Mean ergodic operators and reflexive Fréchet lattices

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Abstract

Connections between (positive) mean ergodic operators acting in Banach lattices and properties of the underlying lattice itself are well understood; see the works of Emel'yanov, Wolff and Zaharopol cited in the references. For Fréchet lattices (or more general locally convex solid Riesz spaces) there is virtually no information available. For a Fréchet lattice E, it is shown here (amongst other things) that every power bounded linear operator on E is mean ergodic if and only if E is reflexive if and only if E is Dedekind σ-complete and every positive power bounded operator on E is mean ergodic if and only if every positive power bounded operator in the strong dual E'_β (no longer a Fréchet lattice) is mean ergodic. An important technique is to develop criteria which detect when E admits a (positively) complemented lattice copy of c_0, ℓ_1 or ℓ_∞.

1 Introduction and statement of results

A continuous linear operator T in a Banach space E (or locally convex Hausdorff space, briefly lcHs) is called *mean ergodic* if the limits

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} T^m x, \quad x \in E, \tag{1}$$

exist (in the topology of E). J. von Neumann (1931) proved that unitary operators in Hilbert space are mean ergodic. Ever since, intensive research has been undertaken concerning mean ergodic operators and their applications; for the period up to the 1980's see [13], Ch. VIII, Section 4, [18], Ch. XVIII, [20], Ch. 2, and the references therein.

It quickly became evident that there was an intimate connection between geometric properties of the underlying Banach space E and mean ergodic operators on E. A continuous linear operator T in E (the space of all such operators is denoted by $\mathcal{L}(E)$) is called *power bounded* if $\sup_{m\geq 0} \|T^m\| < \infty$. The space E itself is called *mean ergodic* if for every power bounded $T \in \mathcal{L}(E)$ the limits

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(1) exist. As a sample, F. Riesz (1938) showed that all L_p-spaces ($1 < p < \infty$) are mean ergodic. In 1938, E.R. Lorch proved that all reflexive Banach spaces are mean ergodic. In the opposite direction, in 2001 V.P. Fonf, M. Lin and P. Wojtaszczyk, [16], established (amongst other things) that a Banach space E *with a basis* is reflexive if and only if E is mean ergodic. For *Banach lattices* the requirement of a basis can be omitted. Indeed, in 1986, R. Zaharopol showed that if E is a Dedekind σ-complete Banach lattice, then E is reflexive if and only if every *positive* power bounded operator $T \in \mathcal{L}(E)$ is mean ergodic, [30]. For an arbitrary Banach lattice E, it was shown by E.Yu. Emel'yanov in 1997, [14], that E is reflexive if and only if every *regular* power bounded operator $T \in \mathcal{L}(E)$ is mean ergodic (regular means that T is the difference of two positive operators). According to E.Yu. Emel'yanov and M.P.H. Wolff, [15], a Banach lattice has order continuous norm if and only if every *power order bounded* operator on E is mean ergodic.

For a lcHs E, the definition of $T \in \mathcal{L}(E)$ being mean ergodic (i.e., via (1)) makes perfectly good sense, as does the notion of power boundedness, now meaning that $\{T^m\}_{m \geq 0}$ is an equicontinuous subset of $\mathcal{L}(E)$. The first "mean ergodic result" for (a special class of) power bounded operators T on certain lcHs' E is due to M. Altman, [6]. The restriction on T that Altman imposed (a weak compactness condition) was later removed by K. Yosida ([28], Ch. VIII). In more recent times, most of the Banach space results mentioned above which connect geometric properties of the underlying space E to mean ergodicity of operators acting on E were extended to the *Fréchet space* setting in [1] and to more general lcHs' E in [2], [3] and [8]. The aim of this article is to extend the above results concerned with mean ergodic operators in Banach lattices to the setting of *Fréchet lattices*. Classical examples of Fréchet lattices to keep in mind include the sequence space $\omega = \mathbb{R}^{\mathbb{N}}$ and all Köthe echelon spaces $\lambda_p(A)$ for $p \in \{0\} \cup [1, \infty]$, with A a Köthe matrix relative to some countable index set (see e.g. [22]). We also mention ℓ^{p+}, $1 \leq p < \infty$ (see [23]) and L^{p-}, $1 < p < \infty$ (see [11]). Further examples are $L^p_{loc}(\Omega)$ for $1 \leq p \leq \infty$ with $\Omega \subseteq \mathbb{R}^N$ open and $C(\Omega)$, equipped with the topology of uniform convergence on compact subsets of Ω. Finally, if m is any Fréchet space valued vector measure, then the spaces $L^p(m)$ (respectively, $L^p_w(m)$) consisting of the p-th power m-integrable (respectively, weakly m-integrable) functions are Fréchet lattices (see [9], [10], [25]).

So, let us formulate some of the main results. General references for the theory of lcHs' are [19], [22] and [26]. If Γ_E is a system of continuous seminorms generating the topology of a lcHs E, then the *strong operator topology* τ_s in $\mathcal{L}(E)$ is determined by the seminorms

$$q_x(S) = q(Sx), \quad S \in \mathcal{L}(E), \tag{2}$$

for each $x \in E$ and $q \in \Gamma_E$ (in which case we write $\mathcal{L}_s(E)$). The *uniform operator topology* τ_b in $\mathcal{L}(E)$ is defined via the seminorms

$$q_B(S) = \sup_{x \in B} q(Sx), \quad S \in \mathcal{L}(X), \tag{3}$$

for each $q \in \Gamma_E$ and bounded set $B \subseteq E$ (in which case we write $\mathcal{L}_b(E)$). If E is a Banach space, then τ_b is the operator norm topology on $\mathcal{L}(E)$. A *Fréchet space* is a complete, metrizable lcHs E, in which case Γ_E may be taken countable. The topological dual space of a lcHs E is denoted by E'. The weak topology induced on E by the pairing $\langle E, E' \rangle$ is written as $\sigma(E, E')$ and the strong topology on E (respectively E') is denoted by $\beta(E, E')$ (respectively $\beta(E', E)$), in which case we write E_β (respectively E'_β). Then, E'_β is called the strong dual of E and $E'' = \left(E'_\beta\right)'_\beta$ is the strong bidual of E. If $E = E''$ as vector spaces (respectively, additionally topologically), then E is called semireflexive (respectively, reflexive). For a Fréchet space E, the strong dual E'_β need not be metrizable but, E'' is again a Fréchet space containing E as a closed subspace ([22], Corollary 25.10).

Relevant references for the theory of Riesz spaces (which will always be over $\mathbb{R}$), with $\leq$ denoting the order, are [21], [24], [29]. For locally convex-solid Riesz spaces E (briefly lc-solid Riesz spaces) we refer to [4], [5], [17], for example. In this case, the seminorms $q \in \Gamma_E$ can all be chosen to be *Riesz seminorms*, that is, $q(x) \leq q(y)$ whenever $|x| \leq |y|$ in E ([4], Theorem 6.1). As usual, a linear operator $T \in \mathcal{L}(E)$ is called *positive* if $Tx \geq 0$ whenever $x \in E^+$, where $E^+ = \{x \in E : x \geq 0\}$ is the positive cone of E. A Riesz space E is said to be Dedekind (σ-) complete if every non-empty (countable) subset of E that is order bounded from above has a supremum. Typical examples of Riesz spaces which fail to be Dedekind σ-complete are the sequence space c and the space of continuous functions $C([0,1])$.

A lc-solid Riesz space which is metrizable and complete is simply called a *Fréchet lattice* ([4], p. 111).

Theorem 1.1 *If E is a Fréchet lattice, then the following assertions are equivalent.*

(i) E is reflexive.

(ii) E is mean ergodic.

(iii) E is Dedekind σ-complete and every positive power bounded operator on E is mean ergodic.

(iv) Every positive, power bounded operator on E'_β is mean ergodic.

The main tool needed to establish Theorem 1.1 is of interest in its own right. Two lc-solid Riesz spaces E and F are called *Riesz homeomorphic* if there exists a *Riesz homeomorphism* $J : E \to F$ (that is, J is a linear lattice homomorphism from E onto F which is also a homeomorphism). If E contains a Riesz subspace which is Riesz homeomorphic to F, then we say that E contains a *lattice copy* of F. As usual, a closed Riesz subspace F of a lc-solid Riesz space E is said to be *(positively) complemented* in E if F is the range of a linear continuous (positive) projection.

Crucial for establishing Theorem 1.1 is the following result.

Theorem 1.2 *If E is a Dedekind σ-complete lc-solid Riesz space which is complete and $\aleph_0$-barrelled, then the following assertions are equivalent.*

(i) E is not semireflexive.

(ii) E contains a lattice copy of either ℓ_∞, ℓ_1 or c_0.

The relevance of the Banach lattices c_0, ℓ_1 and ℓ_∞ is that each one admits a positive power bounded operator which *fails* to be mean ergodic. Indeed, denoting the elements in these sequence spaces by $x = (x_1, x_2, \ldots)$, it can be verified that the operators

$$\begin{array}{ll} T_0 : x \longmapsto (x_1, x_1, x_2, x_3, \ldots), & x \in c_0, \\ T_1 : x \longmapsto (0, x_1, x_2, x_3, \ldots), & x \in \ell_1, \\ T_\infty : x \longmapsto (x_2, x_3, x_4, \ldots), & x \in \ell_\infty, \end{array} \tag{4}$$

on the spaces c_0, ℓ_1 and ℓ_∞ respectively, have the stated properties.

Given a Riesz space E, a linear map $T : E \to E$ is called *power order bounded* if, for every $x \in E^+$ there exists $z \in E^+$ such that

$$\bigcup_{m=0}^{\infty} T^m([-x, x]) \subseteq [-z, z],$$

where $[-u, u]$ denotes the order interval $\{y \in E : -u \leq y \leq u\}$, for each $u \in E^+$. Order intervals in a lc-solid Riesz space E are always topologically bounded ([4], Theorem 5.4). Hence, if E is barrelled and $T \in \mathcal{L}(E)$ is power order bounded, then the uniform boundedness principle implies that T is power bounded ([22], Proposition 23.27).

Recall that a lc-solid Riesz space E has a *(σ-) Lebesgue topology* if, for every decreasing (sequence) net $x_\alpha \downarrow_\alpha 0$ it follows that $x_\alpha \to_\alpha 0$ with respect to the given topology ([4], p. 52). For Banach lattices this notion corresponds to (σ-) order continuity of the norm ([24], Ch. 2, Section 4). The extension of the above mentioned result of Emel'yanov and Wolff can now be formulated.

Theorem 1.3 *For a Fréchet lattice E the following assertions are equivalent.*

(i) E has a Lebesgue topology.

(ii) Every power order bounded operator on E is mean ergodic.

We mention that Theorems 1.1-1.3 will actually be established in somewhat more generality than the (more transparent) versions formulated above.

For a Banach space E *with a basis* it is known that E is uniformly mean ergodic if and only if E is finite dimensional ([16], Corollary 3). Here, a lcHs E is called *uniformly mean ergodic* if every power bounded operator T on E has the property that its Cesàro means

$$T_{[n]} = \frac{1}{n} \sum_{m=1}^{n} T^m, \quad n \in \mathbb{N}, \tag{5}$$

form a convergent sequence in $\mathcal{L}_b(E)$. For Banach lattices the requirement of a basis can be omitted.

Theorem 1.4 *A Banach lattice E is uniformly mean ergodic if and only if E is finite dimensional.*

It is known that every Montel Fréchet lattice (e.g. ω or $\lambda_p(A)$, $1 \leq p \leq \infty$, for those Köthe matrices A such that $\lambda_1(A)$ is reflexive; [22], Theorem 27.9) is necessarily uniformly mean ergodic ([1], Proposition 2.8). Our final result may be viewed as an analogue of Theorem 1.4 for non-normable Fréchet lattices. We point out that every Montel Fréchet lattice is necessarily discrete ([4], Corollary 21.13).

Theorem 1.5 *A Fréchet lattice E is Montel if and only if E is discrete and uniformly mean ergodic.*

2 Some preliminary results

For a Riesz space E, we recall that the *order dual* $E^{\sim}$ is always a Dedekind complete Riesz space ([4], Theorem 3.3). A classical result of F. Riesz states that in any Dedekind complete Riesz space E, every band B is a *projection band*, that is, $E = B \oplus B^d$ or, equivalently, there exists a linear projection $P : E \to E$ with range $\mathrm{Im}\,(\mathrm{P}) = \mathrm{B}$ and satisfying $Px \in [0, x]$, $x \in E^+$ ([4], Theorem 2.12; [24], Theorem 1.2.9). Such a projection P is called a *band projection* in E (note that if E is a lc-solid Riesz space, then every band projection P is continuous, because $|Px| \leq |x|$ for $x \in E$). Here, $B^d = \{x \in E : |x| \wedge |y| = 0, \ \forall\ y \in B\}$.

If E is a lc-solid Riesz space, then E'_β is also a lc-solid Riesz space whose topology is given by the family of Riesz seminorms

$$q_B(x') = \sup_{x \in B} |\langle x, x' \rangle|, \quad x' \in E'_\beta, \tag{6}$$

as B runs through the collection $\mathcal{B}_s$ of all bounded, solid subsets of E ([4], pp. 59, 129). Moreover, E'_β is an ideal in $E^{\sim}$ and so, in particular, E'_β is Dedekind complete ([4], Theorem 5.7). If E happens to be barrelled, then E'_β is a *band* in $E^{\sim}$ ([4], Theorem 6.4). Consequently, E'_β is then topologically complete ([4], Theorem 19.13).

We will require the following result on extending linear functionals ([29], Theorem 83.17).

Theorem 2.1 *Let E be a Riesz space, $F \subseteq E$ be a Riesz subspace (i.e., vector sublattice) and $\theta : E \to \mathbb{R}$ be a sublinear functional which is absolute (i.e., $\theta(x) = \theta(|x|)$, $x \in E$) and monotone on E^+ (i.e., $\theta(x) \leq \theta(y)$ whenever $0 \leq x \leq y$ in E). If $\varphi : F \to \mathbb{R}$ is a positive linear functional satisfying $|\langle x, \varphi \rangle| \leq \theta(x)$ for $x \in F$, then there exists a positive linear functional $\psi : E \to \mathbb{R}$ such that $\psi|_F = \varphi$ and $|\langle x, \psi \rangle| \leq \theta(x)$ for $x \in E$.*

As an immediate application, we present a result which is well known in the Banach lattice setting ([24], Proposition 2.3.11).

Proposition 2.2 *If F is a lattice copy of ℓ_1 in a lc-solid Riesz space E, then*

(i) F is positively complemented in E, and

(ii) E'_β contains a lattice copy of ℓ_∞.

Proof. (i). Let $\|\cdot\|_1$ be a Riesz norm on F such that the topology of F induced by E is given by $\|\cdot\|_1$ and $(F, \|\cdot\|_1)$ is Riesz isometric to ℓ_1. In particular, there exists a continuous Riesz seminorm r on E such that

$$\|x\|_1 \leq r(x), \quad x \in F.$$

Let $\{v_n\}_{n=1}^{\infty} \subseteq F$ correspond to the standard unit basis vectors of ℓ_1 (so that $\|v_n\|_1 = 1$ for all $n \in \mathbb{N}$).

For each $x \in F$ there exists a unique sequence $\{\alpha_n(x)\}_{n=1}^{\infty} \in \ell_1$ satisfying $x = \sum_{n=1}^{\infty} \alpha_n(x) v_n$, with the series convergent in $(F, \|\cdot\|_1)$. Note that $x \in F^+$ if and only if $\alpha_n(x) \geq 0$ for all $n \in \mathbb{N}$. Since $|\alpha_n(x)| \leq \|x\|_1 \leq r(x)$ for $x \in F$, it is clear that $\alpha_n \in (F')^+$ for all $n \in \mathbb{N}$, where $\langle x, \alpha_n \rangle = \alpha_n(x)$, $x \in F$. Define the positive linear functional $y_1' \in F'$ by setting

$$\langle x, y_1' \rangle = \sum_{k=1}^{\infty} \langle x, \alpha_k \rangle, \quad x \in F,$$

in which case

$$|\langle x, y_1' \rangle| \leq \langle |x|, y_1' \rangle \leq r(|x|) = r(x), \quad x \in F.$$

Evidently, $0 \leq \alpha_n \leq y_1'$ for all $n \in \mathbb{N}$. By Theorem 2.1, applied to $\theta = r$ and $\varphi = y_1'$, there exists $0 \leq x_1' \in E^{\sim}$ with $x_1' \mid_F = y_1'$ such that $|\langle x, x_1' \rangle| \leq r(x)$, $x \in E$. In particular, $x_1' \in (E')^+$. Since, for each $n \in \mathbb{N}$,

$$|\langle x, \alpha_n \rangle| \leq \langle |x|, \alpha_n \rangle \leq \langle |x|, y_1' \rangle = \langle |x|, x_1' \rangle, \quad x \in F,$$

it follows from Theorem 2.1 that there exists $0 \leq z_n' \in E^{\sim}$ with $z_n' \mid_F = \alpha_n$ such that $|\langle x, z_n' \rangle| \leq \langle |x|, x_1' \rangle$ for $x \in E$. In particular, $z_n' \in (E')^+$ and $0 \leq z_n' \leq x_1'$ for all $n \in \mathbb{N}$. Let φ_n be the minimal positive extension of the restriction of z_n' to the principal ideal E_{v_n} generated by v_n in E (see [29], Theorems 83.7 and 83.8). It follows from $0 \leq \varphi_n \leq z_n'$ that $\varphi_n \in (E')^+$ and $0 \leq \varphi_n \leq x_1'$ for all n. Since $\langle v_m, \varphi_n \rangle = \delta_{n,m}$ for all $n, m \in \mathbb{N}$, it is clear that $\varphi_n \mid_F = \alpha_n$. Furthermore, since $v_n \wedge v_m = 0$ $(n \neq m)$, it can be verified (see [5], Exercise 2.3) that $\varphi_n \wedge \varphi_m = 0$ in E' whenever $n \neq m$.

If $x \in E$, then

$$\sum\nolimits_{k=1}^{n} |\langle x, \varphi_k \rangle| \leq \sum\nolimits_{k=1}^{n} \langle |x|, \varphi_k \rangle = \left\langle |x|, \bigvee\nolimits_{k=1}^{n} \varphi_k \right\rangle \leq \langle |x|, x_1' \rangle$$

for all $n \in \mathbb{N}$ and so, $\sum_{k=1}^{\infty} |\langle x, \varphi_k \rangle| < \infty$. Consequently,

$$\sum\nolimits_{n=1}^{\infty} \|\langle x, \varphi_n \rangle v_n\|_1 = \sum_{n=1}^{\infty} |\langle x, \varphi_n \rangle| < \infty, \quad x \in E.$$

Hence, the series

$$Px = \sum\nolimits_{n=1}^{\infty} \langle x, \varphi_n \rangle v_n, \quad x \in E, \tag{7}$$

converges in the complete space F. It is now clear that P is a positive projection in E onto F.

(ii). Using the notation introduced in the proof of (i), define the map $\Phi_0 : \ell_\infty^+ \to (E')^+$ by setting

$$\Phi_0(\lambda) = \bigvee\nolimits_{n=1}^{\infty} \lambda_n \varphi_n, \quad 0 \leq \lambda = (\lambda_n) \in \ell_\infty^+.$$

Since $0 \leq \varphi_n \leq x_1'$ for all n and E' is Dedekind complete, this map is well defined and satisfies $0 \leq \Phi_0(\lambda) \leq \|\lambda\|_\infty x_1'$ for $\lambda \in \ell_\infty^+$. Since $\{\varphi_n\}_{n=1}^{\infty}$ is a disjoint system in $(E')^+$, it is clear that Φ_0 is additive, positive homogeneous and that $\Phi_0(\lambda) \wedge \Phi_0(\mu) = 0$ whenever $\lambda \wedge \mu = 0$ in ℓ_∞^+. Therefore, Φ_0 has a unique extension to a Riesz homomorphism $\Phi : \ell_\infty \to E'$ ([4], Theorem 1.17

and Lemma 3.1). We claim that Φ is a linear homeomorphism from ℓ_∞ onto its range in E'_β. Indeed, if p is *any* Riesz seminorm on E', then

$$p(\Phi(\lambda)) = p(\Phi(|\lambda|)) \leq \|\lambda\|_\infty p(x'_1), \quad \lambda \in \ell_\infty.$$

On the other hand, if B is the convex solid hull in E of the bounded set $\{v_n\}_{n=1}^\infty$, then the continuous Riesz seminorm q_B on E'_β, defined by (6), satisfies $q_B(\varphi_n) = 1$ for all $n \in \mathbb{N}$. If $\lambda = (\lambda_n) \in \ell_\infty$, then

$$|\Phi(\lambda)| = \Phi(|\lambda|) \geq |\lambda_n| \varphi_n$$

and so, $q_B(\Phi(\lambda)) \geq |\lambda_n|$ for all $n \in \mathbb{N}$. This implies that $q_B(\Phi(\lambda)) \geq \|\lambda\|_\infty$, $\lambda \in \ell_\infty$, and we may conclude that Φ is a linear homeomorphism. The proof is complete. ■

Remark 2.3 (a) It can be verified that the adjoint $P' \in \mathcal{L}\left(E'_\beta\right)$ of the projection P in E defined by (7) is a positive projection in E'_β onto the lattice copy $\Phi(\ell_\infty)$ of ℓ_∞ in E'_β, as constructed in (ii) of the above proof.

(b) Any lattice copy of ℓ_∞ in a lc-solid Riesz space E is positively complemented. Indeed, suppose that F is a Riesz subspace of E and let $J : \ell_\infty \to F$ be a Riesz homeomorphism. For every continuous Riesz seminorm p on E there exists a constant $C_p \geq 0$ such that $p(J\lambda) \leq C_p \|\lambda\|_\infty$ for $\lambda \in \ell_\infty$. There also exists a continuous Riesz seminorm q on E such that $\|\lambda\|_\infty \leq q(J\lambda)$ for $\lambda \in \ell_\infty$. For each $n \in \mathbb{N}$, define the positive linear functional φ_n on F by $\langle x, \varphi_n \rangle = \left(J^{-1}x\right)(n)$, $x \in F$, where $\left(J^{-1}x\right)(n)$ denotes the n-th coordinate of $J^{-1}x$. Note that $J^{-1}x = (\langle x, \varphi_n \rangle)$ and hence, $J((\langle x, \varphi_n \rangle)) = x$ for all $x \in F$. Since

$$|\langle x, \varphi_n \rangle| = \left|\left(J^{-1}x\right)(n)\right| \leq \left\|J^{-1}x\right\|_\infty \leq q(x), \quad x \in F,$$

it follows from Theorem 2.1 that, for each $n \in \mathbb{N}$, there exists a positive linear functional ψ_n on E such that $\psi_n |_F = \varphi_n$ and $|\langle x, \psi_n \rangle| \leq q(x)$, $x \in E$ (and so, $0 \leq \psi_n \in E'$). It is clear that $(\langle x, \psi_n \rangle) \in \ell_\infty$ for all $x \in E$. Defining the map $P : E \to E$ via

$$Px = J((\langle x, \psi_n \rangle)), \quad x \in E,$$

it can be checked that P is a positive continuous projection in E onto F. This proves the claim.

Proposition 2.4 *Suppose that E is a sequentially complete, lc-solid Riesz space, with the property that countable, bounded subsets of E'_β are equicontinuous. If E'_β contains a lattice copy of c_0, then E contains a positively complemented lattice copy of ℓ_1.*

Remark 2.5 (i) Every $\aleph_0$-barrelled lcHs E has the property that countable, bounded subsets of E'_β are equicontinuous ([26], Observation 8.2.2 (a)). All barrelled (hence, all Fréchet) lcHs' are $\aleph_0$-barrelled ([26], Observation 8.2.2 (b)); the same is true for all complete (DF)-spaces ([26], Observation 8.2.2 (c)), which include the strong duals of Fréchet spaces.

(ii) For E a Banach lattice, Proposition 2.4 occurs in [24], Proposition 2.3.12.

Proof. (of Proposition 2.4) The idea of the proof follows the lines of that of implication (iii)⇒(i) in [24], Proposition 2.3.12, with various modifications required due to the new setting.

Let F be a lattice copy of c_0 in E'_β and let $\{x'_n\}_{n=1}^\infty \subseteq F$ correspond to the standard unit basis vectors of c_0, in which case $x'_n \geq 0$, $n \in \mathbb{N}$. Then $\{x'_n\}_{n=1}^\infty$ is a bounded subset of E'_β which is *not* a null sequence. Hence, there exists a set $B \in \mathcal{B}_s$ such that $q_B(x'_n) \nrightarrow 0$ as $n \to \infty$, with q_B given by (6). So, by passing to a subsequence if necessary, there exists $\delta > 0$ such that $q_B(x'_n) \geq \delta$, $n \in \mathbb{N}$. It follows from (6) that there exists a sequence $\{x_n\}_{n=1}^\infty \subseteq B$ satisfying

$$|\langle x_n, x'_n \rangle| \geq \delta/2, \quad n \in \mathbb{N}. \tag{8}$$

Since $|\langle x, x'_n \rangle| \leq \langle |x|, x'_n \rangle$, $x \in E$, it is clear from (8) that $\langle |x_n|, x'_n \rangle \geq \delta/2$ for all $n \in \mathbb{N}$, with $\{|x_n|\}_{n=1}^\infty \subseteq B$ (as B is solid). Accordingly, replacing x_n by $|x_n|$, we may assume that $\{x_n\}_{n=1}^\infty \subseteq B^+$. Moreover, since $\{x'_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ are bounded in E'_β and E, respectively, there exists a constant $C > 0$ such that $\langle x_n, x'_n \rangle \leq C$ for all $n \in \mathbb{N}$, that is,

$$\delta/2 \leq \langle x_n, x'_n \rangle \leq C, \quad n \in \mathbb{N}.$$

By replacing x_n with $(4/\delta)\, x_n$, B with $(4/\delta)\, B$ and C with $(4/\delta)\, C$, we may assume that the sequence $\{x_n\}_{n=1}^\infty$ satisfies

$$2 \leq \langle x_n, x'_n \rangle \leq C, \quad n \in \mathbb{N}.$$

Since $\{x_n\}_{n=1}^\infty$ is bounded, given any continuous Riesz seminorm p on E, we have $\sup_n p(x_n) < \infty$ and hence, $\sum_{n=1}^\infty p(2^{-n} x_n) < \infty$. By the sequential completeness of E, it follows that there exists $e \in E^+$ such that $\sum_{n=1}^\infty 2^{-n} x_n = e$, as a convergent series in E. Consequently, $\{x_n\}_{n=1}^\infty$ is contained in the principal ideal E_e generated by e in E. Applying [24], Theorem 2.3.1, to the principal ideal E_e and the sequences $\{x_n\}_{n=1}^\infty$ and $\{x'_n|_{E_e}\}_{n=1}^\infty$, and passing to a subsequence if necessary, it follows that there exists a pairwise disjoint sequence $\{v_n\}_{n=1}^\infty$ in E^+ such that

$$0 \leq v_n \leq x_n, \quad \textit{and} \quad \langle v_n, x'_n \rangle \geq 1, \quad n \in \mathbb{N}. \tag{9}$$

Since B is solid, it is clear that $\{v_n\}_{n=1}^\infty \subseteq B^+$.

Define the *countable* set $A \subseteq (E')^+$ by

$$A = \left\{ \sum\nolimits_{j=1}^n x'_j : n \in \mathbb{N} \right\}.$$

Since A is bounded in $F \cong c_0$, it is also bounded in E'_β. By hypothesis, A is then *equicontinuous*. Consequently, there exists a continuous Riesz seminorm p_0 on E such that

$$|\langle x, x' \rangle| \leq p_0(x), \quad x \in E, \quad x' \in A. \tag{10}$$

Fix $a = (a_1, \ldots, a_n, 0, 0, \ldots) \in c_{00}$. The elements $\{v_j\}_{j=1}^n$ are pairwise disjoint and so $\left|\sum_{j=1}^n a_j v_j\right| = \sum_{j=1}^n |a_j| v_j$. Since $\sum_{j=1}^n x'_j \in A$, it follows from (10) and

(9) that

$$\begin{aligned} p_0\left(\sum\nolimits_{j=1}^{n} a_j v_j\right) &= p_0\left(\sum\nolimits_{j=1}^{n} |a_j|\, v_j\right) \geq \left\langle \sum\nolimits_{j=1}^{n} |a_j|\, v_j, \sum\nolimits_{k=1}^{n} x_k' \right\rangle \\ &\geq \sum\nolimits_{j=1}^{n} |a_j| \left\langle v_j, x_j' \right\rangle \geq \sum\nolimits_{j=1}^{n} |a_j| = \|a\|_1 . \end{aligned}$$

This shows that

$$\|a\|_1 \leq p_0\left(\sum\nolimits_{j=1}^{n} a_j v_j\right). \tag{11}$$

On the other hand, given any continuous Riesz seminorm p on E, we have

$$p\left(\sum\nolimits_{j=1}^{n} a_j v_j\right) \leq C_p \|a\|_1 , \tag{12}$$

where $C_p = \sup_{k\in\mathbb{N}} p(v_k) < \infty$, as $\{v_k\}_{k=1}^{\infty} \subseteq B$ is bounded. Estimates (11) and (12) suffice to conclude that the closed Riesz subspace generated by $\{v_n\}_{n=1}^{\infty}$ is Riesz homeomorphic to ℓ_1.

The conclusion now follows from Proposition 2.2 (i). ■

For a *Banach space* E it is a classical result of C. Bessaga and A. Pelczynski that the dual Banach space E'_β contains an isomorphic copy of the Banach space c_0 if and only if E contains a complemented copy of ℓ_1 ([12], p. 48). The extension of this result to *Fréchet spaces* can be found in [7], Lemma 10. The following corollary, which is an immediate consequence of Proposition 2.2 (ii) and Proposition 2.4, may be considered as a lattice version of these results.

Corollary 2.6 *If E is a sequentially complete lc-solid Riesz space with the property that countable, bounded subsets of E'_β are equicontinuous, then E'_β contains a lattice copy of c_0 if and only if E contains a (positively complemented) lattice copy of ℓ_1.*

Corollary 2.6 is known for Banach lattices; see Propositions 2.3.11 and 2.3.12 in [24].

The following simple fact will be required in the sequel. Recall that the topology in a locally solid Riesz space E is said to be *pre-Lebesgue* whenever every increasing, order bounded sequence in E^+ is Cauchy (see [4], Definition 8.1).

Lemma 2.7 *Let E be a lc-solid Riesz space with a pre-Lebesgue topology. If $\{x_n'\}_{n=1}^{\infty}$ is an equicontinuous, pairwise disjoint sequence in E', then $x_n' \to 0$ with respect to $\sigma(E', E)$.*

Proof. Since $\{x_n'\}_{n=1}^{\infty}$ is equicontinuous, there exists a continuous Riesz seminorm r on E such that $|\langle x, x_n'\rangle| \leq r(x)$ for all $x \in E$ and $n \in \mathbb{N}$. Since $\langle |x|, |x_n'|\rangle = \sup\{|\langle y, x_n'\rangle| : |y| \leq |x|\}$, the disjoint sequence $\{|x_n'|\}_{n=1}^{\infty}$ is also equicontinuous. Therefore, we may assume, without loss of generality, that $x_n' \geq 0$ for all n.

Suppose that $x_n' \nrightarrow 0$ relative to $\sigma(E', E)$, i.e., there exists $x \in E$ such that $\langle x, x_n'\rangle \nrightarrow 0$ as $n \to \infty$. Since $|\langle x, x_n'\rangle| \leq \langle |x|, x_n'\rangle$, we may assume that $x \in E^+$. By passing to a subsequence, if necessary, there exists $\delta > 0$ such that $\langle x, x_n'\rangle \geq \delta$ for all $n \in \mathbb{N}$. Since $\sup_{n\in\mathbb{N}} \langle x, x_n'\rangle \leq r(x) < \infty$, it follows from [24], Theorem 2.3.1 (applied in the principal ideal E_x) that, by passing to a subsequence if necessary, there exist a pairwise disjoint sequence $\{v_n\}_{n=1}^{\infty}$ in $[0,x]$ and $\varepsilon > 0$ such that $\langle v_n, x_n' \rangle \geq \varepsilon$ for all n. The topology in E is pre-Lebesgue and so $v_n \to 0$ in E (see [4], Theorem 10.1). In particular, $r(v_n) \to 0$ as $n \to \infty$. Since $\langle v_n, x_n' \rangle \leq r(v_n)$, $n \in \mathbb{N}$, this yields a contradiction. Therefore, we may conclude that $\{x_n'\}_{n=1}^{\infty}$ is a null sequence relative to $\sigma(E', E)$. ■

Remark 2.8 Let E be a lc-solid Riesz space and suppose that $F \subseteq E$ is a lattice copy of c_0. Let $J : c_0 \to F$ be a Riesz homeomorphism. For every continuous Riesz seminorm p on E there exists a constant $C_p \geq 0$ such that $p(J\lambda) \leq C_p \|\lambda\|_\infty$ for $\lambda \in c_0$. There also exists a continuous Riesz seminorm q on E such that $\|\lambda\|_\infty \leq q(J\lambda)$ for $\lambda \in c_0$. For each $n \in \mathbb{N}$, define the positive linear functional φ_n on F by $\langle x, \varphi_n \rangle = (J^{-1}x)(n)$, $x \in F$. Since

$$|\langle x, \varphi_n \rangle| = \left|\left(J^{-1}x\right)(n)\right| \leq \left\|J^{-1}x\right\|_\infty \leq q(x), \quad x \in F,$$

it follows from Theorem 2.1 that, for each $n \in \mathbb{N}$, there exists a positive linear functional ψ_n on E such that $\psi_n |_F = \varphi_n$ and $|\langle x, \psi_n \rangle| \leq q(x)$ for $x \in E$ (and so, $0 \leq \psi_n \in E'$). If it is possible to choose the functionals ψ_n ($n \in \mathbb{N}$) such that $\langle x, \psi_n \rangle \to 0$ as $n \to \infty$ for all $x \in E$, then F is positively complemented in E. Indeed, if this is the case, then the linear map $P : E \to E$ defined by

$$Px = J\left((\langle x, \psi_n \rangle)\right), \quad x \in E,$$

is easily verified to be a positive continuous projection onto F.

Corollary 2.9 *If E is a lc-solid Riesz space with pre-Lebesgue topology, then any lattice copy of c_0 in E is positively complemented.*

Proof. Suppose that F is a Riesz subspace of E for which there exists a Riesz homeomorphism $J : c_0 \to F$. Let the vectors $e_n \in F$ ($n \in \mathbb{N}$) correspond to the unit basis vectors in c_0. Using the notation introduced in Remark 2.8, let $0 \leq x_n' \in E'$ be the minimal positive extension of the restriction of $\psi_n |_{E_{e_n}}$. The sequence $\{x_n'\}_{n=1}^{\infty}$ is pairwise disjoint and $x_n' |_F = \varphi_n$ for all n (cf. the proof of Proposition 2.2 (i)). Since $|\langle x, x_n' \rangle| \leq \langle |x|, x_n' \rangle \leq \langle |x|, \psi_n \rangle \leq q(x)$ for all $x \in E$ and $n \in \mathbb{N}$, it follows that $\{x_n'\}_{n=1}^{\infty}$ is equicontinuous. Hence, Lemma 2.7 applied to the sequence $\{x_n'\}_{n=1}^{\infty}$ implies that $(\langle x, x_n' \rangle) \in c_0$ for all $x \in E$. As observed in Remark 2.8, this implies that the map $P : E \to E$ given by $Px = J((\langle x, x_n' \rangle))$, $x \in E$, is a linear positive continuous projection onto F. ■

3 Proofs of Theorems 1.1 and 1.2

In this section we will present the proofs of Theorems 1.1 and 1.2. Actually, we will prove the results in a more general setting (so that these results also apply to the duals of Fréchet lattices).

For lc-solid Riesz spaces the following characterization of semireflexivity is relevant (see [4], Theorem 22.4). It should be recalled that a lc-solid Riesz space E has a *Levi topology* (or, has the *Levi property*) if every upwards directed, topologically bounded system in E^+ has a supremum in E^+ ([4], p.61). In this case, E is necessarily Dedekind complete.

Proposition 3.1 *If E is a lc-solid Riesz space, then E is semireflexive if and only if the topology in E is both Lebesgue and Levi and the topology in E'_β is Lebesgue.*

Theorem 1.2 is a special case of the following result.

Proposition 3.2 *Suppose that E is a lc-solid Riesz space such that:*

(a) E is Dedekind σ-complete;

(b) E is topologically complete;

(c) countable bounded subsets of E'_β are equicontinuous.

The following statements are equivalent.

(i) E is not semireflexive.

(ii) E contains a positively complemented lattice copy of ℓ_∞, or c_0, or ℓ_1.

Proof. Only (i)⇒(ii) requires a proof. Assuming that E is not semireflexive, Proposition 3.1 yields three possibilities:

(I) the topology of E is not Lebesgue;

(II) the topology of E is Lebesgue but, not Levi;

(III) the topology of E'_β is not Lebesgue.

In case (I) it follows from [4], Theorem 10.3, that the topology of E is not pre-Lebesgue. Since E is Dedekind σ-complete, it follows from [4], Theorem 10.7, that E contains a lattice copy of ℓ_∞, which is positively complemented by Remark 2.3 (b).

In case (II) it follows from [27], Theorem 1, that E contains a lattice copy of c_0. Since the topology in E is Lebesgue (and hence, pre-Lebesgue; see [4], Theorem 10.3), it follows from Corollary 2.9 that this copy of c_0 is positively complemented.

Finally, consider case (III). Since order intervals in E'_β are always topologically complete ([4], Theorem 19.13), it follows that E'_β does not have the pre-Lebesgue property (an inspection of the proof of [4], Theorem 10.3, shows that topological completeness of order intervals suffices). Since E'_β is Dedekind complete, it follows from [4], Theorem 10.7, that E'_β contains a lattice copy of ℓ_∞ and hence, a lattice copy of c_0. Proposition 2.4 now implies that E contains a positively complemented lattice copy of ℓ_1. The proof is complete. ■

Observe that, for cases (I) and (II) in the proof of Proposition 3.2, condition (c) on the space E is not required.

Before proving our next result, the following observations will be useful. A lcHs E is semireflexive if and only if every bounded subset of E is relatively $\sigma(E, E')$-compact (see [22], Proposition 23.18). In particular, if E is semireflexive, then every bounded sequence $\{x_n\}_{n=1}^\infty$ in E has a $\sigma(E, E')$-cluster point (that is, there exists $y \in E$ such that every $\sigma(E, E')$-neighbourhood of y contains x_n for infinitely many values of n). If $T \in \mathcal{L}(E)$ is power bounded and $x \in E$, then it follows via an argument analogous to the one used in the proof of [20], Chapter 2, Theorem 1.1 (replacing the norm by seminorms), that $\lim_{n\to\infty} T_{[n]}x$ exists in E if and only if the sequence $\left\{T_{[n]}x\right\}_{n=1}^{\infty}$ has a $\sigma(E, E')$-cluster point in E (where $T_{[n]}$ is defined by (5)). Since $\{T^n\}_{n=1}^{\infty}$ is equicontinuous, for each $x \in E$ the set $\left\{T_{[n]}x : n \in \mathbb{N}\right\}$ is bounded in E. Consequently, if E is semireflexive, then for all $x \in E$ the sequence $\left\{T_{[n]}x\right\}_{n=1}^{\infty}$ *has* a $\sigma(E, E')$-cluster point in E and so, $\lim_{n\to\infty} T_{[n]}x$ exists in E. This establishes the following result.

Proposition 3.3 *Every semireflexive lcHs is mean ergodic.*

Remark 3.4 (a) Proposition 3.3 improves Proposition 2.3 of [2], where the assumptions on the lcHs E are that it should be reflexive and have the property that relatively $\sigma(E, E')$-compact sets are relatively sequentially $\sigma(E, E')$-compact (in which case E is mean ergodic). Accordingly, several other results in [2], namely Proposition 2.4 and Theorems 3.5 and 3.7, can also be extended by removing the requirement that "relatively $\sigma(E, E')$-compact sets are relatively sequentially $\sigma(E, E')$-compact" and replacing the use of Proposition 2.3 of [2] in their proofs with Proposition 3.3 above.

(b) Suppose that E is a lcHs and F is a closed complemented subspace of E (that is, F is the range of a continuous projection P in E). Let $T \in \mathcal{L}(F)$. Considering TP as a continuous operator from E into itself, it is clear that $(TP)^n = T^nP$ and $(TP)_{[n]} = T_{[n]}P$ for all $n \in \mathbb{N}$. Evidently, the operator TP is power bounded in E whenever T is power bounded in F. Moreover, if T is not mean ergodic in F, then TP is not mean ergodic in E. Note that, if E is a lc-solid Riesz space and F is a Riesz subspace, then TP is positive whenever both T and P are positive.

Proposition 3.5 *Let E be a lc-solid Riesz space satisfying conditions (a), (b) and (c) of Proposition 3.2. The following statements are equivalent.*

(i) E is semireflexive.

(ii) E is mean ergodic.

(iii) Every positive power bounded linear operator in E is mean ergodic.

Proof. Implication (i)⇒(ii) is Proposition 3.3 and (ii)⇒(iii) is trivial. To show that (iii)⇒(i), suppose that E is *not* semireflexive. It follows from Proposition 3.2 that E contains a positively complemented lattice copy F of ℓ_∞, or c_0, or ℓ_1. As observed before (see (4)), there then exists a *positive* power bounded linear operator T in F which is not mean ergodic. Via Remark 3.4 (b), this implies that E does not satisfy statement (iii). The proof is complete. ■

To treat the case where the space E is not Dedekind σ-complete, the following result will be needed. In [1], Theorem 1.6, it is shown that if a Fréchet space E contains a copy of the Banach space c_0, then E is not mean ergodic. The same conclusion holds in any sequentially complete lcHs E; see [2], Theorem 3.8. The next proposition exhibits a similar result for lc-solid Riesz spaces, without any sequential completeness requirement. The proof of this result uses some ideas from the proof of [14], Proposition 1.

Proposition 3.6 *Suppose that E is a lc-solid Riesz space. If E contains a lattice copy of c_0, then there exists a regular power bounded operator on E which is not mean ergodic. In particular, E is not mean ergodic.*

Proof. Suppose that F is a lattice copy of c_0 in E and let $J : c_0 \to F$ be a Riesz homeomorphism. Let the vectors $\{u_n\}_{n=1}^{\infty} \subseteq F$ correspond to the unit basis vectors in c_0, so that

$$J(\lambda) = \sum_{n=1}^{\infty} \lambda_n u_n, \qquad \lambda = (\lambda_n) \in c_0.$$

Since J is a linear homeomorphism, there exists a continuous Riesz seminorm q on E such that

$$\|\lambda\|_\infty \leq q(J\lambda), \quad \lambda \in c_0. \tag{13}$$

Moreover, for every continuous Riesz seminorm p on E there exists a constant $C_p \geq 0$ such that

$$p(J\lambda) \leq C_p \|\lambda\|_\infty, \quad \lambda \in c_0. \tag{14}$$

For $n \in \mathbb{N}$, define the linear functional $0 \leq \varphi_n \in F'$ by

$$\langle x, \varphi_n \rangle = \left(J^{-1}x\right)(n), \quad x \in F.$$

It follows from (13) that

$$|\langle x, \varphi_n \rangle| \leq \left\|J^{-1}x\right\|_\infty \leq q(x), \quad x \in F,$$

and so Theorem 2.1 implies that there exists $0 \leq \psi_n \in E^\sim$ such that $\psi_n|_F = \varphi_n$ and $|\langle x, \psi_n \rangle| \leq q(x)$ for $x \in E$ (so, in particular, $\psi_n \in E'$). If $\langle x, \psi_n \rangle \to 0$ as $n \to \infty$ for all $x \in E$, then F is positively complemented in E (see Remark 2.8), in which case it follows that there exists a positive power bounded operator on E which is not mean ergodic (see the proof of Proposition 3.5). In this case we are done.

So, assume that there exists $0 < u \in E$ such that $\langle u, \psi_n \rangle \nrightarrow 0$ as $n \to \infty$. Fix a sequence $\{\alpha_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ satisfying $0 < \alpha_n < 1$ for all n with $\alpha_n \to 0$ as $n \to \infty$. Define the positive linear operator $B_1 : E \to E$ by

$$B_1 x = J\left(\left(\alpha_n \langle x, \psi_n \rangle\right)\right) = \sum_{n=1}^{\infty} \alpha_n \langle x, \psi_n \rangle u_n, \quad x \in E. \tag{15}$$

Note that B_1 is well defined as $|\langle x, \psi_n \rangle| \leq q(x)$ for all $n \in \mathbb{N}$ implies that $(\alpha_n \langle x, \psi_n \rangle) \in c_0$ for all $x \in E$. If p is any continuous Riesz seminorm on E, then it follows from (14) that

$$p(B_1 x) \leq C_p \|(\alpha_n \langle x, \psi_n \rangle)\|_\infty \leq C_p \|(\langle x, \psi_n \rangle)\|_\infty \leq C_p q(x), \quad x \in E,$$

and so B_1 is continuous. Define the regular operator $T \in \mathcal{L}(E)$ by $T = I - B_1$. Using (15) and the identities $\langle u_n, \psi_k \rangle = \delta_{kn}$ for all $k, n \in \mathbb{N}$, it follows that $T^k = I - B_k$, where

$$B_k x = \sum_{n=1}^{\infty} \beta_n^{(k)} \langle x, \psi_n \rangle u_n, \quad x \in E, \tag{16}$$

and $\beta_n^{(k)} = 1 - (1-\alpha_n)^k$ for all $k, n \in \mathbb{N}$. Observe that $0 \leq \beta_n^{(k)} \uparrow_k 1$ for all n and that

$$0 \leq B_1 \leq B_2 \leq \cdots. \tag{17}$$

Furthermore, if p is any continuous Riesz seminorm on E, then

$$\begin{aligned} p\left(T^k x\right) &\leq p(x) + p(B_k x) \leq p(x) + C_p \left\| \left(\beta_n^{(k)} \langle x, \psi_n \rangle \right) \right\|_\infty \\ &\leq p(x) + C_p \left\| (\langle x, \psi_n \rangle) \right\|_\infty \leq p(x) + C_p q(x) \end{aligned}$$

for all $x \in E$ and $k \in \mathbb{N}$. Hence, the operator T is power bounded. Defining $S_n = n^{-1}(B_1 + \cdots + B_n)$, it is clear that

$$T_{[n]} = I - S_n, \quad n \in \mathbb{N},$$

and $0 \leq S_1 \leq S_2 \leq \cdots$. Hence, if $x \in E$, then $\lim_{n\to\infty} T_{[n]}x$ exists if and only if $\lim_{n\to\infty} S_n x$ exists.

Suppose that $0 \leq x \in E$ is such that $y = \lim_{n\to\infty} S_n x$ exists (in which case $S_n x \uparrow_n y$; [4], Theorem 5.6 (iii)). Using (17) and the fact that any *increasing* Cesàro convergent sequence is itself convergent, it follows that $B_n x \to y$ as $n \to \infty$ (and so $0 \leq B_n x \uparrow_n y$). Observe that

$$\begin{aligned} y &= \bigvee\nolimits_{k=1}^\infty B_k x = \bigvee\nolimits_{k=1}^\infty \bigvee\nolimits_{n=1}^\infty \beta_n^{(k)} \langle x, \psi_n \rangle u_n \\ &= \bigvee\nolimits_{n=1}^\infty \bigvee\nolimits_{k=1}^\infty \beta_n^{(k)} \langle x, \psi_n \rangle u_n = \bigvee\nolimits_{n=1}^\infty \langle x, \psi_n \rangle u_n. \end{aligned}$$

On the other hand, $B_n x \in F^+$ for all n and so $y \in F^+$. Hence, there exists $\lambda = (\lambda_n) \in c_0^+$ such that $y = J\lambda = \bigvee_{n=1}^\infty \lambda_n u_n$ and so, $\bigvee_{n=1}^\infty \langle x, \psi_n \rangle u_n = \bigvee_{n=1}^\infty \lambda_n u_n$. Since $\{u_n\}_{n=1}^\infty$ is a disjoint sequence, it follows that $\langle x, \psi_n \rangle = \lambda_n$ for all n and hence, $(\langle x, \psi_n \rangle) \in c_0$.

We thus have shown that if $0 \leq x \in E$ is such that $\lim_{n\to\infty} T_{[n]}x$ exists, then $(\langle x, \psi_n \rangle) \in c_0$. Since we assumed that there exists $0 < u \in E$ such that $\langle u, \psi_n \rangle \nrightarrow 0$ as $n \to \infty$, this shows that the regular power bounded operator T is not mean ergodic. The proof is complete. ■

As observed in [27], Lemma 1, any sequentially complete lc-solid Riesz space which is not Dedekind σ-complete contains a lattice copy of c_0. Together with Proposition 3.6, this yields the following result.

Corollary 3.7 *If E is a sequentially complete lc-solid Riesz space which is mean ergodic, then E is Dedekind σ-complete.*

In combination with Propositions 3.3 and 3.5 we obtain the following consequence.

Corollary 3.8 *If E is a topologically complete lc-solid Riesz space such that countable bounded subsets of E'_β are equicontinuous, then the following statements are equivalent.*

(i) E is semireflexive.

(ii) E is mean ergodic.

(iii) E is Dedekind σ-complete and every positive power bounded linear operator in E is mean ergodic.

The proof of Theorem 1.1 is now a simple consequence of the previous results. **Proof.** (of Theorem 1.1) Let E be a Fréchet lattice. The equivalence of statements (i), (ii) and (iii) is a special case of Corollary 3.8. The strong dual E'_β of E is a topologically complete and Dedekind complete lc-solid Riesz space for which countable bounded subsets of $\left(E'_\beta\right)'_\beta$ are equicontinuous (see the discussion prior to Theorem 2.1 and Remark 2.5 (i)). Consequently, Proposition 3.5 may be applied to E'_β and so (iv) holds if and only if E'_β is semireflexive.

If E is reflexive, then E'_β is reflexive (see [22], Corollary 25.11) and hence, semireflexive. Assume now that E'_β is semireflexive. Since E is a Fréchet space, its topology coincides with the Mackey topology ([19], p. 261, (4)). Since E is complete, it follows from [19], p. 303, (6), that E is reflexive. This shows that statements (iv) and (i) in Theorem 1.1 are equivalent. The proof is complete. ∎

Remark 3.9 Let E be a topologically complete lc-solid Riesz space. If E is mean ergodic, then it follows from Corollary 3.7 and the proof of Proposition 3.2 that E has both the Lebesgue and the Levi property. In particular, E is Dedekind complete.

4 Proof of Theorem 1.3

Recall that a linear map T on a Riesz space E is called power order bounded if for every $x \in E^+$ there exists $z \in E^+$ such that

$$\bigcup_{n=0}^{\infty} T^n\left([-x, x]\right) \subseteq [-z, z]\,. \tag{18}$$

Note that (18) is equivalent to saying that $|T^n y| \leq z$ for all $n = 0, 1, \ldots$ whenever $|y| \leq x$.

Proposition 4.1 *If E is a complete barrelled lc-solid Riesz space, then the following statements are equivalent.*

(i) The topology of E is Lebesgue.

(ii) Every power order bounded operator on E is mean ergodic.

Proof. (i)⇒(ii). Let T be a power order bounded operator in E. As observed in Section 1, this implies that T is power bounded (as E is assumed to be barrelled). Given $x \in E$, let $z \in E^+$ satisfy (18) for $|x|$, which implies, in particular, that $|T^n x| \leq z$ for all $n \in \mathbb{N}$. Consequently, $\left|T_{[n]}x\right| \leq z$ for all $n \in \mathbb{N}$, that is, the sequence $\left\{T_{[n]}x\right\}_{n=1}^{\infty}$ is contained in the order interval $[-z, z]$. Since E is topologically complete and its topology is Lebesgue, it follows that E is Dedekind complete (see [4], Theorem 10.3). Hence, by [4], Theorem 22.1, the order interval $[-z, z]$ is $\sigma\left(E, E'\right)$-compact and so, the set $\left\{T_{[n]}x : n \in \mathbb{N}\right\}$ is relatively $\sigma\left(E, E'\right)$-compact. Therefore, the sequence $\left\{T_{[n]}x\right\}_{n=1}^{\infty}$ has a $\sigma\left(E, E'\right)$ cluster point in E, which implies that $\lim_{n\to\infty} T_{[n]}x$ exists in E (see the discussion prior to Proposition 3.3). Hence, T is mean ergodic.

(ii)⇒(i). Suppose that the topology in E is not Lebesgue. Since E is complete it follows from [4], Theorem 10.3, that the topology is not pre-Lebesgue. Therefore, there exist $0 < u \in E$ and a disjoint sequence $\{u_n\}_{n=1}^{\infty}$ in $[0, u]$ such that $u_n \not\to 0$ as $n \to \infty$ ([4], Theorem 10.1). Hence, there exists a continuous Riesz seminorm r on E such that $r(u_n) \not\to 0$ as $n \to \infty$. By passing to a subsequence, if necessary, we may assume that $r(u_n) \geq \delta$ for all n and some $\delta > 0$. Define the injective Riesz homomorphism $J_0 : c_{00} \to E$ by

$$J_0\lambda = \sum_{k=1}^{n} \lambda_k u_k, \quad \lambda = (\lambda_1, \ldots, \lambda_n, 0, 0, \ldots) \in c_{00}.$$

Since $|J_0\lambda| \leq \|\lambda\|_\infty u$, it follows that $p(J_0\lambda) \leq p(u)\|\lambda\|_\infty$, $\lambda \in c_{00}$, for all continuous Riesz seminorms p on E. The inequalities $|J_0\lambda| \geq |\lambda_k| u_k$ for all k, imply that $r(J_0\lambda) \geq \delta \|\lambda\|_\infty$. Since E is complete, it follows that J_0 extends continuously to a Riesz homeomorphism J from c_0 onto a closed Riesz subspace F of E satisfying $p(J\lambda) \leq p(u)\|\lambda\|_\infty$, $\lambda \in c_0$, for all continuous Riesz seminorms p on E. Moreover, $\|\lambda\|_\infty \leq q(J\lambda)$ for $\lambda \in c_0$, where $q = \delta^{-1} r$. Consequently, we are in the situation of Proposition 3.6. Let $\{\psi_n\}_{n=1}^{\infty}$ be the sequence in $(E')^+$ as defined in the proof of Proposition 3.6 and observe that $\langle u, \psi_n \rangle \geq \langle u_n, \psi_n \rangle = 1$ for all n. Therefore, if we define $T = I - B_1$, where the positive linear operator B_1 is given by (15), then T is power bounded but not mean ergodic (see the proof of Proposition 3.6).

We claim that T is power order bounded. Recall that $T^k = I - B_k$, $k \in \mathbb{N}$, where the positive operators B_k are given by (16). Let $x \in E^+$ and $y \in E$ satisfy $|y| \leq x$. Using $0 \leq \langle x, \psi_n \rangle \leq q(x)$ for all n, it follows easily that

$$\left|T^k y\right| \leq |y| + |B_k y| \leq x + B_k x \leq x + q(x) u, \quad k \in \mathbb{N},$$

which establishes the claim. We thus have shown that if the topology of E is not Lebesgue, then there exists a power order bounded operator on E which is not mean ergodic. The proof is complete. ∎

Since Fréchet lattices are complete barrelled lc-solid Riesz spaces, Theorem 1.3 is a special case of Proposition 4.1.

5 Uniform mean ergodicity

Recall that a power bounded linear operator $T \in \mathcal{L}(E)$, with E a lcHs, is called *uniformly mean ergodic* if the Cesàro means $T_{[n]}$,$n \in \mathbb{N}$, (as defined by (5)) are convergent in $\mathcal{L}(E)$ with respect to the *uniform operator topology* τ_b (defined via the seminorms q_B in $\mathcal{L}(E)$ given by (3)). If E is a lc-solid Riesz space, then the topology τ_b is generated by the seminorms q_B, where q is a continuous Riesz seminorm on E and $B \in \mathcal{B}_s$.

In the sequel, we denote by $Z(E)$ the *centre* of a lc-solid Riesz space E (see [29], Chapter 20, or [24], Section 3.1). The Boolean algebra $\mathcal{P}(E)$ of all band projections in E (cf. [21], Section 30) coincides with the Boolean algebra of all idempotents in $Z(E)$ (equivalently, the Boolean algebra of all components of the identity operator I in $Z(E)$). If $T \in Z(E)$, then, by definition, there exists $0 \leq \lambda \in \mathbb{R}$ such that $|Tx| \leq \lambda |x|$, $x \in E$, and so $Z(E) \subseteq \mathcal{L}(E)$. Furthermore, $Z(E)$ is a commutative subalgebra of $\mathcal{L}(E)$ and $Z(E)$ is an f-algebra (see e.g.

[24], Section 3.1 or [29], Section 140, for the definition). If q is a continuous Riesz seminorm on E and $B \in \mathcal{B}_s$, then q_B is a Riesz seminorm on $Z(E)$. Indeed, if $|S| \leq |T|$ in $Z(E)$, then $|Sx| = |S|\,|x| \leq |T|\,|x| = |Tx|$ and hence, $q(Sx) \leq q(Tx)$ for all $x \in E$, which implies that $q_B(S) \leq q_B(T)$. Consequently, equipped with the topology τ_b, $Z(E)$ is a lc-solid Riesz space. It should also be observed that an operator $T \in Z(E)$ is power bounded if and only if $|T| \leq I$.

Theorem 5.1 *If E is a Dedekind σ-complete lc-solid Riesz space in which order intervals are topologically complete, then the following statements are equivalent.*

- *(i) Every power bounded $T \in Z(E)$ is uniformly mean ergodic.*
- *(ii) Every topologically bounded, disjoint sequence in E converges to zero.*
- *(iii) Every disjoint sequence of band projections in E converges to zero with respect to τ_b.*
- *(iv) $\mathcal{P}(E)$ is a τ_b-Bade complete Boolean algebra of projections, that is, $\mathcal{P}(E)$ is a complete Boolean algebra and $P_\alpha \uparrow_\alpha P$ in $\mathcal{P}(E)$ implies that $P_\alpha \rightarrow_\alpha P$ with respect to τ_b.*

Proof. (i)⇒(ii). Suppose that $\{u_n\}_{n=1}^{\infty}$ is a topologically bounded, disjoint sequence in E^+. Let P_n denote the band projection in E onto the principal band $\{u_n\}^{dd}$ generated by u_n (recall that a Dedekind σ-complete Riesz space has the principal projection property; see [21], Section 25) and observe that $P_m P_n = 0$ whenever $m \neq n$. Fix a sequence $\{\alpha_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ satisfying $0 < \alpha_n < 1$ for all n and $\alpha_n \uparrow_n 1$. If $x \in E^+$, then

$$0 \leq \sum_{n=1}^{N} \alpha_n P_n x \uparrow_N \leq x$$

and so, $Tx = \sum_{n=1}^{\infty} \alpha_n P_n x = \sup_N \sum_{n=1}^{N} \alpha_n P_n x$ exists in E (as E is Dedekind σ-complete). Consequently,

$$Tx = \sum_{n=1}^{\infty} \alpha_n P_n x, \quad x \in E,$$

exists as an order convergent series in E. Since $0 \leq T \leq I$, it is clear that $T \in Z(E) \subseteq \mathcal{L}(E)$ and T is power bounded. It is easily verified that

$$T^k x = \sum_{n=1}^{\infty} \alpha_n^k P_n x, \quad x \in E, \tag{19}$$

for all $k \in \mathbb{N}$. Note that $0 \leq T^k \downarrow_k$. We claim that $T^k x \downarrow_k 0$ in E for all $x \in E^+$. Indeed, suppose that $w \in E$ is such that $0 \leq w \leq T^k x$ for all $k \in \mathbb{N}$ and some $x \in E^+$. It follows from (19) that $0 \leq P_n w \leq \alpha_n^k P_n x$ for all $k, n \in \mathbb{N}$ and so, $P_n w = 0$ for all n (as $\alpha_n^k \downarrow_k 0$). This implies that $w \wedge P_n x = 0$ for all n, so $w = w \wedge Tx = 0$, which proves the claim.

It is now easy to see that $0 \leq T_{[k]} x \downarrow_k 0$ for all $x \in E^+$. By hypothesis, there exists $S \in \mathcal{L}(E)$ such that $T_{[k]} \rightarrow S$ with respect to τ_b and so, in particular, $T_{[k]} x \rightarrow Sx$ for all $x \in E^+$. Via [4], Theorem 5.6 (iii), it follows that $Sx = 0$

for all $x \in E^{+}$ and hence, $S = 0$. Consequently, $T_{[k]} \to 0$ with respect to τ_b. Since $0 \leq T^k \leq T^j$ $(1 \leq j \leq k)$, it follows that $0 \leq T^k \leq T_{[k]}$ and so, $q_B\left(T^k\right) \leq q_B\left(T_{[k]}\right)$ for every continuous Riesz seminorm q and every $B \in \mathcal{B}_s$. Consequently, $T^k \to 0$ with respect to τ_b. This implies, in particular, that $\lim_{k\to\infty} \sup_n p\left(T^k u_n\right) = 0$ for every continuous Riesz seminorm p on E. Since $T^k u_n = \alpha_n^k u_n$, it follows that $\lim_{k\to\infty} \sup_n \alpha_n^k p\left(u_n\right) = 0$. Given $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ such that $\alpha_n^k \geq 1/2$ for all $n \geq N_k$ and so, $\sup_n \alpha_n^k p\left(u_n\right) \geq 2^{-1} \sup_{n \geq N_k} p\left(u_n\right)$. Therefore, $p\left(u_n\right) \to 0$ as $n \to \infty$, which shows that $u_n \to 0$ as $n \to \infty$ in E. If $\{x_n\}_{n=1}^{\infty} \subseteq E$ is any topologically bounded, disjoint sequence, then $\{|x_n|\}_{n=1}^{\infty}$ has the same properties and so $|x_n| \to 0$, which implies that $x_n \to 0$ as $n \to \infty$.

(ii)⇒(iii). Let $\{P_n\}_{n=1}^{\infty}$ be a disjoint sequence in $\mathcal{P}(E)$ and suppose that $P_n \nrightarrow 0$ with respect to τ_b. Then there exists a continuous Riesz seminorm q on E and $B \in \mathcal{B}_s$ such that $q_B\left(P_n\right) \nrightarrow 0$ as $n \to \infty$ (with q_B given by (3)). By passing to a subsequence, if necessary, we may assume that $q_B\left(P_n\right) \geq \delta$ for all $n \in \mathbb{N}$ and some $\delta > 0$. Hence, for each n there exists $x_n \in B$ such that $q\left(P_n x_n\right) \geq \delta/2$. Since $|P_n x_n| = P_n |x_n| \leq |x_n|$, the sequence $\{P_n x_n\}_{n=1}^{\infty}$ is bounded and disjoint and so, by hypothesis, $P_n x_n \to 0$ as $n \to \infty$. This contradicts the fact that $q\left(P_n x_n\right) \geq \delta/2$ for all n. Hence, we may conclude that $P_n \to 0$ with respect to τ_b.

(iii)⇒(iv). First, observe that the topology in E is pre-Lebesgue. Indeed, suppose that $x \in E^{+}$ and that $\{x_n\}_{n=1}^{\infty}$ is a disjoint sequence in $[0, x]$. Denoting by P_n the band projection in E onto the principal band $\{x_n\}^{dd}$, it is clear that $\{P_n\}_{n=1}^{\infty}$ is a disjoint sequence in $\mathcal{P}(E)$ and so, by hypothesis, $P_n \to 0$ as $n \to \infty$ with respect to τ_b. Since $0 \leq x_n = P_n x_n \leq P_n x$ for all n, it is now clear that $x_n \to 0$ as $n \to \infty$. Hence, the topology of E is pre-Lebesgue. Since, by hypothesis, order intervals in E are complete, this implies that E has the Lebesgue property and E is Dedekind complete (see [4], Theorem 10.3, and its proof, where it is only required that order intervals are complete). Consequently, $\mathcal{P}(E)$ is a complete Boolean algebra (see [21], Theorem 24.9 (i) and Theorem 30.6 (ii)).

We shall show next that τ_b is a pre-Lebesgue topology on $Z(E)$. Since I is a strong order unit in $Z(E)$, it suffices to show that any disjoint sequence $\{T_n\}_{n=1}^{\infty}$ in $[0, I]$ converges to zero with respect to τ_b. Denoting by $P_n \in \mathcal{P}(E)$ the component of I in the principal band $\{T_n\}^{dd}$, it follows that $\{P_n\}_{n=1}^{\infty}$ is a disjoint sequence in $\mathcal{P}(E)$ satisfying $0 \leq T_n = T_n P_n \leq P_n$ for all n. By hypothesis, $P_n \to 0$ and so also $T_n \to 0$ as $n \to \infty$ with respect to τ_b. Hence, τ_b is a pre-Lebesgue topology on $Z(E)$. Since order intervals in E are complete, it is easily verified that order intervals in $Z(E)$ are complete with respect to τ_b. Consequently, τ_b is a Lebesgue topology on $Z(E)$. In particular, if $P_\alpha \uparrow_\alpha P$ in $\mathcal{P}(E)$, then $P_\alpha \to_\alpha P$ with respect to τ_b. We may conclude that $\mathcal{P}(E)$ is τ_b-Bade complete.

(iv)⇒(i). We start with the following simple observation. If $T \in Z(E)$ satisfies $-I \leq T \leq \alpha I$ for some $\alpha < 1$, then $T_{[k]} \to 0$ as $k \to \infty$ with respect to τ_b. Indeed, $I - T \geq (1 - \alpha) I$ and so $(I - T)^{-1}$ exists in $Z(E)^{+}$ (see [29], Theorem 146.3). This implies that

$$T + T^2 + \cdots + T^k = (I - T)^{-1}\left(T - T^{k+1}\right)$$

and so,

$$\left|T_{[k]}\right| \leq (1/k)\,(I-T)^{-1}\left(|T| + |T|^{k+1}\right) \leq (2/k)\,(I-T)^{-1}$$

for all $k \in \mathbb{N}$. From this estimate it is clear that $T_{[k]} \to 0$ with respect to τ_b.

Let $T \in Z(E)$ satisfy $|T| \leq I$ and fix $0 < \alpha < 1$. Let $P \in \mathcal{P}(E)$ be the component of I in the band $\left\{(I-T)^+\right\}^d$, in which case $PT \leq P$. On the other hand, since $(P - PT)^+ = P(I-T)^+ = 0$, it follows that $P \leq PT$ and hence, $P = PT$. This implies, in particular, that $PT_{[k]} = P$ for all $k \in \mathbb{N}$ (as T and P commute). Let $Q_\alpha \in \mathcal{P}(E)$ be the component of I in the band $\left\{(\alpha I - T)^+\right\}^{dd}$. Note that $(\alpha I - T)^+ \leq (I-T)^+$ and so, $PQ_\alpha = 0$. Since $(\alpha Q_\alpha - Q_\alpha T)^- = Q_\alpha(\alpha I - T)^- = 0$, it follows that $Q_\alpha T \leq \alpha Q_\alpha$ and so $-I \leq Q_\alpha T \leq \alpha I$. The previous paragraph implies that $Q_\alpha T_{[k]} = (Q_\alpha T)_{[k]} \to 0$ as $k \to \infty$ with respect to τ_b. Writing

$$T_{[k]} - P = (I-P)\,T_{[k]} = Q_\alpha T_{[k]} + (I - P - Q_\alpha)\,T_{[k]},$$

it follows that

$$\limsup_{k\to\infty} q_B\left(T_{[k]} - P\right) \leq \limsup_{k\to\infty} q_B\left((I-P-Q_\alpha)\,T_{[k]}\right)$$

whenever q is a continuous Riesz seminorm on E and $B \in \mathcal{B}_s$. Furthermore, $|T| \leq I$ yields $\left|T_{[k]}\right| \leq I$ and hence, $\left|(I-P-Q_\alpha)\,T_{[k]}\right| \leq I - P - Q_\alpha$. So, $q_B\left((I-P-Q_\alpha)\,T_{[k]}\right) \leq q_B(I-P-Q_\alpha)$ for all k. Accordingly,

$$\limsup_{k\to\infty} q_B\left(T_{[k]} - P\right) \leq q_B(I-P-Q_\alpha).$$

Now observe that $I - P - Q_\alpha$ is the component of I in the band $\left\{(\alpha I - T)^+\right\}^d \cap \left\{(I-T)^+\right\}^{dd}$. Furthermore, if $\alpha \uparrow 1$, then $\left\{(\alpha I - T)^+\right\}^{dd} \uparrow \left\{(I-T)^+\right\}^{dd}$ and so, $\left\{(\alpha I - T)^+\right\}^d \downarrow \left\{(I-T)^+\right\}^d$. Consequently, $I - P - Q_\alpha \downarrow 0$ as $\alpha \uparrow 1$. By the τ_b-Bade completeness of $\mathcal{P}(E)$, this implies that $q_B(I-P-Q_\alpha) \downarrow 0$ as $\alpha \uparrow 1$ and so we may conclude that $\lim_{k\to\infty} q_B\left(T_{[k]} - P\right) = 0$. This shows that $T_{[k]} \to P$ as $k \to \infty$ with respect to τ_b. The proof is complete. ■

Remark 5.2 If E is a lc-solid Riesz space, then E'_β is always Dedekind complete (see Section 2) and order intervals in E'_β are topologically complete ([4], Theorem 19.13). Consequently, Theorem 5.1 may always be applied in E'_β.

An immediate consequence of the above theorem is the following result.

Corollary 5.3 *If E is topologically complete lc-solid Riesz space which is uniformly mean ergodic, then every topologically bounded, disjoint sequence in E converges to zero.*

Proof. Since E is, in particular, mean ergodic, it follows that E is Dedekind complete (see Remark 3.9) and so, Theorem 5.1 applies. ■

Theorem 1.4 follows immediately from Corollary 5.3. Indeed, if in a Banach lattice every norm bounded, disjoint sequence converges to zero, then every disjoint system in E must be finite. This implies that E is finite dimensional (see [21], Theorem 26.10).

It should be observed that any lcHs E in which bounded sets are relatively compact is necessarily uniformly mean ergodic. Indeed, bounded subsets of E are, in particular, relatively weakly compact and so, E is semireflexive. This implies that E is mean ergodic (see Proposition 3.3). Now, if $T \in \mathcal{L}(E)$ is power bounded, then the sequence $\{T_{[k]}\}_{k=1}^{\infty}$ is equicontinuous and convergent in $\mathcal{L}_s(E)$. Accordingly, the sequence $\{T_{[k]}\}_{k=1}^{\infty}$ also converges uniformly on all relatively compact subsets and hence, on all bounded subsets of E, that is, in $\mathcal{L}_b(E)$. Therefore, E is uniformly mean ergodic.

If E is a *discrete* and complete lc-solid Riesz space in which every bounded disjoint sequence converges to zero, then it follows from [4], Theorem 21.15, that every bounded set in E is relatively compact. This observation, together with the previous paragraph, Corollary 5.3 and Theorem 5.1, yields the following result.

Corollary 5.4 *If E is a topologically complete, discrete, lc-solid Riesz space, then the following statements are equivalent.*

(i) E is Dedekind σ-complete and every power bounded operator $T \in Z(E)$ is uniformly mean ergodic.

(ii) Every topologically bounded, disjoint sequence in E converges to zero.

(iii) Bounded subsets of E are relatively compact.

(iv) E is uniformly mean ergodic.

It should be observed that a lc-solid Riesz space E in which bounded sets are relatively compact, is necessarily discrete ([4], Corollary 21.13). *Therefore, Theorem 1.5 is an immediate consequence of Corollary 5.4.* The next example shows that the discreteness condition cannot be omitted in Corollary 5.4.

Example 5.5 Fix $1 < p \leq \infty$ and define

$$L^{p-} = \bigcap_{1 \leq q < p} L^q(0,1),$$

where the interval $(0,1)$ is equipped with Lebesgue measure λ. Fixing a sequence $1 < p_1 < p_2 < \cdots \uparrow p$, the Fréchet lattice topology in L^{p-} is generated by the sequence $\left\{\|\cdot\|_{p_k}\right\}_{k=1}^{\infty}$ of Riesz norms. Moreover, L^{p-} is reflexive. We claim that every bounded, disjoint sequence in L^{p-} converges to zero. Indeed, let $\{u_n\}_{n=1}^{\infty}$ be a disjoint sequence in L^{p-} such that $\sup_n \|u_n\|_{p_k} = C_k < \infty$ for all k. Defining $A_n = \{t \in (0,1) : |u_n(t)| > 0\}$, it is clear that $\{A_n\}_{n=1}^{\infty}$ consists of pairwise disjoint sets and so $\lambda(A_n) \to 0$ as $n \to \infty$. Given $k \in \mathbb{N}$, it follows from Hölder's inequality that

$$\|u_n\|_{p_k} \leq \|u_n\|_{p_{k+1}} \lambda(A_n)^{\frac{1}{p_k} - \frac{1}{p_{k+1}}} \leq C_k \lambda(A_n)^{\frac{1}{p_k} - \frac{1}{p_{k+1}}}, \quad n \in \mathbb{N}.$$

Hence, $\|u_n\|_{p_k} \to 0$ as $n \to \infty$, which proves the claim. Consequently, the Dedekind complete Fréchet lattice L^{p-} satisfies all (equivalent) statements of Theorem 5.1 (and so, in particular, statements (i) and (ii) of Corollary 5.4). It should be observed that the centre $Z(L^{p-})$ may be identified with $L^\infty(0,1)$, acting on L^{p-} via multiplication. Evidently, L^{p-} is not discrete and hence (as is well known; [11]), it is not a Montel space (that is, L^{p-} does not satisfy condition (ii) of Corollary 5.4). According to [1], Proposition 2.11, the space L^{p-} is not uniformly mean ergodic. We point out that L^{p-} cannot contain any closed Riesz subspace which is lattice isomorphic to an infinite dimensional Banach lattice X. For, all norm bounded, disjoint sequences in X would converge to zero. As noted after Corollary 5.3, X would then be finite dimensional. On the other hand, L^{p-} does have a closed subspace which is topologically isomorphic to the Banach lattice ℓ^2 ([1], Lemma 2.10).

It remains an interesting question whether every uniformly mean ergodic Fréchet lattice is actually discrete (and hence, Montel).

We end this paper with two observations concerning lc-solid Riesz spaces in which topologically bounded, disjoint sequences converge to zero.

Remark 5.6 Recall that a locally solid Riesz space E is called (sequentially) monotone complete if every increasing Cauchy (sequence) net in E is convergent.

(a) If E is a monotone complete lc-solid Riesz space and all topologically bounded, disjoint sequences in E converge to zero, then E is semireflexive. Indeed, it follows from [4], Theorem 21.8, that all bounded subsets of E are relatively weakly compact and hence, E is semireflexive.

(b) If E is a sequentially monotone complete lc-solid Riesz space, then the following two statements are equivalent.

   (i) Every topologically bounded, disjoint sequence in E converges to zero.

   (ii) Every equicontinuous, disjoint sequence in E'_β converges to zero.

   Indeed, this equivalence follows immediately from a result of Burkinshaw and Dodds ([4], Theorem 21.7, equivalence of (i) and (ii)). An inspection of the proof of [4], Theorem 21.7, shows that it actually suffices to assume that the space E is sequentially monotone complete.

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