On Yager and Hamacher t-norms and fuzzy metric spaces

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Abstract

Recently, V. Gregori, S. Morillas and A. Sapena have discussed [On a class of completable fuzzy metric spaces, Fuzzy Sets and Systems **161** (2011), 2193-2205] the so-called strong fuzzy metrics when looking for a class of completable fuzzy metric spaces in the sense of A. George and P. Veeramani and state the question of finding a non-strong fuzzy metric space for a continuous t-norm different from the minimum. Later on, J. Gutiérrez García and S. Romaguera solved this question [Examples of non-strong fuzzy metrics, Fuzzy Sets and Systems **162** (2011), 91-93] by means of two examples for the product and the Lukasiewicz t-norm, respectively. In this direction they posed to find further examples of non-strong fuzzy metrics for continuous t-norm that are greater than the product but different from minimum. In this paper we obtain an example of this kind. On the other hand, P. Tirado established in [Contraction mappings in fuzzy quasi-metric spaces and [0,1]-fuzzy posets, Fixed Point Theory 13:273-283, 2012] a fixed point theorem in fuzzy metric spaces which was successfully used to prove the existence and uniqueness of solution for the recurrence equation associated to the Probabilistic Divide and Conquer Algorithms. Here we generalize this result by using a class of continuous t-norms known as ω -Yager t-noms.

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1 Introduction

I. Kramosil and J. Michalek introduced in [9] their celebrated notion of fuzzy metric space (KM-fuzzy metric space, in the sequel) which constitutes a reformulation of the concept of probabilistic metric spaces to the fuzzy setting. In particular, they showed the equivalence between KM-fuzzy metric spaces and Menger spaces. Sherwood proved in [10] that every Menger space has a completion which is unique up to isometry, so one can easily deduce that every KM-fuzzy metric space has a completion which is unique up to isometry. Later on, A. George and P. Veeramani gave a slight modification of KM-fuzzy metric space (GV-fuzzy metric space, in the sequel). However it is well known [4] that, contrarily to the KM-case, there exist GV-fuzzy metric spaces that are not completable. Motivated by this fact, Gregori, Morillas and Sapena have discussed in [3] the so-called strong GV-fuzzy metrics when looking for a class of completable GV-fuzzy metric spaces. In particular, they state the question of finding a non-strong GV-fuzzy metric space for a continuous t-norm different from the minimum. Answering that question, Gutiérrez García and Romaguera presented in [5] two examples of non-strong GV-fuzzy metrics for the product and the Lukasiewicz t-norm, respectively. In this direction they became interested in finding further examples for continuous t-norms that are greater than the product but different from minimum.
In this paper we obtain an example of this kind by means of a class of continuous t-norms known as Hamacher t-norms ([7]). On the other hand, another class of celebrated continuos t-norms known as Yager t-norms ([12]) is used to generalize a fixed point theorem in KM-fuzzy metric spaces. Several basic results in this paper were presented by the authors at the 10th International Conference of Numerical Analysis and Applied Mathematics, ICNAAM 2012 ([1]).

Throughout this paper the letters ω and $\mathbb{N}$ will denote the set of nonnegative integer numbers and the set of positive integer numbers, respectively.

Recall that by a metric on a nonempty set X we mean a nonnegative real valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $x = y$ if and only if $d(x, y) = 0$; (ii) $d(x, y) = d(y, x)$; (iii) $d(x, y) \leq d(x, z) + d(z, y)$. If d satisfies the conditions (i), (ii) and (iii') $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ then, d is called an ultrametric.

A t-norm is a binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ that satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $x * 1 = x$ for every $x \in [0, 1]$; (iii) $r * x \leq y * z$ whenever $r \leq y$ and $x \leq z$, for $r, x, y, z \in [0, 1]$. If, in addition, $*$ is continuous, then $*$ is called a continuous t-norm.

Paradigmatic examples of continuous t-norms are the minimum, denoted by $\wedge$, the usual product, denoted by $\cdot$ and the Lukasiewicz t-norm, denoted by $*_L$, where $x *_L y = \max\{x + y - 1, 0\}$. They satisfy the following well-known inequalities: $x *_L y \leq x \cdot y \leq x \wedge y$. In fact, $x * y \leq x \wedge y$ for each t-norm $*$.

Definition 1 [9]. A KM-fuzzy metric on a (non-empty) set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, \infty)$ such that for all $x, y, z \in X$: (KM1) $M(x, y, 0) = 0$; (KM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$; (KM3) $M(x, y, t) = M(y, x, t)$ for all $t > 0$; (KM4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s > 0$; (KM5) $M(x, y, \_)$ is left continuous on $[0, \infty)$.

A triple $(X, M, *)$ where X is a (non-empty) set and $(M, *)$ is a KM-fuzzy metric on X, is said to be a KM-fuzzy metric space. It is well known that condition (KM4) implies that $M(x, y, \_)$ is non-decreasing for all $x, y \in X$.

Definition 2 [2]. A GV-fuzzy metric on a (non-empty) set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ such that for all $x, y, z \in X$; $t, s > 0$: (GV1) $M(x, y, t) > 0$; (GV2) $M(x, y, t) = 1$ if and only if $x = y$; (GV3) $M(x, y, t) = M(y, x, t)$; (GV4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$; (GV5) $M(x, y, \_)$ is continuous on $(0, \infty)$.

A triple $(X, M, *)$ where X is a (non-empty) set and $(M, *)$ is a GV-fuzzy metric on X, is said to be a GV-fuzzy metric space. Obviously, each GV-fuzzy metric can be considered as a KM-fuzzy metric by defining $M(x, y, 0) = 0$ for all $x, y \in X$, so if $(X, M, *)$ is a GV-fuzzy metric space we have that $M(x, y, \_)$ is non-decreasing for all $x, y \in X$.

2 Hamacher t-norms and non-strong fuzzy metrics

A special type of GV-fuzzy metrics has been recently considered in [3] under the name of strong fuzzy metric, whose definition is given as it follows.

Definition 3 [3]. Let $(X, M, *)$ be a GV-fuzzy metric space. Then $(M, *)$ is said to be *strong* if it satisfies the following additional axiom: $M(x, z, t) \geq M(x, y, t) * M(y, z, t)$ for all $x, y, z \in X$ and all $t > 0$.

Note that this definition can be also given in the KM-sense. Note that a KM-fuzzy metric is strong if and only if it is a non-Archimedean fuzzy metric in the sense of [6].

Remark 1. Clearly (see for instance [3]), if d is a metric on a set X, then the GV-fuzzy metric $(M_d, *)$ is strong for every continuous t-norm $*$ such that $* \leq \cdot$, where M_d is defined by $M_d(x, y, t) = t/(t+d(x, y))$, for all $x, y \in X$ and $t > 0$. In Section 3 of [3] the authors observed that, however, the GV-fuzzy metric $(M_d, \wedge)$ is strong if and only if d is an ultrametric. Then, they posed the natural question of finding a non-strong GV-fuzzy metric $(M, *)$ where $*$ is not $\wedge$. Gutiérrez García and Romaguera solved this questions by means of two examples. Here we give one of them.

Example 1. Let $X = \{x, y, z\}$, $* = \cdot$ and $M : X \times X \times (0, \infty) \to [0, 1]$ defined for each $t > 0$ as: $M(x, x, t) = M(y, y, t) = M(z, z, t) = 1$; $M(x, z, t) = M(z, x, t) = M(y, z, t) = M(z, y, t) = \frac{t}{t+1}$; $M(x, y, t) = M(y, x, t) = \frac{t^2}{(t+2)^2}$; then $(M, \cdot)$ is a non-strong GV-fuzzy metric on X.

Then, they posed the natural question of finding further examples for continuous t-norms that are greater than the product but different from minimum. The following example solves this question. First we recall some well-known facts and definitions.

In [7] (see also [8, page 106]) H. Hamacher presented a family of continuous t-norms $(*_\lambda)_{\lambda \in [0,\infty)}$ given by: $x *_\lambda y = 0$ if $x = y = \lambda = 0$ and $x *_\lambda y = \frac{xy}{\lambda+(1-\lambda)(x+y-xy)}$ otherwise. Note that $x \cdot y \leq x *_\lambda y$ for each $\lambda \in [0, 1]$. In particular for $\lambda = 1$ we have the following equality: $x *_1 y = x \cdot y$. In the sequel $*_\lambda, \lambda \in [0, \infty)$ will be called a Hamacher t-norm.

Example 2. Let $X = \{x, y, z\}$, $* = *_{1/2}$, i.e $a *_{1/2} b = \frac{2ab}{1+a+b-ab}$ for all $a, b \in [0,1]$, and $M : X \times X \times (0, \infty) \to [0,1]$ defined for each $t > 0$ as: $M(x,x,t) = M(y,y,t) = M(z,z,t) = 1$; $M(x,z,t) = M(z,x,t) = M(y,z,t) = M(z,y,t) = \frac{t}{t+3}$; $M(x,y,t) = M(y,x,t) = \frac{t^2}{(t+3)^2}$. It is easy to check that $(M,*)$ satisfies (GV1)-(GV3) and (GV5). With respect to (GV4), we have:

$$M(x,y,t+s) = \frac{(t+s)^2}{(t+s+3)^2} = \frac{t^2+s^2+2st}{t^2+s^2+2st+6t+6s+9}$$

and

$$M(x,z,t) * M(z,y,s) = \frac{t}{t+3} * \frac{s}{s+3} = \frac{2ts}{2ts+6t+6s+9},$$

so

$$M(x,y,t+s) - M(x,z,t) * M(z,y,s) \geq 0 \Longleftrightarrow$$

$$\frac{t^2+s^2+2st}{t^2+s^2+2st+6t+6s+9} - \frac{2ts}{2ts+6t+6s+9} \geq 0.$$

Since

$$\frac{t^2+s^2+2st}{t^2+s^2+2st+6t+6s+9} - \frac{2ts}{2ts+6t+6s+9}$$
$$= \frac{6t^3+6st^2+9t^2+6s^2t+6s+9s^2}{(t^2+s^2+2st+6t+6s+9)(2ts+6t+6s+9)} \geq 0,$$

it follows $M(x,y,t+s) \geq M(x,z,t) * M(z,y,s)$. Moreover,

$$M(x,z,t+s) = \frac{t+s}{t+s+3} \geq \frac{s}{s+3} = M(y,z,s) \geq M(x,y,t) * M(y,z,s),$$

and

$$M(y,z,t+s) = \frac{t+s}{t+s+3} \geq \frac{s}{s+3} = M(x,z,s) \geq M(y,x,t) * M(x,z,s).$$

Consequently, $(M,*)$ is a GV-fuzzy metric. However, for each $t > 0$ we have:

$$M(x,y,t) = \frac{t^2}{(t+3)^2} = \frac{t^2}{t^2+6t+9} \geq \frac{t^2}{t^2+6t+\frac{9}{2}} = \frac{t}{t+3} * \frac{t}{t+3} = M(x,z,t) * M(z,y,t).$$

It follows that $(M,*)$ is non-strong.

Remark 2. It is possible to generalize the above example, showing that for $X = \{x,y,z\}$, $(M_n, *_{n/n+1})$ is a non-strong GV-fuzzy metric on X, where, for each $n \in \mathbb{N}$, $*_{n/n+1}$ is the Hamacher t-norm for $\lambda_n = \frac{n}{n+1}$, and $M_n : X \times X \times (0,\infty) \to [0,1]$ is defined for each $t > 0$ as: $M_n(x,x,t) = M_n(y,y,t) = M_n(z,z,t) = 1$; $M_n(x,z,t) = M_n(z,x,t) = M_n(y,z,t) = M_n(z,y,t) = \frac{t}{t+1+\frac{4}{n+1}}$; $M_n(x,y,t) = M_n(y,x,t) = \frac{t^2}{(t+2+\frac{2}{n+1})^2}$. It is tedious but not hard to show that, indeed, $(M_n, *_{n/n+1})$ is non-strong.

We conclude this section solving the following question which arises in a natural way in light of Remark 1 above: is there any $* > \cdot$ such that $(M_d, *)$ is strong ?. We answer in the positive, because for each $\lambda \in [0,1)$, $(M_d, *_\lambda)$ is strong.

Indeed, we want to prove that $M_d(x,y,t) \geq M_d(x,z,t) *_\lambda M_d(z,y,t)$, for all $x,y,z \in X$ and $t > 0$, i.e.,

$$\frac{t}{t+d(x,y)} \geq \frac{t}{t+d(x,z)} *_\lambda \frac{t}{t+d(z,y)}$$

$$= \frac{t^2}{\lambda(t+d(x,z))(t+d(z,y)) + (1-\lambda)(t^2+td(x,z)+td(z,y))},$$

that is equivalent to prove:

$$t^2+td(x,y) \leq \lambda(t+d(x,z))(t+d(z,y))+t^2+td(x,z)+td(z,y)-\lambda(t^2+td(x,z)+td(z,y)),$$

which is, indeed, satisfied because, obviously we have $(t+d(x,z))(t+d(z,y)) \geq t^2 + td(x,z) + td(z,y)$.

3 Yager t-norms and fixed point theorems in fuzzy metric spaces

Recall that a sequence $\{x_n\}_n$ in a KM-fuzzy metric space $(X,M,*)$ converges to a point $x \in X$ if $\lim_{n\to\infty} M(x,x_n,t) = 1$, for all $t > 0$. A sequence $\{x_n\}_n$ in a KM-fuzzy metric space $(X,M,*)$ is called a Cauchy sequence if for each $\varepsilon \in (0,1)$, $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n,x_m,t) > 1-\varepsilon$ for all

$n, m > n_0$. A KM-fuzzy metric space $(X, M, *)$ is called complete provided that every Cauchy sequence in X is convergent.

In this section we generalize a fixed point theorem established in [11] which was used to prove the existence and uniqueness of solution for the recurrence equation associated to the Probabilistic Divide and Conquer Algorithms. This is made by using a class of continuous t-norms known as Yager t-norms.
This class of continuous t-norms (see for instance [12] and [8]), that cover the full ranges of these operations, are defined for all $x, y \in [0, 1]$ by $x *_\alpha y = 1 - \min\{1, [(1-x)^{1/\alpha} + (1-y)^{1/\alpha}]^\alpha\}$ where α is a parameter whose range is $(0, \infty)$. A particular continuous t-norm is obtained for each value of the parameter α. It is easy to see that $x *_{\alpha_1} y \geq x *_{\alpha_2} y$ whenever $\alpha_1 \leq \alpha_2$, with $x, y \in [0, 1]$. In particular $x *_{n_1} y \geq x *_{n_2} y$ whenever $n_1 \leq n_2$, with $n_1, n_2 \in \omega$ and $x, y \in [0, 1]$. A subclass of Yager continuous t-norms is $\{*_\alpha\}_{\alpha \in \omega}$. In particular we have that $*_0$ is $\wedge$ and $*_1$ is the Lukasiewicz t-norm. We will call these subclasses as the ω-Yager continuous t-norms.

Theorem 1. *Let $(X, M, *)$ be a complete KM-fuzzy metric space such that $* \geq *_\alpha$ for some $\alpha \in \omega$. If f is a self map on X such that there is $k \in (0, 1)$ satisfying $M(fx, fy, t) \geq 1 - k + kM(x, y, t)$ for all $x, y \in X$, then f has a unique fixed point.*
Proof. Fix $x \in X$. We first show that $M(f^n x, f^{n+1} x, t) \geq 1 - k^n$ for all $n \in \mathbb{N}$ and $t > 0$.

Indeed, for $n = 1$ we have

$$M(fx, f^2 x, t) \geq 1 - k + kM(x, fx) \geq 1 - k.$$

So

$$\begin{aligned} M(f^2 x, f^3 x, t) &\geq 1 - k + kM(fx, f^2 x, t) \\ &\geq 1 - k + k(1 - k) \\ &= 1 - k^2. \end{aligned}$$

Now assume that the inequality holds for $n - 1$, with $n > 3$. Then

$$\begin{aligned} M(f^n x, f^{n+1} x, t) &\geq 1 - k + kM(f^{n-1} x, f^n x, t) \\ &\geq 1 - k + k(1 - k^{n-1}) \\ &= 1 - k^n. \end{aligned}$$

Next we show that $(f^n x)_n$ is a Cauchy sequence in $(X, M, *)$.

Indeed, for each $n, m \in \mathbb{N}$ (we assume without loss of generality that $m = n + j$ for some $j \in \mathbb{N}$), we obtain

$$M(f^n x, f^m x, t) = M(f^n x, f^{n+j} x, t)$$

$$\begin{aligned}
&\geq M(f^n x, f^{n+1} x, t/j) * M(f^{n+1} x, f^{n+2} x, t/j) * ... * M(f^{n+j-1} x, f^{n+j} x, t/j) \\
&\geq (1 - k^n) * (1 - k^{n+1}) * ... * (1 - k^{n+j-1}) \\
&\geq (1 - k^n) *_\alpha (1 - k^{n+1}) *_\alpha ... *_\alpha (1 - k^{n+j-1}).
\end{aligned}$$

Given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty} k^{n/\alpha} < \varepsilon^{1/\alpha}.$$

Therefore, for $n, m \geq n_0$, with $n = m + j$, it follows that:

$$k^{n/\alpha} + k^{(n+1)/\alpha} + ... + k^{(n+j-1)/\alpha} < \varepsilon^{1/\alpha},$$

and hence

$$\begin{aligned}
M(f^n x, f^m x, t) &\geq (1 - k^n) *_\alpha (1 - k^{n+1}) *_\alpha ... *_\alpha (1 - k^{n+j-1}) \\
&= 1 - (k^{n/\alpha} + k^{(n+1)/\alpha} + ... + k^{(n+j-1)/\alpha})^\alpha \\
&> 1 - \varepsilon.
\end{aligned}$$

Consequently $(f^n x)_n$ is a Cauchy sequence in $(X, M, *)$. Then, there is $y \in X$ such that $(f^n x)_n$ converges to y with respect to τ_M.

Since

$$\begin{aligned}
M(f^n y, f^{n+1} x, t) &\geq 1 - k + kM(y, f^n x, t), \quad \text{and} \\
\lim_{n\to\infty} M(y, f^n x, t) &= 1,
\end{aligned}$$

it is follows that

$$\lim_{n\to\infty} M(f^n y, f^{n+1}x, t) = 1,$$

therefore $(f^n x)_n$ converges to fy, so $fy = y$.

Finally, suppose that $z \in X$ satisfies $fz = z$, then:

$$M(y, z, t) = M(fy, fz, t) \geq 1 - k + kM(y, z, t),$$

so

$$(1 - k)M(y, z, t) \geq (1 - k),$$

and, thus

$$M(y, z, t) = 1,$$

for all $t > 0$. We conclude that $z = y$. We have shown that y is the unique fixed point for f.■

In the light of this facts a natural question arises: Given a continuous t-norm $*$, is it possible to find some $n \in \omega$ such that $* \geq *_n$?. If the answer is in the positive way, the previous theorem can be generalized to any continuous t-norm. Unfortunately this is not the case as the following example shows. First we recall well-known facts and definitions.

A prominent subset of $[0,1]^2$ is the diagonal $\{(x, x) : x \in [0, 1]\}$. To simplify notations, for a given continuous t-norm $*$, its diagonal section will be denoted $\delta_* : [0, 1] \to [0, 1]$, specified by $\delta_*(x) = x * x$. For each continuous t-norm $*$, its diagonal section δ_* is a non-decreasing function satisfying $\delta_*(0) = 0$, $\delta_*(1) = 0$, and $\delta_*(x) \leq x$, for all $x \in [0, 1]$. Obviously, δ_* is continuous. Let us write, for convenience: $\mathcal{D} = \{\delta \in [0, 1]^{[0,1]} : \delta$ is continuous non-decreasing and $\delta \leq id_{[0,1]}$, $\delta(0) = 0$, $\delta(1) = 1\}$

In [8, Proposition 7.17] we can find the following characterization of diagonal sections of continuous t-norms.

Proposition 1. *For $\delta \in \mathcal{D}$ the following are equivalent: i) There exists a continuous t-norm $*$ with $\delta_* = \delta$. ii) δ is continuous and the restriction*

$\delta_{[0,1]\setminus\delta^{-1}(x\in[0,1[|\delta(x)=x\})}$ *is strictly increasing.*

The following example shows that we can find a continuous t-norm for which the condition of the previous theorem is not satisfied.

Example 3. We have for each $n \in \omega$, $\delta_{*_n}(x) = \max\{0, 2^n x + 1 - 2^n\}$ for all $x \in [0,1]$, or equivalently $\delta_{*_n}(x) = \max\{0, (x - \frac{4^n-1}{4^n})2^n + \frac{2^n-1}{2^n}\}$ for all $x \in [0,1]$. Note that $\delta_{*_n}(\frac{4^n-1}{4^n}) = \frac{2^n-1}{2^n}$. Let $\delta \in \mathcal{D}$ given in the following way:

$$\delta(x) = (x - \frac{4^n-1}{4^n})\frac{2^{n+1}}{3} + \frac{2^n-1}{2^n}, \quad \forall x \in [\frac{4^n-1}{4^n}, \frac{4^{n+1}-1}{4^{n+1}}], \quad n \in \omega$$

Note that $\delta(\frac{4^n-1}{4^n}) = \frac{2^n-1}{2^n}$, $\delta(\frac{4^{n+1}-1}{4^{n+1}}) = \frac{2^{n+1}-1}{2^{n+1}}$. It is easy to see that $\delta_{*_n}(x) > \delta(x)$ for all $x \in (\frac{4^n-1}{4^n}, 1)$, $n \in \omega$. From the previous proposition there exists a continuous t-norm $*$ such that $\delta_* = \delta$. Obviously there does not exist any $n \in \omega$ such that $* \geq *_n$.

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