DISTRIBUTIONAL CHAOS FOR OPERATORS WITH FULL SCRAMBLED SETS

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Abstract. In this article we answer in the negative the question of whether hypercyclicity is sufficient for distributional chaos for a continuous linear operator (we even prove that the mixing property does not suffice). Moreover, we show that a extremal situation is possible: There are (hypercyclic and non-hypercyclic) operators such that the whole space consists, except zero, of distributionally irregular vectors.

1. Introduction

In recent years many researchers were looking for conditions that yield complex, nontrivial dynamics of linear operators (note that, to admit such behaviour, the space must be infinite dimensional). Probably the most studied is the notion of the hypercyclicity, that is, the existence of vectors $x \in X$ such that the orbit of this vector $x, T(x), T^2(x), \dots$ under action of a continuous and linear operator $T\colon X \to X$ on a topological vector space (most often Banach or Fréchet space) X forms a dense subset of X. We refer the reader to the recent books [5] and [10] for an accessible introduction into the topic together with the review on the history of this problem. After publication of the book of Devaney [7] studies on hypercyclicity became essential tool for understanding of chaos in the sense of Devaney, since definition of chaos from [7] in our context requires hypercyclicity and density of the set of periodic points of T in X (see [1]).

Another definition of chaos was born a few years before [7], when Li and Yorke published their famous paper on the study of complicated dynamics of maps on the unit interval [13]. In contrast to the definition of Devaney, the definition derived from [13] concentrates rather on local aspects of dynamics of pairs than a complex global behavior induced by hypercyclicity.

Distributional chaos was introduced by Schweizer and Smital in [18] as a natural extension of the notion of chaos given several years before by Li and Yorke in [13]. This concept will be the main notion considered in this paper. Following [14], we will consider only the definition of uniform distributional chaos, which is one of the strongest possibilities (see [16]).

Recall that, if $A \subset \mathbb{N}$, then its upper density is the number

$$\overline{\text{dens}}(A) = \limsup_{n\to\infty} \frac{1}{n} \left|\{i < n \; ; \; i \in A\}\right| ,$$

where $|S|$ denotes cardinality of the set S. Using this notation, distributional chaos can be defined as follows:

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Definition 1.1. Let f be a continuous self map on a metric space (X, d). If there exists an uncountable set $D \subset X$ and $\varepsilon > 0$ such that for every $t > 0$ and every distinct $x, y \in D$ the following conditions hold:

$$\overline{\text{dens}}\left\{i \in \mathbb{N}\ ;\ d(f^i(x), f^i(y)) \geq \varepsilon\right\} = 1,$$
$$\overline{\text{dens}}\left\{i \in \mathbb{N}\ ;\ d(f^i(x), f^i(y)) < t\right\} = 1,$$

then we say that f exhibits *uniform distributional chaos*. The set D is called a *distributionally ε-scrambled set.*

Let X be a Banach space and let $T \colon X \to X$ be an operator. While there is lack of full characterization of distributional chaos for operators, it is possible to provide effective criteria ensuring this property [4, 11, 14]. It was proved in [14] that if a weighted backward shift on ℓ^p-space is chaotic in the sense of Devaney then it also exhibits distributional chaos. There were also examples of distributionally chaotic operators which are not hypercyclic. So the only uncertain possibility left in [14] is the question whether hypercyclicity is sufficient for distributional chaos. The negative answer to this question is given in Section 2. Even more is shown, that is, mixing is not enough for distributional chaos. Recall that T is mixing if the set $\{n \in \mathbb{N}\ ;\ T^n(U) \cap V \neq \emptyset\}$ is co-finite for every pair of non-empty open sets U, V.

We recall from [3] that a vector $x \in X$ is said to be *irregular* for an operator T if $\liminf_{n\to\infty} \|T^n x\| = 0$, while $\limsup_{n\to\infty} \|T^n x\| = \infty$. Inspired by this definition, Bermúdez et al. introduced in [4] the following notion:

Definition 1.2. A vector $x \in X$ is said to be *distributionally irregular* for T if there are increasing sequences of integers $A = \{n_k\ ;\ k \in \mathbb{N}\}$ and $B = \{m_k\ ;\ k \in \mathbb{N}\}$ such that $\overline{\text{dens}}(A) = \overline{\text{dens}}(B) = 1$, $\lim_{k\to\infty} \|T^{n_k} x\| = 0$ and $\lim_{k\to\infty} \|T^{m_k} x\| = \infty$.

In [4] the authors proved that the existence of irregular vectors is equivalent to admitting Li-Yorke pairs. The same paper shows that every infinite dimensional separable Banach space admits a hypercyclic and distributionally chaotic operator which has a dense distributionally irregular scrambled set D, that is distributionally scrambled set such that each vector is distributionally irregular. Then the natural question which immediately arises is whether it is possible that $D = X$, i.e. D is not only dense but equals to the whole space.

Definition 1.3. We say that an operator T is *completely distributionally irregular* if every vector $x \in X \setminus \{0\}$ is distributionally irregular.

Remark 1.4. Note that if T is completely distributionally irregular then the whole space is distributionally ε-scrambled, for any $\varepsilon > 0$.

In Section 3 we will show that there exist completely distributionally irregular operators (even with completely distributionally irregular inverse) and that this property is independent of hypercyclicity or distributional irregularity of the inverse.

We refer the reader to [2, 12, 17, 19] for other recent works on distributional chaos for operators. Also, in [15] it is shown that there are completely distributionally irregular operators T such that the sequence $(\|T^n\|)_n$ is increasing.

2. Mixing and distributional chaos

In this section we will consider the unilateral backward shift $(Bx)_i = x_{i+1}$ on the weighted ℓ^p-space

$$\ell^p(v) := \{x = (x_j)_{j\in\mathbb{N}} \ ; \ \|x\|^p = \sum_{j\in\mathbb{N}} |x_j|^p \, v_j < \infty\}.$$

The map $B : \ell^p(v) \to \ell^p(v)$ is well-defined (and, equivalently, continuous) if and only if $\sup_j \frac{v_j}{v_{j+1}} < \infty$. We will show that there are sequences of weights so that unilateral shift is mixing but not distributionally chaotic. We should point out that it is easy to construct non-hypercyclic but distributionally chaotic unilateral shifts (see [14]).

Theorem 2.1. *Let $n_k := (k!)^3$, $k \in \mathbb{N}$, and let $v = (v_j)_j$ be the sequence of weights given by $v_j = k^{-1}$ for $n_k \leq j < n_{k+1}$, $k \in \mathbb{N}$. Then the operator B is mixing on $X := \ell^p(v)$, $1 \leq p < \infty$, but T is not distributionally chaotic.*

Proof. The fact that B is mixing can be deduced from [6] (see also Chapter 4 in [10] for more details), since $\lim_j v_j = 0$.

We will show that, for each $x \in X$ and for every $\varepsilon \in]0,1[$,

$$\lim_{n\to\infty} \frac{|\{j \leq n \ ; \ \|B^j x\| < \varepsilon\}|}{n} = 1,$$

that excludes the possibility of existence of distributionally chaotic pairs.

We fix an integer $k_0 > 6$ satisfying

$$\sum_{j\geq n_{k_0}} |x_j|^p v_j < \varepsilon/4 \quad \text{and} \quad k_0^{-1} < \varepsilon/4.$$

If $n \geq n_{k_0+1}$, let $k \geq k_0$ with $n_{k+1} \leq n < n_{k+2}$. We can write $n = Nn_k + m$ with $m, N \in \mathbb{N}$, $m \leq n_k$, $N > k^3$. Since

$$\sum_{i=1}^{N-1} \left(\sum_{j=in_k}^{(i+1)n_k-1} |x_j|^p v_j \right) \leq \sum_{j\geq n_k} |x_j|^p v_j < \varepsilon/4 < 1,$$

then, for

$$I := \{i < N \ ; \ \sum_{j=in_k}^{(i+1)n_k-1} |x_j|^p v_j \geq k^{-2}\},$$

we have $|I| \leq k^2$. Thus,

$$|\{1, \ldots, N-1\} \setminus I| \geq N - 1 - k^2.$$

If $i \in J := \{1, \ldots, N-1\} \setminus I$, then

$$\sum_{j=in_k}^{(i+1)n_k-1} |x_j|^p \leq (k+1) \sum_{j=in_k}^{(i+1)n_k-1} |x_j|^p v_j < (k+1)k^{-2} < \varepsilon/2,$$

by the definition of I, and since $v_j = (k+1)^{-1}$ for $n_{k+1} \leq j < n_{k+2}$. This implies that, if we fix $i \in J$ and $j \in [in_k, (i+1)n_k - n_{k-1}]$, then

$$\left\|B^{j-1}x\right\|^p = \sum_{l=j}^{(i+1)n_k-1} |x_l|^p v_{l-j+1} + \sum_{l\geq(i+1)n_k} |x_l|^p v_{l-j+1}$$

$$\leq \sum_{l=in_k}^{(i+1)n_k-1} |x_l|^p + \sum_{l\geq (i+1)n_k} |x_l|^p v_l \frac{v_{l-j+1}}{v_l} < \frac{\varepsilon}{2} + \frac{3\varepsilon}{8} < \varepsilon,$$

since $j < n_{k+2}$ and $l-j+1 \geq n_{k-1}$ whenever $l \geq (i+1)n_k$, and by taking into account that $v_s/v_r \leq (k+2)/(k-1) \leq 3/2$ if $r > s \geq n_{k-1}$ and $r - s < n_{k+2}$. Therefore,

$$\frac{|\{j \leq n \ ; \ \left\|B^j x\right\|^p < \varepsilon\}|}{n} \geq \frac{\sum_{i\in J}(n_k - n_{k-1})}{n} \geq \frac{(N-1-k^2)(n_k-n_{k-1})}{n}$$

$$\geq \left(\frac{N-1-k^2}{N+1}\right)\left(\frac{n_k-n_{k-1}}{n_k}\right) > \left(1-\frac{k^2+2}{k^3+1}\right)\left(1-\frac{1}{k^3}\right) \underset{k\to\infty}{\longrightarrow} 1.$$

□

Remark 2.2. Observe that we have shown in Theorem 2.1 that

$$\overline{\text{dens}}\left\{j : d(B^j x, B^j y) \geq \varepsilon\right\} = 0$$

for every $x, y \in X$, $x \neq y$, and for every $\varepsilon > 0$. In other words, the mixing rate of B is very slow.

3. Full distributionally scrambled sets and hypercyclicity

In this section we will consider the bilateral forward shift $(Tx)_i = x_{i-1}$ and the backward shift $(Bx)_i = x_{i+1}$ on the weighted ℓ^p-space

$$\ell^p(v,\mathbb{Z}) := \{x = (x_j)_{j\in\mathbb{Z}} \ ; \ \|x\|^p = \sum_{j\in\mathbb{Z}} |x_j|^p \, v_j < \infty\},$$

where the weight sequence $v = (v_j)_{j\in\mathbb{Z}}$ will be constructed satisfying certain general assumptions. We also recall that $B : \ell^p(v,\mathbb{Z}) \to \ell^p(v,\mathbb{Z})$ (respectively, $T : \ell^p(v,\mathbb{Z}) \to \ell^p(v,\mathbb{Z})$) is well-defined (equivalently, continuous) if and only if $\sup_{j\in\mathbb{Z}} \frac{v_j}{v_{j+1}} < \infty$ (respectively, $\sup_{j\in\mathbb{Z}} \frac{v_j}{v_{j-1}} < \infty$). Our purpose is to provide examples of weight sequences v such that every non-zero vector is distributionally irregular for $T\colon \ell^p(v,\mathbb{Z}) \to \ell^p(v,\mathbb{Z})$, and for some of these examples $T^{-1} = B$ will have the same property. We also study its relation to hypercyclicity. In what follows, we will simply write $a_j^i = (a_j)^i$. We hope it will cause no confusion to the reader.

Theorem 3.1. *Let $v = (v_j)_{j\in\mathbb{Z}}$ be a weight sequence that satisfies the following conditions:*

(1) there are sequences of integers $(n_j)_{j\in\mathbb{Z}}$ and $(m_j)_{j\in\mathbb{Z}}$ with $n_j < m_j < n_{j+1}$, $j \in \mathbb{Z}$, and $M > 1$ such that $Mv_{m_{-k}} \geq v_j$ for every $j \in [m_{-k}, m_{k-1}]$, $k \in \mathbb{N}$, and if we consider

$$S_k := \sup\{\frac{v_j}{v_{j-1}} \ ; \ j \notin]m_{-k}, m_{k-1}]\}, \quad k \in \mathbb{N},$$

then for every $\varepsilon > 0$ we find $k \in \mathbb{N}$ with $v_{n_k} < \varepsilon$ and

$$S_k^{k(n_k - m_{-k})} \leq \min\left\{M, \frac{\min\{v_i ; \ m_{-k} \leq i \leq m_{k-1}\}}{v_{n_k}}\right\},$$

(2) for every $N \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $v_j > N$, for $k \leq j \leq Nk$.

Then the forward shift $T\colon \ell^p(v,\mathbb{Z}) \to \ell^p(v,\mathbb{Z})$ is completely distributionally irregular.

Proof. Let $x \in \ell^p(v,\mathbb{Z})$ be an arbitrary non-zero vector. Given $M > 1$ satisfying condition *(1)*, and an arbitrary $\delta > 0$, we fix $m \in \mathbb{N}$ such that

$$\sum_{|j|>m} |x_j|^p v_j < \frac{\delta}{2(1+M^2)}.$$

Again by condition *(1)* there exists $k \in \mathbb{N}$ with $v_{n_k} < \varepsilon := \delta \frac{\min\{v_j;\ |j| \leq m\}}{2M(1+\|x\|^p)}$, and such that $[-m,m] \subset]m_{-k}, m_{k-1}[$. For any $l \in [n_k - m_{-k}, k(n_k - m_{-k})]$ we have

$$\left\|T^l x\right\|^p = \sum_{j\in[m_{-k},m_{k-1}]} |x_j|^p v_{j+l} + \sum_{j\notin[m_{-k},m_{k-1}]} |x_j|^p v_{j+l}. \tag{3.1}$$

For the first summand we obtain

$$\begin{aligned}
\sum_{j\in[m_{-k},m_{k-1}]} |x_j|^p v_{j+l} &\leq \sum_{|j|\leq m} |x_j|^p v_j \frac{v_{j+l}}{v_j} + \sum_{j\in[m_{-k},m_{k-1}]\setminus[-m,m]} |x_j|^p v_j \frac{v_{j+l}}{v_j} \\
&\leq \sum_{|j|\leq m} |x_j|^p v_j \frac{v_{n_k}\left(\prod_{i=n_k}^{j+l-1} \frac{v_{i+1}}{v_i}\right)}{v_j} + \sum_{j\in[m_{-k},m_{k-1}]\setminus[-m,m]} |x_j|^p v_j \frac{v_{j+l}}{v_j} \\
&\leq \|x\|^p \frac{v_{n_k} S_k^{m+l-n_k}}{\min\{v_j;\ |j|\leq m\}} + \frac{\delta}{2(1+M^2)} \max\{\frac{v_{j+l}}{v_j};\ j\in[m_{-k},m_{k-1}]\setminus[-m,m]\} \\
&< \|x\|^p \frac{\delta M}{2M(1+\|x\|^p)} + \frac{\delta}{2(1+M^2)} \frac{v_{n_k} S_k^{k(n_k-m_{-k})}}{\min\{v_j;\ m_{-k}\leq j\leq m_{k-1}\}} \leq \frac{\delta(2+M^2)}{2(1+M^2)}.
\end{aligned}$$

by the selection of m and ε. With respect to the second summand in (3.1), we get

$$\begin{aligned}
\sum_{j\notin[m_{-k},m_{k-1}]} |x_j|^p v_{j+l} &= \sum_{m_{-k}-l\leq j<m_{-k}} |x_j|^p v_j \frac{v_{j+l}}{v_j} + \sum_{j\notin[m_{-k}-l,m_{k-1}]} |x_j|^p v_j \frac{v_{j+l}}{v_j} \\
&\leq \left(\sum_{m_{-k}-l\leq j<m_{-k}} |x_j|^p v_j\right) M S_k^l + \left(\sum_{j\notin[m_{-k}-l,m_{k-1}]} |x_j|^p v_j\right) S_k^l < \frac{\delta M^2}{2(1+M^2)}.
\end{aligned}$$

Therefore we have $\left\|T^l x\right\|^p < \delta$, and we obtain the existence of $A = \{l_j;\ j\in\mathbb{N}\} \subset \mathbb{N}$ such that $\overline{\text{dens}}(A) = 1$ and $\lim_j \left\|T^{l_j} x\right\| = 0$.

On the other hand, since $x \neq 0$, there is $i_0 \in \mathbb{Z}$ such that $x_{i_0} \neq 0$. By condition *(2)*, given any $N \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that $v_j > N$ for all $j \in [k, Nk]$. Without loss of generality we may assume $N > \max_{j\leq|i_0|} v_j$, so that $k > i_0$, therefore $\|T^r x\| \geq |x_{i_0}| N^{1/p}$ for each $r \in [k - i_0, Nk - i_0]$, which yields that there is $B = \{r_j;\ j \in \mathbb{N}\} \subset \mathbb{N}$ such that $\overline{\text{dens}}(B) = 1$ and $\lim_j \|T^{r_j} x\| = \infty$. □

As a consequence of Theorem 3.1 we obtain the analogous result for the backward shift.

Corollary 3.2. *Let $v = (v_j)_{j\in\mathbb{Z}}$ be a weight sequence that satisfies the following conditions:*

(1) there are sequences of integers $(n_j)_{j\in\mathbb{Z}}$ and $(m_j)_{j\in\mathbb{Z}}$ with $n_j < m_j < n_{j+1}$, $j \in \mathbb{Z}$, and $M > 1$ such that $M v_{m_{k-1}} \geq v_j$ for every $j \in [m_{-k}, m_{k-1}]$, $k \in \mathbb{N}$, and if we consider

$$s_k := \inf\{\frac{v_j}{v_{j-1}}\ ;\ j \notin]m_{-k}, m_{k-1}]\},\ k \in \mathbb{N},$$

then for every $\varepsilon > 0$ we find $k \in \mathbb{N}$ with $v_{n_{-k}} < \varepsilon$ and

$$s_k^{k(n_{-k}-m_{k-1})} \leq \min\left\{M, \frac{\min\{v_i;\ m_{-k} \leq i \leq m_{k-1}\}}{v_{n_{-k}}}\right\},$$

(2) for every $N \in \mathbb{N}$, there exits $k \in \mathbb{N}$ such that $v_j > N$, for $-Nk \leq j \leq -k$.

Then the backward shift $B = T^{-1}\colon \ell^p(v, \mathbb{Z}) \to \ell^p(v, \mathbb{Z})$ is completely distributionally irregular.

Proof. It suffices to consider the isomorphism $(x_j)_{j\in\mathbb{Z}} \mapsto (x_{-j})_{j\in\mathbb{Z}}$ that conjugates B to the forward shift $T\colon \ell^p(v', \mathbb{Z}) \to \ell^p(v', \mathbb{Z})$, where the weight sequence v' is defined as $v'_j = v_{-j}$ and the necessary sequences to apply Theorem 3.1 are defined by $n'_j := -n_{-j}$, $m'_j := -m_{-j-1}$. □

For convenience of the constructions that we will provide in the final examples, we need a further result which is a consequence of Corollary 3.2 by simply applying a shift $m'_k = m_{k+1}$, $n'_k = n_{k+1}$, on the right part of the sequences $(m_k)_k$ and $(n_k)_k$ $(k \geq 0)$, and leaving invariant the left parts $m'_k = m_k$, $n'_k = n_k$, $k < 0$.

Corollary 3.3. *Let $v = (v_j)_{j\in\mathbb{Z}}$ be a weight sequence that satisfies the following conditions:*

(1) there are sequences of integers $(n_j)_{j\in\mathbb{Z}}$ and $(m_j)_{j\in\mathbb{Z}}$ with $n_j < m_j < n_{j+1}$, $j \in \mathbb{Z}$, and $M > 1$ such that $Mv_{m_k} \geq v_j$ for every $j \in [m_{-k}, m_k]$, $k \in \mathbb{N}$, and if we consider

$$s_k := \inf\{\frac{v_j}{v_{j-1}}\ ;\ j \notin]m_{-k}, m_k]\},\ k \in \mathbb{N},$$

then for every $\varepsilon > 0$ we find $k \in \mathbb{N}$ with $v_{n_{-k}} < \varepsilon$ and

$$s_k^{k(n_{-k}-m_k)} \leq \min\left\{M, \frac{\min\{v_i;\ m_{-k} \leq i \leq m_k\}}{v_{n_{-k}}}\right\},$$

(2) for every $N \in \mathbb{N}$, there exits $k \in \mathbb{N}$ such that $v_j > N$, for $-Nk \leq j \leq -k$.

Then the backward shift $B = T^{-1}\colon \ell^p(v, \mathbb{Z}) \to \ell^p(v, \mathbb{Z})$ is completely distributionally irregular.

With the above results, we are ready to provide the desired examples. In all of them the sequences of integers $(n_j)_{j\in\mathbb{Z}}$ and $(m_j)_{j\in\mathbb{Z}}$ with $n_j < m_j < n_{j+1}$, $j \in \mathbb{Z}$ are such that for every $k \in \mathbb{Z}$ we have:

$$v_{j-1} \leq v_j \text{ when } n_k < j \leq m_k, \text{ and } v_{j-1} \geq v_j \text{ when } m_k < j \leq n_{k+1}.$$

Generally speaking, the positions v_{m_k} represent "hills" of the weight sequence, and the positions v_{n_k} are "valleys".

Example 3.4. We will select v such that T and T^{-1} are completely distributionally irregular on $\ell^p(v, \mathbb{Z})$, but T is not hypercyclic. First, we put some general conditions which will lead to inductive construction of sequences of integers $(m_k)_{k\in\mathbb{Z}}$ and $(n_k)_{k\in\mathbb{Z}}$ with the desired properties. We will require that sequences $(m_k)_{k\in\mathbb{Z}}$, $(n_k)_{k\in\mathbb{Z}}$ increase fast enough so that they satisfy the following conditions:

(a) $m_0 = 1$, $n_1 = 4$, $v_{m_0} = 2$, $v_{n_1} = 2^{-2}$,
(b) $v_{n_k} = 2^{-2k}$, $v_{m_k} = 2^{2k+1}$, $k \in \mathbb{N}$, $v_i/v_{i-1} = v_j/v_{j-1}$ if $i, j \in]n_k, m_k]$, or if $i, j \in]m_{k-1}, n_k]$, $k \in \mathbb{N}$,
(c) $m_k - n_k > 2(m_{k-1} - n_{k-1})$, $n_{k+1} - m_k > 2(n_k - m_{k-1})$, $k \in \mathbb{N}$, and
(d) $v_{-j} = v_j^{-1}$, $j \in \mathbb{N}$, $m_k = -n_{-k}$, $k \in \mathbb{Z}$.

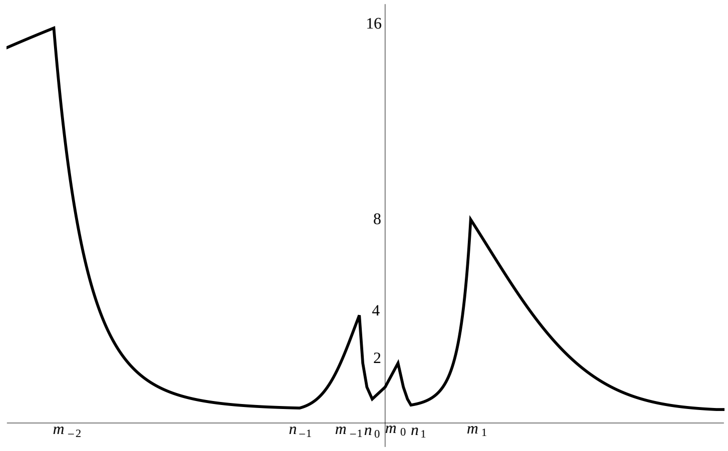

Figure 1. Example 3.4

Observe that conditions (b) and (d) give that $\dfrac{\min\{v_i;\ m_{-k} \le i \le m_{k-1}\}}{v_{n_k}} = \dfrac{v_{n_{-k+1}}}{v_{n_k}} = 2$ for every $k \in \mathbb{N}$, and the supremum of the slope of v outside the interval $[m_{-k}, m_{k-1}]$ is $S_k = v_j/v_{j-1}$ for any $n_k < j \le m_k$, $k \in \mathbb{N}$, by the all the assumptions. In order to fulfill condition *(1)* in Theorem 3.1, we set $M = 2$ and $S_k = 2^{1/k(n_k - m_{-k})} = 2^{1/(2kn_k)}$, $k \in \mathbb{N}$. Thus, we get $S_k^{m_k - n_k} = v_{m_k}/v_{n_k} = 2^{4k+1}$, which implies $m_k = (8k^2 + 2k + 1)n_k$, $k \in \mathbb{N}$.

Analogously, $\dfrac{\min\{v_i;\ m_{-k} \le i \le m_k\}}{v_{n_{-k}}} = \dfrac{v_{n_k}}{v_{n_{-k}}} = 2$ for every $k \in \mathbb{N}$, and the infimum of the slope of v outside the interval $[m_{-k}, m_k]$ is $s_k = v_j/v_{j-1}$ for any $m_k < j \le n_{k+1}$, $k \in \mathbb{N}$. Again, to have condition *(1)* in Corollary 3.3, we set $M = 2$ and $s_k = 2^{1/k(n_{-k} - m_k)} = 2^{-1/(2km_k)}$, $k \in \mathbb{N}$. As a consequence, $s_k^{n_{k+1} - m_k} = v_{n_{k+1}}/v_{m_k} = 2^{-4k-3}$, which yields $n_{k+1} = (8k^2 + 6k + 1)m_k$, $k \in \mathbb{N}$. This allows us the inductive construction of $(n_j)_{j\in\mathbb{Z}}$ and $(m_j)_{j\in\mathbb{Z}}$.

To check condition *(2)* in Theorem 3.1 and Corollary 3.3, we notice that

$$v_r = 2^{2k+1} s_k^{r - m_k} = 2^{2k+1} 2^{-\frac{r - m_k}{2km_k}} > 2^{2k} \quad \text{if} \quad m_k \le r \le 2km_k,$$

and that

$$v_r = 2^{2k} S_k^{m_{-k} - r} = 2^{2k} 2^{\frac{-n_k - r}{2kn_k}} > 2^{2k-1} \quad \text{if} \quad 2km_{-k} \le r \le m_{-k}.$$

This implies that all the conditions in Theorem 3.1 and Corollary 3.3 are satisfied and so T and T^{-1} are completely distributionally irregular.

Finally, since $v_j = v_{-j}^{-1}$ for all $j \in \mathbb{Z}$, there is no increasing sequence $(j_k)_k$ in $\mathbb{N}$ such that $\lim_k v_{j_k} = \lim_k v_{-j_k} = 0$, which avoids the hypercyclicity of T (See, e.g., Theorem 2 in [9]).

Example 3.5. Our purpose now is to construct v such that T and T^{-1} are completely distributionally irregular on $\ell^p(v, \mathbb{Z})$, and T is hypercyclic. To do so, we set up the following preliminary conditions on the weights

(a) $n_1 = 4$, $m_{-1} = -4$, $v_{n_0} = 1$, $v_{m_0} = 2$, $v_{n_1} = 2^{-1}$, $v_{m_{-1}} = 2$, and

(b) $m_k = -n_{-k}$, $k \in \mathbb{Z}$, $v_{n_k} = 2^{-2k+1}$, $v_{m_k} = 2^{k+1}$, $v_{n_{-k}} = 2^{-2k}$, $v_{m_{-k}} = 2^k$, $k \in \mathbb{N}$, $v_i/v_{i-1} = v_j/v_{j-1}$ if $i, j \in]n_k, m_k]$, or if $i, j \in]m_{k-1}, n_k]$, $k \in \mathbb{Z}$, and
(c) $m_k - n_k > 2(m_{k-1} - n_{k-1})$, $n_{k+1} - m_k > 2(n_k - m_{k-1})$, $k \in \mathbb{N}$.

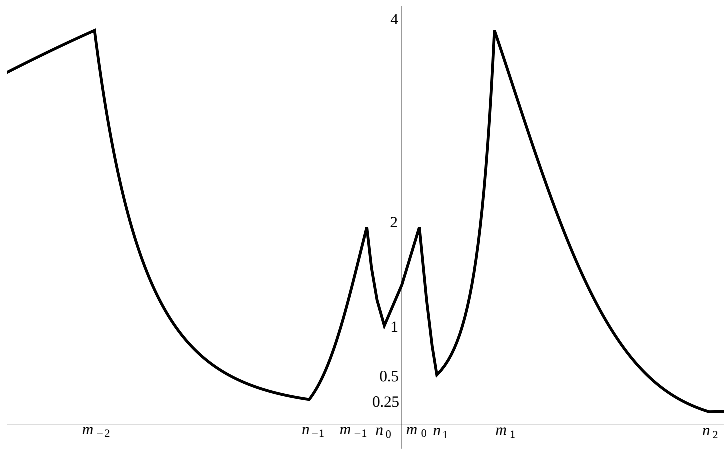

FIGURE 2. Example 3.5

We will check again that the hypothesis of Theorem 3.1 and Corollary 3.3 are satisfied. Condition (b) gives $\dfrac{\min\{v_i;\ m_{-k} \le i \le m_{k-1}\}}{v_{n_k}} = \dfrac{v_{n_{-k+1}}}{v_{n_k}} = 2$ for every $k \in \mathbb{N}$, and the supremum of the slope of v outside the interval $[m_{-k}, m_{k-1}]$ is $S_k = v_j/v_{j-1}$ for any $n_k < j \le m_k$, $k \in \mathbb{N}$. As in the previous example, we set $M = 2$ and $S_k = 2^{1/k(n_k - m_{-k})} = 2^{1/(2kn_k)}$, $k \in \mathbb{N}$. Thus, $S_k^{m_k - n_k} = v_{m_k}/v_{n_k} = 2^{3k}$, which implies $m_k = (6k^2+1)n_k$, $k \in \mathbb{N}$.

Analogously, $\dfrac{\min\{v_i;\ m_{-k} \le i \le m_k\}}{v_{n_{-k}}} = \dfrac{v_{n_k}}{v_{n_{-k}}} = 2$ for every $k \in \mathbb{N}$, and the infimum of the slope of v outside the interval $[m_{-k}, m_k]$ is $s_k = v_j/v_{j-1}$ for any $m_k < j \le n_{k+1}$, $k \in \mathbb{N}$. Again, we set $M = 2$ and $s_k = 2^{1/k(n_{-k} - m_k)} = 2^{-1/(2km_k)}$, $k \in \mathbb{N}$. In consequence, $s_k^{n_{k+1} - m_k} = v_{n_{k+1}}/v_{m_k} = 2^{-3k-2}$, that yields $n_{k+1} = (6k^2+4k+1)m_k$, $k \in \mathbb{N}$.

To check the final conditions for T and T^{-1} being completely distributionally irregular, we notice that

$$v_r = 2^{k+1} s_k^{r-m_k} = 2^{k+1} 2^{-\frac{r-m_k}{2km_k}} > 2^k \quad \text{if } m_k \le r \le 2km_k,$$

and that

$$v_r = 2^k S_k^{m_{-k}-r} = 2^k 2^{\frac{-n_k - r}{2kn_k}} > 2^{k-1} \quad \text{if } 2km_{-k} \le r \le m_{-k}.$$

For the hypercyclicity of T, since it is invertible, it suffices to show that there is an increasing sequence $(j_k)_k$ in $\mathbb{N}$ such that $\lim_k v_{j_k} = \lim_k v_{-j_k} = 0$ (See Theorem 3.2 in [8]). Let $j_k := (m_k + n_k)/2 = (3k^2+1)n_k$. We have

$$v_{j_k} = S_k^{j_k - n_k} v_{n_k} = 2^{-k/2+1},$$

for any $k \in \mathbb{N}$. Note that $R_k = v_j/v_{j-1}$ has the same value for any $j \in]n_{-k}, m_{-k}]$, and thus for all $k \in \mathbb{N}$ we have $R_k < S_k$ and

$$v_{-j_k} = R_k^{-j_k - n_{-k}} v_{n_{-k}} < S_k^{m_k - j_k} 2^{-2k} = 2^{-k/2+1/2}.$$

Indeed, $\lim_k v_{j_k} = \lim_k v_{-j_k} = 0$ which concludes the hypercyclicity of T.

Example 3.6. We will provide v such that T is hypercyclic and completely distributionally irregular on $\ell^p(v, \mathbb{Z})$, but T^{-1} is not completely distributionally irregular. We define

(a) $n_0 = 0$, $m_0 = 1$, $n_1 = 3$, $m_{-1} = -1$, $v_{n_0} = 1$, $v_{m_0} = 2$, $v_{n_1} = 2^{-1}$, $v_{m_{-1}} = 2$, and
(b) $m_{-k} = -m_{k-1}$, $n_{-k} = -m_k + 1$, $n_{k+1} = m_k + 2k^2 + k + 1$, $v_{n_{-k}} = v_{n_k} = 2^{-k}$, $v_{m_k} = 2^{2k^2}$, $v_{m_{-k}} = 2^{-k+2}$, $k \in \mathbb{N}$, $v_i/v_{i-1} = v_j/v_{j-1}$ if $i, j \in]n_k, m_k]$, or if $i, j \in]m_{k-1}, n_k]$, $k \in \mathbb{Z}$, and
(c) $m_k - n_k > 2(m_{k-1} - n_{k-1})$, $n_{k+1} - m_k > 2(n_k - m_{k-1})$, $k \in \mathbb{N}$.

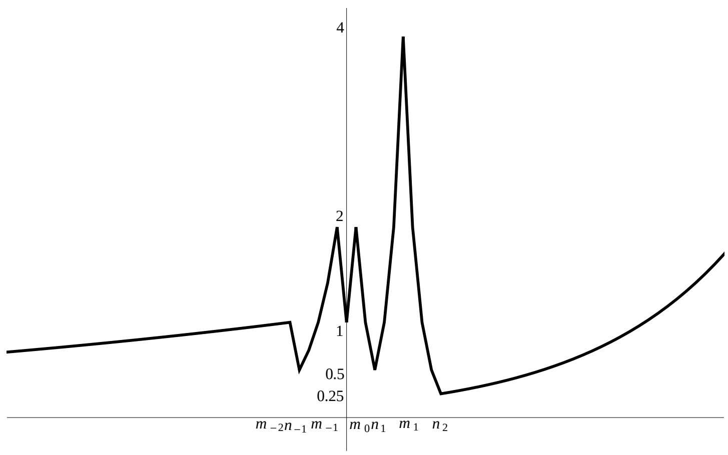

FIGURE 3. Example 3.6

Condition (b) gives $\dfrac{\min\{v_i;\ m_{-k} \le i \le m_{k-1}\}}{v_{n_k}} = \dfrac{v_{n_{k-1}}}{v_{n_k}} = 2$. We set again $M = 2$ and $S_k = 2^{1/k(n_k - m_{-k})} = 2^{1/k(n_k + m_{k-1})}$, $k \in \mathbb{N}$. Thus, $S_k^{m_k - n_k} = 2^{2k^2+k}$, which implies $m_k = n_k + (2k^3 + k)(n_k + m_{k-1})$, $k \in \mathbb{N}$. We then have the formulas to construct $(m_k)_{k\in\mathbb{Z}}$, $(n_k)_{k\in\mathbb{Z}}$, the values of the weight sequence v, and condition *(1)* of Theorem 3.1. For condition (2), pick $r_k := n_k + 2k^2(n_k + m_{k-1})$, $k \in \mathbb{N}$. If $j \in [r_k, m_k]$, then $v_j = v_{n_k} S_k^{j-n_k} \ge 2^k$. Since $(m_k - r_k)/r_k > (k-1)/2$, for each $k \in \mathbb{N}$, we obtain that T is completely distributionally irregular. Moreover, $\lim_{j\to-\infty} v_j = 0$, thus $\lim_k v_{n_k} = \lim_k v_{-n_k} = 0$, and we get the hypercyclicity of T. Finally, if we consider $x = (x_j)_{j\in\mathbb{Z}}$ such that $x_j = 1$ for $j = n_k$, $k \in \mathbb{N}$, and $x_j = 0$ otherwise, we have $x \in \ell^p(v, \mathbb{Z})$ and since $n_k - m_{k-1} \ge k$ we also get that

$$\left\|T^{-k}x\right\| \ge 2^k v_{n_k} \ge 1$$

for all $k \in \mathbb{N}$. This shows that x is not a distributionally irregular vector for T^{-1}.

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