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ON SUBGROUPS OF ZJ TYPE OF AN $\mathfrak{F}$-INJECTOR FOR FITTING CLASSES $\mathfrak{F}$ BETWEEN $\mathfrak{E}_{p^*p}$ AND $\mathfrak{E}_{p^*}\mathfrak{S}_p$

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Abstract

Let G be a finite group and p a prime. We consider an $\mathfrak{F}$-injector K of G, being $\mathfrak{F}$ a Fitting class between $\mathfrak{E}_{p^*p}$ and $\mathfrak{E}_{p^*}\mathfrak{S}_p$, and we study the structure and normality in G of the subgroups $ZJ(K)$ and $ZJ^*(K)$, provided that G verify certain conditions, extending some results of G. Glauberman (A characteristic subgroup of a p-stable group, *Canad. J. Math.* **20** (1968), 555–564).

1. Introduction and notation

In this paper we consider a finite group G verifying certain conditions of stability and constraint, and we study the structure and normality in G of the subgroups $ZJ(K)$ and $ZJ^*(K)$, being K and $\mathfrak{F}$-injector of G and $\mathfrak{F}$ a Fitting class such that $\mathfrak{E}_{p^*p} \subseteq \mathfrak{F} \subseteq \mathfrak{E}_{p^*}\mathfrak{S}_p$, extending some results of Glauberman [**6**].

All groups in this paper are assumed to be finite. Given a fixed prime p, $\mathfrak{S}_p$ will denote the class of all p-groups, $\mathfrak{E}_{p^*}$, the class of all p^*-groups, $\mathfrak{E}_{p^*p}$ the class of all p^*p-groups and $\mathfrak{E}_{p^*}\mathfrak{S}_p$ that of all p^*-by-p-groups. The corresponding radicals in a group G are denoted by $O_p(G)$, $O_{p^*}(G)$, $O_{p^*p}(G)$ and $O_{p^*,p}(G)$ respectively. For all definitions we refer to Bender [**3**].

The notation for Fitting classes is taken from [**4**]. The remainder of the notation is standard and it is taken mainly from [**7**] and [**8**]. In particular, $E(G)$ is the semisimple radical of G and $F^*(G) = F(G)E(G)$ the quasinilpotent radical of G. If H is a subgroup of G, $C^*_G(H)$ is the generalized centralizer of H in G (see [**3**]). Note that $C^*_G(F^*(G)) \leq$

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$F(G)$, in every group G. A group G is said to be $\mathfrak{N}$-constrained if $C_G(F(G)) \leq F(G)$, that is, if $E(G) = 1$.

Moreover, $\pi(G)$ is the set of primes dividing the order of G, $d(G)$ is the maximum of the orders of the abelian subgroups of G, $\mathfrak{A}(G)$ is the set of all abelian subgroups of order $d(G)$ in G and $J(G)$ is the subgroup generated by $\mathfrak{A}(G)$, that is, the Thompson subgroup of G. We set $ZJ(G) = Z(J(G))$.

In **[6]** G. Glauberman proves his well-known ZJ-Theorem and also introduces the subgroup $ZJ^*(P)$ proving the following: "Let p be an odd prime and let P be a Sylow p-subgroup of a group G. Suppose that $C_G(O_p(G)) \leq O_p(G)$ and that $SA(2,p)$ is not involved in G. Then $ZJ^*(P)$ is a characteristic subgroup of G and $C_G(ZJ^*(P)) \leq ZJ^*(P)$".

On the other hand, Arad and Glauberman study in **[2]** the structure and normality of the subgroup $ZJ(H)$, H being a Hall π-subgroup of a π-soluble group G with abelian Sylow 2-subgroups and $O_{\pi'}(G) = 1$.

Some related results were obtained by Arad in **[1]**, by Ezquerro in **[5]** and by Pérez Ramos in **[11]** and **[12]**.

Here we study the structure of the subgroups $ZJ(K)$ and $ZJ^*(K)$ where K is an $\mathfrak{F}$-injector of G, being $\mathfrak{F}$ a Fitting class such that $\mathfrak{E}_{p^*p} \subseteq \mathfrak{F} \subseteq \mathfrak{E}_{p^*}\mathfrak{S}_p$, and we obtain that it depends only of G. Also, we obtain some analogous to Glauberman's ZJ and ZJ^* Theorems for such Fitting classes. Recall that such a Fitting class $\mathfrak{F}$ is dominant in the class of all finite groups, so every finite group G has a unique conjugacy class of $\mathfrak{F}$-injectors (see **[10]**). Moreover, for such $\mathfrak{F}$ every finite group is $\mathfrak{F}$-constrained in the sense of **[9]** (see **[3]**).

In the following $\mathfrak{F}$ will be a Fitting class such that $\mathfrak{E}_{p^*p} \subseteq \mathfrak{F} \subseteq \mathfrak{E}_{p^*}\mathfrak{S}_p$.

2. Preliminary results

Remark 1.

Let K be an $\mathfrak{F}$-injector of a group G. By **[10]** we know that

$$K = (O_{p^*}(G)P)_{\mathfrak{F}}$$

where P is a Sylow p-subgroup of G. Moreover, $O_{p^*}(K) = O_{p^*}(G)$, so $O_{p'}(K) = O_{p'}(G)$ and $O_{p'}(F(K)) = O_{p'}(F(G))$. On the other hand, since $F^*(G) \leq K$, we have $E(K) = E(G)$.

Remark 2.

Suppose that K is an $\mathfrak{E}_{p^*}\mathfrak{S}_p$-group, that is, $K = O_{p^*}(K)S$ where S is a Sylow p-subgroup of K. Since $[O_{p^*}(K), O_p(K)] = 1$, it is clear that K

acts nilpotently on $O_p(K)$, i.e. $K = C^*_K(O_p(K))$. In particular, we can deduce that

$$C^*_K(E(K)O_{p'}(F(K))) = C^*_K(F^*(K)) \leq F(K).$$

Lemma 2.1.

Let G be a group and let K be an $\mathfrak{E}_{p^*}\mathfrak{S}_p$-subgroup of G containing $F^*(G)$. Then $\pi(ZJ(K)) \subseteq \pi(F(G)) = \pi(F(K))$. Moreover if the prime p belongs to $\pi(F(G))$ then $p \in \pi(ZJ(K))$.

Proof:

Since $\pi(F(K)) = \pi(Z(F(K))$ and $Z(F(K)) \leq C_G(F^*(G)) \leq F(G)$, the first statement can be easily obtained. On the other hand if $p \in \pi(F(G))$ and P is a Sylow p-subgroup of K we have $1 \neq Z(P) \cap O_p(K) \leq Z(K) \leq ZJ(K)$ since $K = PO_{p^*}(K)$, and so the result holds. ■

Lemma 2.2.

Let G be a group and let K be an $\mathfrak{E}_{p^*}\mathfrak{S}_p$-subgroup of G containing $O_p(G)$. Let B be a nilpotent normal subgroup of G and let A be any nilpotent subgroup of K. Then $AO_p(B)$ is nilpotent.

Proof:

By the Remark 2 A acts nilpotently on $O_p(B) \leq O_p(K)$, so the result follows. ■

Next we will deal with the subgroup $ZJ^*(K)$ of an arbitrary group K and its properties:

Definition 2.3. [5].

For any group K define two sequences of characteristic subgroups of K as follows. Set $ZJ^0(K) = 1$ and $K_0 = K$. Given $ZJ^i(K)$ and K_i, $i \geq 0$, let $ZJ^{i+1}(K)$ and K_{i+1} the subgroups of K that contain $ZJ^i(K)$ and satisfy:

$$ZJ^{i+1}(K)/ZJ^i(K) = ZJ(K_i/ZJ^i(K))$$
$$K_{i+1}/ZJ^i(K) = C_{K_i/ZJ^i(K)}(ZJ^{i+1}(K)/ZJ^i(K)).$$

Let n be the smallest integer such that $ZJ^n(K) = ZJ^{n+1}(K)$, then $ZJ^n(K) = ZJ^{n+r}(K)$ and $K_n = K_{n+r}$ for every $n \geq 0$. Set $ZJ^*(K) = ZJ^n(K)$ and $K_* = K_n$.

Example.

In general, the subgroups $ZJ(K)$ and $ZJ^*(K)$ of a group K are different. To see this, we can consider, as an example, the group $K = [Q_8 \times C_3]S_3$ generated by the elements a, b, c, x, y with the following relations:

$$a^4=1,\ a^2 = b^2,\ a^b = a^{-1},\ c^3 = 1,\ a^c = a,\ b^c = b,\ x^3 = y^2 = 1,\ x^y = y^{-1},$$
$$a^x = ba,\ b^x = a^{-1},\ c^x = c,\ a^y = b,\ b^y = a,\ c^y = c^{-1}.$$

Then we can get check that $d(K) = 18$, $Z(K) = Z(Q_8) = \langle a^2\rangle$, $ZJ(K) = Z(Q_8)\times C_3$, $K_1 = [Q_8\times C_3]\langle x\rangle = J(K)$ and $ZJ^*(K) = ZJ^2(K) = K_2 = [Q_8 \times C_3]$.

Remark 3.

For every group K:

i) $ZJ(K_i/ZJ^i(K)) = ZJ(K_{i+1}/ZJ^i(K)) = Z(K_{i+1}/ZJ^i(K))$, for every $i \geq 0$.

ii) $Z(K_i) \leq Z(K_{i+1})$, for every $i \geq 0$.

Lemma 2.4.

For any group K and for every $i \geq 0$:

i) *$ZJ^i(K)$ is nilpotent.*

ii) *$F(K_i/ZJ^i(K)) = F(K_i)/ZJ^i(K)$.*

Proof:

i) By induction on i, assume that $ZJ^i(G)$ is nilpotent, for every group G. By ([**5**, Prop. II 3.6]) we have that $ZJ^{i+1}(K)/ZJ^1(K) = ZJ^i(K_1/ZJ^1(K))$, so this is a nilpotent group. Now, by the previous remark, $ZJ^1(K) = ZJ(K) \leq Z(K_1) \leq Z(K_i)$, and $ZJ^{i+1}(K) \leq K_i$, hence $ZJ^{i+1}(K)$ is nilpotent.

ii) By induction on i. The assertion is clear for $i = 0$. Assume now that $F(K_i/ZJ^i(K)) = F(K_i)/ZJ^i(K)$. We have:

$$F(K_{i+1}/ZJ^{i+1}(K)) \cong F(K_{i+1}/ZJ^i(K)/ZJ^{i+1}(K)/ZJ^i(K))$$

and since $ZJ^{i+1}(K)/ZJ^i(K) = Z(K_{i+1}/ZJ^i(K))$, it follows

$$F(K_{i+1}/ZJ^i(K)/ZJ^{i+1}(K)/ZJ^i(K)) = F(K_{i+1}/ZJ^i(K))/ZJ^{i+1}(K)/ZJ^i(K).$$

But applying the inductive hypothesis we have:

$$F(K_{i+1}/ZJ^i(K)) = F(K_i/ZJ^i(K)) \cap K_{i+1}/ZJ^i(K) = F(K_i)/ZJ^i(K) \cap K_{i+1}/ZJ^i(K) = F(K_{i+1}/ZJ^i(K)$$

and so we can conclude that $F(K_{i+1}/ZJ^{i+1}(K)) = F(K_{i+1})/ZJ^{i+1}(K)$. ∎

3. The structure of the ZJ-subgroup and the ZJ^*-subgroup

In this section we will study the structure of the subgroups $ZJ(K)$ and $ZJ^*(K)$ being K an $\mathfrak{E}_{p^*}\mathfrak{S}_p$-subgroup of a group G containing $O_p(G)$ and satisfying that $O_{p^*}(K) = O_{p^*}(G)$, properties that hold for an $\mathfrak{F}$-injector of G, as we have seen.

Theorem 3.1.

Let G be an $\mathfrak{N}$-constrained group and let K be an $\mathfrak{E}_{p^*}\mathfrak{S}_p$-subgroup of G containing $O_p(G)$ and such that $O_{p^*}(K) = O_{p^*}(G)$. Assume that at least one of the following conditions hold:

i) $O_{p'}(F(G)) \leq ZJ(K)$,

ii) $F(G)$ *is abelian,*

iii) $d(K)$ *is odd and* $O_2(G)$ *is abelian.*

Then:

a) $\{O_p(A) | A \in \mathfrak{A}(K)\} = \mathfrak{A}(O_p(K))$.

b) $O_p(ZJ(K)) = ZJ(O_p(K))$.

c) $\{O_{p'}(A) | A \in \mathfrak{A}(K)\} = \mathfrak{A}(O_{p^*}(G))$.

d) $O_{p'}(ZJ(K)) = ZJ(O_{p^*}(G))$.

In particular, if we assume $O_{p'}(F(G)) \leq ZJ(K)$ then for every $A \in \mathfrak{A}(K)$

$$O_{p'}(A) = O_{p'}(ZJ(K)) = O_{p'}(F(G)).$$

Moreover the prime numbers divisors of $d(K)$, $|ZJ(K)|$, $|F(K)|$ and $|F(G)|$ coincide.

Proof:

Let $A \in \mathfrak{A}(K)$. Since $F^*(G) \leq K$ we know that $E(K) = E(G) = 1$, so K is an $\mathfrak{N}$-constrained group. Leading from our assumptions we can obtain that $AF(G)$ is nilpotent (if we assume i) Lemma 2.2 applies; if we assume ii) or iii) Proposition 1 of [**2**] applies). Moreover, since $O_{p^*}(K) = O_{p^*}(G)$ we have $O_{p'}(F(K)) = O_{p'}(F(G))$.

a) Let $A \in \mathfrak{A}(K)$. Since $AF(G)$ is nilpotent $O_p(A)$ centralizes $O_{p'}(F(G))$ and so applying Remark 2 we obtain

$$O_p(A) \leq C_K(O_{p'}(F(K))) \leq F(K)$$

so $O_p(A) \leq O_p(K)$.

Let $B \in \mathfrak{A}(O_p(K))$. Since $AO_p(K)$ is nilpotent by Lemma 2.2, $O_{p'}(A)$ centralizes $O_p(K)$, so $O_{p'}(A)B$ is an abelian subgroup of K and then

$$|O_{p'}(A)B| \leq |A| = |O_{p'}(A)O_p(A)|.$$

Hence $d(O_p(K)) \leq |O_p(A)|$. Since $O_p(A) \leq O_p(K)$ the equality $d(O_p(K)) = |O_p(A)|$ holds.

Thus, for every $B \in \mathfrak{A}(O_p(K))$, $O_{p'}(A) \times B \in \mathfrak{A}(K)$. So we have

$$\{O_p(A) | A \in \mathfrak{A}(K)\} = \mathfrak{A}(O_p(K)).$$

b) This follows easily from a):

$$O_p(ZJ(K)) = O_p(\cap\{A | A \in \mathfrak{A}(K)\}) \\ = \cap\{O_p(A) | A \in \mathfrak{A}(K)\} = ZJ(O_p(K)).$$

c) Let $A \in \mathfrak{A}(K)$. By a) we know that $O_p(A) \leq O_p(K)$. On the other hand, since K is an $\mathfrak{E}_{p^*}\mathfrak{S}_p$-group we have $O_{p'}(A) \leq O^p(K) = O_{p^*}(K) = O_{p^*}(G)$.

Let $B \in \mathfrak{A}(O_{p^*}(G))$. Since $[O_{p^*}(G), O_p(K)] = 1$, $O_p(A)$ centralizes B so $O_p(A)B$ is an abelian subgroup of K and then

$$|O_{p'}(A)B| \leq |A| = |O_p(A)O_{p'}(A)|.$$

Hence $d(O_{p^*}(G)) \leq |O_{p'}(A)|$. Since $O_{p'}(A) \leq O_{p^*}(G)$ it follows $d(O_{p^*}(G)) = |O_{p'}(A)|$. Therefore, for every $B \in \mathfrak{A}(O_{p^*}(G)$, $O_p(A) \times B \in \mathfrak{A}(K)$. This proves c).

d) This follows from c) as in b).

If we assume $O_{p'}(F(G)) \leq ZJ(K)$ then it is clear that $O_{p'}(ZJ(K)) = O_{p'}(F(K)) = O_{p'}(F(G))$. Let $A \in \mathfrak{A}(K)$. Since $ZJ(K) = \cap\{A | A \in \mathfrak{A}(K)\}$ and $AF(G)$ is nilpotent we obtain that $O_{p'}(A) \leq C_G(F(G)) \leq F(G)$ and so the equality $O_{p'}(F(G)) = O_{p'}(ZJ(K)) = O_{p'}(A)$ holds.

Now since $F^*(G) \leq K$ we can apply Lemma 2.1 and our assumptions to obtain $\pi(ZJ(K)) = \pi(F(G)) = \pi(F(K))$. Moreover, if $A \in \mathfrak{A}(K)$ it is clear that $\pi(ZJ(K)) \subseteq \pi(A) = \pi(d(K))$. On the other hand, if q is a prime number such that $q \neq p$ and $q \in \pi(A)$, then $q \in \pi(F(G))$, by the foregoing assertion. Finally, if we assume that $p \in \pi(A)$, then $p \in \pi(F(K)) = \pi(F(G))$ because of a), and so the result follows. ■

Corollary 3.2.

Let G be an $\mathfrak{N}$-constrained group, H an $\mathfrak{E}_{p^*}\mathfrak{S}_p$-injector of G and $K = H_{\mathfrak{F}}$ its associated $\mathfrak{F}$-injector of G. If one of the following conditions holds:

i) $O_{p'}(F(G)) \leq ZJ(K)$,
ii) $F(G)$ *is abelian,*
iii) $d(K)$ *is odd and* $O_2(G)$ *is abelian,*

then

$$ZJ(K) = ZJ(O_{p^*}(G)) \times ZJ(O_p(H)) = ZJ(H).$$

So, in particular, $ZJ(K)$ does not depend on the Fitting class $\mathfrak{F}$.

Proof:

Given A in $\mathfrak{A}(H)$, by Remark 2 we see that $O_p(A) \leq O_p(H) = O_p(K)$. On the other hand, due to the structure of the injectors considered here, one has $O_{p'}(A) \leq O^p(H) = O_{p^*}(H) = O_{p^*}(G) \leq K$. Therefore $\mathfrak{A}(H) = \mathfrak{A}(K)$. Then apply Theorem 3.1 parts b) and d) to the subgroups H and K. ■

Corollary 3.3.

If G is an $\mathfrak{N}$-constrained group and K and $\mathfrak{F}$-injector of G such that $O_{p'}(F(G)) \leq Z(K)$, then

$$K = O_{p'}(F(G)) \times P$$

where P is a Sylow p-subgroup of G. In particular,

$$\mathfrak{A}(K) = \{O_{p'}(F(G))A | A \in \mathfrak{A}(P)\}.$$

Proof:

Since $K = PO_{p^*}(G)$, P a Sylow p-subgroup of K and $O_{p'}(F(G)) \leq Z(K)$, due to 6.11 in [**3**], we can write $[P, O_{p^*}(G)] = 1$. Now by $\mathfrak{N}$-constraint, K is nilpotent and hence it is an $\mathfrak{E}_{p'}\mathfrak{S}_p$-injector of G (see [**10**]); therefore P is a Sylow p-subgroup of G and $K = O_{p'}(F(G)) \times P$. ■

Our next goal is to study the structure of the ZJ^*-subgroup.

Theorem 3.4.

Let G be an $\mathfrak{N}$-constrained group. Let K be an $\mathfrak{E}_{p^*}\mathfrak{S}_p$-subgroup of G containing $O_p(G)$ and such that $O_{p^*}(K) = O_{p^*}(G)$. Assume that $O_{p'}(F(G)) \leq ZJ(K)$. Denote $P = O_p(K)$. Then for every $i \geq 1$, $O_{p'}(ZJ^i(K)) = O_{p'}(F(K_i)) = O_{p'}(F(G))$, K_i is a nilpotent group and

$$O_p(ZJ^i(K)) = ZJ^i(P) \quad O_p(K_i) = P_i$$

with the notation given in Definition 2.3. In particular $O_p(ZJ^*(K)) = ZJ^*(P)$, $O_p(K_*) = P_*$ and

$$ZJ^*(K) = ZJ^*(P) \times O_{p'}(F(G)).$$

Proof:

Since $O_{p'}(ZJ(K)) \leq O_{p'}(ZJ^i(K)) \leq O_{p'}(F(K_i)) \leq O_{p'}(F(K)) = O_{p'}(F(G))$, the first statement is clear.

Notice that $O_{p'}(F(G)) \leq ZJ(K) \leq Z(K_1)$, so $O_{p^*}(K_1) \leq C_G(F(G)) \leq F(G)$. Hence $O_{p^*}(K_1) = O_{p'}(F(K_1)) \leq Z(K_1)$ and K_1 is a nilpotent gorup. Now apply that for every $i \geq 1$, $K_i \leq K_1$.

We will prove that $O_p(ZJ^i(K)) = ZJ^i(P)$ and $O_p(K_i) = P_i$ by induction on i. By Propositiom 3.2 we have $ZJ(P) = O_p(ZJ(K))$. On the other hand $P = O_p(K)$ centralizes $O_{p'}(ZJ(K))$, so $C_P(ZJ(P)) \leq C_K(ZJ(K))$ and then we obtain

$$O_p(K_1) = P \cap K_1 = P \cap C_K(ZJ(K)) = C_P(ZJ(P)) = P_1.$$

Thus, the statement is clear for $i = 1$.

Now suppose that $O_p(ZJ^i(K)) = ZJ^i(P)$ and $O_p(K_i) = P_i$. Applying Lemma 2.4 and the fact that $O_{p'}(F(K_i)) = O_{p'}(ZJ^i(K))$, we get that $K_i/ZJ^i(K) = F(K_i)/ZJ^i(K)$ is a p-group. Then it follows that

$$K_i/ZJ^i(K) = P_iZJ^i(K)/ZJ^i(K) \cong P_i/ZJ^i(K) \cap P_i = P_i/ZJ^i(P)$$

by the inductive hypothesis. Thus

$$\begin{aligned} ZJ^{i+1}(K)/ZJ^i(K) = ZJ(K_i/ZJ^i(K)) &\cong ZJ(P_i/ZJ^i(P)) \\ &= ZJ^{i+1}(P)/ZJ^i(P). \end{aligned}$$

and since $ZJ^{i+1}(K) = ZJ^i(K)(ZJ^{i+1}(K) \cap P_i)$ we can conclude

$$O_p(ZJ^{i+1}(K)) = ZJ^{i+1}(K) \cap O_p(K_i) = ZJ^{i+1}(K) \cap P_i = ZJ^{i+1}(P).$$

Now we will prove that $O_p(K_{i+1}) = P_{i+1}$. It is clear that $O_p(K_{i+1}) \leq O_p(K_i) = P_i$ and

$$\begin{aligned} [O_p(K_{i+1}), ZJ^{i+1}(P)] &\leq [O_p(K_{i+1}), ZJ^{i+1}(K)] \\ &\leq O_p(K_{i+1}) \cap ZJ^i(K) = ZJ^i(P). \end{aligned}$$

Hence by the definition of P_{i+1} it follows that $O_{P'}(K_{i+1}) \leq P_{i+1}$. On the other hand, $P_{i+1} \leq P_i \leq K_i$ and since $O_{p'}(F(G)) \leq ZJ(K) \leq Z(K_i)$, we have

$$[P_{i+1}, ZJ^{i+1}(K)] = [P_{i+1}, ZJ^{i+1}(P)] \leq ZJ^i(P) \leq ZJ^i(K).$$

Thus, by the definition of K_{i+1} we obtain $P_{i+1} \leq K_{i+1}$. Now, since $O_p(K_{i+1})$ is the Sylow p-subgroup of K_{i+1} the result follows. ■

Corollary 3.5.

Let G be an $\mathfrak{N}$-constrained group. Let H be an $\mathfrak{E}_{p^*}\mathfrak{S}_p$-injector of G and assume that $O_{p'}(F(G)) \leq ZJ(H)$. Let $K = H_{\mathfrak{F}}$ be an $\mathfrak{F}$-injector of G. Then

$$ZJ^*(K) = O_{p'}(F(G)) \times ZJ^*(O_p(H)) = ZJ^*(H).$$

In particular, $ZJ^*(K)$ does not depend on $\mathfrak{F}$.

Proof:

Because of Corollary 3.2 we have $ZJ(K) = ZJ(H)$. Now Theorem 3.4 is applied, keeping in mind that $O_p(K) = O_p(H)$. ■

4. The normality of the ZJ-subgroup and the ZJ^*-subgroup

In this section we prove some results related to the normality of the ZJ-subgroup and the normality and self-centrality of the ZJ^*-subgroup of an $\mathfrak{F}$-injector K of a group G, provided that G verifies certain conditions of stability. Concretely, we will use the following version of p-stability:

Definition 4.1.

A group G is said to be *p-stable* if whenever A is a subnormal p-subgroup of G and B is a p-subgroup of $N_G(A)$ satisfying $[A, B, B] = 1$, then

$$B \leq O_p(N_G(A) \bmod C_G(A)).$$

Proposition 4.2.

Let G be a p-stable group. Let K be an $\mathfrak{E}_{p^*}\mathfrak{S}_p$-subgroup of G containing the $\mathfrak{E}_{p^*p}$-radical of G, $O_{p^*p}(G)$. If N is an abelian normal subgroup of K then $N \trianglelefteq\trianglelefteq G$ and $N \leq F(G)$. In particular $ZJ(K) \leq F(G)$.

Proof:

First notice that $O_{p^*p}(G) \leq K$ implies $O_{p^*}(K) = O_{p^*}(G)$ (see [**3**, 4.22]). Thus, $O_{p'}(N) \leq O_{p^*}(G) \leq O_{p^*p}(G) \leq K$, and so $O_{p'}(N) \trianglelefteq O_{p^*p}(G)$.

On the other hand, it holds $[O_p(G), O_p(N), O_p(N)] = 1$ and so applying the p-stability of G we have:

$$\begin{aligned} O_p(N)C_G(O_p(G))/C_G(O_p(G)) &\leq O_p(G/C_G(O_p(G))) \\ &= C_G^*(O_p(G))/C_G(O_p(G)) \end{aligned}$$

(see [**3**, 3.8]). Then we obtain

$$O_p(N) \leq C_G^*(O_p(G)) \cap C_G(E(G)O_{p'}(F(G))) \leq C_G^*(F^*(G)) \leq F(G)$$

so $O_p(N) \trianglelefteq O_{p^*p}(G)$ and the result follows. ■

Theorem 4.3.

Let G be a p-stable group, p and odd prime and assume that $O_p(G) \neq 1$. If K is an $\mathfrak{F}$-injector of G then

$$1 \neq O_p(ZJ(K)) \trianglelefteq G.$$

Moreover, if $O_{p'}(F(G)) \leq ZJ(K)$, then $1 \neq ZJ(K) \trianglelefteq G$.

Proof:

First note that $O_p(ZJ(K)) \trianglelefteq G$ implies $O_p(ZJ(K))$ char G, because of the conjugacy of the $\mathfrak{F}$-injectors.

By Proposition 4.2, we know that $O_p(ZJ(K)) \leq O_p(G)$, and by Lemma 2.1 $O_p(ZJ(K)) \neq 1$. Now, to obtain the theorem it is enough to prove that if B is a normal p-subgroup of G, then $B \cap O_p(ZJ(K))$ is normal in G.

Assume the result false and suppose that G is a minimal counterexample. Suppose that B is a normal p-subgroup of G of least order such that $B \cap O_p(ZJ(K))$ is not normal in G.

Set $Z = O_p(ZJ(K))$ and let B^* be the normal closure of $B \cap Z$ in G, then $B \cap Z = B^* \cap Z$ and by our minimal choice of B we obtain $B = B^*$.

Moreover, since $B' < B$ we have that $B' \cap Z$ is a normal subgroup of G. Thus, for any g in G we have $[(B \cap Z)^g, B] = [B \cap Z, B]^g \leq B' \cap Z$. Since B is generated by all such $(B \cap Z)^g$, it follows that $B' \leq Z$. In particular $B \cap Z$ centralizes B', and applying the foregoing argument we get $[B, B, B] = 1$.

Let $A \in \mathfrak{A}(K)$. By Lemma 2.2 we know that AB is nilpotent, so there exists some positive integer n such that $[B, A; n] = 1$. Moreover, since p is an odd prime $[A, B]' \leq B'$ has odd order.

Now by Glauberman's replacement Theorem ([**1**, Corollary 2.8]) we can conclude that there exists an element A in $\mathfrak{A}(K)$ such that $B \leq N_G(A)$, and therefore $[B, A, A] = 1$.

In particular, $[B, O_p(A), O_p(A)] = 1$. Since G is p-stable we have:

$$O_p(A)C/C \leq O_p(G/C) = T/C \trianglelefteq G/C$$

where $C = C_G(B)$ and $T = C_G^*(B)$. Moreover, since $O_{p'}(A) \leq C_G(B)$ we get

$$A \leq T.$$

If $T = G$, then G/C is a p-group, so KC is a subnormal subgroup of G. Since KC normalizes $B \cap Z$, $KC < G$. Let M be a normal proper subgroup of G such that $KC \leq M$. Clearly M verifies the hypothesis of the theorem, K being an $\mathfrak{F}$-injector of M, so by our minimal choice of G, we get $Z \trianglelefteq M$, and then Z char M. Therefore, $Z \trianglelefteq G$, contrary to our choice of G.

Thus, we have $T < G$. Since $A \leq K \cap T$, it follows that $\mathfrak{A}(K \cap T) \subseteq \mathfrak{A}(K)$, $J(K \cap T) \leq J(K)$ and $ZJ(K) \leq ZJ(K \cap T)$. It is clear that T verifies the hypothesis of the theorem, being $K \cap T$ an $\mathfrak{F}$-injector of T. Thus, by the minimal choice of G, $O_p(ZJ(K \cap T))$ char T and then $O_p(ZJ(K \cap T)) \trianglelefteq G$. Since B is the normal closure of $B \cap Z$ in G we obtain $B \leq O_p(ZJ(K \cap T))$. In particular, B is abelian.

If $J(K) = J(K \cap T)$ then $O_p(ZJ(K)) = O_p(ZJ(K \cap T)) \trianglelefteq G$, contrary to the choice of G. Thus, there exists an element $A_1 \in \mathfrak{A}(K)$ such that A_1 is not a subgroup of T. Then we must have $[B, A_1, A_1] \neq 1$. Among all such A_1, choose A_1 such that $|A_1 \cap B|$ is maximal. As B does not normalize A_1, by Thompson's replacement Theorem ([**1**, Theorem 2.5], there exists an element A_2 in $\mathfrak{A}(K)$ such that $A_1 \cap B < A_2 \cap B$ and A_2 normalizes A_1. The maximal choice of A_1 implies that $[B, A_2, A_2] = 1$ and $A_2 \leq T$. Hence, $B \leq ZJ(K \cap T) \leq A_2 \leq N_G(A_1)$ and this is the last contradiction.

Finally, if in addition we assume $O_{p'}(F(G)) \leq ZJ(K)$, then $O_{p'}(F(G)) = ZJ(K)$ and the result follows. ■

Corollary 4.4 (compare with Glauberman's ZJ-Theorem [**6**]).

Let G be a p-stable group such that $C_G(O_p(G)) \leq O_p(G)$, p and odd prime. If P is a Sylow p-subgroup of G then $ZJ(P) \trianglelefteq G$.

Proof:

Leading from our assumptions we have $O_{p^*}(G) = O_{p'}(G) = 1$, so P is actually an $\mathfrak{E}_{p^*}\mathfrak{S}_p$-injector of G and Theorem 4.3 applies. ■

Theorem 4.5.

Let p be an odd prime and K an $\mathfrak{F}$-injector of a group G, being $\mathfrak{F}$ a Z-extensible and Q_Z-closed Fitting class. Assume that $SA(2,p)$ is not involved in G and that $O_{p'}(F(G)) \leq ZJ(K)$. Then $ZJ^i(K)$ is a characteristic subgroup of G for every $i \geq 0$.

Proof:

Assume the result to be false and let G be a minimal counterexample. Since $SA(2,p)$ is not involved in G, we know that G is p-stable (using Definition 4.1 above, proceed as in [**6**]). Therefore applying Theorem 4.3

we have $ZJ(K)$ char G. Because of the choice of G we can assume $1 \neq ZJ(K)$.

Set $C = C_G(ZJ(K))$. Assume that $C < G$. Then for every $i \geq 0$ we have $ZJ^i(K \cap C)$ char C, and so $ZJ^i(K \cap C) \trianglelefteq G$. Now since $J(K) \leq K \cap C$, it follows that $J(K) = J(K \cap C)$ and $ZJ(K) = ZJ(K \cap C)$. Also $K_1 = C_K(ZJ(K)) = C_{K\cap C}(ZJ(K \cap C))$ and applying induction on i we can obtain $ZJ^i(K) = ZJ^i(K \cap C) \trianglelefteq G$, contrary to the choice of G.

Therefore $C = G$ and then $ZJ(K) = Z(G)$. Since $|G/Z(G)| < G$ and $K/Z(G)$ is an $\mathfrak{F}$-injector of $G/Z(G)$ we obtain $ZJ^i(K/Z(G))$ char $G/Z(G)$, for every $i \geq 0$. Now since $K_1 = C_K(ZJ(K)) = K$, using ([**5**, Prop. II.3.6]) we can deduce $ZJ^i(K/Z(G)) = ZJ^{i+1}(K)/Z(G)$, and so $ZJ^{i+1}(K)$ char G for every $i \geq 0$, which is the last contradiction. ■

Remark 4.

Recall that for any group K, $C_K(ZJ^*(K)) \leq K_*$ and $K_*/C_K(ZJ^*(K))$ is nilpotent (by [**5**, Prop. II 3.7]). Using this facts it is easy to see that for any group K the following statements are equivalent:

i) $C_K(ZJ^*(K)) \leq ZJ^*(K)$ ii) $K_* = ZJ^*(K)$.

Also, we know that $C_K(K_*) \leq C_K(ZJ^*(K)) \leq K_*$, using ([**5**, Prop. II 3.7]).

Remark 5.

Let K be an $\mathfrak{F}$-injector of a group G. Then K is also an $\mathfrak{F}$-injector of any subgroup of G containing K (see [**10**]). In particular, K is an $\mathfrak{F}$-injector of $N_G(K_*)$, and so by the previous remark $Z(K_*) = C_K(K_*) = C_G(K_*) \cap K$ is an $\mathfrak{F}$-injector of $C_G(K_*)$. Thus if $x \in C_G(K_*)$, since $\langle x, Z(K_*)\rangle$ is an abelian subgroup of $N_G(K_*)$ with $Z(K_*) \leq \langle x, Z(K_*)\rangle \leq C_G(K_*)$, we can conclude that $Z(K_*) = \langle x, Z(K_*)\rangle$. Therefore, we have proved that $C_G(K_*) \leq K_*$.

Proposition 4.6.

Let K be an $\mathfrak{F}$-injector of a group G and assume $O_{p'}(F(G)) \leq ZJ(K)$. Then the following are equivalent:

i) *G is an $\mathfrak{N}$-constrained group.*
ii) $K_* = ZJ^*(K)$.
iii) $C_G(ZJ^*(K) \leq ZJ^*(K)$.

Proof:

First notice that, applying Lemma 2.1, since $K_*/ZJ^*(K)$ is an $\mathfrak{E}_{p^*}\mathfrak{S}_p$-group, $ZJ(K_*/ZJ^*(K)) = 1$ implies $O_p(K_*/ZJ^*(K)) = 1$. Now applying Lemma 2.4 and the fact that $O_{p'}(F(G)) \leq ZJ(K)$ we obtain that $F(K_*/ZJ^*(K)) = F(K_*)/ZJ^*(K)$ is a p-group and so we conclude $ZJ^*(K) = F(K_*)$.

i) $\Rightarrow$ ii) Since $F(G) \leq K$ it follows that $C_K(F(K)) \leq F(K)$, and so on $C_{K_*}(F(K_*)) \leq F(K_*)$. Bearing in mind that $ZJ^*(K) = F(K_*)$ and $C_K(ZJ^*(K)) = C_{K_*}(ZJ^*(K))$, ii) follows from Remark 4.

ii) $\Rightarrow$ iii) By the Remark 5.

iii) $\Rightarrow$ i) Since $ZJ^*(K)$ is nilpotent we have $E(G) \leq C_G(ZJ^*(K)) \leq ZJ^*(K)$, and then $E(G) = 1$, that is, G is an $\mathfrak{N}$-constrained group. ■

Corollary 4.7.

Let p be an odd prime and K an $\mathfrak{F}$-injector of an $\mathfrak{N}$-constrained group G, being $\mathfrak{F}$ a Z-extensible and Q_Z-closed Fitting class. Assume that $SA(2,p)$ is not involved in G and that $O_{p'}(F(G)) \leq ZJ(K)$. Then $ZJ^*(K)$ is a characteristic subgroup of G and $C_G(ZJ^*(K)) \leq ZJ^*(K)$.

Recall that both the classes $\mathfrak{E}_{p^*p}$ and $\mathfrak{E}_{p^*p}\mathfrak{S}_p$ are Z-extensible and Q_Z-closed Fitting classes (see [**3**] and [**10**]), so the previous result applies for such classes. Moreover, as in the case of the ZJ-theorem we can also recover the Glauberman's ZJ^*-Theorem quoted at the beginning as a consequence of the above corollary.

5. Final remarks

Remark 6.

There exist $\mathfrak{N}$-constrained groups G such that $O_{p'}(F(G)) \leq ZJ(K)$, being K an $\mathfrak{E}_{p^*}\mathfrak{S}_p$-injector of G, verifying that $SA(2,p)$ is not involved in G, p odd, and however with $O_{p'}(G) \neq 1$.

Proof:

It is enough to take the group $G = SA(3,3) = [N]H$, with $N \cong C_3 \times C_3 \times C_3$ and $H \cong SL(3,3)$ and the prime $p = 13$. Really, G is an $\mathfrak{N}$-constrained group with $O_{p'}(F(G)) = N$, an $\mathfrak{E}_{p^*}\mathfrak{S}_p$-injector of G is $K = O_{p^*}(G)P = NP$ where $P \cong C_{13}$, and $ZJ(K) = N$. Moreover, it is clear that $SA(2,13)$ is not involved in G, bearing the orders in mind. ■

Remark 7.

In [**2**] and [**12**], the authors consider a π-soluble group G with abelian Sylow 2-subgroups and $O_{\pi'}(G) = 1$, and they study the structure of the subgroup $ZJ(H)$, where H is a Hall π-subgroup of G, or H is an $\mathfrak{F}$-injector of G for certain Fitting classes $\mathfrak{F}$, respectively. Recall that such a group is an $\mathfrak{N}$-constrained group (see [**2**]), and moreover it is a p-stable group for any prime number p (see [**12**]).

Moreover, since the p-nilpotent groups are $\mathfrak{E}_{p^*}\mathfrak{S}_p$-groups, we can easily generalizes Lemma 4 of [**2**], as follows:

"Let G be a group and let P be a p-subgroup of $K = O_{p^*,\ p}(G)$. Assume that P centralizes $E(G)O_{p'}(F(G))$. Then $P \leq O_p(G)$".

For the proof, let $K = O_{p^*,\ p}(G)$; since $F^*(K) = F^*(G)$, applying Remark 2 it follows that $P \leq C_K(E(K)O_{p'}(F(K))) \leq F(K)$, and hence $P \leq O_p(K) = O_p(G)$.

References

1. Z. ARAD, A characteristic subgroup of π-stable groups, *Canad. J. Math.* **26** (1974), 1509–1514.
2. Z. ARAD AND G. GLAUBERMAN, A characteristic subgroup of a group of odd order, *Pacific J. Math.* **56(2)** (1975), 305–319.
3. H. BENDER, On the normal p-structure of a finite group and related topics I, *Hokkaido Math. J.* **7** (1978), 271–288.
4. K. DOERK AND T. HAWKES, "*Finite soluble groups*," De Gruyter, Expositions in Mathematics, Berlin, 1992.
5. L. M. EZQUERRO, $\mathfrak{F}$-estabilidad, constricción y factorización de grupos finitos, Tesis doctoral, Univ. de Valencia, 1983.
6. G.GLAUBERMAN, A characteristic subgroup of a p-stable group, *Canad. J. Math.* **20** (1968), 1101–1135.
7. D. GORENSTEIN, "*Finite groups*," Harper & Row, New York, 1968.
8. B. HUPPERT AND N. BLACKBURN, "*Finite groups*," Springer-Verlag, Berlin, 1982.
9. M. J. IRANZO AND F. PÉREZ MONASOR, $\mathfrak{F}$-constraint with respect to a Fitting class, *Arch. Math.* **46** (1986), 205–210.
10. M. J. IRANZO AND M. TORRES, The p^*p-injectors of a finite group, *Rend. Sem. Mat. Univ. Padova* **82** (1989), 233–237.
11. M. D. PÉREZ RAMOS, A self-centralizing characteristic subgroup, *J. Austral Math. Soc. (serie A)* **46** (1989), 302–307.

12. M. D. PÉREZ RAMOS, Stability with respect to a Fitting class, *J. Algebra* **128(2)** (1990), 276–286.

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