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OPTIMAL RANGE THEOREMS FOR OPERATORS WITH p-TH POWER FACTORABLE ADJOINTS

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ABSTRACT. Consider an operator $T : E \to X(\mu)$ from a Banach space E to a Banach function space $X(\mu)$ over a finite measure μ such that its dual map is p-th power factorable. We compute the optimal range of T that is defined to be the smallest Banach function space such that the range of T lies in it and the restricted operator has p-th power factorable adjoint. For the case $p = 1$, the requirement on T is just continuity, so our results give in this case the optimal range for a continuous operator. We give examples from classical and harmonic analysis, as convolution operators, Hardy type operators and the Volterra operator.

1. INTRODUCTION AND NOTATION

In recent years, vector measures and vector valued integration has been used for characterizing optimal domains for operators that are defined on Banach function spaces. This technique has shown to be a useful tool in this setting, and nice description of such optimal domains (i.e. the largest Banach function space having some concrete properties to which the operator can be extended) are nowadays known due to the application of this tool. The Volterra and Hardy operators, for instance, have been intensively studied from this point of view, but more examples can be found in the literature (see for instance [4, 5, 7, 8]). In this paper we adapt this technique for the analysis of the optimal range of operators

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with values in Banach function spaces. Of course, the natural way of doing that is by dualizing the results on optimal domains: in this sense, the starting point is to study optimal domains for Köthe adjoint operators.

On the other hand, p-th power factorable operators where introduced in [15] in order to find specialized versions of the optimal domain theorems for the case of operators satisfying stronger properties than continuity (norm inequalities). This technique is based on the theory of L^p spaces of a vector measure (see [11, 10, 15, 16]). There are plenty of examples of such operators in classical and harmonic analysis, as the ones related to L^p-improving measures (see [15, Ch.7]). The space that is optimal for an operator with respect to this property (i.e. the bigger Banach function space in which the operator still preserves the particular property considered) satisfies in this case more specific geometric properties, like p-convexity in the case of p-th power factorable operators.

The aim of this paper is to provide some results that allow to compute optimal ranges for operators with values in Banach function spaces and to show some applications for interesting operators, as Hardy type operators or convolution operators. For improving the characterization that can be given for the optimal range of such operators, we will impose stronger requirements on the adjoint operators related with their p-th power factorability.

Our notation is standard. If E is a Banach space, we denote by E^* its dual space. If $1 \leq p \leq \infty$, we will write p' for the extended real number satisfying $1/p + 1/p' = 1$. Throughout the paper, (Ω, Σ, μ) will be a finite measure space and $X(\mu)$ a Banach function space in the sense of [12, p.28]. $L^0(\mu)$ is the space of (classes of) measurable functions. We will simply write X instead of $X(\mu)$ if no explicit reference to the measure is needed. If $A \in \Sigma$, we write $\mu|_A$ and $X|_A(\mu|_A)$ to the restriction of the measure and the space to the set A. We denote by X' the Köthe dual of X, i.e. the Banach function space of all integrals in X^*. For some particular results, we will consider quasi-Banach function spaces, i.e. lattices of measurable functions whose topology is provided by a quasi-norm instead of a norm. The same definitions that in the case of Banach function spaces make sense. Recall that a Banach function space is order continuous if and only if $X^* = X'$, and has the Fatou property if and only if it is perfect, i.e. if $X'' = X$. We mainly refer to [12, 15] for definitions and basic results regarding Banach function spaces; some aspects of these spaces that are used in this paper can also be found in [1, 14, 17]. A Banach function space is p-convex if there is a constant $K > 0$ such that for every finite set of functions $f_1, \cdots, f_n \in X$, the following inequality holds.

$$\Big\| (\sum_{i=1}^{n} |f_i|^p)^{1/p} \Big\|_X \leq K \Big(\sum_{i=1}^{n} \|f_i\|_X^p \Big)^{1/p}.$$

If $T : X \to E$ is an operator we write T^* for its adjoint and T' for its Köthe adjoint, i.e. for the restriction of T^* to the Köthe dual X'.

Let $X(\mu)$ and $Y(\mu)$ be a couple of Banach function spaces over the same measure μ. Following the notation of [13] (see also [2]), the space of multiplication operators from X to Y is defined as

$$X^Y := \{g \in L^0(\mu) : g \cdot X \subseteq Y\}.$$

The expression $\|f\|_{X^Y} := \sup_{g \in B_X} \|gf\|_Y$ is a norm when X^Y is a Banach space. Note that $X' = X^{L^1(\mu)}$. This space can be trivial, depending on the properties of the spaces involved. In other case, it is a Banach function space over μ. More information on these spaces, including sufficient conditions to assure that they are Banach function spaces can be found in [3], [13] and [15, Ch.2]. Notice that the definition still make sense if Y is a Banach function space over ν, where ν is absolutely continuous with respect to μ.

If $X(\mu)$ is a Banach function space, its p-th power can be defined as

$$X_{[p]} := \{f \in L^0(\mu) : |f|^{1/p} \in X\}$$

that is a quasi-Banach function space over μ when endowed with the seminorm $\|f\|_{X_{[p]}} := \||f|^{1/p}\|_X^p$. In fact it is a Banach space and the above expression defines a norm if and only if X is p-convex with p-convexity constant 1. For example, $(L^p[0,1])_{[p]} = L^1[0,1]$ isometrically. We use the symbol $i_{[p]}$ to denote the inclusion map $i_{[p]} : X \hookrightarrow X_{[p]}$. Following this definition of p-th power of a Banach function space, in the case that X^Y is a Banach function space, it is easy to see that $(X^Y)_{[p]} = {X_{[p]}}^{Y_{[p]}}$. An operator $T : X(\mu) \to E$ is p-th power factorable if there is a constant $K > 0$ such that $\|T(f)\| \leq K\||f|^{1/p}\|^p$ for all $f \in X$.

Regarding vector measures, we consider in this paper spaces $L^p(m)$ of p integrable functions with respect to a vector measure $m \colon \Sigma \to E$, where Σ is a σ-algebra and E a Banach space. The main reference for vector measures is [9], and [15] for integration with respect to vector measures and the properties of the integration map. If m is a vector measure, we write $\|m\|$ for its semivariation and $|m|$ for its variation. Consider the space $L^0(\|m\|)$ of equivalence classes of measurable functions which differ only in a $\|m\|$-null set, i.e. in a set of null m-semivariation. An element $f \in L^0(\|m\|)$ is integrable with respect to m if it is integrable for each scalar measure $\langle m, e'\rangle$, $e' \in E'$ (that are defined as $\langle m, e'\rangle(A) := \langle m(A), e'\rangle$, $A \in \Sigma$), and for each $A \in \Sigma$ there is a vector $\int_A f dm \in E$ such that $\int_A f d\langle m, e'\rangle = \langle \int_A f dm, e'\rangle$ for each $e' \in E^*$. Such a function is p-integrable with respect to m if $|f|^p$ is integrable with respect to m, $1 \leq p < \infty$. The expression

$$\|f\|_{L^p(m)} := \sup_{e' \in B_{E'}} \Big(\int |f|^p \, d|\langle m, e'\rangle| \Big)^{1/p},$$

that is well defined for each integrable function f, defines in fact a function norm on the linear space of classes of measurable functions. It is equivalent to the expression

$$|||f|||_{L^p(m)} := \sup_{A \in \Sigma} \Big\| \int_A |f|^p \, dm \Big\|^{1/p}.$$

The space $(L^p(m), \|\cdot\|_{L^p(m)})$ is a p-convex order continuous Banach function space over each Rybakov measure for m; recall that a Rybakov measure is a scalar measure as $|\langle m, e'\rangle|$ with the same null sets that $\|m\|$ (see [9]). The integration map $f \rightsquigarrow \int f dm \in E$ is always continuous.

In this paper we will consider what we call a inclusion/quotient maps that appears in the following context. If $X(\mu)$ is an order continuous Banach function space and $T: X \to E$ an operator, the expression $m_T(A) := T(\chi_A)$ always defines a vector measure. Then, if m_T has the same null sets that μ, the space $X(\mu)$ is included in $L^1(m_T)$ (in $L^p(m_T)$ for the case of p-th power factorable operators, see [15, Ch. 5]) and the operator T can be extended to X. In this case it is said that T is μ-determined, which is equivalent to $[i]$ to be injective (see [15, Ch.4]). However, it is not needed for getting a factorization of T through $L^1(m_T)$, since the map $[i] : X \to L^1(m_T)$ (to $L^p(m_T)$), given by $f \rightsquigarrow [i](f) = [f]$ (where $[f]$ denotes the equivalence class of f with respect to $\|m_T\|$) is still well defined and continuous. We call such a map a inclusion/quotient map. Notice that the Köthe adjoint map $[i]'$ is injective, since $\|m_T\|$ is always absolutely continuous with respect to μ. The reader can find this general point of view in [3], where it is shown that the same factorization technique works without the injectivity assumption. In this case, the definition of the map $[i]'$ depends on the Rybakov measure ν that is taken for considering $L^p(m_T)$ as a Banach function space over ν. For the aim of simplicity, we will assume that we have fixed a Rybakov measure when we consider the map $[i]'$, and no explicit reference to this measure will be done in the notation of $[i]'$. In general, and specially in the case that the measure μ of $X(\mu)$ is not equivalent to $\|m_T\|$, the injective map $[i]'$ is not defining an inclusion, in the sense that it is not sending a function g in $(L^p(m_T))'$ to the same function g in $X(\mu)'$. If ν is the fixed Rybakov measure for m_T and $d\nu/d\mu$ is the Radon-Nikodým derivative of ν with respect to μ, the equalities

$$\Big\langle f, [i]'(g) \Big\rangle = \int f\, [i]'(g) d\mu = \int [i](f)\, g d\nu = \int [i](f)\, g\, \big(\frac{d\nu}{d\mu}\big)\, d\mu,$$

(where $f \in X(\mu)$ and $g \in (L^p(m_T))')$ that are given by the duality relation, do not provide a proper inclusion map. However, this is the "inclusion" map that allows to prove an optimal range theorem, in the sense that for every Banach function space $Z(\mu) \subseteq X(\mu)$ and to which the range of the operator T can be restricted, the map $[i]'$ take its values in $Z(\mu)$. We will denote that special "inclusion" relation by means of the symbol $(L^p(m_{T'}))' \Subset Z(\mu)$. Notice that if $[i]$ is injective (in other words, μ is equivalent to $\|m_T\|$), then $[i]'(g) = (d\nu/d\mu) \cdot g$, i.e. $[i]$ is a multiplication map.

2. The Köthe p-adjoint of an operator

In this section we define and characterize a concrete extension (in the range) of an operator on Banach function spaces that will provide in a sense the canonical example of operator satisfying that its adjoint is p-th power factorable. In order to do it, some properties of the spaces of multiplication operators will be given for the particular case that the space where the functions take values are L^p-spaces. Some references where these results can be found are [13] and [2] (see also [15, Ch.2]).

The following equality will be useful in what follows. For a p-convex Banach function space $X(\mu)$, $(X_{[p]})' = (X^{L^p(\mu)})_{[p]}$ for all $0 < p < \infty$. The next calculations using elementary properties of the p-th power of Banach function spaces proves it; it can be also obtained as a direct consequence of Proposition 2.29(ii),(iv) in [15].

$$(X^{L^p(\mu)})_{[p]} = (X_{[p]})^{(L^p(\mu))_{[p]}} = (X_{[p]})^{(L^1(\mu)_{[1/p]})_{[p]}}$$
$$= (X_{[p]})^{L^1(\mu)} = (X_{[p]})' \, .$$

The central role that plays the space $X_{[p]}$ in the paper and the representation for the dual that provides the formula above motivates the following definition.

Definition 2.1. If X is a quasi-Banach space of measurable functions, we define its *Köthe p-dual* X^p by

$$X^p := X(\mu)^{L^p(\mu)}.$$

Notice that $X^1 = X'$ and also that X^p can be the trivial space, or just a quasi-Banach space. However, it is a Banach function space whenever X is p-convex (see for instance [2]). A direct computation shows that X^p is always p-convex (see Lemma 5.1 in [2]).

If $T\colon E \to X$ is an operator, we can define the operator

$$T_p := i_{[p]} \circ T\colon E \xrightarrow{T} X \overset{i_{[p]}}{\hookrightarrow} X_{[p]} \, ,$$

Its adjoint map (that is a continuous operator between Banach function spaces whenever $X_{[p]}$ is a Banach function space, equivalently, X is p-convex) is then given by

$$(T_p)^* = T^* \circ i^*_{[p]} \colon (X_{[p]})^* \overset{i^*_{[p]}}{\hookrightarrow} X^* \xrightarrow{T^*} E^* \, ,$$

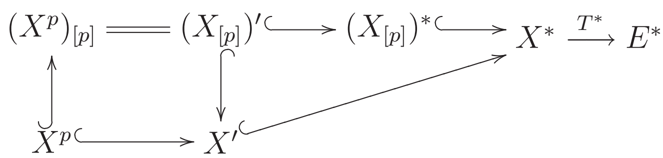
and so its Köthe adjoint is

$$(T_p)' := T^* \circ i^*_{[p]} \circ \iota \colon (X_{[p]})' \overset{\iota}{\hookrightarrow} (X_{[p]})^* \overset{i^*_{[p]}}{\hookrightarrow} X^* \xrightarrow{T^*} E^* \, .$$

Note also that the inclusion map $i^{p,1} : X^p \hookrightarrow (X^{L^p})_{[p]}$ is well defined and take values in $(X_{[p]})'$ whenever X is p-convex. This motivates the following definition. If X is a Banach function space, E is a Banach space and $T\colon X \to E$ is an operator, we define the Köthe p-adjoint operator T^p of T by

$$T^p := (T_p)'|_{X^p} : X^p \to X^* \to E^*.$$

The following scheme shows the factorizations for T^p. All the arrows as "$\hookrightarrow$" denote canonical inclusion maps.



The next result gives sufficient conditions for an operator to satisfy that its Köthe p-adjoint is p-th power factorable. This is the main result of this section, since it provides what is in a sense the canonical example of operators satisfying such property.

Proposition 2.2. *Let $T\colon E \to X(\mu)$ be an operator, where $X(\mu)$ is a σ-order continuous p-convex Banach function space. Then for $p \geq 1$, the Köthe p-adjoint operator T^p is p-th power factorable.*

Proof. Let $f' \in X^p \subseteq X'$ and $T^p = T'|_{X^p}$. Then

$$\|T'(f')\|_{E^*} = \sup_{e\in B_E} |\langle e, T'(f')\rangle| = \sup_{e\in B_E} |\langle T(e), f'\rangle| .$$

We know that $\frac{T}{\|T\|}(B_E) \subseteq B_X$ and since μ is finite, $X \subseteq X_{[p]}$ and by the equality $(X_{[p]})' = \big(X^p\big)_{[p]}$ we obtain

$$\sup_{e\in B_E} |\langle T(e), f'\rangle| \leq \|T\| \sup_{h\in B_X} |\langle h, f'\rangle| \leq \|T\| \sup_{h\in B_{X_{[p]}}} |\langle h, f'\rangle|$$
$$\leq \|T\|\big(\|f'\|_{(X_{[p]})'}\big) = \|T\|\|f'\|_{(X^p)_{[p]}} .$$

Since $X_{[p]}$ is σ-o.c., we obtain that $T^p\colon X^p \to E^*$ is p-th power factorable.

□

3. Optimal range for operators with p-th power factorable adjoint

Consider an operator $T : E \to X(\mu)$ from a Banach space to a Banach function space $X(\mu)$ with the Fatou property and with order continuous dual, such that T' is p-th power factorable. In this section we obtain a representation of the optimal Fatou Banach function space Y in which the range of T is included, in the sense that for each Banach function space $Z(\mu)$ with the Fatou property and with order continuous dual in which $T(E)$ is continuously contained, the relation $Y \Subset Z$ holds, whenever the restriction satisfy the p-th factorability property. Notice that for $p = 1$, this result will provide an optimal range theorem, since in this case this condition is just continuity of the adjoint map.

Let us start by showing in the following examples that some relevant operators satisfy that their adjoint maps are p-th power factorable for some $p > 1$.

Example 3.1. (A Hardy type operator). Let $s > 0$ and consider the kernel operator H_s with kernel function

$$K(x, y) := \frac{1}{x^s}\chi_{[0,x]}(y) .$$

If $H_s\colon L^u[0, 1] \to L^v[0, 1]$ ($u \geq v \geq 1$), the operator is clearly well defined and continuous when $s < \frac{1}{v}$ (in other case it is also sometimes continuous, for instance in the case of the Hardy operator, see [1, Theorem 3.10]). We have that

$$H_s(f)(x) = \int_0^1 K(x, y)f(y)\, dy = \int_0^1 \frac{1}{x^s} f(y)\chi_{[0,x]}(y)\, dy = \frac{1}{x^s}\int_0^x f(y)\, dy .$$

Since for $x, y \in [0, 1]$, $\chi_{[0,x]}(y) = \chi_{[y,1]}(x)$ the adjoint map $H'_s\colon L^{v'}[0, 1] \to L^{u'}[0, 1]$ is given by

$$H'_s(g)(y) = \int_0^1 \frac{1}{x^s}\chi_{[0,x]}(y)g(x)\, dx = \int_0^1 \frac{1}{x^s} g(x)\chi_{[y,1]}(x)\, dy = \int_y^1 \frac{1}{x^s}\, g(x)\, dx .$$

If $g \in L^{v'}[0, 1]$, using Minkowski's integral inequality and Hölder's inequality, we have that

$$\|H'_s(g)\|_{L^{u'}} = \Big(\int_0^1 \Big|\int_y^1 \frac{g(x)}{x^s}\,dx\Big|^{u'} dy\Big)^{1/u'} \leq \int_0^1 \Big(\int_0^1 \Big|\frac{g(x)}{x^s}\Big|^{u'} dy\Big)^{1/u'} dx$$
$$= \int_0^1 |g(x)||x^{-s}|\,dx \leq \|x^{-s}\|_{L^{(v'/q)'}}\|g\|_{L^{v'/q}} = \|x^{-s}\|_{L^{(v'/q)'}}\|g\|_{(L^{v'})_{[q]}}.$$

Thus, H'_s is q-th power factorable if $s(v'/q)' < 1$, i.e. $s < 1 - q/v'$.

The case H_0 gives the Volterra operator. It is well-known when this operator is p-th power factorable (see Example 5.9 in [15]); we have shown in this Example when this condition holds for the adjoint map H'_0. We will come back to this operator in the last section of the paper.

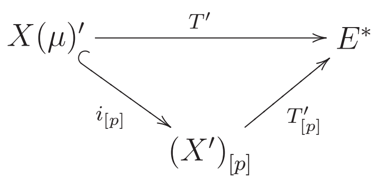
Example 3.2. (Convolution operators). Let G be a compact Hausdorff abelian group with normalized Haar measure μ defined on the Borelian sets of G ($\mathcal{B}(G)$). Let λ be a regular measure on $\mathcal{B}(G)$. We say that λ is L^q-improving ($q \geq 1$) if there exists $r \in (q, \infty)$ such that $f * \lambda \in L^r(G)$ for all $f \in L^q(G)$. It is well known that there is a direct relation between L^q-improving measures and p-th power factorable convolution operators (see [15, Ch.7]). If $h \in L^1(G)$ we can always consider the measure $\mu_h(A) := \int_A h\,d\mu$. For this kind of measures, the fact that h belongs to a particular $L^s(G)$-space determines if it is L^q-improving, and also that the corresponding convolution operator is p-th power factorable for a certain p.

Let $1 < p < \infty$ and consider the convolution operator $C_h^{(p)} : L^p(G) \to L^p(G)$ given by $C_h^{(p)}(f) := f * \mu_h$, that is continuous, and the reflection measure of λ defined as $R\lambda(A) := \lambda(-A)$. Note that for measures $\lambda(A) := \int_A h(x)\,d\mu$ we always have $R\lambda(A) = \int_A h(-x)\,d\mu$. Using Fubini's Theorem, we obtain that the adjoint operator $(C_h^{(p)})' : L^{p'}(G) \to L^{p'}(G)$ is given by $(C_h^{(p)})'(g) = g * R\mu_h$. Thus, we can apply Proposition 7.96 in [15] taking into account that all $L^s(G)$ are rearrangement invariant: for $h \in L^r(G) \setminus L^{p'}(G)$ (where $1 < r < p'$) and $u \in (1, p')$ such that $\frac{1}{u} + \frac{1}{r} = \frac{1}{p'} + 1$, $(C_h^{(p)})'$ is (p'/u)-th power factorable.

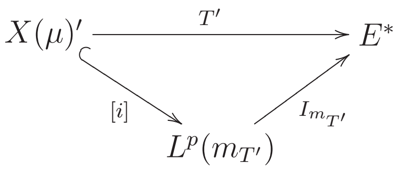
Let us show now the main result of this section. The assumption on T is the following: T' must be p-th power factorable, i.e. there is a constant $K > 0$ such that for every $e \in E$,

$$|\langle T(e), x'\rangle| \leq K\|e\|_E\|x'\|_{(X(\mu)')_{[p]}}$$

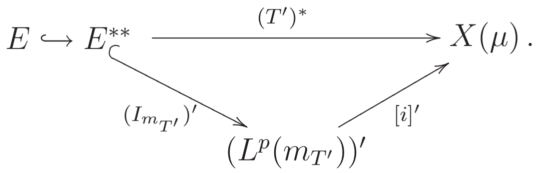
for all $e \in E$ and $x' \in X'$. For order continuous spaces $X(\mu)'$, this implies that the (Köthe) adjoint map T' factorizes as



where $i_{[p]}$ is the natural continuous inclusion and $T'_{[p]}$ the extension of T'. The order continuity of X' gives also that the expression $m_{T'}(A) = T'(\chi_A)$, $A \in \Sigma$, defines a vector measure. An application of the optimal domain theorem for p-th power factorable operators gives that it factorizes also as



where $[i]$ is the inclusion/quotient map and $I_{m_{T'}}$ is the integration map (see Ch.5 in [15], see also [3] for the case when $[i]$ is not injective). Dualizing the factorization scheme again and taking into account that X has the Fatou property we obtain



Theorem 3.3. *Let $X(\mu)$ be a Banach function space over (Ω, Σ, μ) with the Fatou property such that X' is order continuous. Let $T : E \to X(\mu)$ be an operator from a Banach space E to $X(\mu)$ with p-th power factorable adjoint. Then T factorizes through $(L^p(m_{T'}))'$, and if the range of T lies into a Banach function space $Z(\mu)$ where $Z(\mu) \subseteq X(\mu)$ and*

(i) *Z' is order continuous and Z has the Fatou property, and*
(ii) *the (range) restriction $S : E \to Z$ of T has p-th power factorable adjoint,*

then $(L^p(m_{T'}))' \Subset Z$.

Proof. The arguments before the theorem give the factorization through $(L^p(m_{T'}))'$. For the optimality of this space, suppose that the range of T lies in $Z(\mu) \subseteq X(\mu)$. Then T' factorizes through Z' and by hypothesis S' is p-th power factorable. This implies that S' factorizes through $L^p(m_{S'})$. But note that $m_{S'} = m_{T'}$. Consequently, by the optimal domain theorem for p-th power factorable operators (see [15, Ch.5] and [3]), $[i](Z') \subseteq L^p(m_{T'})$, and so $(L^p(m_{T'}))' \Subset Z'' = Z$. $\square$

The following result provides some structure information for the space $(L^p(m_{T'}))'$ without any assumption on the p-th power factorability of T'.

Corollary 3.4. *Assume that X is an order continuous p-convex Banach function space and X^p has the Fatou property. Consider an operator $T : E \to X$. Then $(L^p(m_{T'}))' \Subset (X^p)'$. Moreover, the optimal range in the sense of Theorem 3.3 of the extension $T_0 : E \to X \hookrightarrow (X^p)'$ of T is the space $(L^p(m_{T'}))'$.*

For the proof just use Proposition 2.2 and Theorem 3.3, taking into account that $m_{T'} = m_{T_0'}$. Notice that the requirement of X^p being a Banach function space is fulfilled if X is p-convex (see the comments after Definition 2.1).

Remark 3.5. Let us write Theorem 3.3 for the case $p = 1$, i.e. when there is no restriction on the adjoint map. In this case, we obtain the optimal range for continuous operators. Let $X(\mu)$ be a Banach function space with the Fatou property such that X' is order continuous. Let $T : E \to X(\mu)$. Then T factorizes through $(L^1(m_{T'}))'$, and if the range of T lies into a Banach function space $Z(\mu) \subseteq X(\mu)$ such that Z has the Fatou property and Z' is order continuous then $(L^1(m_{T'}))' \Subset Z$.

For instance, if μ is a Rybakov measure for $m_{T'}$ then we obtain directly that $(L^1(m_{T'}))' \subseteq Z$. In the case that μ is equivalent to $\|m_{T'}\|$ (i.e. if T' is μ-determined) then $[i]$ is an inclusion map and then the formulas of the duality given at the end of Section 1 gives that there is a measurable function h (the Radon-Nikodým derivative $d\nu/d\mu$ of a Rybakov measure ν for $m_{T'}$) such that $h \cdot (L^1(m_{T'}))' \subseteq Z$.

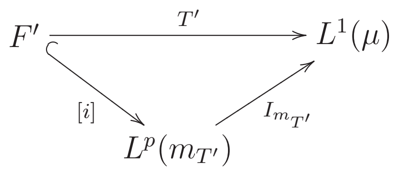
4. Applications and examples

4.1. **Operators from $L^\infty(\mu)$.** We will show in this section that the optimal range of an operator from an AM-space into a Banach function space which adjoint operator is p-th factorable can be described in reasonable terms.

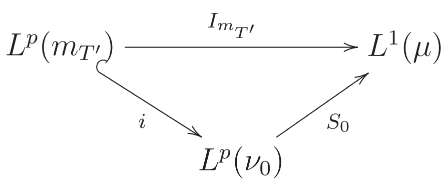
In this paper we will say that a Banach function space $X(\mu)$ is *almost an L^p-space* if for every $\varepsilon > 0$ there is a measurable set $A_\varepsilon \in \Sigma$ such that $\mu(A_\varepsilon) < \varepsilon$ and the restriction $X(\mu|_{\Omega \setminus A_\varepsilon})$ is order isomorphic to an L^p-space.

Theorem 4.1. *Let $p > 1$. Consider a finite measure space (Ω, Σ, ν), a Banach function space $F(\nu)$ and an operator $T : L^\infty(\mu) \to F$, where μ is a σ-finite measure. Suppose that F has the Fatou property and F' is order continuous, T' is positive, ν-determined, p-th power factorable and $T'(F') \subseteq L^1(\mu)$. Then the optimal range $(L^p(m_{T'}))'$ of T is almost an $L^{p'}$-space.*

Proof. Under the requirements above, the Köthe adjoint map can be written as $T' : F' \to L^1(\mu)$ and so $m_{T'}$ is a countably additive vector measure. Since F' is order continuous and T' is p-th power factorable, it can be extended to the space $L^p(m_{T'})$ by means of a inclusion/quotient map $[i]$ (see the explanation at the end of Section 1, [15, Ch.5] and [3]) as follows



Step 1. The integration operator $I_{m_{T'}}$ that appears in the factorization above is a positive map (since $[i](F')$ is dense in the p-convex space $L^p(m_{T'})$) and $L^1(\mu)$ is p-concave for every $p \geq 1$, we have that T' can be extended to $L^p(\nu_0)$ as $T' = S_0 \circ i$ where ν_0 is a Rybakov measure for $m_{T'}$ and S_0 is the extension of the integration map (see the variant of the Maurey-Rosenthal Theorem given by Theorem 6.41 in [15, Ch.6]). Thus this gives an extension of $I_{m_{T'}}$ as



Step 2. Let us show that a restriction of S_0 to a set as small in measure as we want is p-th power factorable. For doing this, just take into account that the vector measure m_{S_0} coincides with $m_{T'}$. In particular, it is positive and 1-concave. Again the variant of the Maurey-Rosenthal Theorem quoted above gives (for $p = 1$) that $S_0 : L^p(\nu_0) \to L^1(\mu)$ can be extended to the space $L^1(\eta)$, where η is a Rybakov measure for m_{S_0} and so for $m_{T'}$. More precisely, it can be factorized through the inclusion map $L^1(m_{S_0}) \hookrightarrow L^1(\eta)$. Consequently there is a constant $0 < Q_1$ and a Radon-Nikodým derivative $v = d\eta/d\nu_0$ such that for every $f \in L^p(\nu_0)$

$$\|S_0(f)\|_{L^1(\mu)} \leq Q_1 \int |f| d\eta = Q_1 \||f|^{1/p}\|^p_{L^p(\eta)} \leq Q_1 \||v|^{1/p}|f|^{1/p}\|^p_{L^p(\nu_0)}.$$

The function $|v|$ is integrable with respect to ν_0 and since this measure is a Rybakov measure for $m_{T'}$, it is equivalent to the semivariation $\|m_{T'}\|$. Fix $\varepsilon > 0$. Thus by the ν_0-integrability of $|v|$ we have that there is a constant K_ε such that $\|m_{T'}\|(A_\varepsilon) < \varepsilon$, where $A_\varepsilon := \{|v| > K_\varepsilon\}$. Then

$$\||v|^{1/p}|f|^{1/p}\|^p_{L^p(\nu_0|_{A^c_\varepsilon})} \leq K_\varepsilon \||f|^{1/p}\|^p_{L^p(\nu_0|_{A^c_\varepsilon})},$$

where $A^c_\varepsilon = \Omega \setminus A_\varepsilon$, i.e. the restriction of S_0 to this set is p-th power factorable (notice that $\|m_{T'}\|$ is equivalent to ν, so the condition $\|m_{T'}\|(A_\varepsilon) < \varepsilon$ can be written in terms of ν). The arguments in Theorem 3.3 on the optimal domain for T' can then be applied. As we said, $m_{T'} = m_{S_0}$ and so

$$L^p(m_{T'}|_{A^c_\varepsilon}) \subseteq L^p(\nu_0|_{A^c_\varepsilon}) \subseteq L^p(m_{S_0}|_{A^c_\varepsilon}) = L^p(m_{T'}|_{A^c_\varepsilon}).$$

The optimal range space given by Theorem 3.3 for the restricted operator

$$P_\varepsilon \circ T \colon L^\infty(\mu) \to F \to F|_{A^c_\varepsilon}(\nu|_{A^c_\varepsilon})$$

(where P_ε is the band projection of F onto $F|_{A^c_\varepsilon}$), gives then

$$(L^p(m_{T'}|_{A^c_\varepsilon}))' = L^{p'}(\nu_0|_{A^c_\varepsilon}).$$

The result is obtained.

□

The next result shows that for the case $p = 1$ (i.e. no restriction on the adjoint map, which provides the limit case), the optimal range is exactly an L^∞-space.

Theorem 4.2. *Consider a finite measure space (Ω, Σ, ν), a Banach function space $F(\nu)$ and an operator $T : L^\infty(\mu) \to F$, where μ is a σ-finite measure. Suppose that F has the Fatou property, F' is order continuous, T' is positive, ν-determined and $T'(F') \subseteq L^1(\mu)$. Then the optimal range of T is $L^\infty(\nu)$.*

Proof. The proof is the same that in the previous theorem, but the second step in the proof is not needed. In this case we obtain

$$L^1(m_{T'}) \subseteq L^1(\nu_0) \subseteq L^1(m_{S_0}) = L^1(m_{T'}).$$

Taking into account that ν and ν_0 are equivalent, the optimal range $(L^1(m_{T'}))'$ given by Theorem 3.3 coincides with $L^\infty(\nu)$. □

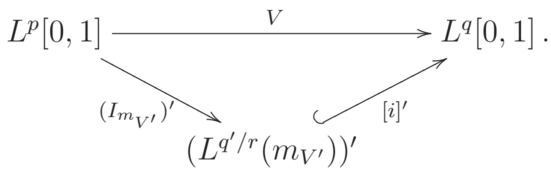
4.2. **Optimal range of operators with compact associated integration map.** Consider an operator $T : X(\mu) \to Y(\nu)$ between Banach function spaces $X(\mu)$ and $Y(\nu)$, where Y' is order continuous. Suppose that

$$R := \{T'(f) : \sup_{A \in \Sigma} \|T'(f\chi_A)\| \leq 1\}$$

is a *relatively compact set.* Let us show that the corresponding optimal range satisfying that the Köthe adjoint of the (range) restricted map is p-th power factorable is order isomorphic to an $L^{p'}$-space.

Since Y' is order continuous, the operator T' defines a countably additive vector measure by $m_{T'}(A) := T'(\chi_A)$, $A \in \Sigma$, and simple functions are dense in both Y' and $L^1(m_{T'})$. This, together with the condition on R implies that the integration map $I_{m_{T'}} : L^1(m_{T'}) \to X'$ is compact (recall the equivalent norm $|||\cdot|||_{L^p(m)}$ for the spaces $L^p(m)$ given in the Introduction). In this case, it is well-known that the space $L^1(m_{T'})$ is order isomorphic to the space $L^1(|m_{T'}|)$ (see Proposition 3.48 in [15] and the references therein), where $|m_{T'}|$ is the variation of $m_{T'}$, that is a scalar measure. Since by Theorem 3.3 the optimal range of T with the p-th power requirement for the dual of the restricted map is the space $(L^p(m_{T'}))'$, we obtain that the optimal range is order isomorphic to $L^{p'}(|m_{T'}|)$. Examples of this situation (i.e. compact integration maps) can be found for instance in Example 3.49 in [15] and the comments after it on the Volterra operator.

4.3. **Optimal range for the Volterra operator.** The spaces of (classes of) p-integrable functions with respect to the Volterra measure (i.e. the one defined by the Volterra operator) are nowadays well known. The reader can find information about in [15, Ch.3] (see for instance Example 3.76 in this book and the references therein). It provides the optimal domain space for this operator. In this section we analyze the structure of the optimal range for this operator. Let $V\colon L^p[0,1] \to L^q[0,1]$ be the Volterra operator for $1 < q \leq p \leq 2$ which adjoint operator is r-th power factorable, $r \geq 1$. Note that $V = H_0$ in Example 3.1, so this condition holds for $r < q'$. From Theorem 3.3 we have the following factorization diagram



Let μ be Lebesgue measure. Let us write the Rybakov measure ν for $m_{V'}$ that is defined by the element $\chi_{[0,1]} \in L^{p'}[0,1]$.

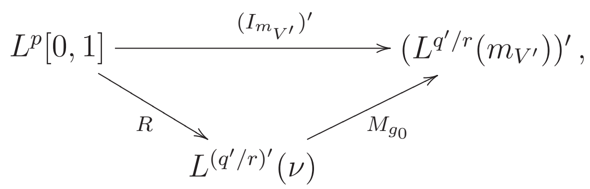
$$\nu(A) := \langle \chi_{[0,1]}, V'(\chi_A) \rangle = \int_0^1 \mu([x,1] \cap A)\, d\mu, \quad A \in \Sigma.$$

(See Example 6.46 in [15] for the corresponding Rybakov measure for the case of the Volterra operator). We denote by h the Radon-Nikodým derivative $\frac{d\nu}{d\mu}$. Recall that $L^{q'/r}(m_{V'})$ is a Banach function space over the measure ν (and so $(L^{q'/r}(m_{V'}))'$ is too). The measure ν has the same null sets that μ. In this case, as was said in the Introduction, $[i]'$ is given by $[i]'(g)(x) := h(x) \cdot g(x) \in L^q[0,1]$, where $g \in (L^{q'/r}(m_{V'}))'$ and $x \in [0,1]$. This allows to write the inclusions

$$V(L^p[0,1]) \subseteq h \cdot (L^{q'/r}(m_{V'}))' \subseteq L^q[0,1],$$

and $(L^{q'/r}(m_{V'}))'$ is the optimal range space, in the sense that was explained in the previous sections. Let us give more information about this space.

We know that $(L^{q'/r}(m_{V'}))'$ is $(q'/r)'$-concave, since $L^{q'/r}(m_{V'})$ is q'/r-convex (see [15, Ch.2]). On the other hand, assume that $r \geq 1$ satisfies that $(q'/r)' \leq p$. Note that in this case $(I_{m_{V'}})'$ is $(q'/r)'$-convex, since $L^p[0,1]$ is p-convex and thus $(q'/r)'$-convex (see [12, Ch.2]), and $(I_{m_{V'}})'$ is positive; to see that, just take into account that the integration map associated to the Volterra operator is again given by the same kernel, and the adjoint map is given by the dual kernel of the Volterra kernel. Using the instance of the Maurey-Rosenthal Theorem given in [6, Corollary 2], we have the following factorization diagram



where R is a continuous operator and $0 < g_0 \in \left[L^{(q'/r)'}(\nu)\right]^{(L^{q'/r}(m_{V'}))'}$ (see [2, Lemma 3.7]). Therefore,

$$V(L^p[0,1]) \subseteq h \cdot g_0 \cdot R(L^p[0,1]) \subseteq h \cdot (L^{q'/r}(m_{V'}))',$$

and $h \cdot (L^{q'/r}(m_{V'}))'$ is the optimal range satisfying the r-th power factorability requirement on the adjoint operator. In the case $r = 1$ we obtain a complete description of the optimal range without assumptions on the adjoint map.

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