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Quadruple fixed point theorems for nonlinear contractions on partial metric spaces

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ABSTRACT

The notion of coupled fixed point was introduced by Guo and Laksmikantham [12]. Later Gnana Bhaskar and Lakshmikantham in [11] investigated the coupled fixed points in the setting of partially ordered set by defining the notion of mixed monotone property. Very recently, the concept of tripled fixed point was introduced by Berinde and Borcut [7]. Following this trend, Karapınar[19] defined the quadruple fixed point. In this manuscript, quadruple fixed point is discussed and some new fixed point theorems are obtained on partial metric spaces.

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1. INTRODUCTION AND PRELIMINARIES

The existence of fixed points in partially ordered metric spaces was considered first by Ran and Reurings [37]. After this remarkable paper, several authors have studied such problems (see e.g. [32, 33, 34, 11, 29, 30, 45, 9, 8]). The notion of coupled fixed point was introduced by Guo and Laksmikantham [12]. After the interesting paper of Gnana Bhaskar and Lakshmikantham [11], many authors focused on coupled fixed point theory and proved several results (see e.g. [29, 30, 45, 9, 8, 18, 17]).

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We recall the basic definitions and results from which our quadruple fixed point is inspired. The triple $(X, d, \leq)$ is called a partially ordered metric spaces if $(X, \leq)$ is a partially ordered set and (X, d) is a metric space. Further, if (X, d) is a complete metric space, then the triple $(X, d, \leq)$ is called partially ordered complete metric spaces.

Definition 1.1 (see [11])**.** Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. We say that F has mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,

$$x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y), \quad \text{for } x_1, x_2 \in X, \quad \text{and}$$
$$y_1 \leq y_2 \Rightarrow F(x, y_2) \leq F(x, y_1), \; \text{for } y_1, y_2 \in X.$$

Definition 1.2 (see [11])**.** An element $(x, y) \in X \times X$ is said to be a couple fixed point of the mapping $F : X \times X \to X$ if

$$F(x, y) = x \text{ and } \; F(y, x) = y.$$

We endow the product space $X \times X$ with the following partial order:

$$(u, v) \leq (x, y) \Leftrightarrow u \leq x, \; y \leq v; \quad \text{for all } \; (x, y), (u, v) \in X \times X. \tag{1.1}$$

Two results of Bhaskar and Lakshmikantham [11] can be unified as follows:

Theorem 1.3. *Let $(X, \leq)$ be a partially ordered set endowed with a metric d on X such that (X, d) is a complete metric spaces. Let $F : X \times X \to X$ have the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)], \; \textit{for all } u \leq x, \; y \leq v. \tag{1.2}$$

Suppose either F is continuous or X has the following properties:

- (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$, $\forall n$;*
- (i) *if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$, $\forall n$.*

If, in addition, there are $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$, then, there exists $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

We notice that Theorem 1.3 was extended to class of cone metric spaces in [17].

Inspired by Definition 1.1, Berinde and Borcut [7] introduced the following definition:

$$(u, v, w) \leq (x, y, z) \text{ if and only if } x \geq u, \; y \leq v, \; z \geq w, \tag{1.3}$$

where $(u, v, w), (x, y, z) \in X^3$.

Definition 1.4 (see [7])**.** Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \to X$. The mapping F is said to has the mixed monotone property if for any $x, y, z \in X$

$$x_1, \; x_2 \in X, \quad x_1 \leq x_2 \Longrightarrow F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1,\ y_2 \in X, \quad y_1 \leq y_2 \Longrightarrow F(x, y_1, z) \geq F(x, y_2, z),$$
$$z_1,\ z_2 \in X, \quad z_1 \leq z_2 \Longrightarrow F(x, y, z_1) \leq F(x, y, z_2),$$

The following is the main tripled fixed point result of Berinde and Borcut [7].

Theorem 1.5. *Let $(X, \leq)$ be partially ordered set and (X, d) be a complete metric space. Let $F : X \times X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist constants $a, b, c \in [0, 1)$ such that $a + b + c < 1$ for which*

$$d(F(x, y, z), F(u, v, w)) \leq ad(x, u) + bd(y, v) + cd(z, w) \tag{1.4}$$

for all $x \geq u,\ y \leq v,\ z \geq w$. If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0), \quad z_0 \leq F(x_0, y_0, z_0)$$

then there exist $x, y, z \in X$ such that

$$F(x, y, z) = x \text{ and } F(y, z, y) = x \text{ and } F(z, y, x) = z$$

The notion of metric space was introduced by Maurice René Fréchet [10] in 1906. Pseudometric space, quasimetric space, semimetric space, partial metric space are some examples of the generalizations of metric space. In this manuscript, we discuss partial metric space, introduced by Matthews (see e.g. [31]).

The concept of the metric space started to apply to computer science around 1970.

By using Baire metric, G. Khan [16] modeled a parallel computation. It consists of a set computing via sending unending streams of information by using infinite sequences. Hence, with this paper, reservoir of the theory of metric space started to be used in the branches of computer science, such as, domain theory and semantics. The handicap of this approaches is, in computer science, infinite sequence corresponding to unterminated programs. But, in computer science, unterminated program is bad. This un-solicited status solved by Matthews with his suggestion of non-zero self distance in metric construction. In the last decade, on partial metric spaces remarkable number of papers were reported (see e.g. [1]-[6],[13]-[15],[24]-[28],[39]-[55])

A mapping $p : X \times X \to [0, \infty)$ is called partial metric (see e.g.[31]) on a nonempty set X if the following conditions are satisfied:

(PM1) $p(x, y) = p(y, x)$ (symmetry)
(PM2) If $p(x, x) = p(x, y) = p(y, y)$ then $x = y$ (equality)
(PM3) $p(x, x) \leq p(x, y)$ (small self-distances)
(PM4) $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$ (triangularity)

The pair (X, p) is called a partial metric space (PMS). Additionally, a triple $(X, p, \leq)$ is called a partially ordered partial metric space if (X, p) is a partial metric space and $(X, \leq)$ is a partially ordered set.

For a partial metric p on X, the functions $d_p, d_m : X \times X \to \mathbb{R}^+$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{1.5}$$

and

$$d_m(x,y) = \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\} \tag{1.6}$$

are (usual) metrics on X. It is clear that d_p and d_m are equivalent. Moreover,

$$\lim_{n\to\infty} d_p(x,x_n) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m) \tag{1.7}$$

Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Example 1.6 (see e.g. [31, 24, 3])**.** Consider $X = [0,\infty)$ with $p(x,y) = \max\{x,y\}$. Then (X,p) is a partial metric space. It is clear that p is not a (usual) metric. Note that in this case $d_m(x,y) = |x-y|$ and $d_p(x,y) = \frac{1}{2}|x-y|$.

Example 1.7 (see [31])**.** Let $X = \{[a,b] : a,b,\in \mathbb{R},\ a \le b\}$ and define $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then (X,p) is a partial metric spaces.

Example 1.8 (see [31])**.** Let $X := [0,1] \cup [2,3]$ and define $p : X \times X \to [0,\infty)$ by

$$p(x,y) = \begin{cases} \max\{x,y\} & \text{if } \{x,y\} \cap [2,3] \neq \varnothing, \\ |x-y| & \text{if } \{x,y\} \subset [0,1]. \end{cases}$$

Then (X,p) is a partial metric space.

Example 1.9 (see [31])**.** Let S be a non-empty set. By S^ω, we denote the set of all infinite sequence $x = \{x_0, x_1, \cdots\}$ over S. For all such sequences $x,y \in S^\omega$ define $d_S(x,y) = 2^{-k}$, where k is the largest number (possibly ∞) such that $x_i = y_i$ for each $i < k$, that is,

$$d_S(x,y) = 2^{-\sup\{n|\forall i<n \ni x_i = y_i\}}.$$

Clearly, (S^ω, d_S) is a metric space which is also known as Baire metric space. Suppose now that the definition d_S is extended to $\bar{S} = S^\omega \cup S^*$, where S^* is the set of all finite sequences. Then $(X, d_{\bar{S}})$ is a partial metric space. But if x is finite then $d_{\bar{S}}(x,x) = \frac{1}{2^k}$ for some $k < \infty$, which is not zero since $x_i = x_i$ can only hold if x_i is defined.

Definition 1.10 (see e.g. [31])**.**

- *(i)* A sequence $\{x_n\}$ in (X,p) converges to $x \in X$ if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$,
- *(ii)* A sequence $\{x_n\}$ in (X,p) is called a Cauchy if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists (and finite),
- *(iii)* (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

(*iv*) Let $P = (x, y, z, w) \in X^4$ and $P_0 = (x_0, y_0, z_0, w_0)$ A mapping $F : X^4 \to X$ is said to be continuous at $(x_0, y_0, z_0, w_0) \in X^4$ with respect to τ_{d_p}, if

$$F(x_0, y_0, z_0, w_0) = \lim_{(x,y,z,w)\to(x_0,y_0,z_0,w_0)} F(x, y, z, w) = F(\lim_{P\to P_0} x, \lim_{P\to P_0} y, \lim_{P\to P_0} z, \lim_{P\to P_0} w).$$

The following lemma plays an important role to give fixed point results on partial metric spaces (See [5], [6], [35], [36]).

Lemma 1.11 (see e.g. [31])**.**

(*A*) *A sequence* $\{x_n\}$ *is Cauchy in* (X, p) *if and only if* $\{x_n\}$ *is Cauchy in the metric space* (X, d_p)*,*

(*B*) (X, p) *is complete if and only if the metric space* (X, d_p) *is complete.*

Lemma 1.12 (see e.g. [3, 26])**.** *Assume* $x_n \to z$ *as* $n \to \infty$ *in* (X, p) *such that* $p(z, z) = 0$*. Then* $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ *for every* $y \in X$*.*

Lemma 1.13 (see e.g. [25, 26])**.** *Let* (X, p) *be a PMS. Then*

(A) *If* $p(x, y) = 0$ *then* $x = y$*,*

(B) *If* $x \neq y$*, then* $p(x, y) > 0$*.*

Remark 1.14. Since d_p and d_m are equivalent, we can take d_m instead of d_p in the above lemma.

Karapınar [19] introduced the concept of quadruple fixed point and proved some quadruple fixed point theorems in partially ordered metric spaces (see also [20]- [23]). The aim of this paper is introduce the concept of quadruple fixed point and prove the related fixed point theorems in the context of partially ordered partial metric spaces.

2. Quadruple Fixed Point Theorems

Let $(X, p, \leq)$ be a partially ordered partial metric spaces. We consider the following partial order on the product space $X^4 = X \times X \times X \times X$:

$$(u, v, r, t) \leq (x, y, z, w) \text{ if and only if } x \geq u,\ y \leq v,\ z \geq r,\ t \leq w \tag{2.1}$$

where $(u, v, r, t), (x, y, z, w) \in X^4$. Regarding this partial order, we state the definition of the following mapping.

Definition 2.1. Let $(X, \leq)$ be partially ordered set and $F : X^4 \to X$. We say that F has the mixed monotone property if $F(x, y, z, w)$ is monotone non-decreasing in x and z, and it is monotone non-increasing in y and w, that is, for any $x, y, z, w \in X$

$$\begin{aligned} x_1, x_2 \in X,\ x_1 \leq x_2 &\Rightarrow\ F(x_1, y, z, w) \leq F(x_2, y, z, w),\\ y_1, y_2 \in X,\ y_1 \leq y_2 &\Rightarrow\ F(x, y_1, z, w) \geq F(x, y_2, z, w),\\ z_1, z_2 \in X,\ z_1 \leq z_2 &\Rightarrow\ F(x, y, z_1, w) \leq F(x, y, z_2, w),\\ w_1, w_2 \in X,\ w_1 \leq w_2 &\Rightarrow\ F(x, y, z, w_1) \geq F(x, y, z, w_2). \end{aligned} \tag{2.2}$$

Definition 2.2. An element $(x, y, z) \in X^4$ is called a quadruple fixed point of $F : X^4 \to X$ if

(2.3)
$$F(x,y,z,w) = x \text{ and } F(y,z,w,x) = y \text{ and } F(z,w,x,y) = z \text{ and } F(w,x,y,z) = w$$

For a metric space (X, d), the function $\rho : X^4 \to [0, \infty)$, given by,

$$\rho((x,y,z,w),(u,v,r,t)) := d(x,u) + d(y,v) + d(z,r) + d(w,t)$$

is a metric space on X^4, that is, (X^4, ρ) is a metric induced by (X, d).

The aim of this paper is to prove the following theorem.

Theorem 2.3. *Let $(X, \leq)$ be partially ordered set and (X, p) be a complete partial metric space. Let $F : X^4 \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a constant $k \in [0, 1)$ such that*

$$p(F(x,y,z,w), F(u,v,r,t)) \leq \frac{k}{4}[p(x,u) + p(y,v) + p(z,r) + p(w,t)] \tag{2.4}$$

for all $x \geq u$, $y \leq v$, $z \geq r$, $w \leq t$. Suppose there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$\begin{aligned} x_0 \leq F(x_0, y_0, z_0, w_0), \quad y_0 \geq F(y_0, z_0, w_0, x_0), \\ z_0 \leq F(z_0, w_0, x_0, y_0), \quad w_0 \geq F(w_0, x_0, y_0, z_0). \end{aligned}$$

Suppose either

(*a*) *F is continuous, or*

(*b*) *X has the following property:*

  (*i*) *if $\{x_n\}$ is a non-decreasing sequence $x_n \to x$ (respectively, $z_n \to z$), then $x_n \leq x$ (respectively, $z_n \leq z$) for all n,*

  (*ii*) *if $\{y_n\}$ is a non-increasing sequence $y_n \to y$(respectively, $w_n \to w$), then $y_n \geq y$ (respectively, $w_n \geq w$) for all n,*

then there exist $x, y, z, w \in X$ such that

$$\begin{aligned} F(x,y,z,w) = x, \quad F(y,z,w,x) = y, \\ F(z,w,x,y) = z, \quad F(w,x,y,z) = w. \end{aligned}$$

Proof. We construct a sequence $\{(x_n, y_n, z_n, w_n)\}$ in the following way: Set

$$\begin{aligned} x_1 &= F(x_0, y_0, z_0, w_0) \geq x_0, \\ y_1 &= F(y_0, z_0, w_0, x_0) \leq y_0, \\ z_1 &= F(z_0, w_0, x_0, y_0) \geq z_0, \\ w_1 &= F(w_0, x_0, y_0, z_0) \leq w_0, \end{aligned}$$

and by the mixed monotone property of F, for $n \geq 1$, inductively we get

$$\begin{aligned} x_n &= F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}) \geq x_{n-1} \geq \cdots \geq x_0, \\ y_n &= F(y_{n-1}, z_{n-1}, w_{n-1}, x_{n-1}) \leq y_{n-1} \leq \cdots \leq y_0, \\ z_n &= F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}) \geq z_{n-1} \geq \cdots \geq z_0, \\ w_n &= F(w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}) \leq w_{n-1} \leq \cdots \leq w_0, \end{aligned} \tag{2.5}$$

Due to (2.4) and (2.5), we have

$$(2.6)\qquad \begin{aligned} p(x_1,x_2) &= p(F(x_0,y_0,z_0,w_0),F(x_1,y_1,z_1,w_1)) \\ &\leq \tfrac{k}{4}[p(x_0,x_1)+p(y_0,y_1)+p(z_0,z_1)+p(w_0,w_1)] \end{aligned}$$

$$(2.7)\qquad \begin{aligned} p(y_1,y_2) &= p(F(y_0,z_0,w_0,x_0),F(y_1,z_1,w_1,x_1)) \\ &\leq \tfrac{k}{4}[p(y_0,y_1)+p(z_0,z_1)+p(w_0,w_1)+p(x_0,x_1)] \end{aligned}$$

$$(2.8)\qquad \begin{aligned} p(z_1,z_2) &= p(F(z_0,w_0,x_0,y_0),F(z_1,w_1,x_1,y_1)) \\ &\leq \tfrac{k}{4}[p(z_0,z_1)+p(w_0,w_1)+p(x_0,x_1)+p(y_0,y_1)] \end{aligned}$$

$$(2.9)\qquad \begin{aligned} p(w_1,w_2) &= p(F(w_0,x_0,y_0,z_0),F(w_1,x_1,y_1,z_1)) \\ &\leq \tfrac{k}{4}[p(w_0,w_1)+p(x_0,x_1)+p(y_0,y_1)+p(z_0,z_1)] \end{aligned}$$

Regarding (2.4) together with (2.6),(2.7),(2.8) we have

$$(2.10)\qquad \begin{aligned} p(x_2,x_3) &= p(F(x_1,y_1,z_1,w_1),F(x_2,y_2,z_2,w_2)) \\ &\leq \tfrac{k}{4}[p(x_1,x_2)+p(y_1,y_2)+p(z_1,z_2)+p(w_1,w_2)] \end{aligned}$$

$$(2.11)\qquad \begin{aligned} p(y_2,y_3) &= p(F(y_1,z_1,w_1,x_1),F(y_2,z_2,w_2,x_2)) \\ &\leq \tfrac{k}{4}[p(y_1,y_2)+p(z_1,z_2)+p(w_1,w_2)+p(x_1,x_2)] \end{aligned}$$

$$(2.12)\qquad \begin{aligned} p(z_2,z_3) &= p(F(z_1,w_1,x_1,y_1),F(z_2,w_2,x_2,y_2)) \\ &\leq \tfrac{k}{4}[p(z_1,z_2)+p(w_1,w_2)+p(x_1,x_2)+p(y_1,y_2)] \end{aligned}$$

$$(2.13)\qquad \begin{aligned} p(w_2,w_3) &= p(F(w_1,x_1,y_2,z_1),F(w_2,x_2,y_2,z_2)) \\ &\leq \tfrac{k}{4}[p(w_1,w_2)+p(x_1,x_2)+p(y_1,y_2)+p(z_1,z_2)] \end{aligned}$$

Recursively we have

(2.14)
$$\begin{aligned} p(x_{n+1},x_{n+2}) &= p(F(x_n,y_n,z_n,w_n),F(x_{n+1},y_{n+1},z_{n+1},w_{n+1})) \\ &\leq \tfrac{k}{4}[p(x_n,x_{n+1})+p(y_n,y_{n+1})+p(z_n,z_{n+1})+p(w_n,w_{n+1})] \end{aligned}$$

(2.15)
$$\begin{aligned} p(y_{n+1},y_{n+2}) &= p(F(y_n,z_n,w_n,x_n),F(y_{n+1},z_{n+1},w_{n+1}),x_{n+1}) \\ &\leq \tfrac{k}{4}[p(y_n,y_{n+1})+p(z_n,z_{n+1})+p(w_n,w_{n+1})+p(x_n,x_{n+1})] \end{aligned}$$

(2.16)
$$\begin{aligned} p(z_{n+1},z_{n+2}) &= p(F(z_n,w_n,x_n,y_n),F(z_{n+1},w_{n+1},x_{n+1},y_{n+1})) \\ &\leq \tfrac{k}{4}[p(z_n,z_{n+1})+p(w_n,w_{n+1})+p(x_n,x_{n+1})+p(y_n,y_{n+1})] \end{aligned}$$

(2.17)
$$\begin{aligned} p(w_{n+1},w_{n+2}) &= p(F(w_n,x_n,y_n,z_n),F(w_{n+1},x_{n+1},y_{n+1},z_{n+1})) \\ &\leq \tfrac{k}{4}[p(w_n,w_{n+1})+p(x_n,x_{n+1})+p(y_n,y_{n+1})+p(z_n,z_{n+1})] \end{aligned}$$

For simplicity, we can use the matrix notation as follow. Set

$$M = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}, \ D_n = \begin{pmatrix} p(x_{n+1}, x_n) \\ p(y_{n+1}, y_n) \\ p(z_{n+1}, z_n) \\ p(w_{n+1}, w_n) \end{pmatrix}$$

and $R = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$. Notice that

$$RM = R \quad \text{and } M^n = M \quad \text{for all } n \in \mathbb{N}. \tag{2.18}$$

So we have,

$$D_1 \leq kD_0, \tag{2.19}$$

$$D_2 \leq kMD_1 \leq k^2M^2D_0 = k^2MD_0, \tag{2.20}$$

and, inductively

$$D_n \leq kMD_{n-1} \leq k^nMD_0. \tag{2.21}$$

$$p(x_{n+1}, x_{n+2}) \leq kRD_n \begin{pmatrix} p(x_n, x_{n+1}) \\ p(y_n, y_{n+1}) \\ p(z_n, z_{n+1}) \\ p(w_n, w_{n+1}) \end{pmatrix} \tag{2.22}$$

Hence, by (2.18),(2.4) and (2.5), we have

$$\begin{aligned} p(x_{n+1}, x_{n+2}) &= p(F(x_n, y_n, z_n, w_n), F(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1})) \\ &\leq \tfrac{k}{4}[p(x_n, x_{n+1}) + p(y_n, y_{n+1}) + p(z_n, z_{n+1}) + p(w_n, w_{n+1})] \\ &\leq kRD_n \leq k^{n+1}RMD_0 \leq k^{n+1}RD_0. \end{aligned} \tag{2.23}$$

We shall show the sequences $\{x_n\}$ are Cauchy easily by using (2.14)-(2.21). Without loss of generality, we may assume that $m > n$. By using (2.14)-(2.21) together with triangle inequality, we obtain that

$$\begin{aligned} p(x_m, x_n) &\leq p(x_m, x_{m-1}) + p(x_{m-1}, x_{m-2}) + \cdots + p(x_{n+1}, x_n) \\ &\leq k^{m-1}RD_0 + \cdots + k^nRD_0 \\ &\leq k^n(1 + \cdots + k^{m-n-1})RD_0 \\ &\leq k^n\tfrac{1}{1-k}RD_0 \end{aligned} \tag{2.24}$$

Letting $n \to \infty$ in (2.24) and recalling that $k \in [0, 1)$, we get that $\lim_{n\to\infty} p(x_n, x_m) = 0$. By definition,

$$d_p(x_n, x_m) = 2p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m) \leq 2p(x_n, x_m).$$

Thus, we have

$$\lim_{n\to\infty} d_p(x_n, x_m) = 0. \tag{2.25}$$

Since (X,p) is a complete partial metric space, then by Lemma 1.11, (X,d_p) is a complete metric space. Thus, $\{x_n\}$ converges in (X,d_p), say x. Again by 1.11, we have

$$p(x,x)=\lim_{n\to\infty}p(x_n,x_m)=\lim_{n\to\infty}p(x_n,x)=0. \tag{2.26}$$

Analogously, one can prove that $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ are Cauchy sequences. Since (X,d_p) is complete metric space, there exists $x,y,z,w\in X$ such that

$$\begin{aligned} p(y,y)&=\lim_{n\to\infty}p(y_n,y_m)=\lim_{n\to\infty}p(y_n,y)=0,\\ p(z,z)&=\lim_{n\to\infty}p(z_n,z_m)=\lim_{n\to\infty}p(z_n,z)=0,\\ p(w,w)&=\lim_{n\to\infty}p(w_n,w_m)=\lim_{n\to\infty}p(w_n,w)=0. \end{aligned} \tag{2.27}$$

Suppose now the assumption (a) holds. Then by (2.26) and (2.27), we have

$$\begin{aligned} x&=\lim_{n\to\infty}x_n=\lim_{n\to\infty}F(x_{n-1},y_{n-1},z_{n-1},w_{n-1})\\ &=F(\lim_{n\to\infty}x_{n-1},\lim_{n\to\infty}y_{n-1},\lim_{n\to\infty}z_{n-1},\lim_{n\to\infty}w_{n-1})\\ &=F(x,y,z,w) \end{aligned} \tag{2.28}$$

Analogously, we also observe that

$$\begin{aligned} y&=\lim_{n\to\infty}y_n=\lim_{n\to\infty}F(x_{n-1},w_{n-1},z_{n-1},y_{n-1})=F(x,w,z,y)\\ z&=\lim_{n\to\infty}z_n=\lim_{n\to\infty}F(z_{n-1},y_{n-1},x_{n-1},w_{n-1})=F(z,y,x,w)\\ w&=\lim_{n\to\infty}w_n=\lim_{n\to\infty}F(z_{n-1},w_{n-1},x_{n-1},y_{n-1})=F(z,w,x,y) \end{aligned} \tag{2.29}$$

Thus, we have

$$\begin{aligned} F(x,y,z,w)&=x, \quad F(x,w,z,y)=y,\\ F(z,y,x,w)&=z, \quad F(z,w,x,y)=w. \end{aligned}$$

Suppose now the assumption (b) holds. Since $\{x_n\}$, $\{z_n\}$ are non-decreasing and $x_n\to x$, $z_n\to z$ and also $\{y_n\}$, $\{w_n\}$ are non-increasing and $y_n\to y$, $w_n\to w$, then by assumption (b) we have

$$x_n\geq x,\quad y_n\leq y,\quad z_n\geq z,\quad w_n\leq w$$

for all n. Due to (2.26) and (2.27), we have

$$p(F(x,y,z,w),F(x,y,z,w))\leq\frac{k}{4}[p(x,x)+p(y,y)+p(z,z)+p(w,w)]=0. \tag{2.30}$$

Consider now,

$$\begin{aligned} p(x_n,F(x,y,z,w))&=p(F(x_{n-1},y_{n-1},z_{n-1},w_{n-1}),F(x,y,z,w))\\ &\leq\tfrac{k}{4}[p(x_{n-1},x)+p(y_{n-1},y)+p(z_{n-1},z)+p(w_{n-1},w)] \end{aligned} \tag{2.31}$$

Letting $n\to\infty$ in (2.31), by Lemma 1.12 we get

$$p(x,F(x,y,z,w))\leq\frac{k}{4}[p(x,x)+p(y,y)+p(z,z)+p(w,w)] \tag{2.32}$$

Regarding (2.26) and (2.27), we conclude that $p(x, F(x, y, z, w)) = 0$. Hence, by (2.26),(2.30),(2.32) and definiton

(2.33)
$$d_p(x, F(x, y, z, w)) = 2p(x, F(x, y, z, w)) - p(F(x, y, z, w), F(x, y, z, w)) - p(x, x) = 0.$$

Thus, we have

$$x = F(x, y, z, w).$$

analogously we we get

$$F(y, z, w, x) = y,$$
$$F(z, w, x, y) = z, \quad F(w, x, y, z) = w.$$

Thus, we proved that F has a quadruple fixed point. □

3. Uniqueness of Quadruple Fixed Point

In this section we shall prove the uniqueness of quadruple fixe point. For a product X^4 of a partial ordered set $(X, \leq)$ we define a partial ordering in the following way: For all $(x, y, z, t), (u, v, r, t) \in X^4$

$$(x, y, z, w) \leq (u, v, r, t) \Leftrightarrow x \leq u, \quad y \geq v, \quad z \leq r, \quad w \geq r. \tag{3.1}$$

We say that (x, y, z, w) is equal (u, v, r, t) if and only if $x = u$, $y = v$, $z = r$ and $w = t$.

Theorem 3.1. *In addition to hypothesis of Theorem 2.3, suppose that for all $(x, y, z, t), (u, v, r, t) \in X \times X \times X \times X$, there exists $(a, b, c, d) \in X \times X \times X \times X$ that is comparable to (x, y, z, t) and (u, v, r, t), then F has a unique quadruple fixed point.*

Proof. The set of quadruple fixed point of F is not empty due to Theorem 2.3. Assume, now, (x, y, z, t) and (u, v, r, t) are the quadruple fixed point of F, that is,

$$\begin{array}{ll} F(x, y, z, w) = x, & F(u, v, r, t) = u, \\ F(y, z, w, x) = y, & F(v, r, t, u) = v, \\ F(z, w, x, y) = z, & F(r, t, u, v) = r, \\ F(w, x, y, z) = w, & F(t, u, v, r) = t, \end{array}$$

We shall show that (x, y, z, w) and (u, v, r, t) are equal. By assumption, there exists $(a, b, c, d) \in X \times X \times X \times X$ that is comparable to (x, y, z, t) and (u, v, r, t). Define sequences $\{a_n\}, \{b_n\}, \{c_n\}$ and $\{d_n\}$ such that

$$a = a_0, \quad b = b_0, \quad c = c_0, \quad d = d_0 \text{ and}$$

$$\begin{aligned} a_n &= F(a_{n-1}, b_{n-1}, z_{n-1}, d_{n-1}), \\ b_n &= F(b_{n-1}, c_{n-1}, d_{n-1}, a_{n-1}), \\ c_n &= F(c_{n-1}, d_{n-1}, a_{n-1}, b_{n-1}), \\ d_n &= F(d_{n-1}, a_{n-1}, b_{n-1}, c_{n-1}). \end{aligned} \tag{3.2}$$

for all n. Since (x,y,z,w) is comparable with (a,b,c,d), we may assume that $(x,y,z,w) \geq (a,b,c,d) = (a_0,b_0,c_0,d_0)$. Recursively, we get that

$$(x,y,z,w) \geq (a_n,b_n,c_n,d_n) \quad \text{for all } n. \tag{3.3}$$

By (3.3) and (2.4), we have

$$\begin{aligned} p(x,a_{n+1}) &= p(F(x,y,z,w),F(a_n,b_n,c_n,d_n)) \\ &\leq \tfrac{k}{4}[p(x,a_n)+p(y,b_n)+p(z,c_n)+p(w,d_n)] \end{aligned} \tag{3.4}$$

$$\begin{aligned} p(b_{n+1},y) &= p(F(b_n,c_n,d_n,a_n),F(y,z,w,x)) \\ &\leq \tfrac{k}{4}[p(b_n,y)+p(c_n,z)+p(d_n,w)+p(a_n,x)] \end{aligned} \tag{3.5}$$

$$\begin{aligned} p(z,c_{n+1}) &= p(F(z,w,x,y),F(c_n,d_n,a_n,b_n)) \\ &\leq \tfrac{k}{4}[p(z,c_n)+p(w,d_n)+p(x,a_n)+p(y,b_n)] \end{aligned} \tag{3.6}$$

$$\begin{aligned} p(d_{n+1},w) &= p(F(c_n,d_n,a_n,b_n),F((w,x,y,z))) \\ &\leq \tfrac{k}{4}[p(d_n,w)+p(a_n,x)+p(b_n,y)+p(c_n,z)] \end{aligned} \tag{3.7}$$

Set $\gamma_n = p(x,a_n)+p(y,b_n)+p(z,c_n)+p(w,d_n)$. Then, due to (3.7)-(3.7), we have

$$\gamma_{n+1} \leq k\gamma_n \leq k^n\gamma_0, \quad \text{for all } n. \tag{3.8}$$

□

Since $0 \leq k < 1$, the sequence $\{\gamma_n\}$ is decreasing and bounded below. Thus, there exists $\gamma \geq 0$ such that

$$\lim_{n\to\infty} \gamma_n = \gamma.$$

Now, we shall show that $\gamma = 0$. Letting $n \to \infty$ in (3.8), and having mind $0 \leq k < 1$, we obtain that

$$\gamma \leq 0.$$

Therefore, $\gamma = 0$. That is,

$$\lim_{n\to\infty} \gamma_n = 0.$$

Consequently, we have

$$\begin{aligned} \lim_{n\to\infty} p(x,a_n) = 0, \quad \lim_{n\to\infty} p(y,b_n) = 0, \\ \lim_{n\to\infty} p(z,c_n) = 0, \quad \lim_{n\to\infty} p(w,d_n) = 0. \end{aligned} \tag{3.9}$$

Analogously, we show that

$$\begin{aligned} \lim_{n\to\infty} p(u,a_n) = 0, \quad \lim_{n\to\infty} p(v,b_n) = 0, \\ \lim_{n\to\infty} p(r,c_n) = 0, \quad \lim_{n\to\infty} p(s,d_n) = 0. \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10) yield ,by uniqueness of the limit, that (x,y,z,w) and (u,v,r,t) are equal. Now, in the following example neither the continuity of the mapping F is satisfied nor the conditions (a) and (b) given in Theorem 2.3 hold, but we still obtain a quadruple fixed point result.

Example 3.2. Let $X = [0, \infty)$, and $p(x, y) = \max\{x, y\}$ be a partial metric. Let "$\leq$" be the usual order on positive half-line. Notice that $d_p(x, y) = |x - y|$ becomes the corresponding metric. It is clear that (X, p) is a complete partial metric space. Now define $F : X^4 \to X$ as

$$F(x, y, z, w) = \begin{cases} \frac{x-y+z-w}{8}, & \text{if } x + z \geq y + w, \\ 0, & \text{otherwise .} \end{cases}$$

Then it is easy to see that F has the mixed monotone property. On the other hand, letting $x \geq u$, $y \leq v$, $z \geq r$, $w \leq t$ we have

$$\begin{aligned} p(F(x, y, z, w), F(u, v, r, t)) &= \max\{F(x, y, z, w), F(u, v, r, t)\} \\ &= \tfrac{x-y+z-w}{8} \\ &\leq \tfrac{k}{4}[p(x, u) + p(y, v) + p(z, r) + p(w, t)] \end{aligned}$$

for $k = \frac{1}{2}$. Hence, the condition (2.4) of Theorem 2.3 holds.
Notice that $(0, 0, 0, 0)$ is the unique quadruple fixed point.

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