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Solving random diffusion models with nonlinear perturbations by the Wiener-Hermite expansion method

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Abstract

This paper deals with the construction of approximate series solutions of random nonlinear diffusion equations where non-linearity is considered by means of a frank small parameter and uncertainty is introduced through the white noise in the forcing term. In the simpler but important case in which diffusion coefficient is time-independent, we provide a Gaussian approximation of the solution stochastic process by taking advantage of the so-called Wiener-Hermite expansion together with the perturbation method. In addition, approximations of the main statistical functions associated with a solution, such as the mean and variance, are computed. Numerical values of these functions are compared with respect to those obtained by applying the Runge-Kutta second order stochastic scheme by means of an illustrative example.

Key words: random differential equation, Wiener-Hermite expansion, perturbation method
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1 Introduction

Deterministic differential equations of the form $\dot{x}(t) = a(t)x(t)$ constitute the basic form of so-called diffusion or transport problems which appear in rele-

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vant models such as: the growth population geometric (or Malthusian) model in Biology, where $a(t)$ represents the *per capita* growth rate; the neutron and gamma ray transport model in Physics, where coefficient $a(t)$ involves the geometry of the cross-sections of the medium; the continuous composed interest rate models for studying the evolution of an investment under time-variable interest rate $r(t)$ in which case $a(t) = 1 + r(t)$; etc. Despite the usefulness of these basic models, often they do not recover all possible situations observed from a practical point of view. In fact, as a simple but illustrative example, if $a(t) = a > 0$, Malthus model predicts unlimited growth of a species despite the fact that resources are always limited. Then, the logistic (or Verhulst) model introduces a nonlinear term in order to overcome this inconvenient by considering the differential equation $\dot{x}(t) = ax(t) - b(x(t))^2$, $a, b > 0$, where the non-linearity intensity is given by parameter b. In many practical situations it is proper to assume that nonlinear term affecting the phenomena under study is small enough, then its intensity is controlled by means of a frank small parameter, say ϵ. Relevant examples in this sense appear for instance in Epidemiology, where the so-called SIS models become nonlinear differential equations where nonlinear term coefficient denoting the contagious rate can be assumed to be a frank small parameter in many situations [1]. In addition of these considerations, diffusion models with nonlinear perturbations can also consider the introduction of a forcing term in order to model external aspects which can become very complex such as environment in Biology, unexpected material changes in the surrounding medium in Physics or foreign political events that can affect the markets where an investment has been ordered in Finance. Stochastic differential equations based on white noise process provide a powerful tool to model dynamically these complex and uncertain aspects. Over the last few years, new and relevant methods for searching the exact solutions of such type of equations have been developed. They include the homotopy perturbation method [2–4] and the Exp-function method [5,6].

This paper deals with the solution of stochastic differential models of the form

$$\dot{x}(t) = a(t)x(t) - \epsilon(x(t))^2 + \lambda n(t), \quad t > 0, \qquad x(0) = x_0, \tag{1}$$

where diffusion coefficient $a(t)$ and initial condition x_0 are deterministic, ϵ is a small parameter and $n(t) = n(t)(\omega)$ is the white noise process, whose intensity is given by parameter λ, and ω is a random outcome of a triple probability space $(\Omega, \mathcal{A}, P)$ where Ω is a sample space, $\mathcal{A}$ is a σ-algebra associated with Ω and P is a probability measure.

The paper is organized as follows. Section 2 summarizes the main results about the Wiener-Hermite expansion (WHE) that provides a powerful technique to represent any stochastic process in terms of certain deterministic kernels to be determined as well as the so-called Wiener-Hermite (WH) polynomials. In Section 3, the WHE is applied in order to obtain two initial integro-differential equations that are satisfied by these kernels. By taking advantage of the perturbation method, the solution of these equations are obtained in Section 4. Previous development is illustrated for the simpler but important case where diffusion coefficient is autonomous. In addition, we compute approximations for its main statistical moments such as the mean and variance. A comparison of the obtained results with respect to the Runge-Kutta second order stochastic scheme for solving stochastic differential equations is also provided. Conclusions are shown in Section 5.

2 The Wiener-Hermite expansion (WHE)

For the sake of clarity in the presentation, we summarize in this section the main ideas of the Wiener-Hermite expansion (WHE) based on the Wiener-Hermite (WH) polynomials. For further details we recommend [7,8,4]. The WH polynomials constitute a complete set of statistically orthogonal stochastic processes, say $H^{(i)} = H^{(i)}(t_1, \ldots, t_i)$, defined in terms of white noise $n(t)$ and the Dirac delta function $\delta(\cdot)$ through the following recurrence relations:

$$\begin{aligned} H^{(i)}(t_1, \ldots, t_i) = & \ H^{(i-1)}(t_1, \ldots, t_{i-1}) H^{(1)}(t_i) \\ & - \sum_{j=1}^{i-1} H^{(i-2)}(t_{i_1}, \ldots, t_{i_{i-2}}) \delta(t_{i-j} - t_i), \quad i \geq 2, \end{aligned} \tag{2}$$

starting from $H^{(0)} = 1$ and $H^{(1)}(t_1) = n(t_1)$. Taking into account the following statistical properties of white noise process

$$\mathrm{E}\left[n(t)\right] = 0, \quad \mathrm{E}\left[n(t_1)n(t_2)\right] = \delta(t_1 - t_2), \tag{3}$$

where $\mathrm{E}\left[\cdot\right]$ denotes the expectation operator, one can establish that WH polynomials are centered with respect to the origin (except $\mathrm{E}\left[H^{(0)}\right] = 1$) and they are statistically orthogonal:

$$\mathrm{E}\left[H^{(i)}\right] = 0, \forall i \geq 1; \quad \mathrm{E}\left[H^{(i)} H^{(j)}\right] = 0, \forall i \neq j. \tag{4}$$

As a consequence of the completeness of the WH set [8], any arbitrary stochastic process, say $x(t) = x(t; \omega)$, $\omega \in \Omega$, can be expanded in terms of a WH polynomials set and this expansion converges to the original stochastic process, i.e.,

$$x(t) = x^{(0)}(t) + \int_{\mathbb{R}} x^{(1)}(t; t_1) H^{(1)}(t_1)\, dt_1 + \int_{\mathbb{R}^2} x^{(2)}(t; t_1, t_2) H^{(2)}(t_1, t_2)\, dt_1 dt_2 + \cdots, \tag{5}$$

where $x^{(0)} = x^{(0)}(t)$, $x^{(i)} = x^{(i)}(t; t_1, \ldots, t_i)$, $i \geq 1$ are called the (deterministic) kernels of the WHE of $x(t)$. The first two terms of the right-hand side define the Gaussian representation of $x(t)$ (begin the zeroth-order term just its mean or average, i.e., $\mathrm{E}[x(t)] = x^{(0)}(t)$), while the second and higher-order terms correspond to the non-Gaussian part. The variance of $x(t)$ can be expressed as follows:

$$\mathrm{Var}[x(t)] = \int_{\mathbb{R}} \left(x^{(1)}(t;t_1)\right)^2 dt_1 + 2\int_{\mathbb{R}^2} \left(x^{(2)}(t;t_1,t_2)\right)^2 dt_1 dt_2 + \cdots . \qquad (6)$$

3 Application of the WHE to approximate the solution of the nonlinear problem

In this section we will apply the WHE in order to analyze the response of the nonlinear model (1) to the Gaussian stochastic process $n(t)$ with intensity λ. The procedure can be described as follows: first, from the original governing equation (1), we expand the unknown $x(t)$ by means of the WHE given by (5), then, integral-differential deterministic equations are derived for the dynamics of the unknown kernel functions $x^{(i)}$ of the WHE of the response. For that, we take advantage of the stochastic orthogonality properties of WH polynomials.

In practice, the WHE series for the response must be truncated after a few terms. Henceforth, we are just interested in obtaining the Gaussian approximation of the response $x(t)$ of problem (1), then two integral-differential equations for $x^{(0)}(t)$ and $x^{(1)}(t;t_1)$ must be established. For the first one, we just follow previous procedure: we substitute the WHE (5) of $x(t)$ in model (1), next we take the expectation operator over the resulting expression and, finally, we take advantage of properties (4) as well as that $\mathrm{E}\left[H^{(1)}(t_1)H^{(1)}(t_2)\right] = \delta(t_1-t_2)$ and $\mathrm{E}\left[H^{(2)}(t_1,t_2)H^{(2)}(t_3,t_4)\right] = \delta(t_1-t_3)\delta(t_2-t_4) + \delta(t_1-t_4)\delta(t_2-t_3)$. This leads to

$$\dot{x}^{(0)}(t) = a(t)x^{(0)}(t) - \epsilon\left\{\left(x^{(0)}(t)\right)^2 + \int_{\mathbb{R}}\left(x^{(1)}(t;t_1)\right)^2 dt_1\right\}, \; x^{(0)}(0) = x_0, \quad (7)$$

where initial condition has been derived by setting $t = 0$ in (5), next applying the expectation operator and again taking advantage of first property given by (4). In order to establish another (deterministic) differential equation for $x^{(1)}(t;t_1)$, firstly we multiply WHE (5) of $x(t)$ by $H^{(1)}(t_5)$, next we take the expectation operator and we again apply above properties together with $\mathrm{E}\left[H^{(1)}(t_1)H^{(1)}(t_2)H^{(1)}(t_3)\right] = 0$, $\mathrm{E}\left[H^{(1)}(t_1)H^{(2)}(t_2,t_3)H^{(2)}(t_4,t_5)\right] = 0$ and $\mathrm{E}\left[H^{(1)}(t_1)H^{(1)}(t_2)H^{(2)}(t_3,t_4)\right] = \delta(t_1-t_3)\delta(t_2-t_4) + \delta(t_1-t_4)\delta(t_2-t_3)$. In this way, one gets

$$\dot{x}^{(1)}(t;t_1) = a(t)x^{(1)}(t;t_1) - 2\epsilon x^{(0)}(t)x^{(1)}(t;t_1) + \lambda\delta(t-t_1), \; x^{(1)}(0;t_1) = 0, \; \forall t_1. \quad (8)$$

In this case, initial condition has been derived multiplying by $H^{(1)}(t_2)$ the

WHE (5), next setting $t = 0$ and taking the expectation operator and, finally, applying first property of (4) as well as $\mathrm{E}\left[H^{(1)}(t_1)H^{(1)}(t_2)\right] = \delta(t_1 - t_2)$.

4 The application of the perturbation method. An illustrative example

In order to compute the Gaussian part of the stochastic process solution of problem (1), we need to solve the nonlinear coupled deterministic problems (7)–(8). Note that both problems depend on small parameter $\epsilon > 0$. Then a reliable technique in order to solve them is the so-called perturbation method under which the deterministic kernels can be represented in first approximation as:

$$x^{(0)}(t) = x_0^{(0)}(t) + \epsilon x_1^{(0)}(t), \quad x^{(1)}(t;t_1) = x_0^{(1)}(t;t_1) + \epsilon x_1^{(1)}(t;t_1). \tag{9}$$

Substituting these representations in equations (7)–(8) and neglecting these powers of ϵ whose exponents are greater than 1, one obtains the following initial value problems:

$$\dot{x}_0^{(0)}(t) = a(t)x_0^{(0)}(t), \quad x_0^{(0)}(0) = x_0, \tag{10}$$

$$\dot{x}_1^{(0)}(t) = a(t)x_1^{(0)}(t) - \left(x_0^{(0)}(t)\right)^2 - \int_0^\infty \left(x_0^{(1)}(t;t_1)\right)^2 dt_1, \quad x_1^{(0)}(0) = 0, \tag{11}$$

$$\dot{x}_0^{(1)}(t;t_1) = a(t)x_0^{(1)}(t;t_1) + \lambda\delta(t - t_1), \quad x_0^{(1)}(0;t_1) = 0, \ \forall t_1 \geq 0, \tag{12}$$

$$\dot{x}_1^{(1)}(t;t_1) = a(t)x_1^{(1)}(t;t_1) - 2x_0^{(0)}(t)x_0^{(1)}(t;t_1), \quad x_1^{(1)}(0;t_1) = 0, \ \forall t_1 \geq 0. \tag{13}$$

Example 1 *Let us consider the important situation where diffusion coefficient does not depend on time, i.e., $a(t) = a$. In this case, we first compute directly the solution of problems (10) and (12), and after that we solve problems (11) and (13). The obtained results are:*

$$x_0^{(0)}(t) = x_0 e^{at}, \quad x_1^{(0)}(t) = -\frac{\left(e^{at} - 1\right)\left(e^{at}(2a(x_0)^2 + \lambda^2) - \lambda^2\right)}{2a^2}, \tag{14}$$

$$x_0^{(1)}(t;t_1) = \begin{cases} \lambda e^{a(t-t_1)} & \text{if } t \geq t_1, \\ 0 & \text{if } t < t_1, \end{cases} \tag{15}$$

$$x_1^{(1)}(t;t_1) = \begin{cases} -\dfrac{2e^{a(t-t_1)}\left(e^{at} - 1\right)\lambda x_0}{a} & \text{if } t \geq t_1, \\ 0 & \text{if } t < t_1. \end{cases} \tag{16}$$

Taking into account that $E[x(t)] = x^{(0)}(t) = x_0^{(0)}(t) + \epsilon x_1^{(0)}(t)$*, one gets the following approximation of the mean of* $x(t)$

$$E[x(t)] = x_0 e^{at} - \epsilon \frac{(e^{at}-1)\left(e^{at}(2a(x_0)^2+\lambda^2)-\lambda^2\right)}{2a^2}. \tag{17}$$

Regarding the variance approximation, note that by perturbation method and (6) one gets

$$Var[x(t)] = \int_{-\infty}^{\infty} \left\{ \left(x_0^{(1)}(t;t_1)\right)^2 + 2\epsilon x_0^{(1)}(t;t_1) x_1^{(1)}(t;t_1) + \epsilon^2 \left(x_1^{(1)}(t;t_1)\right)^2 \right\} dt_1, \tag{18}$$

that leads in our case to

$$\begin{aligned} Var[x(t)] = {} & \frac{\lambda^2}{2a}\left(e^{2at}-1\right) - 2\epsilon \frac{\lambda^2 x_0}{a^2}\left(e^{at}-1\right)^2\left(e^{at}+1\right) \\ & + 2\epsilon^2 \frac{\lambda^2 (x_0)^2}{a^3}\left(e^{at}-1\right)^3\left(e^{at}+1\right). \end{aligned} \tag{19}$$

Columns $E\left[x^{WHE}(t)\right]$ *and* $Var\left[x^{WHE}(t)\right]$ *of Table 1 show the results for the average and variance obtained from (17) and (19), respectively, for* $\lambda = 1$, $\epsilon = 10^{-2}$, $a = 1/2$, $x_0 = 0.5$*. In Figures 1 and 2, we compare these results with respect to those obtained by using a Runge-Kutta second order stochastic scheme [9], where the involved Brownian motion has been simulated taking* $m = 100000$ *simulations and step* $h = 10^{-4}$*. In addition, third and fifth columns of Table 1 show the relative errors for the average (RelErrE) and variance (RelErrVar) with respect to Runge-Kutta scheme. Note that the approximations obtained from both approaches are agree.*

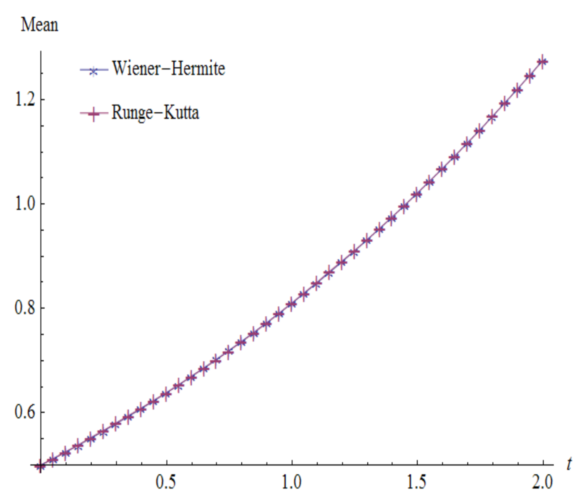

Fig. 1. Comparison of the expectation by using the Wiener-Hermite expansion technique for problem (1) with $a(t) = 1/2$, $\lambda = 1$, $\epsilon = 10^{-2}$ and $x_0 = 0.5$ on the interval $[0, 2]$ and a Runge-Kutta stochastic scheme by considering $m = 100000$ simulations and taking as step $h = 10^{-4}$

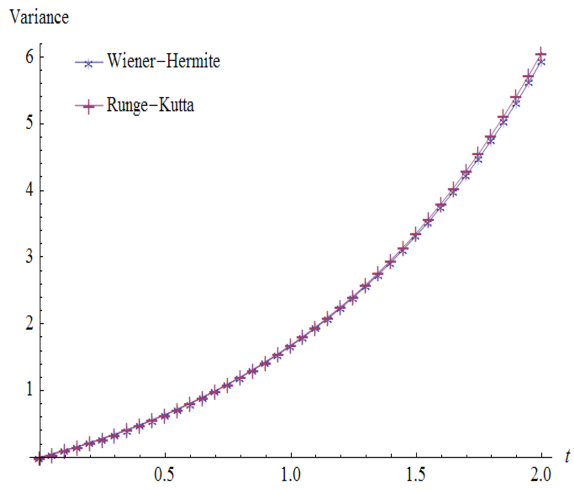

Fig. 2. Comparison of the variance by using the Wiener-Hermite expansion technique for problem (1) with $a(t) = 1/2$, $\lambda = 1$, $\epsilon = 10^{-2}$ and $x_0 = 0.5$ on the interval $[0, 2]$ and a Runge-Kutta stochastic scheme by considering $m = 100000$ simulations and taking as step $h = 10^{-4}$

t	$\mathrm{E}\left[x^{WHE}(t)\right]$	RelErrE	$\mathrm{Var}\left[x^{WHE}(t)\right]$	RelErrVar
0.00	0.5	0	0	0
0.25	0.565465	0.001495642	0.282515	0.004461187
0.50	0.638576	0.000480526	0.641372	0.004179709
0.75	0.720045	0.0009411	1.09676	0.00440265
1.00	0.810596	0.00063616	1.67398	0.00537721
1.25	0.9110935	0.00100017	2.4046	0.00656068
1.50	1.02172	0.00034244	3.32786	0.01095191
1.75	1.14352	0.0006563	4.49228	0.01562586
2.00	1.27674	0.0006662	5.95747	0.01874725

Table 1
Numerical values of the expectation and variance as well as its relative errors by using the Wiener-Hermite expansion technique for problem (1) with $a(t) = 1/2$, $\lambda = 1$, $\epsilon = 10^{-2}$ and $x_0 = 0.5$ on the interval $[0, 2]$ and a Runge-Kutta stochastic scheme by considering $m = 100000$ simulations and taking as step $h = 10^{-4}$

5 Conclusions

This paper shows that WHE technique constitutes a powerful tool to construct approximate solution stochastic process of random diffusion models with non-linear perturbations where uncertainty is considered by means of an additive term defined by white noise. As it has been highlighted, these type of models appear in important applications of fields such as Physics and Epidemiology, for example. Although the success of WHE method depends heavily on complexity encountered in dealing with integro-differential equations, a large of deterministic techniques to solve them are available including mathematical software [10]. Besides computing the Gaussian approximation of the solution, we have also provided approximations of its average and variance. As we have shown, these results are agree with respect to those obtained by applying other stochastic numerical methods. Finally, we remark that in the near future, we will report the corresponding results to the non-Gaussian approximation.

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References

[1] O. Diekmann, J.A.P. Heesterbeek, Mathematical Epidemiology of Infectious Diseases: Model Building, Analysis and Interpretation, Wiley Series in Mathematical and Computational Biology, New York, 2000.

[2] M.A. El–Tawil, The application of the WHEP technique on partial differential equations, Intern. J. Diff. Equat. and its Appl. 7 (2003) 325–337.

[3] M.A. El–Tawil, A.S. Al-Jihany, On the solution of stochastic oscillatory quadratic nonlinear equaitons using different tecniques, a comparison study, Topol. Methods Nonlinear Anal. 31(2) (2008) 315–330.

[4] M.A. El–Tawil, N.A. Al-Mulla, Using homotopy WHEP technique for solving a stochastic nonlinear diffusion equation, Math. Comput. Modell. 51 (2010) 1277–1284.

[5] C.Q. Dai, J.F. Zhang, Application of He's Exp-function method to the stochastic mKdV equation, Intern. Nonlin. Sci. Numer. Simul. 10 (5) (2009) 675–680.

[6] C.Q. Dai, J.F. Zhang, Stochastic exact solutions and two-soliton solution of the Wick–type stochastic KdV equation, Europhysics Letters 86 (2009) Art. No. 40006.

[7] R.G. Ghanem, P.D. Spanos, Stochastic Finite Elements: A Spectral Approach, Dover Publications, New York, 2003.

[8] R.H. Cameron, W.T. Martin, The orthogonal development of non-linear functionals in series of Fourier-Hermite functionals, J. Ann. Math. 48 (1947) 385–392.

[9] E. Kloeden, E. Platen, Numerical Solution of Stochastic Differential Equations, Springer, Berlin 1992.

[10] P. Linz, Analytical and Numerical Methods for Volterra Equations, SIAM, Philadelphia, 1985.