Some questions about subspace-hypercyclic operators

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Abstract

A bounded linear operator T on a Banach space X is called subspace-hypercyclic for a subspace M if $\mathrm{Orb}(T,x) \cap M$ is dense in M for a vector $x \in M$. We show examples that answer some questions posed by H. Rezaei [7]. In particular, we provide an example of an operator T such that $\mathrm{Orb}(T,x) \cap M$ is somewhere dense in M, but it is not everywhere dense in M.

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1. Introduction

An operator on a Banach space is called hypercyclic if there is a vector whose orbit under the operator is dense in the space; such a vector is called a hypercyclic vector for the operator. Hypercyclic operators have been studied for more than twenty years (for more information see [1] and [4]). Recently, B. F. Madore and R. A. Martínez-Avendaño introduced in [6] the concept of subspace-hypercyclic operators.

Definition 1. A bounded linear operator $T : X \to X$ is called subspace-hypercyclic for a nonzero subspace M of X if there exists a vector $x \in X$ such that $\mathrm{Orb}(x,T) \cap M$ is dense in M. The vector x is then called a subspace-hypercyclic vector for T.

Rezaei shows in [7] that, if a bounded linear operator T acting on a Banach space X is subspace-hypercyclic for some subspace M of X and p is complex polynomial, then $\ker(p(T^*)) \subseteq M^{\perp}$, which provides an affirmative answer to question (v) of [6]. Also, he proves as a consequence that, under general additional conditions, a subspace-hypercyclic operator T for a subspace M of X has a dense linear manifold of M consisting entirely, except for zero, of vectors that are subspace-hypercyclic for T. More examples and results of subspace-hypercyclic operators can be found in [5, 6, 7].

In the present paper we answer negatively the following questions from [7]:

> **Question 1.** Let M be a nontrivial subspace of a Banach space X and $x \in M$. Does $\mathrm{Orb}(x,T) \cap M$ being somewhere dense in M imply that it is everywhere dense in M?

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Question 3. Let $T \in L(X)$, M be an infinite dimensional subspace and $x \in M$. Does the density of $\mathrm{Orb}(x,T)$ in M, i.e., $M \subseteq \overline{\mathrm{Orb}(x,T)}$ imply that T is subspace-hypercyclic for M?

Question 4. Does there exist a subspace-hypercyclic operator T for a nontrivial subspace M such that we have neither $T^n(M) \subseteq M$ nor $M \subseteq T^n(M)$ for each $n \geq 1$?

We introduce some notation. We denote by $\ell^2(v)$ the Hilbert space defined by

$$\ell^2(v) := \{(x_i)_{i\in\mathbb{N}} : \sum_{i=1}^{\infty} |x_i|^2 v_i < \infty\},$$

where v is the weight sequence $v = (v_j)_{j\in\mathbb{N}} = (2^{-j})_{j\in\mathbb{N}}$. We denote, as usual, by B the unilateral backward shift on $\ell^2(v)$ and by S the unilateral forward shift on $\ell^2(v)$.

Observe that

$$\|Sx\| = \frac{1}{\sqrt{2}}\|x\|,$$

and hence for every $n \geq 0$

$$\|S^n x\| = \frac{1}{2^{n/2}}\|x\|. \tag{1}$$

2. Counterexamples

We know from [2] that, for linear operators, any somewhere dense orbit is everywhere dense. The following is an example of a subspace-hypercyclic operator T for certain subspace M so that there exists a vector $y \in M$ such that its orbit is somewhere dense in M but it is not everywhere dense in M; i.e. we construct a subspace M, a vector $y \in M$, and a subset $U \subset M$ with non empty interior, such that $\mathrm{Orb}(y,T)$ is dense in U but it is not dense in M, which provides a negative answer to Question 1. This operator will have the additional property that

$$T^n(M) \not\subset M \text{ and } M \not\subset T^n(M)$$

for every $n \in \mathbb{N}$, which provides a negative answer to Question 4.

Example 2. Let B be the backward shift on $\ell^2(v)$, and let

$$A := \{j \in \mathbb{N} \ ; \ \exists n \in \mathbb{N} \text{ with } 2^n - 2 < j \leq 3\cdot 2^{n-1} - 2\} = \{1, 3, 4, 7, 8, 9, 10, 15, \ldots\}$$

and

$$M := \{(x_j)_{j\in\mathbb{N}} \in \ell^2(v) \ ; \ x_j = 0 \text{ if } j \in A\}.$$

We consider the increasing sequence $(B_j)_{j\in\mathbb{N}}$ of subsets of A^c given by

$$B_j = A^c \cap [1, 2^{j+1} - 2], \quad j \in \mathbb{N}.$$

Let

$$U := M \cap \{x \in \ell^2(v) \ ; \ |x_2| \leq 1\}.$$

We will construct a vector $y \in U$ which has dense orbit in U but it is not dense in M.

We can find vectors $y(j) \in U$, for every $j \in \mathbb{N}$, satisfying

(i) $\overline{\{y(j)\ :\ j \in \mathbb{N}\}} = U$,
(ii) $y(j)_k \neq 0$ if and only if $k \in B_j$,
(iii) $|y(j)_k| \leq j$ for all $k, j \in \mathbb{N}$.

To do this, e.g., let $\{z(j)\ ;\ j \in \mathbb{N}\}$ be a countable dense subset of U. For each $j \in \mathbb{N}$, we define $x(j)$ as

$$x(j)_k = \begin{cases} z(j)_k, & \text{if } k \in B_j \text{ and } |z(j)_k| \leq j, \\ 0, & \text{otherwise.} \end{cases}$$

We have that $\{x(j)\ ;\ j \in \mathbb{N}\}$ is also a dense subset of U. Indeed, let $x \in U$ and $\varepsilon \in]0,1[$. We fix $j_0 \in \mathbb{N}$ such that $\sum_{k \geq j_0} |x_k|^2 v_k < \varepsilon^2/2$. Let $\alpha := \max\{|x_k|\ ;\ k < j_0\}$ and $j > 2^{j_0} + \alpha$ such that $\|z(j) - x\| < \varepsilon/2$.

Observe that if $k \notin B_j$ it follows that $k \in A$ or $k \geq 2^{j+1} - 1$. If $k \in A$ then $x_k = 0$, and if $k \geq 2^{j+1} - 1$ then $k > j_0$ since $j > 2^{j_0} + \alpha > j_0$.

Also $|z(j)_k| > j$ implies $k \geq j_0$; otherwise,

$$\|z(j) - x\| \geq |z(j)_k - x_k|\sqrt{v_k} > (|z(j)_k| - \alpha)2^{-j_0/2} > 1,$$

which is not possible.

We then have

$$\|x(j) - x\|^2 = \sum_{k \in B_j \text{ and } |z(j)_k| \leq j} |z(j)_k - x_k|^2 v_k + \sum_{k \notin B_j \text{ or } |z(j)_k| > j} |x_k|^2 v_k$$

$$\leq \|z(j) - x\|^2 + \sum_{k \geq j_0} |x_k|^2 v_k < \varepsilon^2,$$

which shows that $\{x(j)\ ;\ j \in \mathbb{N}\}$ is a dense subset of U.

Finally, we can set $(y(j))_{j \in \mathbb{N}}$ as

$$y(j)_k = \begin{cases} 2^{-jk}, & \text{if } k \in B_j \text{ and } x(j)_k = 0, \\ x(j)_k, & \text{otherwise,} \end{cases}$$

and conditions (i), (ii), and (iii) are satisfied.

We define the vector

$$y = \sum_{j=1}^{\infty} S^{n_j} y(j),$$

where $n_j := \sum_{i=1}^{j} m_i$, and $m_i := 3 \cdot 2^{3i} - 2$, for $i, j \in \mathbb{N}$. We will show that $y \in U$ and

$$\overline{\mathrm{Orb}(y, B) \cap M} = U.$$

Observe that for every $j \in \mathbb{N}$, the vector $y(j)$ has its nonzero elements contained in a block of size $2^{j+1} - 2$. On the other hand a vector in M, starting at position $m_j + 1$, has a block of size at most 2^{3j} of nonzero elements.

A calculation shows that

$$n_{j-1} = \sum_{i=1}^{j-1} m_i = \frac{3}{7}2^{3j} - 2j - \frac{10}{7}.$$

Observe that

$$n_{j-1} + 2^{j+1} - 2 = \frac{3}{7}2^{3j} - 2j - \frac{10}{7} + 2^{j+1} - 2 < 2^{3j}.$$

We claim that $S^{n_j}y(j) \in M$ since it has zeros in the first m_j positions and starting at position $m_j + 1$ it has at most $n_{j-1} + 2^{j+1} - 2$ (possibly) nonzero elements. Since $n_{j-1} + 2^{j+1-2} - 2 < 2^{3j}$ the claim follows. Condition (iii) ensures that the series expressed in y converges, therefore $y \in U$.

We will show that

(a) $B^{n_j}y \in U$ and $||B^{n_j}y - y(j)|| < 1/2^j$ for all $j \in \mathbb{N}$.
(b) If $n_{j-1} < k < n_j$ for some $j \in \mathbb{N}$ ($n_0 := 0$), then, either $B^k y \notin M$, or $(B^k y)_2 = 0$ (and, thus, $B^k y \in U$).

Observe that

$$B^{n_j}y = B^{n_j}\sum_{i=1}^{\infty} S^{n_i}y(i) = \sum_{n_i<n_j} B^{n_j-n_i}y(j) + y(j) + \sum_{n_i>n_j} S^{n_i-n_j}y(i).$$

We know that for $j \in \mathbb{N}$ the largest index where a nonzero element occurs in $y(j)$ is 2^{j+1}-2. If $n_i < n_j$ then $n_j - n_i \geq m_j > 2^{j+1}$, therefore

$$\sum_{n_i<n_j} B^{n_j-n_i}y(j) = 0. \tag{2}$$

If $n_j < n_i$ then $m_i \leq n_i - n_j < n_i$ and therefore, using the same argument as above, each vector $S^{n_i-n_j}y(i)$ is in M, since its only nonzero elements are in a block of size 2^{3j} starting at position $m_j + 1$, therefore

$$\sum_{n_i>n_j} S^{n_i-n_j}y(i) \in M,$$

and thus $B^{n_j}y \in U$, for every $j \in \mathbb{N}$.

By (1), (2) and by (iii) in the construction of $(y(j))_{j\in\mathbb{N}}$ we have

$$||B^{n_j}y - y(j)|| = ||\sum_{i=j+1}^{\infty} S^{n_i-n_j}y(i)|| \leq \sum_{i=j+1}^{\infty} \frac{\sqrt{(2^{i+1}-2)i}}{2^{n_i-n_j}}.$$

Also, a simple calculation shows that

$$\sum_{i=j+1}^{\infty} \frac{\sqrt{(2^{i+1}-2)i}}{2^{n_i-n_j}} \leq \sum_{i=j+1}^{\infty} \frac{\sqrt{(2^{i+1}-2)i}}{2^{m_k+\ldots+m_{j+1}}} \leq \sum_{i=j+1}^{\infty} \frac{\sqrt{(2^{i+1}-2)i}}{2^{2i}\cdot 2^{m_{k-1}+\ldots+m_{j+1}}} \leq \sum_{i=j+1}^{\infty} \frac{1}{2^i}.$$

Therefore we have (a).

Now observe that if $n_{j-1} < k < n_j$ for some $j \in \mathbb{N}$ then $k = m + n_{j-1}$ for some $1 \leq m < m_j$ and

$$B^k y = \sum_{i=j-1}^{\infty} B^k S^{n_i}y(i) = B^m y(j-1) + \sum_{i=j+1}^{\infty} S^{n_i-k}y(i).$$

In case that $B^k y \in M$, suppose $(B^k y)_2 = 0$. Thus $B^k y = (0, a, 0, \dots)$ with $a \neq 0$. But this impossible since the string $[0, a, 0]$ with $a \neq 0$ only happens in $B^m y(j-1)$ when $m = 0$, by condition (ii), and $S^p y(i) = (0, a, 0, \dots)$ with $a \neq 0$ only when $p = 0$. We then obtain (b).

By (a) and (b) we have that $\overline{\mathrm{Orb}(y, B) \cap M} = U$. Therefore B has an orbit that is somewhere dense but is not everywhere dense.

Observe that B is subspace-hypercyclic for M, since we can repeat the same construction as above: Let $(y(j))_{j \in \mathbb{N}} \in M$ be a sequence of vectors satisfying

(i) $\overline{\{y(j)\ ;\ j \in \mathbb{N}\}} = M$,
(ii) $y(j)_k \neq 0$ if and only if $k \in B_j$,
(iii) $|y(j)_k| \leq j$ for all $k, j \in \mathbb{N}$.

Then the vector

$$y = \sum_{j=1}^{\infty} S^{n_j} y(j),$$

where $n_j := \sum_{i=1}^{j} m_i$, and $m_j := 3 \cdot 2^{3j} - 2$, for $j \in \mathbb{N}$, satisfies

$$\overline{\mathrm{Orb}(y, B) \cap M} = M.$$

This provides a negative answer to Question 1.

Now observe that the set A contains intervals of arbitrary length, therefore $B^n(M) \not\subset M$ for any $n \in \mathbb{N}$. Moreover, A^c contains intervals of arbitrary length too, so $M \not\subset B^n(M)$ for each $n \in \mathbb{N}$, which yields a negative answer to Question 4.

The following is an example where the density of $\mathrm{Orb}(T, x)$ in a subspace does not imply that T is subspace-hypercyclic for the subspace, which means that the answer to Question 3 is negative too.

Example 3. Grivaux constructed in [3] a hypercyclic operator $T : H \to H$ on the Hilbert space H such that $Tx = x$ for every $x \in H_1$, where H_1 is certain infinite dimensional closed subspace of H. Let z be a hypercyclic vector for T, and let $M := H_1 \oplus \langle z \rangle$. It is clear that $M \subset \overline{\mathrm{Orb}(z, T)}$. In contrast, we will show that $\mathrm{Orb}(z, T) \cap M = \{z\}$. Let $y \in \mathrm{Orb}(z, T) \cap M$, then $y = \lambda z + x = T^n z$ for some λ, n, and $x \in H_1$. Thus, $(T^n - \lambda I)z = x$, which is not possible since the left-hand vector, being the image of z under a dense range operator that commutes with T, is hypercyclic for T [4], and the right-hand vector is fixed for T. This implies that T is not subspace-hypercyclic for M.

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