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# PRIMITIVE SUBGROUPS AND PST-GROUPS

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ABSTRACT. All groups are finite. A subgroup  $H$  of a group  $G$  is called a primitive subgroup if it is a proper subgroup in the intersection of all subgroups of  $G$  containing  $H$  as its proper subgroup. He, Qiao and Wang [7] proved that every primitive subgroup of a group  $G$  has index a power of a prime if and only if  $G/\Phi(G)$  is a solvable PST-group. Let  $\mathfrak{X}$  denote the class of groups  $G$  all of whose primitive subgroups have prime power index. It is established here that a group  $G$  is a solvable PST-group if and only if every subgroup of  $G$  is an  $\mathfrak{X}$ -group.

## 1. INTRODUCTION AND STATEMENTS OF RESULTS.

All groups considered here are finite. A subgroup  $H$  of a group  $G$  is called primitive if it is a proper subgroup in the intersection of all subgroups containing  $H$  as a proper subgroup. All maximal subgroups of  $G$  are primitive. Some properties of primitive subgroups are given in Lemma 2.1 and include:

- (a) Every proper subgroup of  $G$  is the intersection of a set of primitive subgroups of  $G$ .
- (b) If  $X$  is a primitive subgroup of a subgroup  $T$  of  $G$ , then there exists a primitive subgroup  $Y$  of  $G$  such that  $X = Y \cap T$ .

Johnson [10] introduced the concept of primitive subgroup of a group. He proved that a group  $G$  is supersolvable if every primitive subgroup of  $G$  has prime power index in  $G$ .

The next results on primitive subgroups of a group  $G$  indicate how such subgroups give information about the structure of  $G$ .

**Theorem 1.1** ([7]). *Let  $G$  be a group. The following statements are equivalent:*

- (1) *Every primitive subgroup of  $G$  containing  $\phi(G)$  has prime power index.*
- (2)  *$G/\phi(G)$  is a solvable PST-group.*

**Theorem 1.2** ([6]). *Let  $G$  be a group. The following statements are equivalent:*

- (1) *Every primitive subgroup of  $G$  has prime power index.*
- (2)  *$G = [L]M$  is a supersolvable group, where  $L$  and  $M$  are nilpotent Hall subgroups of  $G$ ,  $L$  is the nilpotent residual of  $G$  and  $G = LN_G(L \cap X)$  for every*

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*primitive subgroup  $X$  of  $G$ . In particular, every maximal subgroup of  $L$  is normal in  $G$ .*

Note that  $G = [L]M$  in Theorem 1.2 means that  $G$  is the semidirect product of  $L$  by  $M$ .

Let  $\mathfrak{X}$  denote the class of groups  $G$  such that the primitive subgroups of  $G$  have prime power index (see [5, pp. 132-137]). By (a) it is clear that  $\mathfrak{X}$  consists of those groups whose subgroups are intersections of subgroups of prime-power indices.

One purpose of this paper is to characterize solvable PST-groups in terms of  $\mathfrak{X}$ -subgroups.

A subgroup  $H$  of a group  $G$  is said to be  $S$ -permutable in  $G$  if it permutes with the Sylow subgroups of  $G$ . Kegel [2, 1.2.14] proved that an  $S$ -permutable subgroup of  $G$  is subnormal in  $G$ .  $S$ -permutability is said to be transitive in  $G$  if  $H$  and  $K$  are subgroups of  $G$  such that  $H$  is  $S$ -permutable in  $K$  and  $K$  is  $S$ -permutable in  $G$ , then  $H$  is  $S$ -permutable in  $G$ . A group  $G$  is said to be a PST-group if  $S$ -permutability is a transitive relation in  $G$ . By Kegel's result  $G$  is a PST-group if and only if every subnormal subgroup of  $G$  is  $S$ -permutable. Agrawal [1] characterized solvable PST-groups. He proved the following theorem.

**Theorem 1.3.** *Let  $G$  be a solvable group.  $G$  is a PST-group if and only if it has an abelian normal Hall subgroup  $N$  such that  $G/N$  is nilpotent and  $G$  acts by conjugation on  $N$  as a group of power automorphisms.*

In Theorem 1.3  $N$  can be taken to be the nilpotent residual of  $G$ . From Theorem 1.3 it follows that subgroups of solvable PST-groups are solvable PST-groups. Many interesting results about PST-groups can be found in Chapter 2 of [2].

**Theorem A.** *Let  $G$  be a group. The following statements are equivalent:*

- (1)  *$G$  is a solvable PST-group.*
- (2) *Every subgroup of  $G$  is an  $\mathfrak{X}$ -group.*

Let  $G$  be an  $\mathfrak{X}$ -group. It follows from Theorem A that if  $G$  is not a solvable PST-group, then  $G$  has a subgroup  $H$  which does not belong to  $\mathfrak{X}$ . See Examples 1 and 2.

A well-known theorem of Lagrange (see [5, Chapter 1, 1.3.6]) states that given a subgroup  $H$  of a group  $G$ , the order of  $G$  is the product of the order  $|H|$  of  $H$  and the index  $|G : H|$  of  $H$  in  $G$ . In particular, the order of any subgroup divides the order of the group. The converse, namely, if  $d$  divides the order of a group  $G$ , then  $G$  has a subgroup of order  $d$ , is not true in general. Groups satisfying this condition are often called CLT-groups. The alternating group of order 12, having no subgroups of order 6, is an example of a non-CLT-group.

On the other hand, abelian groups contain subgroups of every possible order, and it is not difficult to prove that a group is nilpotent if and only if it contains a normal

subgroup of each possible order [8]. Ore [11] and Zappa [15] obtained a similar characterization for supersolvable groups:

**Theorem 1.4.** *A group  $G$  is supersolvable if and only if each subgroup  $H \leq G$  contains a subgroup of order  $d$  for each divisor  $d$  of  $|H|$ .*

Of course, we can state Theorem 1.4 in the following equivalent way, more easily treated:

**Theorem 1.5.** *A group  $G$  is supersolvable if and only if each subgroup  $H \leq G$  contains a subgroup of index  $p$  for each prime divisor  $p$  of  $|H|$ .*

A proof of this theorem can be found in [5, Chapter 1, 4.2]. It must be noted that CLT-groups are not necessarily supersolvable, as the symmetric group of order 4 shows.

The condition on a group  $G$  given in Theorem 1.5, namely

for all  $H \leq G$  and for all primes  $q$  dividing  $|H|$ , there exists a subgroup  $K$  of  $G$  such that  $K \leq H$  and  $|H : K| = q$ ,

has a dual formulation:

for all  $H \leq G$  and for all primes  $q$  dividing  $|G : H|$ , there exists a subgroup  $K$  of  $G$  such that  $H \leq K$  and  $|K : H| = q$ .

Groups satisfying the latter condition have been studied by some authors. Following [5, Chapter 1, 4], we will call them  $\mathcal{Y}$ -groups.

A group  $G$  is said to be a  $\mathcal{Y}$ -group if for all subgroups  $H$  of  $G$  and all primes  $q$  dividing the index  $|G : H|$  of  $H$  in  $G$ , there exists a subgroup  $K$  of  $G$  with  $H \leq K$  and  $|K : H| = q$ .

Note that a group  $G$  is a  $\mathcal{Y}$ -group if and only if for every subgroup  $H$  of  $G$  and for every natural number  $d$  dividing  $|G : H|$  there exists a subgroup  $K$  of  $G$  such that  $H \leq K$  and  $|K : H| = d$ . The following characterization of  $\mathcal{Y}$ -groups appears in [5, Chapter 1, 4.3].

**Theorem 1.6.** *Let  $L = G^{\mathfrak{N}}$  be the nilpotent residual of the group  $G$ . Then  $G$  is a  $\mathcal{Y}$ -group if and only if  $L$  is a nilpotent Hall subgroup of  $G$  such that for all subgroups  $H$  of  $L$ ,  $G = LN_G(H)$ .*

From Theorem 1.6, we see that if  $G \in \mathcal{Y}$  and  $X$  is a normal subgroup of  $L$ , then  $X$  is normal in  $G$ . In particular,  $\mathcal{Y}$ -groups are supersolvable. Moreover, if  $G \in \mathcal{Y}$ , then  $L$  must have odd order.

Further results on  $\mathcal{Y}$ -groups can be found in [5, Chapter 4, 5.2, 5.3]. For example, a solvable group  $G$  is a  $\mathcal{Y}$ -group if and only if every subgroup of  $G$  can be written as an intersection of subgroups of  $G$  of coprime prime-power indices.

From Theorem 1.3 and Theorem 1.6 we obtain

**Theorem 1.7.** *Let  $G$  be a  $\mathcal{Y}$ -group with nilpotent residual  $L$ .*

- (1)  $G$  is a solvable PST-group if and only if  $L$  is abelian.
- (2)  $G/\phi(G)$  is a solvable PST-group.

We note that the class  $\mathcal{Y}$  is a subclass of the class  $\mathfrak{X}$  by Theorems 1.2 and 1.7. The example of Humphreys on p. 136 of [5] (see also [9]) shows that  $\mathcal{Y}$  is a proper subclass of  $\mathfrak{X}$ .

**Theorem B.** *Let  $G$  be a group. The following statements are equivalent:*

- (1)  $G$  is a solvable PST-group.
- (2) Every subgroup of  $G$  is a  $\mathcal{Y}$ -group.
- (3) Every subgroup of  $G$  is an  $\mathfrak{X}$ -group.

Let  $\mathfrak{F}$  be a class of groups. Denote by  $\mathcal{S}\mathfrak{F}$  (resp.  $\mathcal{S}(\mathfrak{F})$ ) the class of groups all of whose subgroups are  $\mathfrak{F}$ -groups (resp. solvable  $\mathfrak{F}$ -groups).

**Theorem C.**  $\mathcal{S}\mathfrak{X} = \mathcal{S}\mathcal{Y} = \mathcal{S}T_0 = \mathcal{S}(T_0) = \mathcal{S}PST = \mathcal{S}(PST) = \mathcal{S}(PST_0) = \mathcal{S}(PT_0)$ .

We mention that  $\mathcal{S}\mathfrak{X} = \mathcal{S}\mathcal{Y}$  of Theorem C follows from Theorem B and is Theorem 5.3 of [5, p. 135]. The proof of Theorem 5.3 in [5] is much different and more difficult than the proof of Theorem B.

## 2. PRELIMINARIES.

**Lemma 2.1** ([6, 7, 10]). *Let  $G$  be a group. The following statements hold:*

- (1) For every proper subgroup  $H$  of  $G$ , there is a set of primitive subgroups  $\{X_i \mid i \in I\}$  in  $G$  such that  $H = \bigcap_{i \in I} X_i$ .
- (2) If  $H \leq G$  and  $T$  is a primitive subgroup of  $H$ , then  $T = H \cap X$  for some primitive subgroup  $X$  of  $G$ .
- (3) If  $K \trianglelefteq G$  and  $K \leq H \leq G$ , then  $H$  is a primitive subgroup of  $G$  if and only if  $H/K$  is a primitive subgroup of  $G/K$ .
- (4) Let  $P$  and  $Q$  be subgroups of  $G$  with  $(|P|, |Q|) = 1$ . Suppose that  $H$  is a subgroup of  $G$  such that  $HP \leq G$  and  $HQ \leq G$ . Then  $HP \cap HQ = H$ . In particular, if  $H$  is a primitive subgroup of  $G$ , then  $P \leq H$  or  $Q \leq H$ .

Let  $G$  be a group.  $G$  is called a T-(resp. PT-)group if  $H \trianglelefteq K \trianglelefteq G$  (resp.  $H$  is permutable in  $K$  and  $K$  is permutable in  $G$ ) then  $H \triangleleft G$  (resp.  $H$  is permutable in  $G$ ). By Kegel's result  $G$  is a PT-group if and only if every subnormal subgroup of  $G$  is permutable. Many results about T- and PT-groups can be found in Chapter 2 of [2].  $G$  is called a  $T_0$ -group if  $G/\phi(G)$  is a T-group where  $\phi(G)$  is the Frattini subgroup of  $G$ .  $T_0$ -groups have been studied in [4, 12, 14]. Several of the results on  $T_0$ -groups given in [4, 12] are contained in the next three lemmas and are needed in the proof of Theorem A.

A group  $G$  is called a  $PT_0$ -(resp.  $PST_0$ -)group provided that  $G/\phi(G)$  is a PT-(resp. PST-)group. For solvable groups we have

**Lemma 2.2** ([12]).  $\mathcal{S}(T_0) = \mathcal{S}(PT_0) = \mathcal{S}(PST_0)$ .

**Lemma 2.3** ([4]). *Let  $G$  be a group.  $G$  is a solvable PST-group if and only if every subgroup of  $G$  is a  $T_0$ -group.*

### 3. PROOFS OF THE THEOREMS.

*Proof of Theorem A.* Let  $G$  be a solvable PST-group and let  $L$  be the nilpotent residual of  $G$ . By Theorem 1.3  $L$  is a normal abelian Hall subgroup of  $G$  on which  $G$  acts by conjugation as a group of power automorphisms. Let  $X$  be a subgroup of  $L$ . Since  $X \triangleleft G$ ,  $G = LN_G(X)$ . Let  $D$  be a system normalizer of  $G$ . By Theorem 9.2.7, p. 264 of [13]  $G = [L]D$ , the semidirect product of  $L$  by  $D$ . It follows by Theorem 1.2 that every primitive subgroup of  $G$  has prime power index and hence  $G$  is an  $\mathfrak{X}$ -group. Since every subgroup of  $G$  is a solvable PST-group, every subgroup of  $G$  is an  $\mathfrak{X}$ -group.

Conversely, assume that every subgroup of  $G$  is an  $\mathfrak{X}$ -group. We are to show that  $G$  is a solvable PST-group. Let  $H$  be a subgroup of  $G$ . Because of Theorem 1.1  $H/\phi(H)$  is a solvable PST-group and hence  $H$  is a solvable  $PST_0$ -group. By Lemma 2.2  $H$  is a  $T_0$ -group. It follows that every subgroup of  $G$  is a solvable  $T_0$ -group and by Lemma 2.3  $G$  is a solvable PST-group.

This completes the proof. □

*Proof of Theorem B.* Let  $G$  be a solvable PST-group. Using the proof of the first part of Theorem A and Theorem 1.6 we see that every subgroup of  $G$  is a  $\mathcal{Y}$ -group and (1) implies (2). Since  $\mathcal{Y} \subseteq \mathfrak{X}$ , (2) implies (3). By Theorem A we see that (3) implies (1). □

*Proof of Theorem C.* By Theorem B,  $\mathcal{S}\mathfrak{X} = \mathcal{S}\mathcal{Y} = \mathcal{S}(PST) = \mathcal{S}PST$ . Note by Theorem 1.1  $\mathcal{S}(T_0) = \mathcal{S}T_0 = \mathcal{S}\mathfrak{X}$ . Finally, it follows that  $\mathcal{S}(T_0) = \mathcal{S}(PST_0) = \mathcal{S}(PT_0)$  by Lemma 2.2. Hence Theorem C holds. □

### 4. EXAMPLES.

**Example 1.** Let  $P = \langle x, y \mid x^5 = y^5 = [x, y]^5 = 1 \rangle$  be an extra-special group of order 125 of exponent 5. Let  $z = [x, y]$  and note  $Z(P) = \Phi(P) = \langle z \rangle$ .  $P$  has an automorphism  $a$  of order 4 given by  $x^a = x^2$ ,  $y^a = y^2$  and  $z^a = z^4 = z^{-1}$ . Put  $G = [P]\langle a \rangle$  and note  $Z(G) = 1$ ,  $\Phi(G) = \langle z \rangle$  and  $G/\Phi(G)$  is a T-group. Thus  $G$  is a solvable  $T_0$ -group. Let  $H = \langle y, z, a \rangle$  and notice  $\Phi(H) = 1$ .  $H$  is not a T-group since the nilpotent residual  $L$  of  $H$  is  $\langle y, z \rangle$  and  $a$  does not act on  $L$  as a power automorphism. Thus  $H$  is not a  $T_0$ -group and hence not a solvable PST-group. By Theorem 1.1  $G$  is an  $\mathfrak{X}$ -group and  $H$  is not an  $\mathfrak{X}$ -group.

**Example 2.** Let  $P = \langle x, y \mid x^3 = y^3 = [x, y]^3 = 1 \rangle$  be an extra-special group of order  $3^3$  and exponent 3.  $P$  has an automorphism  $b$  of order 2 given by  $x^b = x^{-1}$ ,  $y^b = y^{-1}$  and  $[x, y]^b = 1$ . Let  $G = [P]\langle b \rangle$  and note  $Z(G) = Z(P) = \langle [x, y] \rangle = \phi(G)$ .  $G/\phi(G)$  is a T-group and hence  $G$  is a  $T_0$ -group. By Lemma 2.3  $G$  has a subgroup which is not a  $T_0$ -group and hence not a solvable PST-group. Note  $G$  is an  $\mathfrak{X}$ -group.

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