The Variational Principle in Transformation Optics Engineering and some Applications

Carlos García-Meca[1] and Michael M. Tung[2]

[1] *Valencia Nanophotonics Technology Center, Universitat Politècnica de València, Valencia, Spain*
e-mail: cargarm2@ntc.upv.es

[2] *Instituto de Matemática Multidisciplinar, Universitat Politècnica de València, 46022 Valencia, Spain*
e-mail: mtung@imm.upv.es

Abstract

Transformation optics specializes in the engineering of advanced optical devices, and in combination with differential geometry it allows to control electromagnetic fields with artificial media in an unprecedented manner. In this work, we model transformation optics in an inherently covariant fashion starting with a fundamental Lagrangian function. As an application, we present the construction of a flat reflectionless immersion lens whose superior performance is important to applications in bio- and nano-technology.

Keywords: *Lagrangian formalism and models, variational principle, differential geometry, transformation optics, nano-technology engineering*

Published in: *Math. Comput. Model.* **57** (2013) 1773–1779. The final publication is available at Elsevier via http://dx.doi.org/10.1016/j.mcm.2011.11.035

1 Introduction

Transformation optics is an emergent field of engineering with a great impact on recent developments in advanced optical devices. With the help of differential geometry, transformation optics allows us to fully control electromagnetic fields in a previously unknown manner [1, 2]. Through a geometric reinterpretation of Maxwell's equations, this technique provides a way to engineer curved spaces for light by using suitable media. On the other hand, variational principles mathematically describe in a concise and elegant way a great variety of natural phenomena, independently of a particular choice of coordinate system. In this work, we extend this concept in a differential-geometric framework to formulate transformation optics in an inherently covariant and coordinate-independent form by postulating a Lagrangian function for the most fundamental description of the optical system. As an application of transformation optics, we present the construction of a novel immersion lens of potential relevance to bio- and nano-technology. In particular, we focus on improving the shape and reflection properties of such optical devices.

The paper is organized as follows. Section 2.1 first introduces the mathematical groundwork, presenting the compact covariant formulation of Maxwell's theory, which is most suitable for a description of electrodynamic phenomena in curved space with arbitrary coordinates. Section 2.2 explains the differential geometric apparatus for finding the appropriate Lagrangian which is to model electrodynamics in media. In Section 2.3 we construct such a Lagrangian, examining all its underlying symmetry properties. We then derive via a variational principle the corresponding equations of motion (Maxwell's macroscopic equations). Next, we arrive at general expressions for transformation optics from this variational principle. As a practical example, Section 3 outlines the construction of an immersion lens with improved features by using a suitable transformation within the elaborated framework.

2 Mathematical basics of transformation optics

2.1 Classical and curved spacetime electrodynamics

Modern optics in its conventional approach is invariably based on Maxwell's macroscopic equations in differential form, describing the electric and magnetic fields produced by matter of charge density ρ and current density $\mathbf{j}$:

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0, \qquad \nabla \cdot \mathbf{B} = 0, \tag{2.1}$$

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c}\mathbf{j}, \qquad \nabla \cdot \mathbf{D} = 4\pi\rho, \tag{2.2}$$

where c denotes the speed of light and we have chosen Gaussian units for practical purposes [3]. As usual, symbols in bold represent vector quantities, whereas symbols in italics represent scalar quantities. The resulting fields are the electric field $\mathbf{E}$, the electric displacement $\mathbf{D}$, the magnetic field $\mathbf{H}$, and the magnetic induction $\mathbf{B}$. Eqs. (2.1) represent the source-free (homogeneous) equations and Eqs. (2.2) the source (inhomogeneous) equations of the electromagnetic field. As an integrability condition for Maxwell's equations, the equation of continuity has to be satisfied

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \tag{2.3}$$

In four-dimensional form, this condition can readily be generalized to any curvilinear coordinate system

$$j^{\mu}{}_{;\mu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\, j^{\mu}\right)_{,\mu} = 0, \tag{2.4}$$

where $j^{\mu} = (c\rho, \mathbf{j})$ is the contravariant current four-vector and $g < 0$ is the determinant of the underlying pseudo-Riemannian metric $g_{\mu\nu}$ with indefinite signature $(-,+,+,+)$. Here and in the following, the Einstein summation convention is implied for equal upper and lower indices with $\mu = 0,1,2,3$, denoting by zero the temporal component and the remaining three indices corresponding to the spatial components. We also employ the standard comma and semicolon notation for partial and covariant derivatives, respectively.

Because of Eqs. (2.1), one can also introduce a scalar potential ϕ and a vector potential $\mathbf{A}$, which both combine to the electromagnetic four-potential $A^{\mu} = (\phi, \mathbf{A})$. In many aspects the four-potential A^{μ} or $A_{\mu} = g_{\mu\nu}A^{\nu}$ describes the electromagnetic field at a more fundamental level than the electric and magnetic field strengths themselves [4]. Its definition is however not unique, but the same physical situation may also be identified by any other field of the form

$$A'_{\mu} = A_{\mu} + \psi_{,\,\mu} \tag{2.5}$$

where ψ is an arbitrary scalar field. This essential symmetry property is the so-called *gauge invariance* [5] and also a guiding principle for conceiving a physical model for electrodynamics in media and thus transformation optics based on a variational approach. In such a model, expressions with $A_\mu A^\mu$ cannot occur due to their lack of gauge invariance. Notwithstanding, the electromagnetic field-strength tensor or Faraday tensor

$$F_{\mu\nu} = 2A_{[\mu;\nu]} = A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu} = 2A_{[\mu,\nu]} \tag{2.6}$$

is easily shown to be gauge-invariant. Moreover, definition Eq. (2.6) makes explicit that $F_{\mu\nu}$ is antisymmetric as is indicated by the usual bracket notation for the anti-symmetrization operator. Any antisymmetric four-tensor of rank 2 has six independent components. In this case, the Faraday tensor just accommodates all six electromagnetic field components:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \tag{2.7}$$

such that

$$E_i = F_{i0} \quad \text{and} \quad B_i = \tfrac{1}{2}\,\epsilon_{ijk} F^{jk}. \tag{2.8}$$

Latin indices only run over spatial values and ϵ_{ijk} denotes the completely antisymmetric three-dimensional Levi-Civita symbol.

With Eq. (2.6) it is a straightforward exercise in tensor analysis to prove that

$$F_{[\lambda\mu;\nu]} = 0. \tag{2.9}$$

Note that by substituting Eq. (2.8) into Eq. (2.9), one recovers the two source-free Maxwell equations. Hence, Eq. (2.9) is just the covariant form of Eq. (2.1).

The remaining Maxwell equations, which link the sources to the electromagnetic fields, are given by

$$F^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{-g}} \left(\sqrt{-g} F^{\mu\nu}\right)_{,\nu} = \frac{4\pi}{c} j^\mu \tag{2.10}$$

and are the covariant form of the inhomogeneous equations (2.2), representing Gauss's and Ampère's law together succinctly. Again, this equivalence may be checked by direct substitution of Eq. (2.8) into Eq. (2.10).

In summary, Eqs. (2.9) and (2.10) comprise Maxwell's equations in an inherently covariant manner and permit to describe electromagnetic phenomena on pseudo-Riemannian manifolds independent of the choice of coordinate frame. Differential-geometric methods provide the ideal framework to recast transformation optics in such a compact and elegant way. In the following subsections we set up the necessary mathematical groundwork for modelling phenomena in the domain of transformation optics.

2.2 Lagrangian framework

In order to describe electromagnetic processes, we introduce the four-potential A^μ on a smooth four-dimensional manifold M endowed with a Lorentzian metric $\mathbf{g}$ with mixed signature $(-,+,+,+)$, where $x^0/c \geq 0$ will denote the time and $x^i \in \mathbb{R}$, $i = 1, 2, 3$, will be the parameters to specify the location. A non-vanishing Faraday $F_{\mu\nu}$ tensor is the driving force of the free electromagnetic field and the current j^μ is related to interaction with charged matter, if present.

Following the terminology of Marsden *et al.* [6], we further identify *base space* $B = M$, which constitutes standard spacetime, and *ambient space* P, given by the four-potential $A^\mu : B \to M$.

Then, $N = B \times P$ is the *configuration space* with coordinates $(x^0, x^i, A^0, A^j) \simeq (ct, \mathbf{x}, \phi, \mathbf{A})$, so that any particular constellation or state of the system is uniquely determined by the mapping $B \to N$.

We seek a general principle to find the evolution of A^μ on the given manifold $(M, \mathbf{g})$. In geometric mechanics [7], the Lagrangian function L completely governs the behaviour of a deterministic system, and it suffices to define L on a tangent bundle TM, which is identical with the corresponding phase space. The tangent bundle TM consists of the manifold M and its tangent spaces T_pM for all $p \in M$.

In this case, however, we will require the partial derivatives of a configuration with respect to all *spacetime* coordinates. This generalization leads to the jet bundle $J^1N = B \times TP$, see *e.g.* [8] and references therein. The corresponding Lagrangian function in the jet-bundle description of first-order classical field theories will therefore be a mapping

$$L : J^1N \to \mathbb{R}. \tag{2.11}$$

2.3 Hamilton's principle and transformation optics

We now proceed with the construction of a feasible Lagrangian function L which will enable us derive Maxwell's macroscopic equations. In the case of electromagnetism, the theory not only has to possess gauge invariance (charge conservation), but also has to possess invariance with respect to time and space translation (energy-momentum conservation). This imposes a considerable constraint on L, and with the additional assumption that we deal with a local field theory, no second- or higher-order derivatives of the field variable A^μ can appear.

Hamilton's principle states that the dynamics or field dynamics of a physical system is governed by a simple variational principle (see *e.g.* Refs. [9, 10] in the case of diffusion), whose solutions of equations of motion are determined by the extremum of the action functional $\mathcal{A}$. For electrodynamics the variation of the following action integral must vanish:

$$\delta\mathcal{A} = \delta \int\limits_{\Omega} d^4x\sqrt{-g}\; L(A_\alpha, A_{\alpha;\beta}) = 0. \tag{2.12}$$

The invariant volume element is $d^4x\sqrt{-g} = dx^0dx^1dx^2dx^3\sqrt{-g}$, indicating that integration occurs over spacetime. We let $\Omega \subset M$ be a bounded, closed set of spacetime [7].

The Lagrangian should contain a quadratic term in $F_{\mu\nu}$, generally regarded as the 'kinetic energy' of the field. It will also contain the usual 'interaction term' between field and charged matter. Considering the simplest possible form of the Lagrangian satisfying all of the aforementioned constraints, we then postulate the following form for the *Lagrangian density*

$$\mathcal{L} = \sqrt{-g}\,L = \sqrt{-g}\left(-\frac{1}{32\pi}X^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} - \frac{1}{c}j^\alpha A_\alpha\right) \tag{2.13}$$

where the coefficients are chosen for later convenience, and $X^{\mu\nu\rho\sigma}$ denotes the so-called *constitutive tensor*, which couples linearly the field to yield the kinetic term. It contains the total dependence of permittivity ε, permeability μ and bianisotropy tensor κ for the macroscopic medium in the most general linear form. On the other hand, it can be interpreted geometrically, which establishes the key concept of transformation optics.

As outlined before, the Faraday tensor Eq. (2.6) by itself is already gauge-invariant under transformation Eq. (2.5), so $X^{\mu\nu\rho\sigma}$ can be assumed to be independent of the field A_α. It is not obvious that the interaction term $j^\alpha A_\alpha$ be gauge-invariant. It will, however, provide a correct gauge-invariant contribution to the action Eq. (2.12) by requiring the source j^μ to vanish on the boundary $\partial\Omega$ and following the standard procedure of integration by parts, which eventually gives a zero surface integral.

It is customary to also introduce the *excitation tensor* defined by

$$G^{\mu\nu} = \tfrac{1}{2} X^{\mu\nu\rho\sigma} F_{\rho\sigma}. \tag{2.14}$$

The six independent equations provided by expression (2.14) are called the *(linear) constitutive relations* for macroscopic media.

By inspecting Eq. (2.13), the constitutive tensor evidently has pairwise antisymmetry

$$X^{\mu\nu\rho\sigma} = X^{[\mu\nu][\rho\sigma]} \tag{2.15}$$

and block symmetry

$$X^{\mu\nu\rho\sigma} = X^{\rho\sigma\mu\nu}. \tag{2.16}$$

The cyclic symmetry property is best seen in void space where

$$X^{\mu\nu\rho\sigma} = g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}. \tag{2.17}$$

This can be verified by substituting (2.17) into (2.13), which results in the well-known Lagrangian for electrodynamics in vacuum. Eq. (2.17) immediately yields $X^{[\mu\nu\rho\sigma]} = g^{[\mu\rho} g^{\nu\sigma]} - g^{[\mu\sigma} g^{\nu\rho]} = 0$ and by general covariance in any reference frame it must hold

$$X^{[\mu\nu\rho\sigma]} = 0. \tag{2.18}$$

Note that these symmetries were also found in Ref. [11], but not directly derived from a fundamental Lagrangian as in our approach. Some properties and restrictions of the Lagrangian density are also addressed, although without giving a complete explicit expression and not in the context of transformation optics. Taking into account all symmetry properties Eqs. (2.15)–(2.18), allows for exactly 20 independent components of $X^{\mu\nu\rho\sigma}$ in spacetime, similar to the Riemann curvature tensor. This already hints at the fact that $X^{\mu\nu\rho\sigma}$ bears some non-trivial geometric content. It is common practice to absorb the metric determinant of the volume element in Eq. (2.12) into the following quantities to yield tensor densities of weight +1:

$$\chi^{\mu\nu\rho\sigma} = \sqrt{-g}\, X^{\mu\nu\rho\sigma} \quad \textit{(constitutive tensor density)} \tag{2.19}$$

$$\mathcal{J}^{\alpha} = \sqrt{-g} j^{\alpha} \quad \textit{(current density)} \tag{2.20}$$

$$\mathcal{G}^{\alpha} = \tfrac{1}{2} \chi^{\mu\nu\rho\sigma} F_{\rho\sigma} \quad \textit{(excitation tensor density)} \tag{2.21}$$

Observe also that in completely analogous manner to Eq. (2.8), we introduce

$$D_i = H_{i0} \quad \text{and} \quad H_i = \tfrac{1}{2}\, \epsilon_{ijk} G^{jk}, \tag{2.22}$$

after exploiting all symmetry properties of $X^{\mu\nu\rho\sigma}$ in Eq. (2.14). This readily gives in matrix form for the excitation tensor density

$$\mathcal{G}^{\mu\nu} = \sqrt{-g}\, G^{\mu\nu} = \sqrt{-g} \begin{pmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & H_z & -H_y \\ -D_y & -H_z & 0 & H_x \\ -D_z & H_y & -H_x & 0 \end{pmatrix}. \tag{2.23}$$

After outlining the physical and geometrical meaning of the macroscopic fields relevant to electrodynamics in media, we now proceed by substituting the chosen Lagrangian density Eq. (2.13) into Eq. (2.12) to obtain the following variational principle

$$\delta\mathcal{A} = \delta \int d^4x \left(-\frac{1}{32\pi} \chi^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} - \frac{1}{c} \mathcal{J}^\alpha A_\alpha \right) = 0, \tag{2.24}$$

where again the variation only takes places in the field, namely δA_μ. The solution is provided by the corresponding Euler-Lagrange equations, which in combination with Eq. (2.9) fully control the dynamics of the underlying physical system. As is well known, they can be determined by computing the associated functional derivative and requiring it to be zero:

$$\frac{\delta\mathcal{L}}{\delta A_\alpha} = \frac{\partial\mathcal{L}}{\partial A_\alpha} - \left(\frac{\partial\mathcal{L}}{\partial A_{\alpha,\beta}} \right)_{,\beta} = 0. \tag{2.25}$$

The explicit calculation is lengthy but straightforward and based on the following partial results using definitions Eqs. (2.20) and (2.21) to yield

$$\frac{\partial\mathcal{L}}{\partial A_\alpha} = -\frac{1}{c}\,\mathcal{J}^\alpha \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial A_{\alpha,\beta}} = -\frac{1}{4\pi}\,\mathcal{G}^{\alpha\beta}. \tag{2.26}$$

Inserting Eqs. (2.26) into Eq. (2.25), one finally arrives at the covariant source expression

$$\mathcal{G}^{\alpha\beta}{}_{,\beta} = \frac{4\pi}{c}\,\mathcal{J}^\alpha \tag{2.27}$$

which are just the two inhomogeneous Maxwell equations for optics in macroscopic media, displaying a structure similar to the vacuum case, Eq. (2.10).

Now we are in the position to derive the fundamental expressions of transformation optics from the Lagrangian formalism we have established. First, imagine some *virtual space* characterized by a metric g', in which light propagates as desired. For simplicity, we assume that this space is void (the procedure can be straightforwardly extended to non-empty virtual spaces), so that the constitutive tensor is given by Eq. (2.17). Thus, its corresponding Lagrangian density reads

$$\mathcal{L}' = \sqrt{-g'} \left[-\frac{1}{32\pi} \left(g'^{\mu\rho} g'^{\nu\sigma} - g'^{\mu\sigma} g'^{\nu\rho} \right) F'_{\mu\nu} F'_{\rho\sigma} - \frac{1}{c} j'^\alpha A'_\alpha \right]. \tag{2.28}$$

On the other hand, Eq. (2.13) represents the *physical space* we live in. We want light to propagate in physical space as it does in virtual space. For this, we reinterpret the coordinates of virtual space as those of physical space (in exceptional cases they may coincide from the onset) and take advantage of the fact that both Lagrangians (and therefore the resulting solutions for the fields) are formally identical,[1] provided that we fill physical space with the following media and sources:[2]

$$X^{\mu\nu\rho\sigma} = \frac{\sqrt{-g'}}{\sqrt{-g}} \left(g'^{\mu\rho} g'^{\nu\sigma} - g'^{\mu\sigma} g'^{\nu\rho} \right) \quad \text{and} \quad j^\alpha = \frac{\sqrt{-g'}}{\sqrt{-g}} j'^\alpha. \tag{2.29}$$

This way we are implementing the desired geometry for light in physical space by using the appropriate medium and sources. This powerful theory allows us to realize both, curved-space

[1]Note that $\mathcal{L}$ and $\mathcal{L}'$ could differ by a boundary term on $\partial\Omega$ in Eq. (2.12) without having any physical consequences.

[2]$X^{\mu\nu\rho\sigma}$ will reverse sign for coordinate transformations that do not preserve orientation.

metrics (*e.g.*, that of black holes) in a flat physical space, and/or any deformation caused by coordinate changes [1,2]. Naturally, for deformations arising exclusively from transformations, g and g' represent the same geometry expressed in different coordinate systems. We will use this last approach in the next section. Expressions similar to (2.29) were obtained in [12] starting from Maxwell's equations instead of from a variational principle. It is worth mentioning that there is an erroneous extra factor of $\frac{1}{2}$ in the expression for $X^{\mu\nu\rho\sigma}$ in [12].

3 Flat reflectionless immersion lens

An immersion lens made up of a dielectric medium with refractive index n can improve the diffraction-limited resolution of free space by a factor of $1/n$, see Ref. [13]. The usual geometry of an immersion lens is that of a hemisphere, although other curved surfaces can be used. Thus, one side of the system is flat, while the other one is curved. For some applications, it would be desirable that both surfaces were flat. In addition, there appear reflections at the boundary of the lens with free space due to the difference in the refractive indices of both media. We want to overcome both drawbacks. Our aim is to use the far-field optical system to reconstruct a certain electromagnetic field distribution in vacuum, for instance in the plane $z = 0$ (in Cartesian coordinates), far away (in terms of wavelength) from the optical system. The spatial resolution of our system will be limited to approximately the free-space wavelength λ_0. However, if the whole system were immersed in a dielectric medium with refractive index n_1,the limiting resolution would be the wavelength in that medium λ_0/n_1. We would like to transfer this higher resolution power to free space in such a way that we can use our external optical system to produce an image with that resolution. We will limit ourselves to a TE (transverse electric) two-dimensional problem (invariant in the y-direction along which $\mathbf{E}$ is polarized), in which $\mathbf{E}$ and $\mathbf{H}$ are then completely determined by $E_y(x, z)$. Let us consider two different cases. In the first one we have a certain field distribution $E_y^{(1)}(x, 0)$ in free space. In the second, we have a distribution $E_y^{(2)}(x, 0)$ in a dielectric medium with index n_1, which is a compressed version of $E_y^{(1)}$ such that $E_y^{(2)}(x, 0) = E_y^{(1)}(n_1 x, 0)$. We can use the angular spectrum decomposition to express E_y in any z-plane as a superposition of plane waves [14]:

$$\hat{E}_y(k_x, z) = \int_{-\infty}^{\infty} dx\, E_y(x, z)\, e^{-ik_x x},\, E_y(x, z) = \int_{-\infty}^{\infty} dk_x \hat{E}_y(k_x, z)\, e^{ik_x x}, \tag{3.1}$$

where k_x is the transverse component of the wave vector $\mathbf{k}$, with modulus $|\mathbf{k}| = k = k_0 n$, with k_0 being the free-space wavenumber. At a distance $z = d$, the disturbance in Fourier space is given by Goodman [14]:

$$\hat{E}_y(k_x, d) = \hat{E}_y(k_x, 0)\, e^{i\sqrt{k^2 - k_x^2}d}. \tag{3.2}$$

With the help of Eqs. (3.1)–(3.2), it can be deduced that

$$E_y^{(2)}(x, d/n_1) = E_y^{(1)}(n_1 x, d). \tag{3.3}$$

If we could stretch by a factor of n_1 the field $E_y^{(2)}(x, d/n_1)$ of the dielectric, it would be equal to that resulting from the propagation of a distance d of $E_y^{(1)}$ in free space. Transformation optics provides a way to design the required medium to achieve this field deformation and obtain $E_y^{(2)}(x/n_1, d/n_1)$ from $E_y^{(2)}(x, d/n_1)$. Specifically, we will employ the transformation

$$t' = t, \qquad x' = x/(1 + Cz), \qquad y' = y, \qquad z' = z, \tag{3.4}$$

where C is a constant, and use (2.29) to calculate the properties of our device (note that $j^\alpha = 0$ in our case). Primed and unprimed coordinates correspond to virtual and physical space, respectively. Moreover, it has been shown that squeezers based on transformation optics are reflectionless for TE waves if the output medium is a dielectric with a refractive index equal to the compression factor [15]. Following a similar reasoning, it can be proved that an expanding device is reflectionless if the medium to be expanded has a refractive index equal to the expansion factor and the output medium is free space. In fact, for TE waves, the reflection coefficient for arbitrary input (the background media to be expanded) and output media, characterized by relative constitutive parameters ε_{in}, μ_{in}, and ε_{out}, μ_{out} is

$$R_{\text{TE}} = \frac{\mu_{\text{out}}\sqrt{k_0^2\varepsilon_{\text{in}}\mu_{\text{in}}F_x^2 - k_x^2} - \mu_{\text{in}}F_y\sqrt{k_0^2\varepsilon_{\text{out}}\mu_{\text{out}} - k_x^2}}{\mu_{\text{out}}\sqrt{k_0^2\varepsilon_{\text{in}}\mu_{\text{in}}F_x^2 - k_x^2} + \mu_{\text{in}}F_y\sqrt{k_0^2\varepsilon_{\text{out}}\mu_{\text{out}} - k_x^2}}, \tag{3.5}$$

where F_x and F_y are the compression factors in the x and y directions, respectively. In our case, $F_x = 1/n_1 = 1 + Cd$ (inverse expansion factor in the x-direction), $F_y = \mu_{\text{in}} = \mu_{\text{out}} = \varepsilon_{\text{out}} = 1$, and $\varepsilon_{\text{in}} = n_1^2$. As a consequence, the device is reflectionless ($R_{\text{TE}} = 0$). With all the previous results in mind, the idea is as follows. We have a certain electromagnetic field distribution that we want to image in the far field with a resolution of λ_0/n_1. For instance, two illuminated punctual objects separated by a distance of λ_0/n_1 in the x-direction. We put this object very near from (or embed it in) a medium with a refractive index of n_1, where the limiting resolution is λ_0/n_1. Now we modify a section of this medium, which will be adjacent to air, in order to expand the fields by a factor of n_1 at a distance $z = d/n_1$ away from the source. We cut the modified dielectric medium exactly at this point $z = d/n_1$, where we have the expanded fields, leaving free space on the right side of the device. We know that there will be no reflections at this interface, since we used the adequate expanding factor. Thus, the fields exiting the device are the same as those that two punctual sources in free space separated by a distance of λ_0 would generate. This way, we can obtain a magnified image (with a magnification factor n_1) of the two original sources with our optical system, whose resolution is limited to λ_0. This magnifying lens is flat and reflectionless.

To verify our theoretical predictions, we performed numerical calculations with COMSOL Multiphysics. Specifically, we designed a lens with a magnifying factor $n_1 = 3$. To test the device, two sources separated by $\lambda_0/2$ are placed inside a medium with n_1. When the fields radiated by these sources directly exit to air, *viz.* Figure 1(a), reflections appear. In addition, the radiation pattern in air is that of two sources separated by $\lambda_0/2$. When we use the magnifying lens between the medium with $n_1 = 3$ and air, no reflections appear and the radiation pattern is that of two sources separated by $1.5\lambda_0$, *viz.* Figure 1(b). By using air's inverse transfer function for propagating waves, we can calculate the fields that our external optical system would reconstruct for the cases with and without lens ($\mathbf{E}_{\text{TO lens}}$ and $\mathbf{E}_{\text{no lens}}$, respectively), see Figure 1(c). In the first case the two sources can be clearly observed, while in the second one, some components are lost and we only detect a broad unique source. Note that in the image obtained with the lens there is a magnification factor of 3. This is equivalent to having the two sources in air and separated by a distance of $1.5\lambda_0$. We verified this by comparing the image reconstructed from the analytically calculated fields radiated by two sources separated by $1.5\lambda_0$ ($\mathbf{E}_{\text{analytic}}$) with $\mathbf{E}_{\text{TO lens}}$, observing an excellent agreement. We also include the fields we would reconstruct if we used a classical hemispherical immersion lens ($\mathbf{E}_{\text{classical lens}}$), whose amplitude is lower than that of $\mathbf{E}_{\text{TO lens}}$ due to reflections. Finally, we show in Figure 1(d) the effect of separating the sources from the lens origin. Clearly, this does not affect the performance of the lens based on transformation optics, while the classical lens introduces some distortion, even though the sources are only 1μm away from the origin. Note that, when used in the reverse direction, the proposed immersion lens works as a perfect coupler from free-space to a dielectric medium of

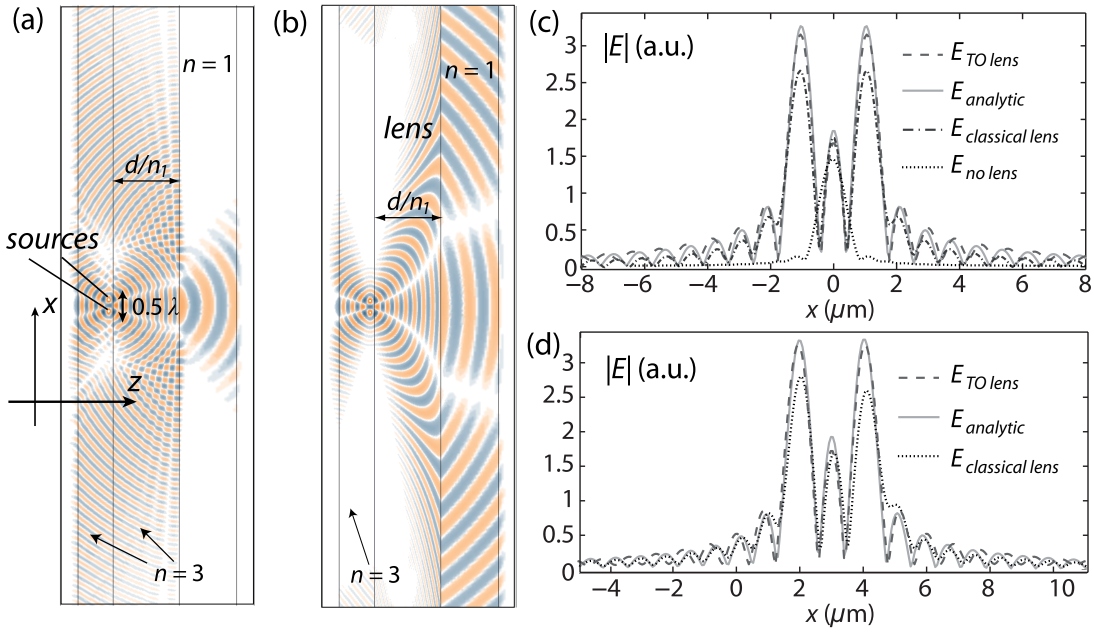

Figure 1: Electric field generated by two sources embedded in a dielectric with $n = 3$ (a) without and (b) with the designed lens between the dielectric and air ($n = 1$). (c) Reconstructed field amplitude for different cases. (d) Same as in (c) but with the sources 1μm away from the lens centre. The working free-space wavelength is $\lambda_0 = 1.5\mu$m.

index n_1. In this sense, it could be used to compress wide light beams and couple them to high-index nanophotonic waveguides.

4 Conclusions

We have developed a conceptual framework to represent macroscopic electrodynamics in media by a Lagrangian on a pseudo-Riemannian manifold, which straightforwardly allows to identify any virtual space possessing especially designed optical properties with real, physical space. This novel approach encompasses all previous formulations of transformation optics in a compact and elegant way. We applied this technique to design an immersion lens with superior properties for possible application in bio- and nano-technology.

Acknowledgements

This work has been supported by the Spanish *Ministerio de Ciencia e Innovación* under the grant MTM2009-08587, contract CSD2008-00066, and the FPU programme.

References

[1] J. B. Pendry, D. Schurig, and D. R. Smith, Controlling Electromagnetic Fields, Science **312** (2006) 1780–1782.

[2] U. Leonhardt and T. G. Philbin, General Relativity in Electrical Engineering, New J. Phys. **8** (2006) 247.

[3] J. D. Jackson, *Classical Electrodynamics*, Wiley & Sons, Inc., New York, 1998.

[4] Y. Aharonov and D. Bohm, Significance of Electromagnetic Potentials in the Quantum Theory, Phys. Rev. **115** (1959) 485–491.

[5] H. Weyl, Elektron und Gravitation, Z. Phys. **56** (1929) 330–352.

[6] A. Lew, J.E. Marsden, M. Ortiz and M. West, Asynchronous Variational Integrators, Arch. Ration. Mech. Anal. **167** (2003) 85–146.

[7] O. Calin and D.-C. Chang, *Geometric Mechanics on Riemannian Manifolds*, Birkhäuser & Springer Science, Boston, 2005.

[8] A. Echeverría-Enríquez, M. C. Muñoz-Lecanda, and N. Román-Roy, Non-standard connections in classical mechanics, J. Phys. **A28** (1995) 5553–5567.

[9] M. M. Tung, Basics of a differential-geometric approach to diffusion: Uniting Lagrangian and Eulerian models on a manifold, Mathematics in Industry **12** (2007) 897–901.

[10] M. M. Tung and A. Hervás, A differential-geometric approach to model isotropic diffusion on circular conic surfaces in uniform rotation, Mathematics in Industry **15** (2010) 1053–1058.

[11] E. J. Post, *Formal Structure of Electromagnetics*, Dover, New York, 1962.

[12] L. Bergamin, Generalized transformation optics from triple spacetime metamaterials, Phys. Rev. A **78** (2008) 43825.

[13] S. M. Mansfield and G. S. Kino, Solid immersion microscope, Appl. Phys. Lett. **57** (1990) 2615–2616.

[14] J. W. Goodman, *Introduction to Fourier Optics*, 2nd edition, McGraw-Hill, New York, 1996.

[15] C. García-Meca, M. M. Tung, J. V. Galán, R. Ortuño, F. J. Rodríguez-Fortuño, J. Martí, and A. Martínez, Squeezing and expanding light without reflections via transformation optics, Opt. Express **19**(4) (2011) 3562–3575.