the *Newton polyhedron of* $h$ is defined as $\Gamma_+(h) = \Gamma_+(\mathrm{supp}(h))$. If $h = 0$ then we set $\Gamma_+(h) = \emptyset$. If $I$ is an ideal of $\mathcal{O}_n$ and $g_1, \dots, g_s$ is a generating system of $I$, then we define the *Newton polyhedron of $I$* as the convex hull of $\Gamma_+(g_1) \cup \cdots \cup \Gamma_+(g_s)$. It is easy to check that the definition of $\Gamma_+(I)$ does not depend on the chosen generating system of $I$. We denote the Newton boundary of $\Gamma_+(I)$ by $\Gamma(I)$.

We say that a proper ideal $I$ of $\mathcal{O}_n$ is *monomial* when $I$ admits a generating system formed by monomials. We recall that if $I$ is a monomial ideal of $\mathcal{O}_n$ of finite colength, then $e(I) = n!\mathrm{V}_n(\mathbb{R}^n_+ \smallsetminus \Gamma_+(I))$, where $\mathrm{V}_n$ denotes $n$-dimensional volume (see for instance [31, p. 239]).

**Definition 2.4.** Let $I_1, \dots, I_n$ be monomial ideals of $\mathcal{O}_n$ with $\sigma(I_1, \dots, I_n) < \infty$. Then we denote by $\mathcal{S}(I_1, \dots, I_n)$ the family of maps $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $g^{-1}(0) = \{0\}$, $g_i \in I_i$, for all $i = 1, \dots, n$, and $\sigma(I_1, \dots, I_n) = e(g_1, \dots, g_n)$, where $e(g_1, \dots, g_n)$ stands for the multiplicity of the ideal of $\mathcal{O}_n$ generated by $g_1, \dots, g_n$. The elements of $\mathcal{S}(I_1, \dots, I_n)$ are characterized in [4, Theorem 3.10].

We denote by $\mathcal{S}_0(I_1, \dots, I_n)$ the set formed by the maps $(g_1, \dots, g_n) \in \mathcal{S}(I_1, \dots, I_n)$ such that $\Gamma_+(g_i) = \Gamma_+(I_i)$, for all $i = 1, \dots, n$.

Let $I_1, \dots, I_n$ be ideals of a local ring $(R, m)$ for which $\sigma(I_1, \dots, I_n) < \infty$. Then we define

$$r(I_1, \dots, I_n) = \min\big\{r \in \mathbb{Z}_+ : \sigma(I_1, \dots, I_n) = e(I_1 + m^r, \dots, I_n + m^r)\big\}. \tag{6}$$

We recall that if $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ is an analytic map germ such that $g^{-1}(0) = \{0\}$, then $\mathcal{L}_0(g)$ denotes the Łojasiewicz exponent of the ideal generated by the components of $g$.

**Theorem 2.5.** [5, p. 398] *Let $I_1, \dots, I_n$ be monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1, \dots, I_n)$ is finite. If $g \in \mathcal{S}_0(I_1, \dots, I_n)$, then $\mathcal{L}_0(g)$ depends only on $I_1, \dots, I_n$ and it is given by*

$$\mathcal{L}_0(g) = \min_{s \geqslant 1} \frac{r(I_1^s, \dots, I_n^s)}{s}. \tag{7}$$

The previous result motivated the following definition.

**Definition 2.6.** [5, p. 399] Let $(R, m)$ be a Noetherian local ring of dimension $n$. Let $I_1, \dots, I_n$ be ideals of $R$ for which $\sigma(I_1, \dots, I_n) < \infty$. We define the *Łojasiewicz exponent of* $I_1, \dots, I_n$ as

$$\mathcal{L}_0(I_1, \dots, I_n) = \inf_{s \geqslant 1} \frac{r(I_1^s, \dots, I_n^s)}{s}.$$

As a consequence of Lemma 2.8, we have that $r(I_1^s, \dots, I_n^s) \leqslant sr(I_1, \dots, I_n)$, for all $s \in \mathbb{Z}_{\geqslant 1}$. Hence $\mathcal{L}_0(I_1, \dots, I_n) \leqslant r(I_1, \dots, I_n)$.

The Łojasiewicz exponent given in Definition 2.6 is coherent with the original definition of Łojasiewicz exponent for an analytic map (see (1)), as is shown in the following result.

**Lemma 2.7.** *Let $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be an analytic map germ such that $g^{-1}(0) = \{0\}$. Then*

$$\mathcal{L}_0(g) = \mathcal{L}_0(\langle g_1 \rangle, \dots, \langle g_n \rangle). \tag{8}$$

*Proof.* Let us fix integers $r, s \geqslant 1$. Let $I$ denote the ideal generated by the components of $g$. It can be proved using Proposition 2.2 and Lemma 2.3 that

$$e(\langle g_1^s \rangle + m^r, \dots, \langle g_n^s \rangle + m^r) = e(\langle g_1^s, \dots, g_n^s \rangle + m^r).$$

Moreover $\overline{I^s} = \overline{\langle g_1^s, \dots, g_n^s \rangle}$ (see for instance [14, p. 344]). Then, by the Rees' multiplicity theorem (see [14, p. 222]) we obtain that $e(g_1^s, \dots, g_n^s) = e(\langle g_1^s \rangle + m^r, \dots, \langle g_n^s \rangle + m^r)$ if and only if $m^r \subseteq \overline{I^s}$. Hence we deduce that

$$r(\langle g_1^s \rangle, \dots, \langle g_n^s \rangle) = \min \left\{ r : m^r \subseteq \overline{I^s} \right\}.$$

Therefore

$$\begin{aligned} \mathcal{L}_0(\langle g_1 \rangle, \dots, \langle g_n \rangle) &= \min_{s \geqslant 1} \frac{\min \left\{ r : m^r \subseteq \overline{I^s} \right\}}{s} \\ &= \min \left\{ \frac{p}{q} : p, q \in \mathbb{Z}_{\geqslant 1},\ m^p \subseteq \overline{I^q} \right\} = \mathcal{L}_0(I), \end{aligned}$$

where the last equality follows from (2). □

Under the hypothesis of Definition 2.6, let us denote by $J$ a proper ideal of $R$. An easy application of Lemma 2.3 shows that

$$\sigma(I_1, \dots, I_n) = \max_{r \in \mathbb{Z}_+} \sigma(I_1 + J^r, \dots, I_n + J^r).$$

Hence, let us define

$$r_J(I_1, \dots, I_n) = \min \left\{ r \in \mathbb{Z}_+ : \sigma(I_1, \dots, I_n) = \sigma(I_1 + J^r, \dots, I_n + J^r) \right\}. \tag{9}$$

Let $I$ be an ideal of $R$ of finite colength. Then we denote by $r_J(I)$ the number $r_J(I, \dots, I)$, where $I$ is repeated $n$ times. We deduce from the Rees' multiplicity theorem (see [14, p. 222]) that if $R$ is quasi-unmixed then $r_J(I) = \min\{r \geqslant 1 : J^r \subseteq \overline{I}\}$.

**Lemma 2.8.** [7, p. 581] *Let $(R, m)$ be a Noetherian local ring of dimension $n$. Let $I_1, \dots, I_n$ be ideals of $R$ such that $\sigma(I_1, \dots, I_n) < \infty$ and let $J$ be a proper ideal of $R$. Then*

$$r_J(I_1^s, \dots, I_n^s) \leqslant s r_J(I_1, \dots, I_n)$$

$$r_{J^s}(I_1, \dots, I_n) \geqslant \frac{1}{s} r_J(I_1, \dots, I_n)$$

*for any integer $s \geqslant 1$.*

We remark that the previous lemma was proven in [7] under the assumption that the ideal $J$ has finite colength, but the same proof works equally for any proper ideal $J$ of $\mathcal{O}_n$.

If $I$ is an ideal of $\mathcal{O}_n$ of finite colength and $J$ is a proper ideal of $\mathcal{O}_n$, then the *Łojasiewicz exponent of $I$ with respect to $J$*, denoted by $\mathcal{L}_J(I)$, is defined as the infimum of those $\alpha > 0$ such that there exist a constant $C > 0$ and an open neighbourhood $U$ of $0 \in \mathbb{C}^n$ for which

$$\sup_j |h_j(x)|^\alpha \leqslant C \sup_i |g_i(x)|, \tag{10}$$

for all $x \in U$, where $\{h_j : j = 1, \dots, r\}$ and $\{g_i : i = 1, \dots, s\}$ are generating systems of $J$ and $I$, respectively. As a consequence of [17, §7] we have that $\mathcal{L}_J(I)$ is a rational number and

$$\mathcal{L}_J(I) = \min\left\{\frac{p}{q} : p, q \in \mathbb{Z}_{\geqslant 1},\ J^p \subseteq \overline{I^q}\right\}. \tag{11}$$

If $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ is an analytic map germ such that $g^{-1}(0) = \{0\}$ and $J$ is a proper ideal of $\mathcal{O}_n$ then we denote by $\mathcal{L}_J(g)$ the Łojasiewicz exponent $\mathcal{L}_J(I)$, where $I$ is the ideal generated by the component functions of $g$.

Now we extend Definition 2.6 by considering $r_J(I_1, \dots, I_n)$ instead of $r(I_1, \dots, I_n)$.

**Definition 2.9.** [7, p. 581] Let $(R, m)$ be a Noetherian local ring of dimension $n$. Let $I_1, \dots, I_n$ be ideals of $R$ such that $\sigma(I_1, \dots, I_n) < \infty$. Let $J$ be a proper ideal of $R$. We define the *Łojasiewicz exponent of* $I_1, \dots, I_n$ *with respect to* $J$, denoted by $\mathcal{L}_J(I_1, \dots, I_n)$, as

$$\mathcal{L}_J(I_1, \dots, I_n) = \inf_{s \geqslant 1} \frac{r_J(I_1^s, \dots, I_n^s)}{s}. \tag{12}$$

Under the conditions of the previous definition, we observe that $\mathcal{L}_J(I_1, \dots, I_n)$ is expressed as a limit inferior (see [7, p. 581] for details), that is:

$$\mathcal{L}_J(I_1, \dots, I_n) = \liminf_{s \to \infty} \frac{r_J(I_1^s, \dots, I_n^s)}{s}.$$

If $I$ is an $m$-primary ideal of $R$, then we denote by $\mathcal{L}_J(I)$ the number $\mathcal{L}_J(I, \dots, I)$, where $I$ is repeated $n$ times. We remark that when $R$ is quasi-unmixed then $\mathcal{L}_J(I)$ is given by

$$\mathcal{L}_J(I) = \min\left\{\frac{p}{q} : p, q \in \mathbb{Z}_{\geqslant 1},\ J^p \subseteq \overline{I^q}\right\}, \tag{13}$$

by a direct application of the Rees' multiplicity theorem.

We point out that we are denoting $\mathcal{L}_m(I_1, \dots, I_n)$ by $\mathcal{L}_0(I_1, \dots, I_n)$ and that $\mathcal{L}_J(I_1, \dots, I_n)$ is not defined when $J$ is the zero ideal. If $I_i$ is an ideal of $\mathcal{O}_n$, for all $i = 1, \dots, n$, then the subscript in $\mathcal{L}_0(I_1, \dots, I_n)$ corresponds to the commonly used notation to refer to Łojasiewicz exponents in a neighbourhood of $0 \in \mathbb{C}^n$, as defined in (1). If $J$ is a proper ideal of $\mathcal{O}_n$, we also remark that the result analogous to Lemma 2.7 obtained by writing $\mathcal{L}_0$ instead of $\mathcal{L}_J$ in equality (8) also holds; it follows by a straightforward reproduction of the proof of Lemma 2.7 consisting of replacing $m$ by $J$.

For the sake of completeness, we recall the following two results, which will be applied in the next section.

**Lemma 2.10.** [7, p. 582] *Let $(R, m)$ be a quasi-unmixed Noetherian local ring of dimension $n$. Let $I_1, \dots, I_n$ be ideals of $R$ for which $\sigma(I_1, \dots, I_n) < \infty$. If $J_1, J_2$ are proper ideals of $R$ such that $J_2$ has finite colength then*

$$\mathcal{L}_{J_1}(I_1, \dots, I_n) \leqslant \mathcal{L}_{J_1}(J_2)\mathcal{L}_{J_2}(I_1, \dots, I_n).$$

**Proposition 2.11.** *Let $(R, m)$ be a Noetherian local ring of dimension $n$. Let $J$ be a proper ideal of $R$. For each $i = 1, \dots, n$ let us consider ideals $I_i$ and $J_i$ of $R$ such that $I_i \subseteq J_i$ and $\sigma(I_1, \dots, I_n) = \sigma(J_1, \dots, J_n) < \infty$. Then*

$$\mathcal{L}_J(I_1, \dots, I_n) \leqslant \mathcal{L}_J(J_1, \dots, J_n), \tag{14}$$

*Proof.* It follows by replacing $m$ by $I$ in the proof of [5, Proposition 4.7] □

We remark that if $(R, m)$ is a Noetherian quasi-unmixed local ring of dimension $n$ and $I_1, I_2$ are ideals of $R$ of finite colength such that $I_1 \subseteq I_2$ then $\mathcal{L}_J(I_1) \geqslant \mathcal{L}_J(J_2)$, as a consequence of (13). However, the analogous inequality for Łojasiewicz exponents of sets of ideals does not hold in general, as the following example shows.

**Example 2.12.** Let us consider the ideals $I_1, I_2$ of $\mathcal{O}_2$ defined by $I_1 = \langle x^3 \rangle$ and $I_2 = \langle y^3 \rangle$. Let $J_1, J_2$ be the ideals of $\mathcal{O}_2$ defined by $J_1 = \langle x^3, xy \rangle$, $J_2 = I_2$. Then we have that $I_i \subseteq J_i$, $i = 1, 2$, but $\mathcal{L}_0(I_1, I_2) = 3$ and $\mathcal{L}_0(J_1, J_2) = 6$.

## 3. Newton filtrations

Let us fix a Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n_+$. If $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$ then we define

$$\ell(v, \Gamma_+) = \min\{\langle v, k \rangle : k \in \Gamma_+\}$$
$$\Delta(v, \Gamma_+) = \big\{k \in \Gamma_+ : \langle v, k \rangle = \ell(v, \Gamma_+)\big\},$$

where $\langle\, , \rangle$ stands for the standard scalar product in $\mathbb{R}^n$. A *face* of $\Gamma_+$ is any set of the form $\Delta(v, \Gamma_+)$, for some $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$. Hence we also say that $\Delta(v, \Gamma_+)$ is the *face of* $\Gamma_+$ *supported by* $v$. The *dimension* of a face $\Delta$ of $\Gamma_+$ is the minimum of the dimensions of the affine subspaces of $\mathbb{R}^n$ containing $\Delta$. If $\Delta$ is a face of $\Gamma_+$ of dimension $n-1$ then we say that $\Delta$ is a *facet* of $\Gamma_+$.

It is easy to observe that a face $\Delta$ of $\Gamma_+$ is compact if and only if it is supported by a vector $v \in (\mathbb{R}_+ \smallsetminus \{0\})^n$. The union of all compact faces of $\Gamma_+$ will be denoted by $\Gamma$; this is also known as the *Newton boundary of* $\Gamma_+$. We remark that $\Gamma_+$ determines and is determined by $\Gamma$, since $\Gamma_+ = \Gamma + \mathbb{R}^n_+$.

We denote by $\Gamma_-$ the union of all segments joining the origin and some point of $\Gamma$. Therefore $\Gamma_-$ is a compact subset of $\mathbb{R}^n_+$. If $\Gamma_+$ is convenient, then $\Gamma_-$ is equal to the closure of $\mathbb{R}^n_+ \smallsetminus \Gamma_+$.

If $\Delta$ is a face of $\Gamma_+$, then $C(\Delta)$ denotes the cone formed by all half-rays emanating from the origin and passing through some point of $\Delta$.

We say that a vector $v \in \mathbb{Z}^n_+ \setminus \{0\}$ is *primitive* when the non-zero coordinates of $v$ are mutually prime integer numbers. Then any facet of $\Gamma_+$ is supported by a unique primitive vector of $\mathbb{Z}^n_+$. Let us denote by $\mathcal{F}(\Gamma_+)$ the set of primitive vectors of $\mathbb{R}^n_+$ supporting some facet of $\Gamma_+$ and by $\mathcal{F}_c(\Gamma_+)$ the set of vectors $v \in \mathcal{F}(\Gamma_+)$ such that $\Delta(v, \Gamma_+)$ is compact. Let us remark that if $\Gamma_+$ is convenient then $\mathcal{F}(\Gamma_+) = \mathcal{F}_c(\Gamma_+) \cup \{e_1, \dots, e_n\}$, where $e_1, \dots, e_n$ is the canonical basis of $\mathbb{R}^n$.

Let us suppose that $\mathcal{F}_c(\Gamma_+) = \{v^1, \dots, v^r\}$. Therefore $\ell(v^i, \Gamma_+) \neq 0$, for all $i = 1, \dots, r$. Let us denote by $M_\Gamma$ the least common multiple of the set of integers $\{\ell(v^1, \Gamma_+), \dots, \ell(v^r, \Gamma_+)\}$. Hence we define the *filtrating map* associated to $\Gamma_+$ as the map $\phi : \mathbb{R}^n_+ \to \mathbb{R}_+$ given by

$$\phi_\Gamma(k) = \min\left\{ \frac{M_\Gamma}{\ell(v^i, \Gamma_+)} \langle k, v^i \rangle : i = 1, \dots, r \right\}, \quad \text{for all } k \in \mathbb{R}^n_+.$$

We observe that $\phi_\Gamma(\mathbb{Z}^n_+) \subseteq \mathbb{Z}^n_+$, $\phi_\Gamma(k) = M_\Gamma$, for all $k \in \Gamma$, and the map $\phi_\Gamma$ is linear on each cone $C(\Delta)$, where $\Delta$ is any compact face of $\Gamma_+$.

Let us define the map $\nu_\Gamma : \mathcal{O}_n \to \mathbb{R}_+ \cup \{+\infty\}$ by $\nu_\Gamma(h) = \min\{\phi_\Gamma(k) : k \in \operatorname{supp}(h)\}$, for all $h \in \mathcal{O}_n$, $h \neq 0$; we set $\nu_\Gamma(0) = +\infty$. We refer to $\nu_\Gamma$ as the *Newton filtration* induced by $\Gamma_+$ (see also [8, 15]).

From now on, we will assume that $\Gamma_+$ is a convenient Newton polyhedron in $\mathbb{R}^n_+$.

Let $h \in \mathcal{O}_n$ and let $h = \sum_k a_k x^k$ be the Taylor expansion of $h$ around the origin. If $A$ is a compact subset of $\mathbb{R}^n$ then we denote by $h_A$ the sum of all terms $a_k x^k$ such that $k \in A$. If $\operatorname{supp}(h) \cap A = \emptyset$, then we set $h_A = 0$. Let $J$ be an ideal of $\mathcal{O}_n$ and let $g_1, \dots, g_s$ be a generating system of $J$. We recall that $J$ is said to be *Newton non-degenerate* (see [3] or [28]) when

$$\big\{x \in \mathbb{C}^n : (g_1)_\Delta(x) = \cdots = (g_s)_\Delta(x) = 0\big\} \subseteq \big\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\big\},$$

as set germs at $0 \in \mathbb{C}^n$, for each compact face $\Delta$ of $\Gamma_+(J)$ (see Theorem 3.6). It is immediate to check that this definition does not depend on the chosen generating system of $J$. In particular, any monomial ideal is Newton non-degenerate.

The next result compares the asymptotic Samuel function and the Newton filtration.

**Proposition 3.1.** [3, p. 26] *Let $J \subseteq \mathcal{O}_n$ be an ideal of finite colength. Let $\Gamma$ denote the Newton boundary of $\Gamma_+(J)$ and let $M = M_\Gamma$. Then $M\overline{\nu}_J \leqslant \nu_\Gamma$ and equality holds if and only if $J$ is Newton non-degenerate.*

As a consequence of the previous result, if $J$ is an ideal of finite colength of $\mathcal{O}_n$ and $r_i = \min\{r : re_i \in \Gamma_+(J)\}$, for all $i = 1, \dots, n$, then $\max\{r_1, \dots, r_n\} \leqslant \mathcal{L}_0(J)$ and equality holds if $J$ is a Newton non-degenerate ideal (see [3, p. 27] for details).

Given an integer $r \in \mathbb{Z}_{\geqslant 0}$, we denote by $\mathcal{A}_r$ the ideal of $\mathcal{O}_n$ generated by the elements $h \in \mathcal{O}_n$ such that $\nu_\Gamma(h) = r$ (we assume that the ideal generated by the empty set is 0). In particular, $\mathcal{A}_{M_\Gamma} = \langle x^k : k \in \Gamma_+ \rangle$.

Moreover, we denote by $\mathcal{B}_r$ the ideal of $\mathcal{O}_n$ generated by the elements $h \in \mathcal{O}_n$ for which $\nu_\Gamma(h) \geqslant r$. Then

$$\mathcal{B}_r = \big\{h \in \mathcal{O}_n : \phi_\Gamma(\mathrm{supp}(h)) \subseteq [r, +\infty[\,\big\} \cup \{0\},$$

for all $r \geqslant 0$ and $\nu_\Gamma(h) = \max\{r \geqslant 0 : h \in \mathcal{B}_r\}$, for all $h \in \mathcal{O}_n$, $h \neq 0$. We will refer indistinctly to the map $\nu_\Gamma$ and to the family of ideals $\{\mathcal{B}_r\}_{r\geqslant 1}$ as the *Newton filtration induced by* $\Gamma_+$.

It is immediate to check that

(a) $\mathcal{B}_r$ is an integrally closed monomial ideal of finite colength, for all $r \geqslant 1$;
(b) $\mathcal{B}_r\mathcal{B}_s \subseteq \mathcal{B}_{r+s}$, for all $r, s \geqslant 1$;
(c) $\mathcal{B}_0 = \mathcal{O}_n$.

If $I$ is an ideal of $\mathcal{O}_n$, then we denote by $\nu_\Gamma(I)$ the maximum of those $r$ such that $I \subseteq \mathcal{B}_r$. Then, if $g_1, \dots, g_s$ denotes any generating system of $I$, we have

$$\nu_\Gamma(I) = \min\{\nu_\Gamma(g_1), \dots, \nu_\Gamma(g_s)\}.$$

Given an integer $r \geqslant 0$, we observe that $\mathcal{A}_r \subseteq \mathcal{B}_r$ and $\overline{\mathcal{A}_r} \neq \mathcal{B}_r$ in general. Moreover it follows easily that $\overline{\mathcal{A}_r} = \mathcal{B}_r$ if and only if $\mathcal{A}_r$ is an ideal of finite colength of $\mathcal{O}_n$.

Let us remark that $\mathrm{supp}(\mathcal{A}_{M_\Gamma}) = \Gamma_+ \cap \mathbb{Z}^n_+$, $\mathcal{A}_{M_\Gamma}$ has finite colength and $e(\mathcal{A}_{M_\Gamma}) = n!\mathrm{V}_n(\Gamma_-)$, since $\Gamma$ is convenient and $\mathcal{A}_{M_\Gamma}$ is a monomial ideal (see the paragraph before Definition 2.4).

**Proposition 3.2.** *Let us fix a family of ideals $J_1, \dots, J_n$ of $\mathcal{O}_n$ such that $\sigma(J_1, \dots, J_n) < \infty$. Let $\nu_\Gamma(J_i) = r_i$, for all $i = 1, \dots, n$, and let $M = M_\Gamma$. Then*

$$\sigma(J_1, \dots, J_n) \geqslant \frac{r_1 \cdots r_n}{M^n} n!\mathrm{V}_n(\Gamma_-). \tag{15}$$

*Proof.* By Proposition 2.2 we have that $\sigma(J_1, \dots, J_n) = e(g_1, \dots, g_n)$, for a sufficiently general element $(g_1, \dots, g_n) \in J_1 \oplus \cdots \oplus J_n$. Then the result arises as a direct application of [8, Theorem 3.3]. ☐

As a consequence of [8, Theorem 3.3], equality in (15) is characterized by means of a condition imposed to any element $(g_1, \dots, g_n) \in J_1 \oplus \cdots \oplus J_n$ such that $e(g_1, \dots, g_n) = \sigma(J_1, \dots, J_n)$ (we refer the reader to [8] for details). By coherence with the nomenclature of [8, Theorem 3.3] we introduce the following definition.

**Definition 3.3.** Let $J_1, \dots, J_n$ be a family of ideals of $\mathcal{O}_n$ such that $\sigma(J_1, \dots, J_n) < \infty$. Let $M = M_\Gamma$. We say that $(J_1, \dots, J_n)$ is *non-degenerate on* $\Gamma_+$, or that $(J_1, \dots, J_n)$ is $\Gamma_+$*-non-degenerate*, when equality holds in (15). That is, when $\sigma(J_1, \dots, J_n) = \frac{r_1 \cdots r_n}{M^n} e(\mathcal{A}_M)$.

Under the hypothesis of the previous definition, let us suppose that $J_1$ is principal, that is, $J_1 = \langle h \rangle$, for some $h \in \mathcal{O}_n$. Then, in order to simplify the notation, we will write $(h, J_2, \dots, J_n)$ instead of $(\langle h \rangle, J_2, \dots, J_n)$. We will adopt the same simplification if any other ideal $J_i$ is principal, for some $i \in \{1, \dots, n\}$. Hence, the previous definition applies to germs of complex analytic maps $(g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $g^{-1}(0) = \{0\}$.

If $r_1,\dots,r_n\in\mathbb{Z}_{\geqslant 1}$, then it is not true in general that $\sigma(\mathcal{A}_{r_1},\dots,\mathcal{A}_{r_n})<\infty$, even if $\mathcal{A}_{r_i}\neq 0$, for all $i=1,\dots,n$. However $\sigma(\mathcal{B}_{r_1},\dots,\mathcal{B}_{r_n})<\infty$, since $\mathcal{B}_{r_i}$ has finite colength, for all $i=1,\dots,n$. If $\sigma(\mathcal{A}_{r_1},\dots,\mathcal{A}_{r_n})<\infty$, then it is also not true in general that $(\mathcal{A}_{r_1},\dots,\mathcal{A}_{r_n})$ is non-degenerate on $\Gamma_+$, as the next example shows. If $\Gamma_+$ has only one compact face of dimension $n-1$ (that is, if the Newton filtration induced by $\Gamma_+$ is a weighted homogeneous filtration), then the condition $\sigma(\mathcal{A}_{r_1},\dots,\mathcal{A}_{r_n})<\infty$ implies that $(\mathcal{A}_{r_1},\dots,\mathcal{A}_{r_n})$ is non-degenerate on $\Gamma_+$ (see [7, Proposition 4.2] for details).

**Example 3.4.** Let $J=\langle x^4,xy,y^4\rangle$ and let $\Gamma_+=\Gamma_+(J)$. We observe that $M_\Gamma=4$ and the map $\phi_\Gamma:\mathbb{R}^2_+\to\mathbb{R}$ is given by $\phi_\Gamma(k_1,k_2)=\min\{k_1+3k_2,3k_1+k_2\}$, for all $(k_1,k_2)\in\mathbb{R}^2_+$. Hence we have that $\mathcal{A}_5=\langle x^5,x^2y,xy^2,y^5\rangle$ and

$$\sigma(\mathcal{A}_5,\mathcal{A}_5)=e(\mathcal{A}_5)=13\neq\frac{5\cdot 5}{4^2}e(J)=\frac{200}{16}=\frac{25}{2}.$$

For the sake of completeness, we show in Proposition 3.5 a reformulation of [8, Theorem 3.3] considering the notion of Rees mixed multiplicity. If $\Delta$ is a compact face of $\Gamma_+$, then we denote by $\mathcal{R}_\Delta$ the subring of $\mathcal{O}_n$ formed by all germs $h\in\mathcal{O}_n$ such that $\operatorname{supp}(h)\subseteq C(\Delta)$. If $\alpha>0$, then $\alpha\Delta$ will denote the set $\{\alpha k: k\in\Delta\}$. Given a function germ $h\in\mathcal{O}_n$, if $h=\sum_k a_kx^k$ is the Taylor expansion of $h$ around the origin, then we denote by $p_\Delta(h)$ the sum of all terms $a_kx^k$ for which $\nu_\Gamma(x^k)=\nu_\Gamma(h)$ and $k\in C(\Delta)$. If no such terms exist, then we set $p_\Delta(h)=0$. We recall that $h_\Delta$ denotes the sum of all terms $a_kx^k$ such that $k\in\Delta$. Hence, if $d=\nu_\Gamma(h)$, we observe that

$$p_\Delta(h)=h_{\frac{d}{M}\Delta}. \tag{16}$$

**Proposition 3.5.** *Let $g=(g_1,\dots,g_n):(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ be a complex analytic map. Then the following conditions are equivalent:*

1. *$g$ is non-degenerate on $\Gamma_+$;*
2. *for each compact facet $\Delta$ of $\Gamma_+$, the ideal of $\mathcal{R}_\Delta$ generated by $p_\Delta(g_1),\dots,p_\Delta(g_n)$ has finite colength in $\mathcal{R}_\Delta$.*

*Proof.* Condition (1) means that $e(g_1,\dots,g_n)=\frac{r_1\cdots r_n}{M^n}n!\mathrm{V}_n(\Gamma_-)$, where $r_i=\nu_\Gamma(g_i)$, for all $i=1,\dots,n$. Therefore the result is an immediate consequence of [8, Theorem 3.3]. $\square$

We remark that the equivalence of the previous result is considered as the definition of non-degeneracy on $\Gamma_+$ given in [8]. We show a result (Corollary 3.7) that helps in the task of testing condition (2) of the previous proposition. First we recall a result of Kouchnirenko [15] that is stated in the context of Laurent series but that we will state here for germs of $\mathcal{O}_n$.

**Theorem 3.6.** [15, Théorème 6.2] *Let $\Delta$ be a compact face of $\Gamma_+$ and let $g_1,\dots,g_s\in\mathcal{O}_n$ such that* $\operatorname{supp}(g_i)\subseteq\Gamma_+$, *for all $i=1,\dots,s$. Then the following conditions are equivalent:*

1. *the ideal of $\mathcal{R}_\Delta$ generated by $(g_1)_\Delta,\dots,(g_s)_\Delta$ has finite colength in $\mathcal{R}_\Delta$;*

(2) *for all compact faces* $\Delta' \subseteq \Delta$, *the set germ at* $0$ *of common zeros of* $(g_1)_{\Delta'}, \dots, (g_s)_{\Delta'}$ *is contained in* $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$.

**Corollary 3.7.** *Under the hypothesis of the previous theorem, the following conditions are equivalent:*

(1) *the ideal of* $\mathcal{R}_\Delta$ *generated by* $p_\Delta(g_1), \dots, p_\Delta(g_s)$ *has finite colength in* $\mathcal{R}_\Delta$*;*
(2) *for all compact faces* $\Delta' \subseteq \Delta$, *the set germ at* $0$ *of common zeros of* $p_{\Delta'}(g_1), \dots, p_{\Delta'}(g_s)$ *is contained in* $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$.

*Proof.* Let $r_i = \nu_\Gamma(g_i)$, for all $i = 1, \dots, s$, and let $r = r_1 \cdots r_s$. Let $I$ denote the ideal of $\mathcal{R}_\Delta$ generated by $p_\Delta(g_1), \dots, p_\Delta(g_s)$ and let $J$ denote the ideal of $\mathcal{R}_\Delta$ generated by $\{p_\Delta(g_1)^{r/r_1}, \dots, p_\Delta(g_s)^{r/r_s}\}$. We observe that $I$ has finite colength in $\mathcal{R}_\Delta$ if and only if $J$ has finite colength in $\mathcal{R}_\Delta$, since $I$ and $J$ have the same radical. It is straightforward to check (see relation (16)) that

$$p_\Delta(g_i)^{r/r_i} = ((g_i)_{\frac{r_i}{M}\Delta})^{r/r_i} = (g_i^{r/r_i})_{\frac{r}{M}\Delta}, \tag{17}$$

for all $i = 1, \dots, s$. Then, by Theorem 3.6, the ideal $J$ has finite colength if and only if the set germ at 0 of common zeros of $(g_1^{r/r_1})_A, \dots, (g_s^{r/r_s})_A$ is contained in $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$, for any compact face $A \subseteq \frac{r}{M}\Delta$. Given a subset $\Delta' \subseteq \mathbb{R}^n_+$, we observe that $\Delta'$ is a compact face of $\Delta$ if and only if $\frac{r}{M}\Delta'$ is a compact face of $\frac{r}{M}\Delta$. Then the result follows immediately from relation (17) and Theorem 3.6. $\square$

**Corollary 3.8.** *Let* $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ *be a complex analytic map such that* $g^{-1}(0) = \{0\}$. *Then* $g$ *is non-degenerate on* $\Gamma_+$ *if and only if the set germ at* $0$ *of common zeros of* $p_\Delta(g_1), \dots, p_\Delta(g_n)$ *is contained in* $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$, *for all compact faces* $\Delta$ *of* $\Gamma_+$.

*Proof.* It follows immediately as a consequence of Proposition 3.5 and Corollary 3.7. $\square$

We will use the following lemma in the proof of the main result (Theorem 3.11).

**Lemma 3.9.** *Let* $J_1, \dots, J_n$ *be ideals of* $\mathcal{O}_n$ *such that* $(J_1, \dots, J_n)$ *is non-degenerate on* $\Gamma_+$. *Then* $(J_1 + \mathcal{A}_M^r, \dots, J_n + \mathcal{A}_M^r)$ *is also non-degenerate on* $\Gamma_+$, *for all* $r \geqslant 1$, *where* $M = M_\Gamma$. *That is*

$$e(J_1 + \mathcal{A}_M^r, \dots, J_n + \mathcal{A}_M^r) = \frac{\min\{r_1, rM\} \cdots \min\{r_n, rM\}}{M^n} n! \mathrm{V}_n(\Gamma_-), \tag{18}$$

*for all* $r \geqslant 1$, *where* $r_i = \nu_\Gamma(J_i)$, *for all* $i = 1, \dots, n$.

*Proof.* Let us fix an integer $r \geqslant 1$. Let $S = \{i : r_i < rM\}$. If $S = \emptyset$, then $J_i \subseteq \mathcal{A}_{rM}$, for all $i = 1, \dots, n$. Thus, since $\overline{\mathcal{A}_M^r} = \overline{\mathcal{A}_{rM}}$ and mixed multiplicities are invariant by integral closures, we have

$$\begin{aligned} e(J_1 + \mathcal{A}_M^r, \dots, J_n + \mathcal{A}_M^r) &= e(J_1 + \mathcal{A}_{rM}, \dots, J_n + \mathcal{A}_{rM}) = e(\mathcal{A}_{rM}, \dots, \mathcal{A}_{rM}) \\ &= e(\mathcal{A}_{rM}) = e(\mathcal{A}_M^r) = r^n n! \mathrm{V}_n(\Gamma_-), \end{aligned}$$

and the result follows.

Let us suppose that $S \neq \emptyset$. After reordering the integers $r_1, \dots, r_n$, we can assume that $S = \{1, \dots, s\}$, for some $s \geqslant 1$. Then, we have

$$e(J_1 + \mathcal{A}_M^r, \dots, J_n + \mathcal{A}_M^r) = e(J_1 + \mathcal{A}_{rM}, \dots, J_s + \mathcal{A}_{rM}, \mathcal{A}_{rM}, \dots, \mathcal{A}_{rM}).$$

By Proposition 2.2, there exists an element $(g_1, \dots, g_n) \in J_1 \oplus \cdots \oplus J_n$ such that $\nu_\Gamma(g_i) = r_i$, for all $i = 1, \dots, n$, and

$$e(g_1, \dots, g_n) = \sigma(J_1, \dots, J_n) = \frac{r_1 \cdots r_n}{M^n} e(\mathcal{A}_M). \tag{19}$$

Let $\Delta$ be a compact facet of $\Gamma_+$. Let us denote by $I$ the ideal of $\mathcal{R}_\Delta$ generated by $p_\Delta(g_1), \dots, p_\Delta(g_s)$. By Proposition 3.5, the ideal of $\mathcal{R}_\Delta$ generated by $p_\Delta(g_1), \dots, p_\Delta(g_n)$ has finite colength in $\mathcal{R}_\Delta$. Then, since $\mathcal{R}_\Delta$ is Cohen-Macaulay of dimension $n$ (see [13] or [15, p. 24]), the ring $\mathcal{R}_\Delta / I$ is Cohen-Macaulay of dimension $n - s$.

Let $\mathcal{A}_{rM,\Delta}$ be the ideal of $\mathcal{R}_\Delta$ generated by all monomials $x^k \in \mathcal{A}_{rM}$ such that $k \in r\Delta$. Let us denote the image of $\mathcal{A}_{rM,\Delta}$ in $\mathcal{R}_\Delta / I$ by $H$.

Since $\Gamma(\mathcal{A}_{rM,\Delta}) = r\Delta$, we have that $\mathcal{A}_{rM,\Delta}$ has finite colength in $\mathcal{R}_\Delta$, by Theorem 3.6. Then *a fortiori* the ideal $H$ has also finite colength in $\mathcal{R}_\Delta / I$ and hence $H$ has analytic spread $n - s$. According to the Northcott-Rees theorem of existence of reductions (see [14, p. 166]), there exist sufficiently general $\mathbb{C}$-linear combinations $h_{s+1}, \dots, h_n$ of the set of monomials $\{x^k : k \in r\Delta\}$ such that the ideal generated by the images of $h_{s+1}, \dots, h_n$ in $\mathcal{R}_\Delta / I$ is a reduction of $H$. By the construction of the elements $h_{s+1}, \dots, h_n$, the image of $h_i$ in $\mathcal{R}_\Delta / I$ equals the image of $(h_i)_{r\Delta}$ in $\mathcal{R}_\Delta / I$, for all $i = s+1, \dots, n$. Moreover $(h_i)_{r\Delta} = p_\Delta(h_i)$, for all $i = s+1, \dots, n$. In particular, the ideal $\{p_\Delta(g_1), \dots, p_\Delta(g_s), p_\Delta(h_{s+1}), \dots, p_\Delta(h_n)\}\mathcal{R}_\Delta$ has finite colength in $\mathcal{R}_\Delta$.

Since $\Gamma_+$ has a finite number of facets we conclude that there exist $\mathbb{C}$-generic linear combinations $h_{s+1}, \dots, h_n$ of $\{x^k : \nu_\Gamma(k) = rM\}$ such that the ideal of $\mathcal{R}_\Delta$ generated by $p_\Delta(g_1), \dots, p_\Delta(g_s), p_\Delta(h_{s+1}), \dots, p_\Delta(h_n)$ has finite colength in $\mathcal{R}_\Delta$, for all compact facets $\Delta$ of $\Gamma_+$. In particular, the map $G = (g_1, \dots, g_s, h_{s+1}, \dots, h_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is non-degenerate on $\Gamma_+$, by Proposition 3.5 and then

$$e(G) = \frac{r_1 \cdots r_s (rM)^{n-s}}{M^n} n! \mathrm{V}_n(\Gamma_-) = \frac{\min\{r_1, rM\} \cdots \min\{r_n, rM\}}{M^n} n! \mathrm{V}_n(\Gamma_-). \tag{20}$$

Moreover we have the following inequalities, as a direct application of Lemma 2.3 and Proposition 3.2:

$$\begin{aligned} e(G) = e(g_1, \dots, g_s, h_{s+1}, \dots, h_n) &\geqslant e(J_1 + \mathcal{A}_{rM}, \dots, J_s + \mathcal{A}_{rM}, \mathcal{A}_{rM}, \dots, \mathcal{A}_{rM}) \\ &\geqslant \frac{\min\{r_1, rM\} \cdots \min\{r_n, rM\}}{M^n} n! \mathrm{V}_n(\Gamma_-). \end{aligned}$$

Then the result follows by applying relation (20). □

The following definition is fundamental in our study of Łojasiewicz exponents via Newton filtrations.

**Definition 3.10.** Let $J_1, \dots, J_n$ be ideals of $\mathcal{O}_n$ such that $\sigma(J_1, \dots, J_n) < \infty$. Let $r_i = \nu_\Gamma(J_i)$, for $i = 1, \dots, n$, let $p = \max\{r_1, \dots, r_n\}$ and let $A = \{i : r_i = p\}$. Let $I$ be a proper ideal of $\mathcal{O}_n$. We say that the pair $(I; J_1, \dots, J_n)$ is $\Gamma_+$*-linked* when there exists some $i_0 \in A$ such that

$$(J_1, \dots, J_{i_0-1}, I, J_{i_0+1}, \dots, J_n)$$

is non-degenerate on $\Gamma_+$

If $g \in \mathcal{O}_n$, then we will write $(I; g, J_2, \dots, J_n)$ instead of $(I; \langle g \rangle, J_2, \dots, J_n)$. We will adopt the same simplification of the notation if any other ideal $J_i$ is generated by only one element, for some $i \in \{1, \dots, n\}$.

Under the conditions of the previous definition, if we assume that $(J_1, \dots, J_n)$ is non-degenerate on $\Gamma_+$, then $(I; J_1, \dots, J_n)$ is $\Gamma_+$-linked if and only if there exists some $i_0 \in A$ such that

$$\sigma(J_1, \dots, J_{i_0-1}, I, J_{i_0+1}, \dots, J_n) = \frac{\nu_\Gamma(I)}{p} \sigma(J_1, \dots, J_n), \tag{21}$$

by a direct application of Definition 3.3. In particular, $p$ must be a divisor of $\nu_\Gamma(I)\sigma(J_1, \dots, J_n)$ in this case (see Example 3.13).

Here we show the main result of the article.

**Theorem 3.11.** *Let $J_1, \dots, J_n$ be a set of ideals of $\mathcal{O}_n$. Let $\nu_\Gamma(J_i) = r_i$, for all $i = 1, \dots, n$. Let us suppose that $(J_1, \dots, J_n)$ is non-degenerate on $\Gamma_+$. Let $I$ be a proper ideal of $\mathcal{O}_n$. Then*

$$\mathcal{L}_I(J_1, \dots, J_n) \leqslant \mathcal{L}_I(\mathcal{B}_{r_1}, \dots, \mathcal{B}_{r_n}) \leqslant \frac{\max\{r_1, \dots, r_n\}}{\nu_\Gamma(I)} \tag{22}$$

*and the above inequalities turn into equalities if $(I; J_1, \dots, J_n)$ is $\Gamma_+$-linked.*

*Proof.* Along this proof we set $M = M_\Gamma$. By a direct application of Lemma 2.3 and Proposition 3.2 we have

$$\sigma(J_1, \dots, J_n) \geqslant \sigma(\mathcal{B}_{r_1}, \dots, \mathcal{B}_{r_n}) \geqslant \frac{r_1 \cdots r_n}{M^n} n! \mathrm{V}_n(\Gamma_+).$$

Since $(J_1, \dots, J_n)$ is non-degenerate on $\Gamma_+$, the previous inequalities show that $\sigma(J_1, \dots, J_n) = \sigma(\mathcal{B}_{r_1}, \dots, \mathcal{B}_{r_n})$. Hence we can apply Proposition 2.11 to deduce the inequality

$$\mathcal{L}_I(J_1, \dots, J_n) \leqslant \mathcal{L}_I(\mathcal{B}_{r_1}, \dots, \mathcal{B}_{r_n}).$$

Let us denote $\max\{r_1, \dots, r_n\}$ and $\nu_\Gamma(I)$ by $p$ and $q$, respectively. Let us see first that $\mathcal{L}_I(\mathcal{B}_{r_1}, \dots, \mathcal{B}_{r_n}) \leqslant \frac{p}{q}$.

Since $\sigma(\mathcal{B}_{r_1},\dots,\mathcal{B}_{r_n})<\infty$, we can compute the number $r_{\mathcal{A}_M}(\mathcal{B}^s_{r_1},\dots,\mathcal{B}^s_{r_n})$, for all $s\geqslant 1$:

$$\begin{aligned}
r_{\mathcal{A}_M}(\mathcal{B}^s_{r_1},\dots,\mathcal{B}^s_{r_n}) &= \min\left\{r\geqslant 1:\sigma(\mathcal{B}^s_{r_1},\dots,\mathcal{B}^s_{r_n})=e(\mathcal{B}^s_{r_1}+\mathcal{A}^r_M,\dots,\mathcal{B}^s_{r_n}+\mathcal{A}^r_M)\right\}\\
&=\min\left\{r\geqslant 1:\frac{sr_1\cdots sr_n}{M^n}n!\mathrm{V}_n(\Gamma_-)=\frac{\min\{sr_1,rM\}\cdots\min\{sr_n,rM\}}{M^n}n!\mathrm{V}_n(\Gamma_-)\right\}\\
&=\min\left\{r\geqslant 1:rM\geqslant\max\{sr_1,\dots,sr_n\}\right\}\\
&=\min\left\{r\geqslant 1:r\geqslant\frac{\max\{sr_1,\dots,sr_n\}}{M}\right\}=\left\lceil\frac{\max\{sr_1,\dots,sr_n\}}{M}\right\rceil,
\end{aligned}$$

where $\lceil a\rceil$ denotes the least integer greater than or equal to $a$, for any $a\in\mathbb{R}$, and the second equality is a direct application of Lemma 3.9. Therefore

$$\begin{aligned}
\mathcal{L}_{\mathcal{A}_M}(\mathcal{B}_{r_1},\dots,\mathcal{B}_{r_n}) &= \inf_{s\geqslant 1}\frac{r_{\mathcal{A}_M}(\mathcal{B}^s_{r_1},\dots,\mathcal{B}^s_{r_n})}{s}\leqslant\inf_{a\geqslant 1}\frac{r_{\mathcal{A}_M}(\mathcal{B}^{aM}_{r_1},\dots,\mathcal{B}^{aM}_{r_n})}{aM}\\
&=\inf_{a\geqslant 1}\frac{1}{aM}\left\lceil\frac{\max\{aMr_1,\dots,aMr_n\}}{M}\right\rceil=\frac{\max\{r_1,\dots,r_n\}}{M}.
\end{aligned}$$

By [17, Théorème 6.3] we have the relation $\mathcal{L}_I(\mathcal{A}_M)=\frac{1}{\overline{\nu}_{\mathcal{A}_M}(I)}$, where $\overline{\nu}_{\mathcal{A}_M}$ is the asymptotic Samuel function of $\mathcal{A}_M$. We observe that $\Gamma_+(\mathcal{A}_M)=\Gamma_+$. Then, since $\mathcal{A}_M$ is a monomial ideal we have

$$\mathcal{L}_I(\mathcal{A}_M)=\frac{M}{\nu_\Gamma(I)},$$

as a consequence of Proposition 3.1. Therefore, by Lemma 2.10 we obtain

$$\begin{aligned}
\mathcal{L}_I(\mathcal{B}_{r_1},\dots,\mathcal{B}_{r_n}) &\leqslant \mathcal{L}_I(\mathcal{A}_M)\mathcal{L}_{\mathcal{A}_M}(\mathcal{B}_{r_1},\dots,\mathcal{B}_{r_n})\\
&\leqslant\frac{M}{\nu_\Gamma(I)}\frac{\max\{r_1,\dots,r_n\}}{M}=\frac{\max\{r_1,\dots,r_n\}}{\nu_\Gamma(I)}=\frac{p}{q}.
\end{aligned}$$

Supposing that $(I;J_1,\dots,J_n)$ is $\Gamma_+$-linked, let us prove that $\mathcal{L}_I(J_1,\dots,J_n)\geqslant\frac{p}{q}$. By the definition of $\mathcal{L}_I(J_1,\dots,J_n)$, this inequality holds if and only if

$$\frac{r_I(J_1^s,\dots,J_n^s)}{s}\geqslant\frac{p}{q}$$

for all $s\geqslant 1$. By Lemma 2.8 we have that $qr_I(J_1^s,\dots,J_n^s)\geqslant r_I(J_1^{sq},\dots,J_n^{sq})$, for all $s\geqslant 1$. Therefore it suffices to show that

$$r_I(J_1^{sq},\dots,J_n^{sq})>sp-1, \tag{23}$$

for all $s\geqslant 1$. Let us fix an integer $s\geqslant 1$. Then relation (23) is equivalent to saying that

$$\sigma(J_1^{sq},\dots,J_n^{sq})>\sigma(J_1^{sq}+I^{sp-1},\dots,J_n^{sq}+I^{sp-1}). \tag{24}$$

By [5, Lemma 2.6] we have

$$\sigma(J_1^{sq},\dots,J_n^{sq})=(sq)^n\sigma(J_1,\dots,J_n). \tag{25}$$

Let us denote the multiplicity $\sigma(J_1,\dots,J_n)$ also by $\sigma$.

Let $A = \{i : r_i = p\}$. By hypothesis, there exists an index $i_0 \in A$ such that

$$(J_1, \dots, J_{i_0-1}, I, J_{i_0+1}, \dots, J_n)$$

is non-degenerate on $\Gamma_+$. Then, we have $\sigma(J_1, \dots, J_{i_0-1}, I, J_{i_0+1}, \dots, J_n) = \frac{q}{p}\sigma$ (see relation (21)) and hence

$$\begin{aligned}(sq)^{n-1}(sp-1)\,\frac{q}{p}\sigma &= \sigma(J_1^{sq}, \dots, J_{i_0-1}^{sq}, I^{sp-1}, J_{i_0+1}^{sq} \dots, J_n^{sq}) \\ &\geqslant \sigma(J_1^{sq} + I^{sp-1}, \dots, J_n^{sq} + I^{sp-1}),\end{aligned} \tag{26}$$

where the second inequality comes from Lemma 2.3. An elementary computation shows that

$$(sq)^n\sigma > (sq)^{n-1}(sp-1)\,\frac{q}{p}\sigma$$

if and only if $sp > sp - 1$, which is the case. Thus, comparing (25) and (26) we conclude that relation (24) holds and hence the result is proven. □

**Example 3.12.** Let us consider the ideals $J_1$ and $J_2$ of $\mathcal{O}_2$ given by $J_1 = \langle x^5, x^2y^2, y^5\rangle$ and $J_2 = \langle x^3y^3\rangle$. Let $\Gamma_+ = \Gamma_+(J_1)$. The filtrating map associated to $\Gamma_+$ is given by $\phi_\Gamma(k_1, k_2) = \min\{2k_1 + 3k_2, 3k_1 + 2k_2\}$, for all $(k_1, k_2) \in \mathbb{R}^2_+$. Hence $\nu_\Gamma(J_1) = M_\Gamma = 10$, $\nu_\Gamma(J_2) = 15$ and $\nu_\Gamma(m) = 2$. Using the program *Singular* [11] we check that $\sigma(J_1, J_2) = 30$. Then we have the relation

$$\sigma(J_1, J_2) = 30 = \frac{\nu_\Gamma(J_1)\nu_\Gamma(J_2)}{M_\Gamma^2}e(J_1),$$

which shows that $(J_1, J_2)$ is non-degenerate on $\Gamma_+$. Moreover we have that relation (21) holds in this context, that is

$$\sigma(J_1, m) = 4 = \frac{\nu_\Gamma(m)}{\max\{\nu_\Gamma(J_1), \nu_\Gamma(J_2)\}}\sigma(J_1, J_2).$$

Then $\mathcal{L}_0(J_1, J_2) = \frac{15}{2}$, by Theorem 3.11.

**Example 3.13.** Let us consider the ideal of $\mathcal{O}_2$ given by $J = \langle x^4, xy, y^5\rangle$ and let $\Gamma_+ = \Gamma_+(J)$. We observe that $e(J) = 9$ and that the filtrating map associated to $\Gamma_+$ is given by $\phi_\Gamma(k_1, k_2) = 20\min\{\frac{k_1+3k_2}{4}, \frac{4k_1+k_2}{5}\}$, for all $(k_1, k_2) \in \mathbb{R}^2_+$, with $M_\Gamma = 20$. Let $J_1 = \langle x^2y^2, x^8\rangle$ and let $J_2 = \langle xy, y^5\rangle$. Then we observe that $\nu_\Gamma(J_1) = 40$, $\nu_\Gamma(J_2) = 20$ and moreover

$$\sigma(J_1, J_2) = 18 = \frac{\nu_\Gamma(J_1)\nu_\Gamma(J_2)}{20^2}e(J).$$

Then $(J_1, J_2)$ is non-degenerate on $\Gamma_+$. If $(m; J_1, J_2)$ were $\Gamma_+$-linked then the multiplicity $\sigma(m, J_2)$ would be equal to $\frac{\nu_\Gamma(m)\nu_\Gamma(J_2)}{20^2}e(J) = \frac{9}{5} \notin \mathbb{Z}$. However the Łojasiewicz exponent $\mathcal{L}_0(J_1, J_2)$ attains the maximum possible value, that is

$$\mathcal{L}_0(J_1, J_2) = 10 = \frac{\max\{\nu_\Gamma(J_1), \nu_\Gamma(J_2)\}}{\nu_\Gamma(m)} = \frac{40}{4} = 10,$$

where the first equality follows from [5, §4] (see also [18]).

**Example 3.14.** Let us fix an integer $a \geqslant 2$. Let us consider the map $g = (g_1, g_2, g_3) : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ given by

$$\begin{aligned} g_1(x,y,z) &= x^6 + y^6 - z^5 + xyz \\ g_2(x,y,z) &= x^2y^2z^2 \\ g_3(x,y,z) &= y^{6a} + z^{5a}. \end{aligned}$$

Let $\Gamma_+ = \Gamma_+(g_1)$. We have that $M_\Gamma = 30$ and the Newton filtration $\nu_\Gamma$ is determined by the filtrating map $\phi_\Gamma : \mathbb{R}^3_+ \to \mathbb{R}_+$ given by

$$\phi_\Gamma(k_1,k_2,k_3) = \min\big\{19k_1 + 5k_2 + 6k_3,\ 5k_1 + 19k_2 + 6k_3,\ 5\big(k_1 + k_2 + 4k_3\big)\big\}.$$

Using Corollary 3.8, it is immediate to check that the map $g$ is non-degenerate on $\Gamma_+$. Moreover $\nu_\Gamma(g_1) = 30$, $\nu_\Gamma(g_2) = 60$ and $\nu_\Gamma(g_3) = 30a$.

Let $m$ denote the maximal ideal of $\mathcal{O}_3$. Let $h$ denote a $\mathbb{C}$-generic linear form. Then $\nu_\Gamma(m) = \nu_\Gamma(h) = 5$. Again by Corollary 3.8, we obtain that the map $(g_1, g_2, h)$ is non-degenerate on $\Gamma_+$. Hence we conclude that $(m; g_1, g_2, g_3)$ is $\Gamma_+$-linked. Then, by Theorem 3.11 we obtain

$$\mathcal{L}_0(g_1,g_2,g_3) = \frac{\max\{\nu_\Gamma(g_1), \nu_\Gamma(g_2), \nu_\Gamma(g_3)\}}{\nu_\Gamma(m)} = 6a.$$

**Definition 3.15.** Let $\Delta$ be a compact facet of $\Gamma_+$. We say that $\Delta$ is an *inner* facet of $\Gamma_+$ when no vertex of $\Delta$ is contained in some coordinate axis of $\mathbb{R}^n$.

**Remark 3.16.** If $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a non-degenerate map on $\Gamma_+$ and $\Delta$ denotes a compact facet of $\Gamma_+$, then the ideal of $\mathcal{R}_\Delta$ generated by $p_\Delta(g_1), \dots, p_\Delta(g_n)$ has finite colength in $\mathcal{R}_\Delta$ (see Proposition 3.5). In particular $p_\Delta(g_i) \neq 0$, for all $i = 1, \dots, n$, since $\dim(\mathcal{R}_\Delta) = n$.

We observe that if $h$ is a $\mathbb{C}$-linear form and $\Delta$ is an inner facet of $\Gamma_+$ then $p_\Delta(h) = 0$. Then, if we suppose that the pair $(m; g_1, \dots, g_n)$ is $\Gamma_+$-linked, we are forcing the Newton polyhedron $\Gamma_+$ to have no inner facets. The same happens when replacing the maximal ideal $m$ by any ideal whose support is contained in the union of the coordinate axis. Therefore, in the aim of applying Theorem 3.11 to obtain the exact value of $\mathcal{L}_0(g_1, \dots, g_n)$ via the Newton filtration induced by $\Gamma_+$, we need the Newton polyhedron $\Gamma_+$ to have no inner facets.

We remark that the Newton polyhedra $\Gamma_+$ that appear in Examples 3.12 and 3.14 do not have inner facets.

## 4. Applications to weighted homogeneous filtrations

Along this section we will denote by $w$ a primitive vector $w = (w_1, \dots, w_n) \in \mathbb{Z}^n_{\geqslant 1}$. Let $w_0 = \min_i w_i$ and let $A_w = \{i : w_i = w_0\}$. If $h \in \mathcal{O}_n$, $h \neq 0$, then we denote by $d_w(h)$ the $w$-degree of $h$, that is, $d_w(h) = \min\{\langle w, k\rangle : k \in \operatorname{supp}(h)\}$, where $\langle\, , \rangle$ denotes the standard scalar product in $\mathbb{R}^n$. If $h = 0$ then we set $d_w(h) = +\infty$. Moreover, if $J$ is an ideal of $\mathcal{O}_n$ then we define $d_w(J) = \min\{d_w(h) : h \in J\}$.

Let us denote by $\Gamma^w_+$ the Newton polyhedron in $\mathbb{R}^n_+$ determined by $\{\frac{w_1\cdots w_n}{w_1}e_1,\dots,\frac{w_1\cdots w_n}{w_n}e_n\}$ and by $\Gamma^w$ the Newton boundary of $\Gamma^w_+$. It is straightforward to see that $\Gamma^w$ has only one compact facet, which is supported by $w$, and that the weighted homogeneous filtration induced by $w$ (see [7, Section 4]) coincides with the Newton filtration of $\mathcal{O}_n$ induced by $\Gamma^w_+$. That is $d_w(h)=\nu_{\Gamma^w}(h)$, for all $h\in\mathcal{O}_n$.

In [7, p. 584] we introduced the following definition, which lead us to formulate a sufficient condition for $\mathcal{L}_0(J_1,\dots,J_n)$ to attain the bound $\frac{1}{w_0}\max\{d_w(J_1),\dots,d_w(J_n)\}$ (see [7, Theorem 4.7] for details) and to derive interesting consequences (see [7, Proposition 4.14 and Corollary 4.16]).

**Definition 4.1.** [7, p. 584] Let $J_1,\dots,J_n$ be a family of ideals of $\mathcal{O}_n$ and let $r_i=d_w(J_i)$, for all $i=1,\dots,n$. We say that the $n$-tuple of ideals $(J_1,\dots,J_n)$ *admits a $w$-matching* if there exist a permutation $\tau$ of $\{1,\dots,n\}$ and an index $i_0\in\{1,\dots,n\}$ such that

(a) $w_{i_0}=\min\{w_1,\dots,w_n\}$,
(b) $r_{\tau(i_0)}=\max\{r_1,\dots,r_n\}$ and
(c) the pure monomial $x_i^{r_{\tau(i)}/w_i}$ belongs to $J_{\tau(i)}$, for all $i\neq i_0$.

**Remark 4.2.** Under the conditions of the above definition, if $(J_1,\dots,J_n)$ admits a $w$-matching, then $J_i$ contains some pure monomial, for all $i$ such that $r_i<\max\{r_1,\dots,r_n\}$.

Let $J_1,\dots,J_n$ be ideals of $\mathcal{O}_n$ and let $r_i=d_w(J_i)$, for all $i=1,\dots,n$. In order to simplify the nomenclature, we will say that $(J_1,\dots,J_n)$ is *$w$-non-degenerate* when $(J_1,\dots,J_n)$ is non-degenerate on $\Gamma^w_+$, which is to say that $\sigma(J_1,\dots,J_n)=\frac{r_1\cdots r_n}{w_1\cdots w_n}$ (see Definition 3.3). Moreover, we will say that the pair $(m;J_1,\dots,J_n)$ *$w$-linked* when it is *$\Gamma^w_+$-linked* (see Definition 3.10).

Let $p=\max\{r_1,\dots,r_n\}$ and let $A=\{i: r_i=p\}$. Then, as a direct application of Definition 3.10, it follows that the pair $(m;J_1,\dots,J_n)$ is $w$-linked if and only if there exists some $i_0\in A$ such that

$$\sigma(J_1,\dots,J_{i_0-1},m,J_{i_0+1},\dots,J_n)=\frac{\min\{w_1,\dots,w_n\}}{\max\{r_1,\dots,r_n\}}\frac{r_1\cdots r_n}{w_1\cdots w_n}. \tag{27}$$

**Lemma 4.3.** *Let $J_1,\dots,J_n$ be a family of ideals of $\mathcal{O}_n$. Let us suppose that $J_1,\dots,J_n$ admits a $w$-matching. Then $(m;J_1,\dots,J_n)$ is $w$-linked.*

*Proof.* Let $r_i=d_w(J_i)$, for all $i=1,\dots,n$. Let $i_0\in\{1,\dots,n\}$ verifying conditions (a), (b) and (c) of Definition 4.1, for some permutation $\tau$ of $\{1,\dots,n\}$. We observe that

$$\begin{aligned}\frac{r_1\cdots r_n}{w_1\cdots w_n}\frac{w_{i_0}}{r_{\tau(i_0)}}&=e\left(x_1^{r_{\tau(1)}/w_1},\dots,x_{i_0-1}^{r_{\tau(i_0-1)}/w_{i_0-1}},x_{i_0},x_{i_0+1}^{r_{\tau(i_0+1)}/w_{i_0+1}},\dots,x_n^{r_{\tau(n)}/w_n}\right)\\ &\geqslant e\left(J_{\tau(1)},\dots,J_{\tau(i_0-1)},m,J_{\tau(i_0+1)},\dots,J_{\tau(n)}\right)\geqslant\frac{r_1\cdots r_n}{w_1\cdots w_n}\frac{w_{i_0}}{r_{\tau(i_0)}},\end{aligned} \tag{28}$$

where the first inequality of (28) is a direct consequence of condition (c) together with Lemma 2.3, and the second inequality of (28) follows immediately from Proposition 3.2. Therefore these inequalities are actually equalities. That is, if we denote $\tau(i_0)$ by $j_0$, then

$$\sigma(J_1, \dots, J_{j_0-1}, m, J_{j_0+1}, \dots, J_n) = \frac{r_1 \cdots r_n}{w_1 \cdots w_n} \frac{w_{i_0}}{r_{j_0}},$$

which is to say that $(J_1, \dots, J_{j_0-1}, m, J_{j_0+1}, \dots, J_n)$ is $w$-non-degenerate. □

The converse of Lemma 4.3 is not true in general, as the following simple example shows.

**Example 4.4.** Let $w = (1, 1, 2)$ and let us consider the ideals of $\mathcal{O}_3$ given by $J_1 = \langle xy \rangle$ and $J_2 = \langle x^4, z^2 \rangle$. We observe that $d_w(J_1) = 2$, $d_w(J_2) = 4$ and

$$\sigma(J_1, J_2, m_3) = 4 = \frac{d_w(J_1)d_w(J_2)d_w(m_3)}{2}. \tag{29}$$

Let $J_3$ denote any ideal of $\mathcal{O}_3$ of $w$-degree strictly greater than 4 such that $(J_1, J_2, J_3)$ is $w$-non-degenerate. Then relation (29) implies that $(m_3; J_1, J_2, J_3)$ is $w$-linked and hence $\mathcal{L}_0(J_1, J_2, J_3) = d_w(J_3)$, by Theorem 3.11. We observe that $(J_1, J_2, J_3)$ does not admit a $w$-matching, since $J_1$ does not contain any pure monomial (see Remark 4.2).

If $f \in \mathcal{O}_n$ and $w \in \mathbb{Z}^n_{\geqslant 1}$ then we denote by $p_w(f)$ the *principal part of $f$ with respect to $w$*. That is, if $f = \sum_k a_k x^k$ is the Taylor expansion of $f$ around the origin, then $p_w(f)$ is the sum of all terms $a_k x^k$ such that $\langle w, k \rangle = d_w(f)$. We say that $f$ is *weighted homogeneous with respect to $w$* when $p_w(f) = f$. Moreover, the function $f$ is termed *semi-weighted homogeneous with respect to $w$* when $p_w(f)$ has an isolated singularity at the origin.

Let $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be an analytic map germ. We say that $g$ is *weighted homogeneous with respect to $w$ of degree* $(r_1, \dots, r_n)$ when $g_i$ is weighted homogeneous with respect to $w$ of degree $r_i$, for all $i = 1, \dots, n$. Moreover, we define the map $p_w(g) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ by $p_w(g) = (p_w(g_1), \dots, p_w(g_n))$ and we also define $d_w(g) = (d_w(g_1), \dots, d_w(g_n))$.

If $g^{-1}(0) = \{0\}$, we recall that we denote by $e(g_1, \dots, g_n)$, or by $e(g)$, the multiplicity of the ideal of $\mathcal{O}_n$ generated by $g_1, \dots, g_n$.

Given a vector $r = (r_1, \dots, r_n) \in \mathbb{Z}^n_{\geqslant 1}$, then we denote by $H(w; r)$ the set of all weighted homogeneous maps $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ for which $d_w(g) = (r_1, \dots, r_n)$ and $g^{-1}(0) = \{0\}$.

We say that $g$ is *semi-weighted homogeneous with respect to $w$* when $p_w(g)^{-1}(0) = \{0\}$. In this case we have $e(g) = e(p_w(g)) = \frac{r_1 \cdots r_n}{w_1 \cdots w_n}$, where $r_i = d_w(g_i)$, for $i = 1, \dots, n$ (see [1, Section 12]). It is straightforward to see that if $f \in \mathcal{O}_n$ then $f$ is semi-weighted homogeneous with respect to $w$ if an only if its gradient map $\nabla f = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is semi-weighted homogeneous with respect to $w$.

Let $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a complex analytic map. In order to simplify the nomenclature, we say that $g$ is *$w$-linked* when the pair $(m; g_1, \dots, g_n)$ is $w$-linked. It follows from Definition 3.10 and Proposition 3.5 that $g$ is $w$-linked if and only if $p_w(g)$ is $w$-linked.

If $J$ is an ideal of $\mathcal{O}_n$ then we denote by $p_w(J)$ the ideal of $\mathcal{O}_n$ generated by $\{p_w(h) : h \in J,\ d_w(h) = d_w(J)\}$. If $J$ is a monomial ideal then we have $p_w(J) \subseteq J$. The following result helps in the task of checking the condition of $w$-linkage.

**Lemma 4.5.** *Let $J_1, \dots, J_n$ be ideals of $\mathcal{O}_n$ such that $\sigma(J_1, \dots, J_n) < \infty$. Then $(J_1, \dots, J_n)$ is $w$-non-degenerate if and only if $\sigma\left(p_w(J_1), \dots, p_w(J_n)\right) < \infty$.*

*Proof.* By Proposition 2.2, we can consider a sufficiently general element $(g_1, \dots, g_n)$ of $J_1 \oplus \cdots \oplus J_n$ such that $\sigma(J_1, \dots, J_n) = e(g_1, \dots, g_n)$. Then $(J_1, \dots, J_n)$ is $w$-non-degenerate if and only if $(g_1, \dots, g_n)$ is also, which is to say that for any compact facet $\Delta$ of $\Gamma_+^w$, the ideal of $\mathcal{R}_\Delta$ generated by $p_\Delta(g_1), \dots, p_\Delta(g_n)$ has finite colength in $\mathcal{R}_\Delta$, by Corollaries 3.7 and 3.8. Moreover $\Gamma_+^w$ has only one compact facet $\Delta$, $\mathcal{R}_\Delta = \mathcal{O}_n$ and $p_\Delta(g_i) = p_w(g_i)$, for all $i = 1, \dots, n$. Then the result follows. □

**Remark 4.6.** Under the conditions of the above result, let $p = \max\{d_w(J_1), \dots, d_w(J_n)\}$. Let $m_w = \langle x_i : i \in A_w \rangle$. Then, as a consequence of Lemma 4.5, have that $(m; J_1, \dots, J_n)$ is $w$-linked if and only if there exists some $i_0 \in \{1, \dots, n\}$ for which $d_w(J_{i_0}) = p$ and

$$\sigma\big(p_w(J_1), \dots, p_w(J_{i_0-1}), m_w, p_w(J_{i_0+1}), \dots, p_w(J_n)\big) < \infty.$$

In particular, if $A_w = \{i_0\}$, for some $i_0 \in \{1, \dots, n\}$, and $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a germ of analytic function with an isolated singularity at the origin, then $\nabla f$ is $w$-linked if and only if

$$\sigma\left(\frac{\partial p_w(f)}{\partial x_1}, \dots, \frac{\partial p_w(f)}{\partial x_{i_0-1}}, x_{i_0}, \frac{\partial p_w(f)}{\partial x_{i_0+1}}, \dots, \frac{\partial p_w(f)}{\partial x_n}\right) < \infty.$$

**Corollary 4.7.** *Let $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a semi-weighted homogeneous map with respect to $w$. Let $r_i = d_w(g_i)$, for all $i = 1, \dots, n$. Then*

$$\mathcal{L}_0(p_w(g)) \leqslant \mathcal{L}_0(g) \leqslant \frac{\max\{r_1, \dots, r_n\}}{w_0} \tag{30}$$

*and both inequalities turn into equalities if $g$ is $w$-linked. In particular if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a semi-weighted homogeneous function of degree $d$ with respect to $w$, then*

$$\mathcal{L}_0(\nabla f) \leqslant \frac{d - w_0}{w_0}, \tag{31}$$

*and equality holds if $\nabla f$ is $w$-linked.*

*Proof.* Let us see the first inequality of (30). Let $I_i = \langle x^k : k \in \Gamma_+(p_w(g_i)) \rangle$ and let $J_i = \langle x^k : k \in \Gamma_+(g_i) \rangle$, for all $i = 1, \dots, n$. Since $\Gamma_+(p_w(g_i)) \subseteq \Gamma_+(g_i)$, we have that $I_i \subseteq J_i$, for all $i = 1, \dots, n$. Thus

$$e(p_w(g)) \geqslant \sigma(I_1, \dots, I_n) \geqslant \sigma(J_1, \dots, J_n) \geqslant \frac{r_1 \cdots r_n}{w_1 \cdots w_n}, \tag{32}$$

by Lemmas 2.3 and 3.2. We know that $e(p_w(g)) = e(g) = \frac{r_1 \cdots r_n}{w_1 \cdots w_n}$ (see [1, Section 12]). Hence $\sigma(I_1, \dots, I_n) = \sigma(J_1, \dots, J_n)$, by relation (32). Thus $\mathcal{L}_0(I_1, \dots, I_n) \leqslant \mathcal{L}_0(J_1, \dots, J_n)$, by Proposition 2.11. From Theorem 2.5 we have $\mathcal{L}_0(I_1, \dots, I_n) = \mathcal{L}_0(p_w(g))$ and $\mathcal{L}_0(J_1, \dots, J_n) = \mathcal{L}_0(g)$. Then $\mathcal{L}_0(p_w(g)) \leqslant \mathcal{L}_0(g)$. This inequality also follows as a consequence of the lower semi-continuity of Łojasiewicz exponents in deformations with constant multiplicity, as is proven by Płoski in [26].

The second inequality of (30) and the equality under the hypothesis that $g$ is $w$-linked are direct consequences of Theorem 3.11. □

Let us consider the case $w = (1, \dots, 1)$. Let $f$ be a semi-weighted homogeneous function of degree $d$ with respect to $w$ and let $h = p_w(f)$. In particular, $h$ is a homogeneous polynomial. Since $h$ has an isolated singularity at the origin and $m_w = m$ in this case, we have clearly that $\nabla f$ is $w$-linked. Hence we obtain the well-known equality $\mathcal{L}_0(\nabla f) = d - 1$ (see for instance [25, Proposition 2.2] or [24, Theorem 3.5]).

The inequality $\mathcal{L}_0(p_w(g)) \leqslant \mathcal{L}_0(g)$ of (30) can be strict, as the following example shows.

**Example 4.8.** Let us consider the maps $g, g' : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ given by

$$g(x, y) = (x^3, y^4 + xy)$$
$$g'(x, y) = (x^3, y^4).$$

By an application of Proposition 3.1 we have $\mathcal{L}_0(g') = 4$. Moreover $\mathcal{L}_0(g) = 9$, by applying the main result of [18] about the computation of Łojasiewicz exponents in dimension 2 (or [5, Section 4]). Then we see that

$$\mathcal{L}_0(g') = 4 < 9 = \mathcal{L}_0(g). \tag{33}$$

Let $w = (w_1, w_2) \in \mathbb{Z}^2_{\geqslant 1}$ be any vector such that $w_1 > 3w_2$. We observe that $g$ is semi-weighted homogeneous with respect to $w$ with $p_w(g) = g'$. Using Remark 4.6 we observe that $g$ is not $w$-linked with respect to such vectors of weights.

Moreover, we have that $g, g' \in H((3, 1); (9, 4))$. Then, relation (33) also shows that if $g, g' \in H(w; r)$, then $g$ and $g'$ do not have the same Łojasiewicz exponent in general.

**Remark 4.9.** We do not know if an example similar to the previous one exists for gradient maps. That is, if we fix a vector of weights $w \in \mathbb{Z}^n_{\geqslant 1}$ and $f, f'$ are weighted homogeneous functions with respect to $w$ with the same $w$-degree and with an isolated singularity at the origin, it is still not known in general if $\mathcal{L}_0(\nabla f) = \mathcal{L}_0(\nabla f')$. Obviously, this equality holds when $\nabla f$ is $w$-linked and $\nabla f'$ is also, by virtue of Corollary 4.7.

On the other hand, if $f$ is a semi-weighted homogeneous function with respect to $w$, it is still an open question to determine if the equality $\mathcal{L}_0(\nabla f) = \mathcal{L}_0(\nabla p_w(f))$ holds. We recall that $f$ and $p_w(f)$ are topologically equivalent (see for instance [9, Corollary 5] or [23, Corollary 2.1] for a more general result). Then, if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ denotes an arbitrary analytic function germ with an isolated singularity at the origin we remark that it is not known in general if $\mathcal{L}_0(\nabla f) = \mathcal{L}_0(\nabla(f \circ \phi))$, where $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ denotes a germ of homeomorphism (see [30, p. 278]).

If $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a complex analytic map, then we denote by $I(g)$ the ideal of $\mathcal{O}_n$ generated by the component functions of $g$. The next result shows a sufficient condition for equality in the first inequality of (30).

**Proposition 4.10.** *Let $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a semi-weighted homogeneous map with respect to $w$. Let $h_1 = p_w(g)$ and let $h_2 = g - h_1$. If $I(h_2) \subseteq \overline{m \cdot I(h_1)}$, then*

$$\mathcal{L}_0(p_w(g)) = \mathcal{L}_0(g).$$

*Proof.* By Corollary 4.7 we have $\mathcal{L}_0(p_w(g)) \leqslant \mathcal{L}_0(g)$. Let us see the reverse inequality. Using the hypothesis $I(h_2) \subseteq \overline{m \cdot I(h_1)}$ we observe that

$$I(h_1) \subseteq I(g) + I(h_2) \subseteq I(g) + \overline{m \cdot I(h_1)} \subseteq \overline{I(g) + m \cdot I(h_1)}.$$

Then, by the integral Nakayama Lemma (see [29, p. 324]) we obtain the inclusion

$$I(h_1) \subseteq \overline{I(g)},$$

which implies that $\mathcal{L}_0(I(h_1)) \geqslant \mathcal{L}_0(I(g))$, that is $\mathcal{L}_0(p_w(g)) \geqslant \mathcal{L}_0(g)$ and hence the result follows. □

**Corollary 4.11.** *Let $J_1, \dots, J_n$ be ideals of $\mathcal{O}_n$ such that $(J_1, \dots, J_n)$ is $w$-non-degenerate. Let $r_i = d_w(J_i)$, for all $i = 1, \dots, n$. If there exists some $i_0 \in \{1, \dots, n\}$ such that $r_{i_0} = \max\{r_1, \dots r_n\}$ and $p_w(J_{i_0}) \subseteq p_w(I)$, then*

$$\mathcal{L}_I(J_1, \dots, J_n) = \frac{\max\{r_1, \dots, r_n\}}{d_w(I)}.$$

*Proof.* Since $(J_1, \dots, J_n)$ is $w$-non-degenerate we have $\sigma\left(p_w(J_1), \dots, p_w(J_n)\right) < \infty$, by Lemma 4.5. Thus the inclusion $p_w(J_{i_0}) \subseteq p_w(I)$ leads to obtain

$$\infty > \sigma\left(p_w(J_1), \dots, p_w(J_n)\right) \geqslant \sigma\left(p_w(J_1), \dots, p_w(J_{i_0-1}), p_w(I), p_w(J_{i_0+1}), \dots, p_w(J_n)\right),$$

by Lemma 2.3. Then the pair $(I; J_1, \dots, J_n)$ is $w$-linked by Lemma 4.5 and thus the result follows by Theorem 3.11. □

Given a function $f \in \mathcal{O}_n$ and an integer $i \in \{1, \dots, n\}$, we denote by $\mathrm{supp}_i(f)$ the set of those $k \in \mathrm{supp}(f)$ such that $k_i \neq 0$. It is clear that if $\mathrm{supp}_i(f) = \emptyset$, for some $i \in \{1, \dots, n\}$, then $f$ does not depend on the variable $x_i$ and therefore $f$ does not have an isolated singularity at the origin. The next result, which is an immediate consequence of Corollary 4.11, allows to determine easily the Łojasiewicz exponent $\mathcal{L}_0(\nabla f)$ for an ample class of functions $f \in \mathcal{O}_n$.

**Corollary 4.12.** *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a semi-weighted homogeneous function of degree $d$ with respect to $w$. Let $h = p_w(f)$. Let us suppose that there exists some $i_0 \in A_w$ such that $\frac{\partial f}{\partial x_{i_0}} \in m_w$. Then*

$$\mathcal{L}_0(\nabla h) = \mathcal{L}_0(\nabla f) = \frac{d - w_0}{w_0}. \tag{34}$$

*In particular, if $A_w = \{i_0\}$, for some $i_0 \in \{1, \dots, n\}$, and $k_{i_0} \geqslant 2$, for all $k \in \mathrm{supp}_{i_0}(f)$, then the above equality holds.*

We remark that a slight modification of the proof of [16, Proposition 2] implies Corollary 4.12 if $f$ is weighted homogeneous.

**Example 4.13.** Let us consider the function $f : (\mathbb{C}^4, 0) \to (\mathbb{C}, 0)$ of [7, Example 4.12], that is $f(x, y, z, t) = z^9 - y^{11}t + yt^5 + x^{27}$. This function is weighted homogeneous of degree 27 with respect to $w$, where $w = (1, 2, 3, 5)$. We observe that $A_w = \{1\}$ and $\operatorname{supp}_1(f) = \{(27, 0, 0, 0)\}$. Then $\mathcal{L}_0(\nabla f) = 26$, by Corollary 4.12. In [7] we arrived to the same conclusion via the notion of $w$-matching.

If $h \in \mathcal{O}_n$, then we denote by $J(h)$ the ideal of $\mathcal{O}_n$ generated by $\frac{\partial h}{\partial x_1}, \dots, \frac{\partial h}{\partial x_n}$.

**Example 4.14.** Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be the function defined by

$$f(x, y, z) = x^{12} + y^4 + z^3 + x^6yz.$$

We have that $f$ is semi-weighted homogeneous of degree $d = 12$ with respect to $w = (1, 3, 4)$. Moreover $f$ satisfies the hypothesis of Corollary 4.12. Then

$$\mathcal{L}_0(p_w(f)) = \mathcal{L}_0(\nabla f) = \frac{d - w_0}{w_0} = 11.$$

We remark that $f = p_w(f) + x^6yz$ and $d_w(x^6yz) = 13 > d$. That is, $f$ is not weighted homogeneous with respect to $w$, hence it is not possible to apply the results of [16] in order to obtain the value of $\mathcal{L}_0(\nabla f)$. Moreover, since $\Gamma_+(x^6z) \nsubseteq \Gamma_+(m \cdot J(p_w(f)))$ we have $J(x^6yz) \nsubseteq \overline{m \cdot J(p_w(f))}$ and therefore the equality $\mathcal{L}_0(\nabla f) = \mathcal{L}_0(\nabla p_w(f))$ is not a consequence of Proposition 4.10.

**Example 4.15.** Let us fix an integer $a \geqslant 3$. Let us consider the function $h : (\mathbb{C}^4, 0) \to (\mathbb{C}, 0)$ given by

$$h(x, y, z, t) = xy^{a+2} + z^a + tx^{a-2} + t^{a-1}.$$

A straightforward computation shows that $h$ has an isolated singularity at the origin and that $h$ is weighted homogeneous of degree $d = a^3 + a^2 - 2a$ with respect to the vector of weights $w = (a^2 + 2a, a^2 - 2a, a^2 + a - 2, a^2 + 2a)$. Hence we have $\mu(h) = a^4 - 3a^3 + 2a^2$, as a consequence of the Milnor-Orlik formula [20] (or more generally, as a consequence of Proposition 3.2 and Lemma 4.5). Since $a \geqslant 3$, the minimum of the components of $w$ is $a^2 - 2a$ and $A_w = \{2\}$. Hence $h$ satisfies the hypothesis of Corollary 4.12 and therefore

$$\mathcal{L}_0(\nabla h) = \frac{a^3 + a^2 - 2a - (a^2 - 2a)}{a^2 - 2a} = \frac{a^2}{a - 2}.$$

Moreover, also from Corollary 4.12, we have $\mathcal{L}_0(\nabla f) = \mathcal{L}_0(\nabla h)$, for all $f \in \mathcal{O}_4$ such that $p_w(f) = h$.

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