APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 15, no. 1 (2014), 93-109
doi:10.4995/agt.2014.2268

Soft set theory and topology

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Abstract

In this paper we study and discuss the soft set theory giving new definitions, examples, new classes of soft sets, and properties for mappings between different classes of soft sets. Furthermore, we investigate the theory of soft topological spaces and we present new definitions, characterizations, and properties concerning the soft closure, the soft interior, the soft boundary, the soft continuity, the soft open and closed maps, and the soft homeomorphism.

2010 MSC: *54A05; 06D72.*

Keywords: *Soft set theory; Soft topology.*

1. Preliminaries

For every set X we denote by $\mathcal{P}(X)$ the power set of X, that is the set of all subsets of X and by $|X|$ the cardinality of X. Also, we denote by ω the first infinite cardinal and by $\mathbb{R}$ the set of real numbers.

In 1999 D. Molodtsov (see [17]) introduced the notion of soft set. Later, he applied this theory to several directions (see [18], [19], and [20]).

The soft set theory has been applied to many different fields (see, for example, [1], [2], [4], [5], [7], [8], [10], [12], [13], [14], [15], [21], [23], [25]).

In 2011 and 2012 few researches (see, for example, [3], [6], [9], [16], [22], [24]) introduced and studied the notion of soft topological spaces which are defined over an initial universe with a fixed set of parameters.

The paper is organized as follows. In section 2 we study and discuss the soft set theory giving new definitions, examples, new classes of soft sets, and properties for mappings between different classes of soft sets. In section 3 we

Received July 2012 – Accepted October 2012

investigate the theory of soft topological spaces and we present new definitions, characterizations, and many properties concerning the soft closure, the soft interior, the soft boundary, the soft continuity, the soft open and closed maps, and the soft homeomorphism.

2. Soft Set Theory

Definition 2.1 (see [17])**.** Let X be an initial universe set and A a set of parameters. A pair (F, A), where F is a map from A to $\mathcal{P}(X)$, is called a *soft set* over X.

In what follows by $\mathrm{SS}(X, A)$ we denote the family of all soft sets (F, A) over X.

Definition 2.2 (see [17])**.** Let $(F, A), (G, A) \in \mathrm{SS}(X, A)$. We say that the pair (F, A) is a *soft subset* of (G, A) if $F(p) \subseteq G(p)$, for every $p \in A$. Symbolically, we write $(F, A) \sqsubseteq (G, A)$. Also, we say that the pairs (F, A) and (G, A) are *soft equal* if $(F, A) \sqsubseteq (G, A)$ and $(G, A) \sqsubseteq (F, A)$. Symbolically, we write $(F, A) = (G, A)$.

Definition 2.3 (see, for example, [17] and [24])**.** Let I be an arbitrary index set and $\{(F_i, A) : i \in I\} \subseteq \mathrm{SS}(X, A)$. The *soft union* of these soft sets is the soft set $(F, A) \in \mathrm{SS}(X, A)$, where the map $F : A \to \mathcal{P}(X)$ defined as follows: $F(p) = \cup\{F_i(p) : i \in I\}$, for every $p \in A$. Symbolically, we write

$$(F, A) = \sqcup\{(F_i, A) : i \in I\}.$$

Example 2.4. Let $X = \mathbb{R}$, $A = \{0, 1\}$, and $I = \{1, 2, \ldots\}$. For every $i \in I$ we consider the soft set (F_i, A), where the map $F_i : A \to \mathcal{P}(X)$ defined as follows:

$$F_i(p) = \begin{cases} (0, i), & \text{if } p = 0, \\ (-i, 0), & \text{if } p = 1. \end{cases}$$

Then, $\sqcup\{(F_i, A) : i \in I\} = (F, A)$, where the map $F : A \to \mathcal{P}(X)$ defined as follows:

$$F(p) = \begin{cases} (0, +\infty), & \text{if } p = 0, \\ (-\infty, 0), & \text{if } p = 1. \end{cases}$$

Definition 2.5 (see, for example, [17] and [24])**.** Let I be an arbitrary index set and $\{(F_i, A) : i \in I\} \subseteq \mathrm{SS}(X, A)$. The *soft intersection* of these soft sets is the soft set $(F, A) \in \mathrm{SS}(X, A)$, where the map $F : A \to \mathcal{P}(X)$ defined as follows: $F(p) = \cap\{F_i(p) : i \in I\}$, for every $p \in A$. Symbolically, we write

$$(F, A) = \sqcap\{(F_i, A) : i \in I\}.$$

Example 2.6. Let $X = \mathbb{R}$, $A = \{0, 1, 2\}$, and $I = \{1, 2, \ldots\}$. For every $i \in I$ we consider the soft set (F_i, A), where the map $F_i : A \to \mathcal{P}(X)$ defined as follows:

$$F_i(p) = \begin{cases} (-\frac{1}{i}, \frac{1}{i}), & \text{if } p = 0, \\ (1 - \frac{1}{i}, 1 + \frac{1}{i}), & \text{if } p = 1, \\ (2 - \frac{1}{i}, 2 + \frac{1}{i}), & \text{if } p = 2. \end{cases}$$

Then, $\sqcap\{(F_i, A) : i \in I\} = (F, A)$, where the map $F : A \to \mathcal{P}(X)$ defined as follows:

$$F(p) = \begin{cases} \{0\}, & \text{if } p = 0, \\ \{1\}, & \text{if } p = 1, \\ \{2\}, & \text{if } p = 2. \end{cases}$$

Definition 2.7 (see, for example, [24])**.** Let $(F, A) \in \mathrm{SS}(X, A)$. The *soft complement* of (F, A) is the soft set $(H, A) \in \mathrm{SS}(X, A)$, where the map $H : A \to \mathcal{P}(X)$ defined as follows: $H(p) = X \setminus F(p)$, for every $p \in A$. Symbolically, we write $(H, A) = (F, A)^c$.

Example 2.8. Let $X = \mathbb{R}$ and $A = \{1, 2, \ldots\}$. We consider the soft set (F, A), where the map $F : A \to \mathcal{P}(X)$ defined as follows: $F(p) = [p, +\infty)$, for every $p \in A$. Then, $(F, A)^c = (H, A)$, where the map $H : A \to \mathcal{P}(X)$ defined as follows: $H(p) = (-\infty, p)$, for every $p \in A$.

Definition 2.9 (see [17])**.** The soft set $(F, A) \in \mathrm{SS}(X, A)$, where $F(p) = \varnothing$, for every $p \in A$ is called the *A-null soft set* of $\mathrm{SS}(X, A)$ and denoted by $\mathbf{0}_A$. The soft set $(F, A) \in \mathrm{SS}(X, A)$, where $F(p) = X$, for every $p \in A$ is called the *A-absolute soft set* of $\mathrm{SS}(X, A)$ and denoted by $\mathbf{1}_A$.

The proofs of the following propositions are straightforward verifications of the above definitions.

Proposition 2.10. *Let $(F, A) \in \mathrm{SS}(X, A)$. The following statements are true:*

(1) $(F, A) \sqcap (F, A) = (F, A)$.
(2) $(F, A) \sqcup (F, A) = (F, A)$.
(3) $(F, A) \sqcap \mathbf{0}_A = \mathbf{0}_A$.
(4) $(F, A) \sqcup \mathbf{0}_A = (F, A)$.
(5) $(F, A) \sqcap \mathbf{1}_A = (F, A)$.
(6) $(F, A) \sqcup \mathbf{1}_A = \mathbf{1}_A$.
(7) $(F, A) \sqcap (F, A)^c = \mathbf{0}_A$.
(8) $(F, A) \sqcup (F, A)^c = \mathbf{1}_A$.
(9) $(\mathbf{0}_A)^c = \mathbf{1}_A$.
(10) $(\mathbf{1}_A)^c = \mathbf{0}_A$.
(11) $((F, A)^c)^c = (F, A)$.
(12) $\mathbf{0}_A \sqsubseteq (F, A) \sqsubseteq \mathbf{1}_A$.

Proposition 2.11. *Let $(F, A), (G, A), (H, A) \in \mathrm{SS}(X, A)$. The following statements are true:*

(1) $(F, A) \sqcap ((G, A) \sqcap (H, A)) = ((F, A) \sqcap (G, A)) \sqcap (H, A)$.
(2) $(F, A) \sqcup ((G, A) \sqcup (H, A)) = ((F, A) \sqcup (G, A)) \sqcup (H, A)$.
(3) $(F, A) \sqcap ((G, A) \sqcup (H, A)) = ((F, A) \sqcap (G, A)) \sqcup ((F, A) \sqcap (H, A))$.
(4) $(F, A) \sqcup ((G, A) \sqcap (H, A)) = ((F, A) \sqcup (G, A)) \sqcap ((F, A) \sqcup (H, A))$.

Proposition 2.12. *Let I be an arbitrary set and $\{(F_i, A) : i \in I\} \subseteq \mathrm{SS}(X, A)$. The following statements are true:*

(1) $(F_i, A) \sqsubseteq \sqcup\{(F_i, A) : i \in I\}$, *for every* $i \in I$.
(2) $\sqcap\{(F_i, A) : i \in I\} \sqsubseteq (F_i, A)$, *for every* $i \in I$.
(3) $(\sqcup\{(F_i, A) : i \in I\})^c = \sqcap\{(F_i, A)^c : i \in I\}$.
(4) $(\sqcap\{(F_i, A) : i \in I\})^c = \sqcup\{(F_i, A)^c : i \in I\}$.

Definition 2.13. Let $(F, A), (G, A) \in \mathrm{SS}(X, A)$. The *soft symmetric difference* of these soft sets is the soft set $(H, A) \in \mathrm{SS}(X, A)$, where the map $H : A \to \mathcal{P}(X)$ defined as follows: $H(p) = (F(p) \setminus G(p)) \cup (G(p) \setminus F(p))$, for every $p \in A$. Symbolically, we write $(H, A) = (F, A) \triangle (G, A)$.

Example 2.14. Let $X = \{1, 2, 3, 4, 5\}$ and $A = \{0, 1, 2, \ldots\}$. We consider the soft sets (F, A) and (G, A), where the maps $F : A \to \mathcal{P}(X)$ and $G : A \to \mathcal{P}(X)$ defined as follows:

$$F(p) = \begin{cases} \{1, 2, 3, 4\}, \text{ if } p = 0, \\ \varnothing, \text{ otherwise}, \end{cases} \qquad G(p) = \begin{cases} \{1, 4, 5\}, \text{ if } p = 0, \\ \varnothing, \text{ otherwise}. \end{cases}$$

Then, $(F, A) \triangle (G, A) = (H, A)$, where the map $H : A \to \mathcal{P}(X)$ defined as follows:

$$H(p) = \begin{cases} \{2, 3, 5\}, \text{ if } p = 0, \\ \varnothing, \text{ otherwise}. \end{cases}$$

The proof of the following proposition is straightforward verification of the Definition 2.13.

Proposition 2.15. *Let $(F, A), (G, A), (H, A) \in \mathrm{SS}(X, A)$. The following statements are true:*

(1) $(F, A) \triangle ((G, A) \triangle (H, A)) = ((F, A) \triangle (G, A)) \triangle (H, A)$.
(2) $(F, A) \triangle (G, A) = (G, A) \triangle (F, A)$.
(3) $(F, A) \triangle \mathbf{0}_A = (F, A)$.
(4) $(F, A) \triangle (F, A) = \mathbf{0}_A$.
(5) $(F, A) \sqcap ((G, A) \triangle (H, A)) = ((F, A) \sqcap (G, A)) \triangle ((F, A) \sqcap (H, A))$.

Remark 2.16. By Proposition 2.15 follows that the pair $(\mathrm{SS}(X, A), \triangle)$ is a group of soft sets. The identity element is the soft set $\mathbf{0}_A$ and the inverse of the element (F, A) is the soft set (F, A). Also, the triad $(\mathrm{SS}(X, A), \triangle, \sqcap)$ is a ring of soft sets.

Let X and Y be two initial universe sets, P_X and P_Y two sets of parameters, $f : X \to Y$, and $e : P_X \to P_Y$. In [11] the authors, using f and e, define the notion of a mapping from the family of all soft sets (F, A) over X, where $A \subseteq P_X$, to the family of all soft sets (G, B) over Y, where $B \subseteq P_Y$.

In [24] the authors gave a mapping from $\mathrm{SS}(X, A)$ to $\mathrm{SS}(Y, B)$ and studied properties of images and inverse images of soft sets. The given below definition is actually the definition of this mapping.

Definition 2.17. Let X and Y be two initial universe sets, A and B two sets of parameters, $f : X \to Y$, and $e : A \to B$. Then, by Φ_{fe} we denote the map from $\mathrm{SS}(X, A)$ to $\mathrm{SS}(Y, B)$ for which:

(1) If $(F, A) \in \mathrm{SS}(X, A)$, then the image of (F, A) under Φ_{fe}, denoted by $\Phi_{fe}(F, A)$, is the soft set $(G, B) \in \mathrm{SS}(Y, B)$ such that

$$G(p_Y) = \begin{cases} \bigcup\{f(F(p)) : p \in e^{-1}(\{p_Y\})\}, \text{ if } e^{-1}(\{p_Y\}) \neq \varnothing, \\ \varnothing, \text{ if } e^{-1}(\{p_Y\}) = \varnothing, \end{cases}$$

for every $p_Y \in B$.

(2) If $(G, B) \in \mathrm{SS}(Y, B)$, then the *inverse image* of (G, B) under Φ_{fe}, denoted by $\Phi^{-1}_{fe}(G, B)$, is the soft set $(F, A) \in \mathrm{SS}(X, A)$ such that

$$F(p_X) = f^{-1}(G(e(p_X))), \text{ for every } p_X \in A.$$

The following propositions are easily proved.

Proposition 2.18. *Let $(F, A), (F_1, A) \in \mathrm{SS}(X, A)$, $(G, B), (G_1, B) \in \mathrm{SS}(Y, B)$. The following statements are true:*

(1) *If $(F, A) \sqsubseteq (F_1, A)$, then $\Phi_{fe}(F, A) \sqsubseteq \Phi_{fe}(F_1, A)$.*
(2) *If $(G, B) \sqsubseteq (G_1, B)$, then $\Phi^{-1}_{fe}(G, B) \sqsubseteq \Phi^{-1}_{fe}(G_1, B)$.*
(3) $(F, A) \sqsubseteq \Phi^{-1}_{fe}(\Phi_{fe}(F, A))$.
(4) *If f is an 1-1 map of X into Y and e is an 1-1 map of A into B, then $(F, A) = \Phi^{-1}_{fe}(\Phi_{fe}(F, A))$.*
(5) $\Phi_{fe}(\Phi^{-1}_{fe}(G, B)) \sqsubseteq (G, B)$.
(6) *If f is a map of X onto Y and e is a map of A onto B, then $\Phi_{fe}(\Phi^{-1}_{fe}(G, B)) = (G, B)$.*
(7) $\Phi^{-1}_{fe}((G, B)^c) = (\Phi^{-1}_{fe}(G, B))^c$.

Proposition 2.19. *Let I be an arbitrary set, $\{(F_i, A) : i \in I\} \subseteq \mathrm{SS}(X, A)$, and $\{(G_i, B) : i \in I\} \subseteq \mathrm{SS}(Y, B)$. The following statements are true:*

(1) $\Phi_{fe}(\sqcup\{(F_i, A) : i \in I\}) = \sqcup\{\Phi_{fe}(F_i, A) : i \in I\}$.
(2) $\Phi_{fe}(\sqcap\{(F_i, A) : i \in I\}) \sqsubseteq \sqcap\{\Phi_{fe}(F_i, A) : i \in I\}$.
(3) $\Phi^{-1}_{fe}(\sqcup\{(G_i, B) : i \in I\}) = \sqcup\{\Phi^{-1}_{fe}(G_i, B) : i \in I\}$.
(4) $\Phi^{-1}_{fe}(\sqcap\{(G_i, B) : i \in I\}) = \sqcap\{\Phi^{-1}_{fe}(G_i, B) : i \in I\}$.

Definition 2.20. Define the *order* of a soft set $(F, A) \in \mathrm{SS}(X, A)$ as follows:

(1) $\mathrm{ord}(F, A) = n$, where $n \in \omega$, if and only if the intersection of any $n + 2$ distinct elements of $\{F(p) : p \in A\}$ is empty and there exist $n + 1$ distinct elements of $\{F(p) : p \in A\}$, whose intersection is not empty.

(2) $\mathrm{ord}(F, A) = \infty$, if and only if for every $n \in \omega$ there exist n distinct elements of $\{F(p) : p \in A\}$, whose intersection is not empty.

Note. Let X be an initial universe set and A a set of parameters. We consider the following subsets of $\mathrm{SS}(X, A)$:

(1) $\mathcal{C}(X, A) = \{(F, A) \in \mathrm{SS}(X, A) : \bigcup\{F(p) : p \in A\} = X\}$.
(2) $\mathrm{SS}(X, A, \nu) = \{(F, A) \in \mathrm{SS}(X, A) : |F(p)| = \nu,$ for every $p \in A\}$, where ν is an ordinal such that $\nu \leq |X|$.
(3) $\mathcal{F}(X, A) = \{(F, A) \in \mathrm{SS}(X, A) : |F(p)| < \omega,$ for every $p \in A\}$.
(4) $\mathcal{O}(X, A, n) = \{(F, A) \in \mathrm{SS}(X, A) : \mathrm{ord}(F, A) = n\}$, where $n \in \omega \cup \{\infty\}$.

Example 2.21.

(1) Let X be a nonempty set. Every cover $\{U_i : i \in I\}$ of X, that is $\cup\{U_i : i \in I\} = X$, can be considered as the soft set $(F, A) \in \mathcal{C}(X, A)$, where $A = I$ and the map $F : A \to \mathcal{P}(X)$ defined as follows: $F(i) = U_i$, for every $i \in A$.
(2) Let X be a set with $|X| = 5$. Then, the family of all subsets Y of X with $|Y| = 3$ can be considered as the element (F, A) of $\mathrm{SS}(X, A, 3)$, where $A = \{1, 2, \ldots, 10\}$ and F is an 1-1 map of A to $\mathcal{P}(X)$.

Proposition 2.22. *Let $(F, A) \in \mathrm{SS}(X, A)$ and $(G, B) \in \mathrm{SS}(Y, B)$. The following statements are true:*

(1) *If $(F, A) \in \mathcal{C}(X, A)$ and the maps $f : X \to Y$ and $e : A \to B$ are onto, then $\Phi_{fe}(F, A) \in \mathcal{C}(Y, B)$.*
(2) *If $(F, A) \in \mathrm{SS}(X, A, \nu)$, the map $f : X \to Y$ is 1-1, and the map $e : A \to B$ is 1-1 and onto, then $\Phi_{fe}(F, A) \in \mathrm{SS}(Y, B, \nu)$.*
(3) *If $(F, A) \in \mathcal{F}(X, A)$, the map $f : X \to Y$ is 1-1, and the map $e : A \to B$ is 1-1 and onto, then $\Phi_{fe}(F, A) \in \mathcal{F}(Y, B)$.*
(4) *If $(F, A) \in \mathcal{O}(X, A, n)$, the map $f : X \to Y$ is 1-1, and the map $e : A \to B$ is 1-1 and onto, then $\Phi_{fe}(F, A) \in \mathcal{O}(Y, B, n)$.*
(5) *If $(G, B) \in \mathcal{C}(Y, B)$, then $\Phi^{-1}_{fe}(G, B) \in \mathcal{C}(X, A)$.*
(6) *If $(G, B) \in \mathrm{SS}(Y, B, \nu)$ and the map $f : X \to Y$ is 1-1 and onto, then $\Phi^{-1}_{fe}(G, B) \in \mathrm{SS}(X, A, \nu)$.*
(7) *If $(G, B) \in \mathcal{F}(Y, B)$ and the map $f : X \to Y$ is 1-1, then $\Phi^{-1}_{fe}(G, B) \in \mathcal{F}(X, A)$.*
(8) *If $(G, B) \in \mathcal{O}(Y, B, n)$ and the map $f : X \to Y$ is onto, then $\Phi^{-1}_{fe}(G, B) \in \mathcal{O}(X, A, n)$.*

Proof. Suggestively we prove the statements (1), (2), (7), and (8).

(1) Let $(F, A) \in \mathcal{C}(X, A)$ and $\Phi_{fe}(F, A) = (G, B)$. Then,

$$\bigcup\{F(p_X) : p_X \in A\} = X.$$

Since the map $f : X \to Y$ is onto, $f(X) = Y$. Therefore,

$$\begin{aligned}\bigcup\{G(p_Y) : p_Y \in B\} &= \bigcup\{\bigcup\{f(F(p)) : p \in e^{-1}(\{p_Y\})\} : p_Y \in B\} \\ &= \bigcup\{f(F(p_X)) : p_X \in A\} = f(\cup\{F(p_X) : p_X \in A\}) \\ &= f(X) = Y.\end{aligned}$$

Thus, $\Phi_{fe}(F,A) \in \mathcal{C}(Y,B)$.

(2) Let $(F,A) \in \mathrm{SS}(X,A,\nu)$ and $\Phi_{fe}(F,A) = (G,B)$. Then, $|F(p)| = \nu$, for every $p \in A$. Let $p_Y \in B$. Since the map $e : A \to B$ is 1-1, $G(p_Y) = f(F(p))$, where $p \in e^{-1}(\{p_Y\})$. Also, since the map $f : X \to Y$ is 1-1, we have $|f(F(p))| = |F(p)|$. Therefore, $|G(p_Y)| = |f(F(p))| = |F(p)| = \nu$. Thus, $\Phi_{fe}(F,A) \in \mathrm{SS}(Y,B,\nu)$.

(7) Let $(G,B) \in \mathcal{F}(Y,B)$ and $\Phi^{-1}_{fe}(G,B) = (F,A)$. Then, $|G(p_Y)| < \omega$, for every $p_Y \in B$. Let $p_X \in A$. Then, $F(p_X) = f^{-1}(G(e(p_X)))$. Since the map $f : X \to Y$ is 1-1, we have $|F(p_X)| = |f^{-1}(G(e(p_X)))| < \omega$. This means that $\Phi^{-1}_{fe}(G,B) \in \mathcal{F}(X,A)$.

(8) Let $(G,B) \in \mathcal{O}(Y,B,n)$ and $\Phi^{-1}_{fe}(G,B) = (F,A)$. Then, the intersection of any $n+2$ distinct elements of $\{G(p_Y) : p_Y \in B\}$ is empty and there exist $n+1$ distinct elements of $\{G(p_Y) : p_Y \in B\}$, whose intersection is not empty. Let $p^1_X, \ldots, p^{n+1}_X \in A$ such that $G(e(p^1_X)) \cap \ldots \cap G(e(p^{n+1}_X)) \neq \varnothing$. Then,

$$\begin{array}{rcl} F(p^1_X) \cap \ldots \cap F(p^{n+1}_X) & = & f^{-1}(G(e(p^1_X))) \cap \ldots \cap f^{-1}(G(e(p^{n+1}_X))) \\ & = & f^{-1}(G(e(p^1_X)) \cap \ldots \cap G(e(p^{n+1}_X))). \end{array}$$

Since the map $f : X \to Y$ is onto, $F(p^1_X) \cap \ldots \cap F(p^{n+1}_X) \neq \varnothing$. This means that there exist $n+1$ distinct elements of $\{F(p_X) : p_X \in A\}$, whose intersection is not empty.

Now, we prove that the intersection of any $n+2$ distinct elements of the set $\{F(p_X) : p_X \in A\}$ is empty. Let $p^1_X, \ldots, p^{n+2}_X \in A$. Then,

$$\begin{array}{rcl} F(p^1_X) \cap \ldots \cap F(p^{n+2}_X) & = & f^{-1}(G(e(p^1_X))) \cap \ldots \cap f^{-1}(G(e(p^{n+2}_X))) \\ & = & f^{-1}(G(e(p^1_X)) \cap \ldots \cap G(e(p^{n+2}_X))) \\ & = & f^{-1}(\varnothing) = \varnothing. \end{array}$$

Thus, $\Phi^{-1}_{fe}(G,B) \in \mathcal{O}(X,A,n)$. □

3. Soft Topology

Definition 3.1 (see, for example, [24]). Let X be an initial universe set, A a set of parameters, and $\tau \subseteq \mathrm{SS}(X,A)$. We say that the family τ defines a *soft topology* on X if the following axioms are true:

(1) $\mathbf{0}_A, \mathbf{1}_A \in \tau$.
(2) If $(G,A), (H,A) \in \tau$, then $(G,A) \sqcap (H,A) \in \tau$.
(3) If $(G_i,A) \in \tau$ for every $i \in I$, then $\sqcup\{(G_i,A) : i \in I\} \in \tau$.

The triplet (X,τ,A) is called a *soft topological space* or *soft space*. The members of τ are called *soft open sets* in X. Also, a soft set (F,A) is called *soft closed* if the complement $(F,A)^c$ belongs to τ. The family of soft closed sets is denoted by τ^c.

Remark 3.2. Let (X,τ,A) be a soft topological space. Then, by Proposition 2.10 (11) we have $(G,A) \in \tau$ if and only if $(G,A)^c \in \tau^c$.

The proof of the following proposition is straightforward verification of the Definition 3.1, Proposition 2.10, and Proposition 2.12.

Proposition 3.3. *Let (X, τ, A) be a soft topological space. The family τ^c has the following properties:*

(1) $\mathbf{0}_A, \mathbf{1}_A \in \tau^c$.
(2) *If $(Q, A), (R, A) \in \tau^c$, then $(Q, A) \sqcup (R, A) \in \tau^c$.*
(3) *If $(Q_i, A) \in \tau^c$ for every $i \in I$, then $\sqcap\{(Q_i, A) : i \in I\} \in \tau^c$.*

Example 3.4. (1) Let $X = \{1, 2, \ldots\}$, $A = \{0, 1\}$, and

$$\tau = \{(G_n, A) : n = 1, 2, \ldots\} \cup \{\mathbf{0}_A, \mathbf{1}_A\},$$

where the map $G_n : A \to \mathcal{P}(X)$ defined as follows:

$$G_n(p) = \begin{cases} \{n, n+1, \ldots\}, \text{ if } p = 0, \\ \varnothing, \text{ if } p = 1. \end{cases}$$

The triplet (X, τ, A) is a soft topological space.

(2) Let (X, t) be a topological space, A a nonempty set, and

$$\tau = \{(G_U, A) : U \in t\},$$

where the map $G_U : A \to \mathcal{P}(X)$ defined as follows: $G_U(p) = U$, for every $p \in A$. The triplet (X, τ, A) is a soft topological space.

Definition 3.5. Let (X, τ, A) be a soft topological space, $a \in A$, and $x \in X$. We say that a soft set $(F, A) \in \tau$ is an *a-soft open neighborhood of x in* (X, τ, A) if $x \in F(a)$.

Proposition 3.6. *Let (X, τ, A) be a soft topological space. Then, $(G, A) \in \tau$ if and only if for every $a \in A$ and $x \in G(a)$ there exists an a-soft open neighborhood $(G_{(a,x)}, A)$ of x in (X, τ, A) such that $(G_{(a,x)}, A) \sqsubseteq (G, A)$.*

Proof. If $(G, A) \in \tau$, then for every $a \in A$ and $x \in G(a)$ we consider the soft set $(G_{(a,x)}, A)$, where $G_{(a,x)} = G$. Obviously, $(G_{(a,x)}, A)$ is an a-soft open neighborhood of x.

Conversely, we suppose that for every $a \in A$ and $x \in G(a)$ there exists an a-soft open neighborhood $(G_{(a,x)}, A)$ of x in (X, τ, A) such that $(G_{(a,x)}, A) \sqsubseteq (G, A)$, that is

$$G_{(a,x)}(p) \subseteq G(p), \text{ for every } p \in A. \tag{1}$$

We prove that $(G, A) \in \tau$. We set $I = \{(a, x) : a \in A,\ x \in G(a)\}$. It suffices to prove that

$$(G, A) = \sqcup\{(G_{(a,x)}, A) : (a, x) \in I\}$$

or equivalently

$$G(p) = \cup\{G_{(a,x)}(p) : (a, x) \in I\}, \text{ for every } p \in A.$$

Let $p \in A$. By relation (1) we have $G_{(a,x)}(p) \subseteq G(p)$, for every $(a, x) \in I$. Therefore, $\cup\{G_{(a,x)}(p) : (a, x) \in I\} \subseteq G(p)$.

We prove that $G(p) \subseteq \cup\{G_{(a,x)}(p) : (a,x) \in I\}$. Let $y \in G(p)$. Then, by assumption there exists a p-soft open neighborhood $(G_{(p,y)}, A)$ of y in (X, τ, A) such that $(G_{(p,y)}, A) \sqsubseteq (G, A)$. Therefore,

$$y \in G_{(p,y)}(p) \subseteq \cup\{G_{(a,x)}(p) : (a,x) \in I\}.$$

Thus, $(G, A) = \sqcup\{(G_{(a,x)}, A) : (a,x) \in I\}$. □

Definition 3.7 (see [22])**.** Let (X, τ, A) be a soft topological space. The *soft closure* $\mathrm{Cl}(F, A)$ of $(F, A) \in \mathrm{SS}(X, A)$ is the soft set

$$\sqcap\{(Q, A) \in \tau^c : (F, A) \sqsubseteq (Q, A)\}.$$

Definition 3.8. Let (X, τ, A) be a soft topological space and $a \in A$. A point $x \in X$ is said to be an *a-cluster point* of $(F, A) \in \mathrm{SS}(X, A)$ if for every a-soft open neighborhood (G, A) of x we have $(F, A) \sqcap (G, A) \neq \mathbf{0}_A$. The set of all a-cluster points of (F, A) is denoted by $\mathrm{cl}(F, a)$. Also, the set of all a-cluster points of $(F, A)^c$ is denoted by $\mathrm{cl}((F, a)^c)$.

Proposition 3.9. *Let (X, τ, A) be a soft space and $(F, A) \in \mathrm{SS}(X, A)$. Then, $\mathrm{Cl}(F, A) = (R_{F,A}, A)$, where the map $R_{F,A} : A \to \mathcal{P}(X)$ defined as follows: $R_{F,A}(p) = F(p) \cup \mathrm{cl}(F, p)$, for every $p \in A$.*

Proof. We need to prove that (a) $(F, A) \sqsubseteq (R_{F,A}, A)$, (b) $(R_{F,A}, A) \in \tau^c$, and (c) $(R_{F,A}, A) \sqsubseteq (Q, A)$, for every $(Q, A) \in \tau^c$ such that $(F, A) \sqsubseteq (Q, A)$.

(a) First we observe that $F(p) \subseteq F(p) \cup \mathrm{cl}(F, p) = R_{F,A}(p)$, for every $p \in A$. Thus, $(F, A) \sqsubseteq (R_{F,A}, A)$.

(b) We prove that $(R_{F,A}, A) \in \tau^c$ or equivalently $(R_{F,A}, A)^c \in \tau$.

Let $a \in A$ and $x \in X \setminus R_{F,A}(a) = X \setminus (F(a) \cup \mathrm{cl}(F, a))$. By Proposition 3.6, it suffices to prove that there exists an a-soft open neighborhood $(G_{(a,x)}, A)$ of x such that $(G_{(a,x)}, A) \sqsubseteq (R_{F,A}, A)^c$ or equivalently

$$G_{(a,x)}(p) \subseteq X \setminus R_{F,A}(p) = X \setminus (F(p) \cup \mathrm{cl}(F, p)), \text{ for every } p \in A.$$

Since $x \notin \mathrm{cl}(F, a)$, there exists an a-soft open neighborhood $(G_{(a,x)}, A)$ of x such that $(F, A) \sqcap (G_{(a,x)}, A) = \mathbf{0}_A$. This means that $F(p) \cap G_{(a,x)}(p) = \varnothing$, for every $p \in A$. Therefore, $G_{(a,x)}(p) \subseteq X \setminus F(p)$, for every $p \in A$.

We prove that $G_{(a,x)}(p) \subseteq X \setminus \mathrm{cl}(F, p)$, for every $p \in A$. Indeed, let $y \in G_{(a,x)}(p)$, where $p \in A$. Then, the soft set $(G_{(a,x)}, A)$ is a p-soft open neighborhood of y such that $(F, A) \sqcap (G_{(a,x)}, A) = \mathbf{0}_A$ and, therefore, $y \in X \setminus \mathrm{cl}(F, p)$.

Thus,

$$G_{(a,x)}(p) \subseteq (X \setminus F(p)) \cap (X \setminus \mathrm{cl}(F, p)) = X \setminus (F(p) \cup \mathrm{cl}(F, p)),$$

for every $p \in A$.

(c) Finally, let $(Q, A) \in \tau^c$ such that

$$(F, A) \sqsubseteq (Q, A). \tag{2}$$

We prove that $(R_{F,A}, A) \sqsubseteq (Q, A)$. Since $R_{F,A}(p) = F(p) \cup \mathrm{cl}(F, p)$ and $F(p) \subseteq Q(p)$, for every $p \in A$, it suffices to prove that $\mathrm{cl}(F, p) \subseteq Q(p)$ or $X \setminus Q(p) \subseteq X \setminus \mathrm{cl}(F, p)$, for every $p \in A$. Indeed, let $y \in X \setminus Q(p)$ and $y \in \mathrm{cl}(F, p)$, where $p \in A$. We observe that the soft set $(Q, A)^c$ is a p-soft open neighborhood of y such that $(F, A) \sqcap (Q, A)^c \neq \mathbf{0}_A$ which contradicts relation (2). Thus, $y \in X \setminus \mathrm{cl}(F, p)$. $\square$

Definition 3.10 (see [24]). Let (X, τ, A) be a soft topological space. The *soft interior* $\mathrm{Int}(F, A)$ of $(F, A) \in \mathrm{SS}(X, A)$ is the soft set

$$\sqcup\{(G, A) \in \tau : (G, A) \sqsubseteq (F, A)\}.$$

Definition 3.11. Let (X, τ, A) be a soft topological space and $a \in A$. A point $x \in X$ is said to be an *a-interior point* of $(F, A) \in \mathrm{SS}(X, A)$ if there exists an a-soft open neighborhood (G, A) of x such that $(G, A) \sqsubseteq (F, A)$. The set of all a-interior points of (F, A) is denoted by $\mathrm{int}(F, a)$.

Proposition 3.12. *Let (X, τ, A) be a soft space and $(F, A) \in \mathrm{SS}(X, A)$. Then, $\mathrm{Int}(F, A) = (R_{F,A}, A)$, where the map $R_{F,A} : A \to \mathcal{P}(X)$ defined as follows: $R_{F,A}(p) = F(p) \cap \mathrm{int}(F, p)$, for every $p \in A$.*

Proof. We need to prove that (a) $(R_{F,A}, A) \sqsubseteq (F, A)$, (b) $(R_{F,A}, A) \in \tau$, and (c) $(G, A) \sqsubseteq (R_{F,A}, A)$, for every $(G, A) \in \tau$ such that $(G, A) \sqsubseteq (F, A)$.

(a) First we observe that $R_{F,A}(p) = F(p) \cap \mathrm{int}(F, p) \subseteq F(p)$, for every $p \in A$. Thus, $(R_{F,A}, A) \sqsubseteq (F, A)$.

(b) We prove that $(R_{F,A}, A) \in \tau$.

Let $a \in A$ and $x \in R_{F,A}(a) = F(a) \cap \mathrm{int}(F, a)$. By Proposition 3.6, it suffices to prove that there exists an a-soft open neighborhood $(G_{(a,x)}, A)$ of x such that $(G_{(a,x)}, A) \sqsubseteq (R_{F,A}, A)$ or equivalently

$$G_{(a,x)}(p) \subseteq F(p) \cap \mathrm{int}(F, p), \text{ for every } p \in A.$$

Since $x \in \mathrm{int}(F, a)$, there exists an a-soft open neighborhood $(G_{(a,x)}, A)$ of x such that $(G_{(a,x)}, A) \sqsubseteq (F, A)$. Therefore, $G_{(a,x)}(p) \subseteq F(p)$, for every $p \in A$.

We prove that $G_{(a,x)}(p) \subseteq \mathrm{int}(F, p)$, for every $p \in A$. Indeed, let $y \in G_{(a,x)}(p)$, where $p \in A$. Then, the soft set $(G_{(a,x)}, A)$ is a p-soft open neighborhood of y such that $(G_{(a,x)}, A) \sqsubseteq (F, A)$ and, therefore, $y \in \mathrm{int}(F, p)$.

Thus, $G_{(a,x)}(p) \subseteq F(p) \cap \mathrm{int}(F, p)$, for every $p \in A$.

(c) Finally, let $(G, A) \in \tau$ such that

$$(G, A) \sqsubseteq (F, A). \tag{3}$$

We must prove that $(G, A) \sqsubseteq (R_{F,A}, A)$. It suffices to prove that

$$G(p) \subseteq F(p) \cap \mathrm{int}(F, p), \text{ for every } p \in A.$$

Indeed, let $y \in G(p)$, where $p \in A$. Then, (G, A) is a p-soft open neighborhood of y such that $(G, A) \sqsubseteq (F, A)$. Therefore, $y \in \mathrm{int}(F, p)$. Also, by relation (3) we have $G(p) \subseteq F(p)$. Hence, $y \in F(p)$. Thus, $y \in F(p) \cap \mathrm{int}(F, p)$. $\square$

Proposition 3.13. *Let (X, τ, A) be a soft space, $a \in A$, and $(F, A) \in \mathrm{SS}(X, A)$. Then, $\mathrm{cl}((F, a)^c) = X \setminus \mathrm{int}(F, a)$.*

Proof. We prove that $\mathrm{cl}((F, a)^c) \subseteq X \setminus \mathrm{int}(F, a)$. Let $x \in \mathrm{cl}((F, a)^c)$. Then, for every a-soft open neighborhood (G, A) of x we have $(F, A)^c \sqcap (G, A) \neq \mathbf{0}_A$. We suppose that $x \in \mathrm{int}(F, a)$. Then, there exists an a-soft open neighborhood

(G, A) of x such that $(G, A) \sqsubseteq (F, A)$. Therefore, $(F, A)^c \sqcap (G, A) = \mathbf{0}_A$, which is a contradiction. Thus, $x \in X \setminus \operatorname{int}(F, a)$.

Now, we prove that $X \setminus \operatorname{int}(F, a) \subseteq \operatorname{cl}((F, a)^c)$. Let $x \in X \setminus \operatorname{int}(F, a)$ and (G, A) be an a-soft open neighborhood of x. We must prove that $(F, A)^c \sqcap (G, A) \neq \mathbf{0}_A$. Since $x \notin \operatorname{int}(F, a)$, there exists $p \in A$ such that $G(p) \nsubseteq F(p)$. This means that there exists $x \in X$ such that $x \in G(p)$ and $x \in X \setminus F(p)$. Hence, $(X \setminus F(p)) \cap G(p) \neq \varnothing$ and, therefore, $(F, A)^c \sqcap (G, A) \neq \mathbf{0}_A$. Thus, $x \in \operatorname{cl}((F, a)^c)$. □

Definition 3.14. Let (X, τ, A) be a soft topological space. The *soft boundary* $\operatorname{Bd}(F, A)$ of $(F, A) \in \operatorname{SS}(X, A)$ is the soft set

$$\operatorname{Cl}(F, A) \sqcap \operatorname{Cl}((F, A)^c).$$

Proposition 3.15. *Let (X, τ, A) be a soft space and $(F, A) \in \operatorname{SS}(X, A)$. Then, $\operatorname{Bd}(F, A) = (R_{F,A}, A)$, where the map $R_{F,A} : A \to \mathcal{P}(X)$ defined as follows: $R_{F,A}(p) = (F(p) \cup \operatorname{cl}(F, p)) \cap ((X \setminus F(p)) \cup (X \setminus \operatorname{int}(F, p)))$, for every $p \in A$.*

Proof. By Propositions 3.9, 3.12, and 3.13 for every $p \in A$ we have

$$\begin{aligned} R_{F,A}(p) &= (F(p) \cup \operatorname{cl}(F, p)) \cap ((X \setminus F(p)) \cup \operatorname{cl}((F, p)^c))) \\ &= (F(p) \cup \operatorname{cl}(F, p)) \cap ((X \setminus F(p)) \cup (X \setminus \operatorname{int}(F, p))). \end{aligned}$$

□

Definition 3.16. Let (X, τ, A) be a soft topological space. A family $\mathcal{B} \subseteq \tau$ is called a *base* for (X, τ, A) if for every soft open set $(G, A) \neq \mathbf{0}_A$, there exist $(G_i, A) \in \mathcal{B}$, $i \in I$, such that $(G, A) = \sqcup\{(G_i, A) : i \in I\}$.

Proposition 3.17. *Let (X, τ, A) be a soft topological space. Then, a family $\mathcal{B} \subseteq \tau$ is a base for (X, τ, A) if and only if for every $a \in A$, $x \in X$, and every a-soft open neighborhood (G, A) of x there exists an a-soft open neighborhood $(G_{(a,x)}, A)$ of x such that $(G_{(a,x)}, A) \in \mathcal{B}$ and $(G_{(a,x)}, A) \sqsubseteq (G, A)$.*

Proof. Let $\mathcal{B}$ be a base for (X, τ, A), $a \in A$, $x \in X$, and (G, A) be an a-soft open neighborhood of x. Then, $x \in G(a)$. Since $\mathcal{B}$ is a base for (X, τ, A), there exist $(G_i, A) \in \mathcal{B}$, $i \in I$, such that $(G, A) = \sqcup\{(G_i, A) : i \in I\}$. Hence, $G(a) = \cup\{G_i(a) : i \in I\}$ and, therefore, $x \in G_{i_0}(a)$ for some $i_0 \in I$. Thus, (G_{i_0}, A) is an a-soft open neighborhood of x such that $(G_{i_0}, A) \in \mathcal{B}$ and $(G_{i_0}, A) \sqsubseteq (G, A)$.

Conversely, let $\mathcal{B} \subseteq \tau$. Suppose that for every $a \in A$, $x \in X$, and every a-soft open neighborhood (G, A) of x there exists an a-soft open neighborhood $(G_{(a,x)}, A)$ of x such that $(G_{(a,x)}, A) \in \mathcal{B}$ and $(G_{(a,x)}, A) \sqsubseteq (G, A)$. We prove that $\mathcal{B}$ is a base for (X, τ, A).

Indeed, let $(G, A) \neq \mathbf{0}_A$ be a soft open set. We consider the set

$$I = \{(a, x) : a \in A,\ x \in G(a)\}.$$

Then, as in the proof of Proposition 3.6 we have

$$(G, A) = \sqcup\{(G_{(a,x)}, A) : (a, x) \in I\}.$$

Since $(G_{(a,x)}, A) \in \mathcal{B}$, for every $(a, x) \in I$, the set $\mathcal{B}$ is a base for (X, τ, A). □

Definition 3.18. Let (X, τ_X, A) and (Y, τ_Y, B) be two soft topological spaces, $x \in X$, and $e : A \to B$. A map $f : X \to Y$ is called *soft e-continuous at the point x* if for every $a \in A$ and every $e(a)$-soft open neighborhood (G, B) of $f(x)$ in (Y, τ_Y, B) there exists an a-soft open neighborhood (F, A) of x in (X, τ_X, A) such that

$$\Phi_{fe}(F, A) \sqsubseteq (G, B).$$

If the map f is soft e-continuous at any point $x \in X$, then we say that the map f is *soft e-continuous*.

Proposition 3.19. *Let (X, τ_X, A) and (Y, τ_Y, B) be two soft topological spaces and $e : A \to B$. Then, the following statements are equivalent:*

(1) *The map $f : X \to Y$ is soft e-continuous.*
(2) $\Phi^{-1}_{fe}(G, B) \in \tau_X$*, for every* $(G, B) \in \tau_Y$*.*
(3) $\Phi^{-1}_{fe}(Q, B) \in \tau^c_X$*, for every* $(Q, B) \in \tau^c_Y$*.*

Proof. (1) $\Rightarrow$ (2) Let $(G, B) \in \tau_Y$. Then, $\Phi^{-1}_{fe}(G, B)$ is the soft set $(F, A) \in \mathrm{SS}(X, A)$ such that $F(p_X) = f^{-1}(G(e(p_X)))$, for every $p_X \in A$.

Let $a \in A$ and $x \in F(a)$. By Proposition 3.6, it suffices to prove that there exists an a-soft open neighborhood $(F_{(a,x)}, A)$ of x in (X, τ_X, A) such that $(F_{(a,x)}, A) \sqsubseteq (F, A)$.

Since $x \in F(a) = f^{-1}(G(e(a)))$, we have $f(x) \in G(e(a))$. This means that the soft set (G, B) is an $e(a)$-soft open neighborhood of $f(x)$ in (Y, τ_Y, B). Since the map $f : X \to Y$ is soft e-continuous, there exists an a-soft open neighborhood $(F_{(a,x)}, A)$ of x in (X, τ_X, A) such that

$$\Phi_{fe}(F_{(a,x)}, A) \sqsubseteq (G, B).$$

Therefore, by Proposition 2.18,

$$(F_{(a,x)}, A) \sqsubseteq \Phi^{-1}_{fe}(\Phi_{fe}(F_{(a,x)}, A)) \sqsubseteq \Phi^{-1}_{fe}(G, B).$$

Thus, $\Phi^{-1}_{fe}(G, B) \in \tau_X$.

(2) $\Rightarrow$ (1) Let $x \in X$, $a \in A$, and (G, B) be an $e(a)$-soft open neighborhood of $f(x)$ in (Y, τ_Y, B). Then, $f(x) \in G(e(a))$ or $x \in f^{-1}(G(e(a)))$. Therefore, by assumption, $\Phi^{-1}_{fe}(G, B)$ is an a-soft open neighborhood of x in (X, τ_X, A). Therefore, by Proposition 2.18, $\Phi_{fe}(\Phi^{-1}_{fe}(G, B)) \sqsubseteq (G, B)$. Thus, the map $f : X \to Y$ is soft e-continuous at the point x.

(2) $\Rightarrow$ (3) Let $(Q, B) \in \tau^c_Y$. Then, $(Q, B)^c \in \tau_Y$. By Proposition 2.18 we have $\Phi^{-1}_{fe}((Q, B)^c) = (\Phi^{-1}_{fe}(Q, B))^c$. Since $\Phi^{-1}_{fe}((Q, B)^c) \in \tau_X$, we have $\Phi^{-1}_{fe}(Q, B) \in \tau^c_X$.

(3) $\Rightarrow$ (2) Let $(G, B) \in \tau_Y$. Then, $(G, B)^c \in \tau^c_Y$. By Proposition 2.18 we have $\Phi^{-1}_{fe}((G, B)^c) = (\Phi^{-1}_{fe}(G, B))^c$. Since $\Phi^{-1}_{fe}((G, B)^c) \in \tau^c_X$, we have $\Phi^{-1}_{fe}(G, B) \in \tau_X$. $\square$

Example 3.20. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$, $A = \{0, 1\}$, and $B = \{0, 1, 2\}$. We consider the following soft sets (F, A), (G, A), and (H, A) over X defined as follows:

$$F(p) = \begin{cases} \{x_3\}, \text{ if } p = 0, \\ \{x_1, x_2\}, \text{ if } p = 1, \end{cases} \quad G(p) = \begin{cases} \varnothing, \text{ if } p = 0, \\ \{x_3\}, \text{ if } p = 1, \end{cases}$$

$$H(p) = \begin{cases} \{x_3\}, \text{ if } p = 0, \\ X, \text{ if } p = 1. \end{cases}$$

Also, we consider the following soft sets (Q, B) and (R, B) over Y defined as follows:

$$Q(p) = \begin{cases} \{y_1\}, \text{ if } p = 0, \\ \{y_3\}, \text{ if } p = 1, \\ \varnothing, \text{ if } p = 2, \end{cases} \quad R(p) = \begin{cases} \{y_1, y_2\}, \text{ if } p = 0, \\ \{y_3\}, \text{ if } p = 1, \\ Y, \text{ if } p = 2. \end{cases}$$

Then, the triplets (X, τ_X, A) and (Y, τ_X, B), where

$$\tau_X = \{\mathbf{0}_A, \mathbf{1}_A, (F, A), (G, A), (H, A)\}$$

and

$$\tau_Y = \{\mathbf{0}_B, \mathbf{1}_B, (Q, B), (R, B)\}$$

are soft topological spaces.

Let $f : X \to Y$ be the map such that $f(x_1) = f(x_2) = y_1$ and $f(x_3) = y_3$ and $e : A \to B$ the map such that $e(0) = 1$ and $e(1) = 0$. Then, the map f is soft e-continuous. Also, if $e' : A \to B$ is the map such that $e'(0) = 1$ and $e'(1) = 2$, then the map f is not soft e'-continuous.

Proposition 3.21. *Let (X, τ_X, A) and (Y, τ_Y, B) be two soft topological spaces, $\mathcal{B}_Y$ a base for (Y, τ_Y, B), and $e : A \to B$. Then, the following statements are equivalent:*

(1) *The map $f : X \to Y$ is soft e-continuous.*
(2) *$\Phi_{fe}^{-1}(G, B) \in \tau_X$, for every $(G, B) \in \mathcal{B}_Y$.*

Proof. (1) $\Rightarrow$ (2) Follows by Proposition 3.19.

(2) $\Rightarrow$ (1) By Proposition 3.19 it suffices to prove that $\Phi_{fe}^{-1}(G, B) \in \tau_X$, for every $(G, B) \in \tau_Y$.

Let $(G, B) \in \tau_Y$. Then, there exist $(G_i, B) \in \mathcal{B}_Y$, $i \in I$, such that

$$(G, B) = \sqcup\{(G_i, B) : i \in I\}.$$

Therefore, by Proposition 2.19 we have

$$\Phi_{fe}^{-1}(G, B) = \Phi_{fe}^{-1}(\sqcup\{(G_i, B) : i \in I\}) = \sqcup\{\Phi_{fe}^{-1}(G_i, B) : i \in I\} \in \tau_X.$$

$\square$

Remark 3.22 (see, for example, [24]). Let (X, τ, A) be a soft topological space and $(F, A) \in \mathrm{SS}(X, A)$. We recall the following properties :

(1) $(F, A) \in \tau^c$ if and only if $\mathrm{Cl}(F, A) = (F, A)$.
(2) $(F, A) \in \tau$ if and only if $\mathrm{Int}(F, A) = (F, A)$.

(3) $\mathrm{Int}(F,A) = (\mathrm{Cl}((F,A)^c))^c$.
(4) $\mathrm{Cl}(F,A) = (\mathrm{Int}((F,A)^c))^c$.
(5) If $(F,A) \sqsubseteq (G,A)$, then $\mathrm{Cl}(F,A) \sqsubseteq \mathrm{Cl}(G,A)$.

Proposition 3.23. *Let (X, τ_X, A) and (Y, τ_Y, B) be two soft topological spaces and $e : A \to B$. Then, the following statements are equivalent:*

(1) *The map $f : X \to Y$ is soft e-continuous.*
(2) $\Phi_{fe}(\mathrm{Cl}(F,A)) \sqsubseteq \mathrm{Cl}(\Phi_{fe}(F,A))$*, for every* $(F,A) \in \mathrm{SS}(X,A)$.
(3) $\mathrm{Cl}(\Phi_{fe}^{-1}(G,B)) \sqsubseteq \Phi_{fe}^{-1}(\mathrm{Cl}(G,B))$*, for every* $(G,B) \in \mathrm{SS}(Y,B)$.
(4) $\Phi_{fe}^{-1}(\mathrm{Int}(G,B)) \sqsubseteq \mathrm{Int}(\Phi_{fe}^{-1}(G,B))$*, for every* $(G,B) \in \mathrm{SS}(Y,B)$.

Proof. (1) $\Rightarrow$ (2) Let $(F,A) \in \mathrm{SS}(X,A)$. Since $\Phi_{fe}(F,A) \sqsubseteq \mathrm{Cl}(\Phi_{fe}(F,A))$, by Proposition 2.18 we have

$$(F,A) \sqsubseteq \Phi_{fe}^{-1}(\Phi_{fe}(F,A)) \sqsubseteq \Phi_{fe}^{-1}(\mathrm{Cl}(\Phi_{fe}(F,A))).$$

Therefore,

$$\mathrm{Cl}(F,A) \sqsubseteq \mathrm{Cl}(\Phi_{fe}^{-1}(\mathrm{Cl}(\Phi_{fe}(F,A)))).$$

Since $\mathrm{Cl}(\Phi_{fe}(F,A)) \in \tau_Y^c$, by Proposition 3.19, $\Phi_{fe}^{-1}(\mathrm{Cl}(\Phi_{fe}(F,A))) \in \tau_X^c$ and, therefore,

$$\mathrm{Cl}(\Phi_{fe}^{-1}(\mathrm{Cl}(\Phi_{fe}(F,A)))) = \Phi_{fe}^{-1}(\mathrm{Cl}(\Phi_{fe}(F,A))).$$

Hence,

$$\mathrm{Cl}(F,A) \sqsubseteq \Phi_{fe}^{-1}(\mathrm{Cl}(\Phi_{fe}(F,A))).$$

Finally, by Proposition 2.18 we have

$$\Phi_{fe}(\mathrm{Cl}(F,A)) \sqsubseteq \Phi_{fe}(\Phi_{fe}^{-1}(\mathrm{Cl}(\Phi_{fe}(F,A)))) \sqsubseteq \mathrm{Cl}(\Phi_{fe}(F,A)).$$

(2) $\Rightarrow$ (3) Let $(G,B) \in \mathrm{SS}(Y,B)$. We apply (2) to $(F,A) = \Phi_{fe}^{-1}(G,B)$ and we obtain the inclusion

$$\Phi_{fe}(\mathrm{Cl}(\Phi_{fe}^{-1}(G,B))) \sqsubseteq \mathrm{Cl}(\Phi_{fe}(\Phi_{fe}^{-1}(G,B))) \sqsubseteq \mathrm{Cl}(G,B).$$

Therefore,

$$\mathrm{Cl}(\Phi_{fe}^{-1}(G,B)) \sqsubseteq \Phi_{fe}^{-1}(\Phi_{fe}(\mathrm{Cl}(\Phi_{fe}^{-1}(G,B)))) \sqsubseteq \Phi_{fe}^{-1}(\mathrm{Cl}(G,B)).$$

(3) $\Rightarrow$ (4) Let $(G,B) \in \mathrm{SS}(Y,B)$. We apply (3) to $(G,B)^c$ and we obtain the inclusion

$$\mathrm{Cl}(\Phi_{fe}^{-1}((G,B)^c)) \sqsubseteq \Phi_{fe}^{-1}(\mathrm{Cl}((G,B)^c)),$$

which gives (see Proposition 2.18)

$$\begin{aligned}
\Phi_{fe}^{-1}(\mathrm{Int}(G,B)) &= \Phi_{fe}^{-1}((\mathrm{Cl}((G,B)^c))^c) = (\Phi_{fe}^{-1}(\mathrm{Cl}((G,B)^c)))^c \\
&\sqsubseteq (\mathrm{Cl}(\Phi_{fe}^{-1}((G,B)^c)))^c = (\mathrm{Cl}((\Phi_{fe}^{-1}(G,B))^c))^c \\
&= \mathrm{Int}(\Phi_{fe}^{-1}(G,B)).
\end{aligned}$$

(4) $\Rightarrow$ (1) By Proposition 3.19 it suffices to prove that $\Phi_{fe}^{-1}(G,B) \in \tau_X$, for every $(G,B) \in \tau_Y$. Let $(G,B) \in \tau_Y$. Then, $\mathrm{Int}(G,B) = (G,B)$. Therefore,

$$\Phi_{fe}^{-1}(G,B) = \Phi_{fe}^{-1}(\mathrm{Int}(G,B)) \sqsubseteq \mathrm{Int}(\Phi_{fe}^{-1}(G,B)).$$

Also,

$$\mathrm{Int}(\Phi_{fe}^{-1}(G,B)) \sqsubseteq \Phi_{fe}^{-1}(G,B).$$

Thus, $\mathrm{Int}(\Phi_{fe}^{-1}(G,B)) = \Phi_{fe}^{-1}(G,B)$, which means that $\Phi_{fe}^{-1}(G,B) \in \tau_X$. □

Definition 3.24. Let (X,τ_X,A) and (Y,τ_Y,B) be two soft topological spaces and $e: A \to B$. A map $f: X \to Y$ is called *soft e-open* (respectively, *soft e-closed*) if for every $(F,A) \in \tau_X$ (respectively, $(F,A) \in \tau_X^c$) we have $\Phi_{fe}(F,A) \in \tau_Y$ (respectively, $\Phi_{fe}(F,A) \in \tau_Y^c$).

Proposition 3.25. *Let (X,τ_X,A) and (Y,τ_Y,B) be two soft topological spaces and $e: A \to B$. Then, the following statements are equivalent:*

(1) *The map $f: X \to Y$ is soft e-open.*
(2) $\Phi_{fe}(\mathrm{Int}(F,A)) \sqsubseteq \mathrm{Int}(\Phi_{fe}(F,A))$*, for every* $(F,A) \in \mathrm{SS}(X,A)$.

Proof. (1) ⇒ (2) Let $(F,A) \in \mathrm{SS}(X,A)$. Since, $\mathrm{Int}(F,A) \sqsubseteq (F,A)$, we have

$$\Phi_{fe}(\mathrm{Int}(F,A)) \sqsubseteq \Phi_{fe}(F,A).$$

Since $\mathrm{Int}(F,A) \in \tau_X$, we have $\Phi_{fe}(\mathrm{Int}(F,A)) \in \tau_Y$. Therefore, by the above inclusion we have

$$\Phi_{fe}(\mathrm{Int}(F,A)) \sqsubseteq \sqcup\{(G,A) \in \tau_Y : (G,A) \sqsubseteq \Phi_{fe}(F,A)\} = \mathrm{Int}(\Phi_{fe}(F,A)).$$

(2) ⇒ (1) We prove that $\Phi_{fe}(F,A) \in \tau_Y$, for every $(F,A) \in \tau_X$. Let $(F,A) \in \tau_X$. Then,

$$\Phi_{fe}(F,A) = \Phi_{fe}(\mathrm{Int}(F,A)) \sqsubseteq \mathrm{Int}(\Phi_{fe}(F,A)).$$

Also,

$$\mathrm{Int}(\Phi_{fe}(F,A)) \sqsubseteq \Phi_{fe}(F,A).$$

Thus, $\mathrm{Int}(\Phi_{fe}(F,A)) = \Phi_{fe}(F,A)$, which means that $\Phi_{fe}(F,A) \in \tau_Y$. □

The proof of the following proposition is similar to the proof of Proposition 3.25.

Proposition 3.26. *Let (X,τ_X,A) and (Y,τ_Y,B) be two soft topological spaces and $e: A \to B$. Then, the following statements are equivalent:*

(1) *The map $f: X \to Y$ is soft e-closed.*
(2) $\mathrm{Cl}(\Phi_{fe}(F,A)) \sqsubseteq \Phi_{fe}(\mathrm{Cl}(F,A))$*, for every* $(F,A) \in \mathrm{SS}(X,A)$.

Definition 3.27. Let (X,τ_X,A) and (Y,τ_Y,B) be two soft topological spaces and e a 1-1 map of A onto B. A soft e-continuous map f of X onto Y is called *soft e-homeomorphism* if the map f is 1-1 and the inverse map $f^{-1}: Y \to X$ is soft e^{-1}-continuous.

Proposition 3.28. *Let $(F,A) \in \mathrm{SS}(X,A)$, e a 1-1 map of A onto B, and f a 1-1 map of X onto Y. Then,*

(1) $\Phi_{fe}(F,A) = \Phi_{f^{-1}e^{-1}}^{-1}(F,A)$.
(2) $\Phi_{fe}((F,A)^c) = (\Phi_{fe}(F,A))^c$.

Proof. (1) Let $\Phi_{fe}(F,A)=(G,B)$, $\Phi^{-1}_{f^{-1}e^{-1}}(F,A)=(G',B)$, and $p_Y\in B$. We must prove that $G(p_Y)=G'(p_Y)$. Let $e^{-1}(p_Y)=p_X$. Since the map $e:A\to B$ is 1-1, $G(p_Y)=f(F(p_X))$. On the other hand,

$$G'(p_Y)=(f^{-1})^{-1}(F(e^{-1}(p_Y)))=f(F(p_X)).$$

Thus, $G(p_Y)=G'(p_Y)$.

(2) Let $\Phi_{fe}((F,A)^c)=(G,B)$, $(\Phi_{fe}(F,A))^c=(G',B)$, and $p_Y\in B$. We must prove that $G(p_Y)=G'(p_Y)$. Let $e^{-1}(p_Y)=p_X$. Since the map $e:A\to B$ is 1-1, $G(p_Y)=f(X\setminus F(p_X))$. Since the map $f:X\to Y$ is 1-1 and onto, $f(X\setminus F(p_X))=Y\setminus f(F(p_X))$. Therefore, $G(p_Y)=Y\setminus f(F(p_X))$. On the other hand, $G'(p_Y)=Y\setminus f(F(p_X))$. Thus, $G(p_Y)=G'(p_Y)$. □

Proposition 3.29. *Let (X,τ_X,A) and (Y,τ_Y,B) be two soft topological spaces, e a map of A onto B, and f a 1-1 map of X onto Y. Then, the following statements are equivalent:*

(1) *The map f is soft e-homeomorphism.*
(2) *The map f is soft e-continuous and soft e-open.*
(3) *The map f is soft e-continuous and soft e-closed.*

Proof. (1) $\Rightarrow$ (2) We prove that $\Phi_{fe}(F,A)\in\tau_Y$, for every $(F,A)\in\tau_X$. Let $(F,A)\in\tau_X$. Since the map f^{-1} is soft e^{-1}-continuous and $(F,A)\in\tau_X$, we have $\Phi^{-1}_{f^{-1}e^{-1}}(F,A)\in\tau_Y$. By Proposition 3.28, $\Phi_{fe}(F,A)\in\tau_Y$.

(2) $\Rightarrow$ (3) We prove that $\Phi_{fe}(F,A)\in\tau^c_Y$, for every $(F,A)\in\tau^c_X$. Let $(F,A)\in\tau^c_X$. Then, $(F,A)^c\in\tau_X$ and, therefore, $\Phi_{fe}((F,A)^c)\in\tau_Y$. By Proposition 3.28, $(\Phi_{fe}(F,A))^c=\Phi_{fe}((F,A)^c)$, which means that $\Phi_{fe}(F,A)\in\tau^c_Y$.

(3) $\Rightarrow$ (1) We prove that the inverse map $f^{-1}:Y\to X$ is soft e^{-1}-continuous. It suffices to prove that $\Phi^{-1}_{f^{-1}e^{-1}}(F,A)\in\tau^c_Y$, for every $(F,A)\in\tau^c_X$. Let $(F,A)\in\tau^c_X$. Then, $\Phi_{fe}(F,A)\in\tau^c_Y$. By Proposition 3.28, $\Phi^{-1}_{f^{-1}e^{-1}}(F,A)=\Phi_{fe}(F,A)$. Thus, $\Phi^{-1}_{f^{-1}e^{-1}}(F,A)\in\tau^c_Y$. □

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