Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 178460, 6 pages
http://dx.doi.org/10.1155/2013/178460

Research Article

Isomorphisms on Weighed Banach Spaces of Harmonic and Holomorphic Functions

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Received 30 May 2013; Accepted 3 August 2013

Academic Editor: Miguel Martin

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For an arbitrary open subset $U \subset \mathbb{R}^d$ or $U \subseteq \mathbb{C}^d$ and a continuous function $v : U \to]0, \infty[$ we show that the space $h_{v_0}(U)$ of weighed harmonic functions is almost isometric to a (closed) subspace of c_0, thus extending a theorem due to Bonet and Wolf for spaces of holomorphic functions $H_{v_0}(U)$ on open sets $U \subset \mathbb{C}^d$. Inspired by recent work of Boyd and Rueda, we characterize in terms of the extremal points of the dual of $h_{v_0}(U)$ when $h_{v_0}(U)$ is isometric to a subspace of c_0. Some geometric conditions on an open set $U \subseteq \mathbb{C}^d$ and convexity conditions on a weight v on U are given to ensure that neither $H_{v_0}(U)$ nor $h_{v_0}(U)$ are rotund.

1. Introduction, Notation, and Preliminaries

Let $U \subseteq \mathbb{R}^d$ or $U \subseteq \mathbb{C}^d$ be a nonempty open set. A *weight* on U is a function $v : U \to \mathbb{R}$ which is strictly positive, continuous, and bounded. Let $h(U)$ denote the space of complex valued harmonic functions on U. For a *weight* v the weighed Banach spaces of harmonic functions with *weight* v are defined by

$$h_v(U) := \left\{ f \in h(U) : \|f\|_v := \sup_{z \in G} v(z) |f(z)| < \infty \right\},$$
$$h_{v_0}(U) := \{ f \in h(U) : vf \text{ vanishes at infinity on } U \}. \quad (1)$$

A function $g : U \to \mathbb{C}$ is said to vanish at infinity on U if for each $\varepsilon > 0$ there exists a compact subset $K \subset U$ such that $|g(x)| \leq \varepsilon$ for each $x \in U \setminus K$.

If $U \subseteq \mathbb{C}^d$ then the weighed Banach spaces of holomorphic functions $H_v(U)$ and $H_{v_0}(U)$ are defined in the same way. The unit disc and the unit circle in $\mathbb{C}$ are denoted by $\mathbb{D}$ and Γ, respectively. U is said to be balanced if $\lambda z \in U$ for each $z \in U$ and each $\lambda \in \mathbb{D}$. A weight v on a balanced open subset $U \subseteq \mathbb{C}^n$ is said to be radial whenever $v(\lambda z) = v(z)$ for each $\lambda \in \Gamma$.

Weighed Banach spaces of holomorphic functions with weighed supremum norms and composition operators between them have been studied by Bierstedt et al., Bonet et al., Contreras and Hernández-Díaz, García et al., Montes Rodríguez, and others; see [1–8] and the references therein.

Spaces of harmonic functions have been investigated by Shields and Williams [9], in connection with the growth of the harmonic conjugate of a function. In [10], they proved results of duality for weighed spaces of harmonic functions on the open unit disk. Lusky also considers weighed spaces of harmonic functions in [11, 12], where the isomorphism classes in the case of radial weights on the disk are determined. In [13] the authors have studied associated weights in weighed Banach spaces of harmonic functions and composition operators with holomorphic symbol between weighed spaces of pluriharmonic functions.

In [14–16] Boyd and Rueda investigate the geometry of $h_{v_0}(U)$ and $Hv_0(U)$ where U is an open set of the complex plane. Moreover they examined the extreme points of $h_{v_0}(U)^*$ and of $H_{v_0}(U)^*$ for $U \subseteq \mathbb{C}^d$.

In this work we prove in the second section that, for any weight v on an open subset U of $\mathbb{R}^d$, the weighed Banach space $h_{v_0}(U)$ of harmonic functions with vf vanishing at infinity on U is always isomorphic to a subspace of c_0 thus

extending a result due to Bonet and Wolf [17] for holomorphic functions. Further, given $0 < \varepsilon < 1$ one can get a linear mapping $T : h_{v_0}(U) \to c_0$ such that $(1-\varepsilon)\|f\|_v \le \|T(f)\| \le \|f\|_v$ for each $f \in h_{v_0}(U)$; that is, $h_{v_0}(U)$ is almost isometric to a subspace of c_0. We show that for radial weights in balanced open subsets the isomorphism cannot be isometric, generalizing [14, Theorem 16, Corollary 17] to spaces of harmonic functions defined on not necessarily bounded sets. We also show in this section that if the unit ball of $h_{v_0}(U)^*$ has a big quantity of extremal points, then $h_{v_0}(U)$ is not isometric to any subspace of c_0.

In the third section we mainly focus ourselves on weighed spaces of holomorphic and harmonic functions defined on balanced subsets $U \subseteq \mathbb{C}^d$ and with radial weights. We study the extremal points of the unit sphere of $H_{v_0}(U)^*$ in the way that Boyd and Rueda initiated in [14, 15]. More precisely, we give conditions on U and v under which one can find in the unit sphere of $H_{v_0}(U)^*$ a big quantity of extremal points. We show that these conditions imply that $H_{v_0}(U)$ is not rotund.

Our notation for functional analysis and Banach space theory is standard. We refer to [18]. For a Banach space X we denote by B_X and S_X its unit ball and its unit sphere, respectively. A (real or complex) Banach space X is said to be rotund whenever $\|(x+y)/2\| < 1$ for $\|x\| = \|y\| = 1$ and $x \neq y$. A vector x in the unit sphere S_X of a Banach space X is called *extremal* if there are not $y, z \in S_X$ such that $y \neq z$ and $x = (1/2)(y+z)$, and it is called *exposed* if there is $x^* \in X^*$ such that $\|x^*\| = 1$, $x^*(x) = 1$, and $x^*(y) \neq 1$ for each $y \in S_X \setminus \{x\}$. If X is complex this last condition is equivalent to $x^*(x) = 1$ and $\operatorname{Re} x^*(y) < 1$ for $y \in B_X \setminus \{x\}$, where $\operatorname{Re} z$ denotes the real part of a complex number z. If x is exposed, then x is also extremal. If X is a dual Banach space and the functional exposing x can be taken in the predual, then x is called *weak* exposed*. For $x, y \in \mathbb{R}^d$ (or $\mathbb{C}^d$) we denote by $\langle x, y\rangle$ its canonical scalar product. The Euclidean norm is denoted by $\|\cdot\|$. When we consider other norms on a finite dimensional space they are denoted by $|\cdot|$, making an abuse of notation because the modulus of a complex number is denoted also by $|\cdot|$.

For a Hausdorff locally compact topological space Q we denote by $\widehat{Q} = \{Q \cup \infty\}$ its Alexandroff compactification. The boundary of a subset U of a topological space is denoted by ∂U. If $U \subseteq \mathbb{R}^d$ is an open subset then the space $h_{v_0}(U)$ is formed by the harmonic functions on U such that vf can be extended continuously to $\widehat{U}$ by $(vf)(\infty) = 0$. For an open set $U \subseteq \mathbb{R}^d$, the isometry between $h_{v_0}(U)$ and a subspace of $C(\widehat{U})$ by means of $f \mapsto vf$ permits to use the same argument as in [14, Proposition 1] to show that the extreme points of $h_{v_0}(U)^*$ are contained in the set

$$\{\lambda v(z)\,\delta_z : \lambda \in \Gamma, z \in U\}, \tag{2}$$

where δ_z denotes the evaluation at z. The following definitions are due to Boyd and Rueda [14–16]. The harmonic v-boundary is defined as follows:

$$b_v(U) := \{z \in U : v(z)\,\delta_z \text{ is extremal in the unit sphere of } h_{v_0}(U)^*\}. \tag{3}$$

The set of harmonic v-peak points is defined as

$$p_v(U) := \{z \in U : v(z)\,\delta_z \text{ is weak}^* \text{exposed in the unit sphere of } h_{v_0}(U)^*\}. \tag{4}$$

Remark 1. The same argument as in [15, Theorem 6] shows that to check that a functional of the form $v(x_0)\delta_{x_0}$ is weak* exposed it is enough to show that there exists f in the unit ball of $h_{v_0}(U)$ such that $v(x_0)f(x_0) = 1$, $v(x)\ \operatorname{Re} f(x) < 1$ for each $x \in U \setminus \{x_0\}$.

If $U \subseteq \mathbb{C}^d$ then the holomorphic v-boundary and the set of holomorphic v-peak points are defined analogously, and they are denoted by $B_v(U)$ and $P_v(U)$, respectively. Since $H_{v_0}(U) \subset h_{v_0}(U)$, in this case, we have

$$P_v(U) \subseteq B_v(U) \subseteq U, \tag{5}$$

and also

$$P_v(U) \subseteq p_v(U) \subseteq b_v(U) \subseteq U. \tag{6}$$

We finish this section with this lemma also due to Boyd and Rueda.

Lemma 2 (see [14, 15]). *Let $U \subseteq \mathbb{R}^d (U \subseteq \mathbb{C}^d)$ be a nonempty open set, and let v be a weight which vanishes at infinity on U.*

(a) *For any $\lambda \in \Gamma$ and $z \in U$ the functional $\lambda v(z)\delta_z$ is extremal in the unit sphere of $h_{v_0}(U)^*$ (is extremal in the unit sphere of $H_{v_0}(U)^*$) if and only if $z \in b_v(U) (z \in B_v(U))$.*

(b) *If $U \subseteq \mathbb{C}^d$ is balanced, v is radial and $\lambda \in \Gamma$ then $\lambda z \in b_v(U)$ for each $z \in b_v(U) (\lambda z \in B_v(U)$ for each $z \in B_v(U))$.*

2. Isomorphisms on $h_{v_0}(U)$

We extend below the main theorem of [17] to weighed spaces of harmonic functions. The proof is analogous to the proof given by Bonet and Wolf. One only has to take in account that there are Cauchy type inequalities which are valid for the derivatives of harmonic functions [19, 2.4]. We include it for the sake of completeness. The argument was inspired by one used by Kalton and Werner [20, Corollary 4.9] to show that the little Bloch space B_0 embeds almost isometrically in c_0. Lusky showed in [21] that whenever v is a radial weight on $\mathbb{D}$, then $h_{v_0}(\mathbb{D})$ is isomorphic to a subspace of c_0.

Theorem 3. *Let $U \subset \mathbb{R}^d$ be a nonempty open set, let v be a weight on U, and let $\varepsilon > 0$. There exists an isomorphism $T : h_{v_0}(U) \to c_0$ such that $(1-\varepsilon)\|f\|_v \le \|T(f)\| \le \|f\|_v$ for each $f \in h_{v_0}(U)$.*

Proof. Let $(K_j)_j$ be a fundamental sequence of compact subsets of U, and let $\varepsilon > 0$. We claim that there exists a sequence of pairwise disjoint finite subsets $(F_j)_j$ such that

$F_j \subset A_j := K_j \setminus K_{j-1}^{\circ}$ and satisfying that for each $f \in h_{v_0}$ with $\|f\|_v = 1$

$$\sup_{x\in A_j} v(x)\left|f(x)\right| \le \varepsilon + \sup_{y\in F_j} v(y)\left|f(y)\right|. \tag{7}$$

This implies that if we denote $F := \cup_j F_j$, then we have

$$\sup_{x\in F} v(x)\left|f(x)\right| \le \|f\|_v \le (1+\varepsilon)\sup_{x\in F} v(x)\left|f(x)\right|, \tag{8}$$

and consequently, if we write F as a sequence $(w_j)_j$, then the linear mapping $T : h_{v_0}(U) \to c_0$, $f \mapsto (v(w_j)f(w_j))_j$ is injective; it has closed range and norm no greater than one, and its inverse T^{-1} defined on a closed subspace of c_0 has norm less than or equal to $1+\varepsilon$. Since this is true for each $\varepsilon > 0$, this is equivalent to our statement.

We proceed to show the claim. For each $j \in \mathbb{N}$ we define

$$M_j := \sup_{x\in K_j} v(x), \qquad a_j := \min\left(\frac{1}{2}d\left(K_j, \mathbb{R}^N \setminus K_{j+1}^{\circ}\right), 1\right), \tag{9}$$

and we select $x^{(j)} \in K_j$ such that $v(x^{(j)}) = \min_{x\in K_j} v(x)$. For each $f \in h_{v_0}(U)$ with $\|f\|_v = 1$ and each $j \in \mathbb{N}$, we have

$$\sup_{x\in K_j}\left|f(x)\right| \le \frac{1}{v\left(x^{(j)}\right)}. \tag{10}$$

If $x \in K_j$ and $\zeta \in D(x, a_j)$ (here $D(x, a_j)$ denotes the closed ball with center x and radius a_j), then from the definition of a_j we deduce that $\zeta \in K_{j+1}$, and hence, together with the previous inequality,

$$\left|f(\zeta)\right| \le \frac{1}{v\left(x^{(j+1)}\right)}. \tag{11}$$

By Cauchy's inequalities for harmonic functions given in [19, 2.4], there exists a positive number C (independent of x) such that, for each $\zeta \in D(x, a_j)$, the ith derivative of f satisfies

$$\left|D_i f(\zeta)\right| \le \frac{C}{a_j v\left(x^{(j+1)}\right)}, \quad i = 1, 2, \ldots, d. \tag{12}$$

By compactness we can get a finite subset F_j of A_j such that

$$A_j \subset \bigcup_{x\in F_j}\left\{x' \in U : \left|x' - x\right| < \delta_j, \left|v\left(x'\right) - v(x)\right| < \delta_j\right\}, \tag{13}$$

where δ_j has been chosen satisfying $\delta_j < a_j$ and

$$\left(\frac{1}{v\left(x^{(j)}\right)} + \frac{M_{j+1}dC}{a_j v\left(x^{(j+1)}\right)}\right)\delta_j < \varepsilon. \tag{14}$$

Consequently, for each $x \in A_j$, there exists $w \in F_j$ with $|w - x| < \delta_j$ and $|v(w) - v(x)| < \delta_j$. On the other hand, $D(x, \delta_j) \subset D(x, a_j) \subset K_{j+1}$ for each $x = (x_1, x_2, \ldots, x_d) \in A_j$. We write $w = (w_1, w_2, \ldots, w_d)$ to get

$$\begin{aligned}\left|f(x)\right| &= \left|f(x_1, \ldots, x_d)\right| \\ &\le \left|f(x_1, \ldots, x_d) - f(w_1, x_2, \ldots, x_d)\right| \\ &\quad + \left|f(w_1, x_2, \ldots, x_d) - f(w_1, w_2, \ldots, x_d)\right| \\ &\quad + \cdots + \left|f(w_1, w_2, \ldots, w_d)\right|.\end{aligned} \tag{15}$$

We apply the mean value theorem to each difference and (12) to get

$$\begin{aligned}\left|f(x)\right| &\le \frac{C}{a_j v\left(x^{(j+1)}\right)} d\left|x - w\right| + \left|f(w)\right| \\ &\le \frac{d\delta_j C}{a_j v\left(x^{(j+1)}\right)} + \left|f(w)\right|.\end{aligned} \tag{16}$$

Now, since $w \in K_{j+1}$,

$$\begin{aligned}v(x)\left|f(x)\right| &\le \left|v(x) - v(w)\right|\left|f(x)\right| + v(w)\left|f(x)\right| \\ &\le \delta_j \frac{1}{v\left(x^{(j)}\right)} + v(w)\left(\frac{d\delta_j C}{a_j v\left(x^{(j+1)}\right)} + \left|f(w)\right|\right) \\ &\le \frac{\delta_j}{v\left(x^{(j)}\right)} + M_{j+1}\frac{d\delta_j C}{a_j v\left(x^{(j+1)}\right)} + v(w)\left|f(w)\right| \\ &\le \varepsilon + v(w)\left|f(w)\right| \le \varepsilon + \max_{w\in F_j} v(w)\left|f(w)\right|.\end{aligned} \tag{17}$$

Therefore,

$$\sup_{x\in A_j} v(x)\left|f(x)\right| \le \varepsilon + \max_{w\in F_j} v(w)\left|f(w)\right|. \tag{18}$$

□

Again as in [17], since each infinite dimensional subspace of c_0 contains a complemented subspace isomorphic to c_0, the following result follows.

Corollary 4. *There are not infinite dimensional subspaces of $h_{v_0}(U)$ which are reflexive.*

In order to get examples where the isomorphism above cannot be an isometry, we will consider a condition which is inspired by the results of Boyd and Rueda [14, Theorem 16, Corollary 17] for weighed spaces defined on bounded open sets of $\mathbb{C}^d$. First we observe that for any weight v for which $h_{v_0}(U)$ contains the polynomials it happens that the functionals $\{\delta_z : z \in U\} \subseteq h_{v_0}(U)^*$ are linearly independent. If U is bounded this is the case for any weight v which tends to zero at the boundary. If $U = \mathbb{R}^d$, then the condition is satisfied when $v = g(|x|)$, with $|\cdot|$ being a norm in $\mathbb{R}^d$ and $g : [0, \infty[\to]0, \infty[$ being a rapidly decreasing continuous function, that is, with $\lim_{t\to\infty} t^n g(t) = 0$ for any $n \in \mathbb{N}$. First we extend easily [14, Lemma 10] to unbounded domains.

Lemma 5. *Let $U \subset \mathbb{R}^d$ be a nonempty open set, and let v be a continuous strictly positive weight which vanishes at infinity on U. If $h_{v_0}(U)$ contains the polynomials, then the map $\mu : U \to (h_{v_0}(U)^*, w^*)$, and $z \mapsto v(z)\delta_z$ is a homeomorphism onto its image.*

Proof. Since the polynomials belong to $h_{v_0}(U)$, we get that $\{\delta_z : z \in U\} \subset h_{v_0}(U)^*$ are linearly independent functionals. We consider the map $\widehat{\mu} : \widehat{U} \to (h_{v_0}(U)^*, w^*)$, $z \to v(z)\delta_z$ if $z \in U$ and $\widehat{\mu}(\infty) = 0$. The very definition of $h_{v_0}(U)$ implies that $\widehat{\mu}$ is continuous, and it is injective by the linear independence of the evaluations. By the compactness of $\widehat{U}$ it follows that $\widehat{\mu}$ is a homeomorphism, and hence also its restriction μ to U. □

Proposition 6. *Let $U \subset \mathbb{R}^d (U \subset \mathbb{C}^d)$ be a nonempty open set, and let v be a weight for which $h_{v_0}(U)(H_{v_0}(U))$ contains the polynomials. Assume that $b_v(U)(B_v(U))$ is not discrete. Then $h_{v_0}(U)(H_{v_0}(U))$ cannot be isometric to any subspace of c_0.*

Proof. We give only the proof for the harmonic case. Denote by $\widehat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ the Alexandroff compactification of $\mathbb{N}$. Let $T : h_{v_0}(U) \to c_0$ be an isometry onto its image. Let $F := T(h_{v_0}(U)) \subseteq c_0$. Since c_0 is the subspace of $C(\widehat{\mathbb{N}})$ formed by the functions which vanish at ∞, the extremal points of the unit sphere of F^* are contained in $\{\lambda\delta_n : \lambda \in \Gamma, n \in \mathbb{N}\}$ (cf. [22, Lemma V.8.6]). Let $Q : F \to h_{v_0}(U)$ be the inverse isometry of T. The transpose linear mapping $Q^t : h_{v_0}(U)^* \to F^*$ is also an isometry, and it maps extremal points to extremal points. Thus for each $z \in b_v(U)$ there exists $\lambda_z \in \Gamma$ and $n(z) \in \mathbb{N}$ such that $Q^t(v(z)\delta_z) = \lambda_z\delta_{n(z)}$. From the linear independence of $\{\delta_z : z \in U\}$ in $h^*_{v_0}$, we conclude that $n(z) \neq n(w)$ for $z, w \in U$ with $z \neq w$, and then $b_v(U)$ is countable and $\{\lambda_z\delta_{n(z)} : z \in b_v(U)\}$ is discrete for the weak* topology, since it is a sequence in the sphere of F^* which is $\sigma(F^*, F)$ convergent to 0. But Q^t is a weak*-weak* homeomorphism. Hence we conclude from Lemma 5 that $b_v(U)$ is discrete. □

Now we apply the set of extremal points of the unit ball of a dual Banach space which is never empty and Lemma 2 to get the following result.

Corollary 7. *If $U \subseteq \mathbb{C}^d$ is a balanced open set and v is a radial weight which vanishes at infinity on U, then neither $H_{v_0}(U)$ nor $h_{v_0}(U)$ is isometric to a subspace of c_0.*

Although we do not have an example of weight v on U such that $b_v(U)$ is a discrete subset of U, we see below that we can reformulate the problem of finding an example of space $h_{v_0}(U)$ which can be isometrically embedded in c_0 in terms of the possible existence of an open set $U \subseteq \mathbb{R}^d$ and a weight v such that $b_v(U)$ is not discrete. Of course the analogue result is also true for the holomorphic case. In view of Lemma 2 this is not possible for radial weights in balanced domains.

Proposition 8. *If $U \subseteq \mathbb{R}^d$ is an open set and v is a weight on U such that $h_{v_0}(U)$ contains the polynomials and $b_v(U)$ is discrete, then $h_{v_0}(U)$ embeds isometrically in c_0.*

Proof. Since the closed unit ball D of $h_{v_0}(U)^*$ is the weak* closure of the absolutely convex hull of the extremal points of the sphere, Lemma 2(a) implies that D is in fact the closure of the absolutely convex hull of $\{v(z)\delta_z : z \in U\}$. Thus this implies that $b_v(U)$ cannot be finite, and since it is discrete we have that $b_v(U) = (x_n)_n$ with x_n tending to infinity on U when n goes to infinity. We get now that D is the weak* closure of the absolutely convex hull of $\{v(x_n)\delta_{x_n} : n \in \mathbb{N}\}$, and hence, for each $f \in h_{v_0}(U)$, it holds:

$$\|f\|_v := \sup_{n\in\mathbb{N}} v(x_n)\,|f(x_n)|. \tag{19}$$

Since x_n tends to infinity on U, we have that the linear map

$$T : h_{v_0}(U) \longrightarrow c_0, \qquad f \longmapsto \left(v(x_n)\, f(x_n)\right)_n \tag{20}$$

is an isometry. □

3. Geometry of Weighed Banach Spaces of Holomorphic and Harmonic Functions and Their Duals

In [15, Theorem 29] Boyd and Rueda showed that if $U \subseteq \mathbb{C}^d$ is a balanced bounded open set and v is a radial weight vanishing at infinity on U such that $P_v(U) = B_v(U)$ then $H_{v_0}(U)$ is not rotund and then neither $h_{v_0}(U)$ is. This condition is trivially satisfied when $P_v(U) = U$. To the best of our knowledge there are not so far concrete examples of open subsets $U \subset \mathbb{C}^d$ and weights v on U such that $B_v(U) \setminus P_v(U) \neq \emptyset$. In the concrete examples of spaces $H_{v_0}(U)$ given in [15] where $B_v(U)$ is calculated the equality $B_v(U) = P_v(U)$ is always satisfied. With a similar proof to the one of [15, Theorem 29], we present below a new condition for spaces $H_{v_0}(U)$ with U balanced and v radial depending only on the size of $P_v(U)$ which ensures that $H_{v_0}(U)$ is not rotund.

Proposition 9. *Let $U \subseteq \mathbb{C}^d$ be the open unit ball for a norm $|\cdot|$ in $\mathbb{C}^d$, and let $v = g(|x|)$ for $g : [0,1] \to [0,\infty[$ be a nonincreasing continuous function with $g(1) = 0$. If there exists $z_0 \in \partial U$ such that the set $\{tz_0 : 0 < t < 1\} \subseteq P_v(U)$, then $H_{v_0}(U)$ is not rotund and then neither $h_{v_0}(U)$ is.*

Proof. Let φ be a linear map on $\mathbb{C}^d$ such that $\|\varphi^2\|_v = 1$. Since $v\varphi^2$ is continuous on $\widehat{U}$ and vanishes at infinity, there exists $z_1 \in U$, $z_1 \neq 0$ such that

$$1 = \left\|\varphi^2\right\|_v = v(z_1)\,\varphi^2(z_1) = v(-z_1)\,\varphi^2(-z_1). \tag{21}$$

Since $|z_0| = 1$, we apply the theorem of Hahn-Banach to get $y \in \mathbb{C}^d$ such that $\langle z_0, y\rangle = 1$ and $|\langle z, y\rangle| \leq |z|$ for each $z \in \mathbb{C}^d$. Let $r_0 := |z_1|$. We define

$$h : U \longrightarrow \mathbb{C}, \qquad z \longrightarrow \varphi^2\left(\frac{\langle z, y\rangle}{r_0} z_1\right). \tag{22}$$

We have

$$v(r_0z_0)\,h(r_0z_0) = g(r_0)\,\varphi^2(z_1) = g(|z_1|)\,\varphi^2(z_1) = v(z_1)\,\varphi^2(z_1) = 1. \tag{23}$$

Since φ is linear we also get

$$v(-r_0z_0)\,h(-r_0z_0) = g(r_0)\,\varphi^2(-z_1) = v(-z_1)\,\varphi^2(-z_1) = 1. \tag{24}$$

For $z \in U$ arbitrary, we use that $|z_1/r_0| = 1$, $|\langle z, y\rangle| \le |z|$, and g is not increasing to compute

$$v(z)\,|h(z)| = g(|z|)\,|h(z)| \le g\left(\left|\langle z, y\rangle\right|\right)|h(z)| = v\left(\frac{\langle z, y\rangle}{r_0}z_1\right)\left|\varphi^2\left(\frac{\langle z, y\rangle}{r_0}z_1\right)\right| \le \left\|\varphi^2\right\|_v = 1. \tag{25}$$

Thus h is in the unit ball of $H_{v_0}(U)$ and

$$v(r_0z_0)\,\delta_{r_0z_0}(h) = v(-r_0z_0)\,\delta_{(-r_0z_0)}(h) = 1. \tag{26}$$

Since r_0z_0 is assumed to be a peak point we can find f in the unit ball of $H_{v_0}(U)$ different from h such that $v(r_0z_0)f(r_0z_0) = 1$ and $v(z)\ \mathrm{Re}(f(z)) < 1$ for each $z \in U \setminus \{r_0z_0\}$. Hence we conclude from

$$\left\|\frac{f+h}{2}\right\|_v = v(r_0z_0)\,\frac{(f(r_0z_0) + h(r_0z_0))}{2} = 1 \tag{27}$$

that $H_{v_0}(U)$ is not rotund. □

Let $g : [0,1] \to [0,\infty[$ be a function such that $g(1) = 0$, g is decreasing, g is twice differentiable in $]0,1[$, and $\log(1/g(t))$ is convex. In [15, Corollary 12] the authors showed that for a weight v defined in the Euclidean unit ball U of $\mathbb{C}^d$ by $v(x) = g(\|x\|)$ the equality $P_v(U) = U$ holds, and then $H_{v_0}(U)$ cannot be rotund by [15, Theorem 29]. Also if $U_1 \subseteq \mathbb{C}^d$ and $U_2 \subseteq \mathbb{C}^k$ are the Euclidean unit balls and v_1 and v_2 satisfy this last condition for certain g_1 and g_2, then $v : U_1 \times U_2 \to]0,\infty[$, $v(x,y) = v_1(x)v_2(y)$ also satisfies $P_v(U) = U$ (cf. [15, Proposition 26]). We present below new examples of spaces $H_{v_0}(U)$ which cannot be rotund. In the required assumptions, besides considering a wider class of open sets, we remove from the weight the condition of differentiability given by Boyd and Rueda as we did in [13, Theorem 1] to give conditions under which $v = \widetilde{v}$, $\widetilde{v}$ being the associated weight introduced by Bierstedt et al. in [23]. The same technique permits us to get concrete radial weights $v(x) = g(\|x\|)$ defined in the Euclidean ball U of $\mathbb{R}^d$ with d odd such that the spaces $h_{v_0}(U)$ are not isometric to any subspace of c_0. If $d = 2k$, then the assertion is a consequence of Corollary 7 since U is the Euclidean unit ball of $\mathbb{C}^k$ and $H_{v_0}(U) \subset h_{v_0}(U)$.

Proposition 10. *Let $g : [0,1] \to [0,\infty[$ be a continuous decreasing function such that $g(1) = 0$ and $\log(1/g)$ is strictly convex in $[0,1[$.*

(a) *Let $U \subseteq \mathbb{C}^d$ be the unit ball for a norm $|\cdot|$ and $v : U \to]0,\infty[$, $x \mapsto g(|x|)$. Then $H_{v_0}(U)$ is not rotund (and then neither $h_{v_0}(U)$ is).*

(b) *Let $U \subseteq \mathbb{R}^d$ be the Euclidean unit ball and $v(x) = g(\|x\|)$ for $x \in U$. Then $h_{v_0}(U)$ is not isometric to any subspace of c_0.*

Proof. Let $h(r) := \log(1/g(r))$, $0 \le r < 1$. Since h is increasing and strictly convex, for each $0 \le r_0 < 1$, the map

$$[0,1[\setminus\{r_0\} \longrightarrow]0,\infty[\,, \qquad r \longmapsto \frac{(h(r) - h(r_0))}{(r - r_0)} \tag{28}$$

is increasing. Hence for each $0 \le r_0 < 1$ there exists $\alpha_0 \ge 0$ which depends on r_0 such that

$$h(r) - h(r_0) > \alpha_0(r - r_0) \tag{29}$$

for all $r \in [0,1[\setminus\{r_0\}$. Now we compute

$$\begin{aligned}\sup_{0\le r<1, r\ne r_0} g(r)\exp(\alpha_0 r) &= \exp\left(\sup_{0<r<1, r\ne r_0}\left(-h(r) + \alpha_0 r\right)\right)\\ &< \exp\left(\sup_{0\le r<1, r\ne r_0}\left(-\alpha_0(r - r_0)\right.\right.\\ &\qquad \left.\left. - h(r_0) + \alpha_0 r\right)\right)\\ &= g(r_0)\exp(\alpha_0 r_0).\end{aligned} \tag{30}$$

We prove (a). Straszewicz's theorem [24, Theorem 7.89] together with the Krein-Milman theorem imply that there exists $z_0 \in \partial U$ which is exposed. Let $y \in \mathbb{C}^d$ such that $\langle z_0, y\rangle = 1$ and $\mathrm{Re}(\langle z, y\rangle) < |z|$ if $z \in U \setminus \{rz_0 : 0 \le r < 1\}$. Let $0 \le r_0 < 1$ be arbitrary, and let α_0 be as above. The function $f : U \to \mathbb{C}$ defined by $f_0(z) := (\exp(\alpha_0\ \langle z, y\rangle))/(g(r_0)\exp(\alpha_0\ r_0))$ is holomorphic and $v(r_0z_0)f_0(r_0z_0) = 1$. It follows immediately from (30) that if $0 \le r < 1$, $r \ne r_0$

$$v(rz_0)\left|f_0(rz_0)\right| = \frac{g(r)\exp(\alpha_0 r)}{g(r_0)\exp(\alpha_0 r_0)} < 1. \tag{31}$$

If $z \in U \setminus \{rz_0 : 0 \le r < 1\}$, then we use (30) and $\mathrm{Re}(\langle z, y\rangle) < |z|$ to get

$$\begin{aligned}v(z)\left|f_0(z)\right| &= g(|z|)\,\frac{\exp(\alpha_0\ \mathrm{Re}(\langle z, y\rangle))}{g(r_0)\exp(\alpha_0 r_0)}\\ &< g(|z|)\,\frac{\exp(\alpha_0|z|)}{g(r_0)\exp(\alpha_0 r_0)} \le 1.\end{aligned} \tag{32}$$

Hence f_0 peaks $v(r_0z_0)\delta_{r_0z_0}$. We conclude that $\{rz_0 : 0 \le r < 1\} \subseteq P_v(U)$ since r_0 is arbitrary. We apply Proposition 9 to get that $H_{v_0}(U)$ is not rotund.

To prove (b) we consider $x_0 \in \mathbb{R}^d$ with $0 < \|x_0\| = r_0 < 1$ and get α_0 again as above. Let A be an orthogonal transformation such that $A(x_0) = (r_0, 0, \ldots, 0)$. For $x = (x_1, \ldots, x_d)$

we take the harmonic function $h(x) := \exp(\alpha_0 x_1)\cos(\alpha_0 x_2)$ and we define $f_0(x) := h \circ A(x)$ for $x \in U$, which is harmonic; see [19, Chapter 1]. A similar argument as in (a) permits us to conclude that $v(x_0)f_0(x_0) = 1$ and $v(x)|f(x)| < 1$ for each $x \in U \setminus \{x_0\}$. Hence $p_v(U) = U$, and therefore $h_{v_0}(U)$ is not isometrically embedded in c_0 by Proposition 6. □

Remark 11. (a) An inspection of the proofs of Propositions 9 and 10 shows that the statements remain true when $U = \mathbb{C}^d$ and $g : [0, \infty[\to]0, \infty[$ is a decreasing continuous function which satisfies that $\log(1/g)$ is strictly convex and $g(t) = o(e^{-at})$ when t goes to infinity for each $a > 0$.

(b) For any increasing function $h : [0, 1[\to [0, \infty[$ which is strictly convex and satisfies $\lim_{t \to 1} h(t) = \infty$ (or $h : [0, \infty[\to]0, \infty[$ with $e^{at} = o(e^{h(t)})$ when t goes to infinity for each $a > 0$), the function $g(t) = e^{-h(t)}$ satisfies the hypothesis of Propositions 9 and 10. The examples given in [15, Example 13] of weights in the Euclidean unit ball can be obtained using this general method.

Acknowledgments

The authors are grateful to J. Bonet for a lot of ideas and discussions during all this work. They are also indebted to P. Rueda, who besides giving them some references provided them some unpublished work which has inspired a big part of this paper. They also thank M. Maestre for his careful reading of the paper and helpful discussions. Finally, they thank the referees for the careful analysis of the paper and the important suggestions which they gave us. The research of the first author was supported by MICINN andFEDER, Project MTM2010–15200. The research of both authors is partially supported by Programa de Apoyo a la Investigación y Desarrollo de la UPV PAID-06-12.

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