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r-Critical Points and Taylor Expansion of the Exponential Map for Smooth Immersions in $\mathbb{R}^{k+n}$

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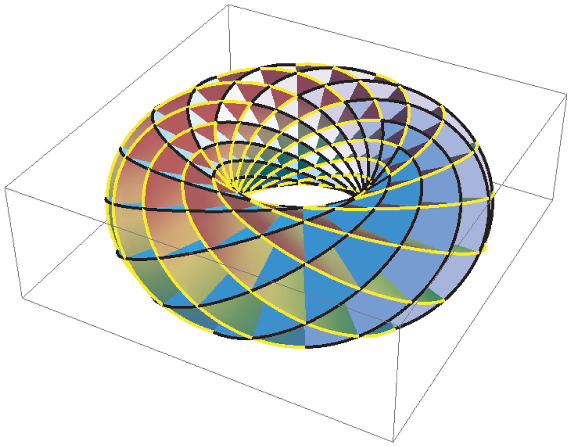


Construcción del toro mediante círculos de Villarceau, [17]

A vosotros

Contents

Chapter 1

Summary

Classically, the study of the contact with hyperplanes and hyperspheres has been realized by using the family of height and distance squared functions ([25],[21]). On the first part of the thesis, we analyze the Taylor expansion of the exponential map up to order three of a submanifold M immersed in $\mathbb{R}^n$. Our main goal is to show its usefulness for the description of special contacts of the submanifolds with geometrical models.

As we analyze the contacts of high order, the complexity of the calculations increases. In this work, through the Taylor expansion of the exponential map, we characterize the geometry of order higher than 3 in terms of invariants of the immersion, so that the effective computations in specific cases become more affordable. It allows also to get new geometric insights.

Let M be a regular surface immersed in $\mathbb{R}^n$ and $\gamma : I \to \mathbb{R}$ be the geodesic defined in M by the initial condition $\gamma'(0) = v \in T_mM$. Let $h : I \to \mathbb{R}^n$ be the geodesic defined in $\mathbb{R}^n$ with the same initial velocity,

that is $h(t) = m+vt$. The difference $\gamma - h$ gives the geodesic deviation of the immersion for the initial condition v. The Taylor expansion of $\gamma(t) - h(t)$ begins with the second order term which is proportional to the second fundamental form of M at m acting upon v, say $\alpha(v, v)$. It is orthogonal to T_mM and its meaning is well known. The third term has in general non-vanishing orthogonal and tangential components with respect to T_mM. The orthogonal component depends essentially on the third order geometry of the surface, that is on the covariant derivative of the second fundamental form. The tangential component, on its part, depends only on the second fundamental form at m and may be decomposed naturally into two components, one tangent to v and the other orthogonal to it. We call the first, the *frontal deviation*, and the second, the *lateral deviation*. We shall distinguish the directions $v \in T_mM$ on which the norm or the frontal deviation (resp. the lateral deviation) are extremal. In the case of M being a surface, there are in general at most four directions of each or these classes. We shall show that the directions where the lateral deviation vanishes are the directions of higher contact of a geodesic with the normal section of the surface.

On the other hand, we obtain an expression for the normal torsion in terms of invariants related to the second fundamental form and its covariant derivative.

Finally, we compute by using the Taylor expansion of the exponential map, the directions of higher contact with hyperspheres of a surface in $\mathbb{R}^4$, defined by J. Montaldi in [22], and characterize the centers of these hyperspheres through the normal curvature and normal torsion. We also characterize the asymptotic directions of a surface in $\mathbb{R}^5$, [29]. In both cases, the results are given in terms of invariants of the immersion, so that the numerical or symbolic computation of those directions becomes affordable, not hampered by the recourse to Monge's, isothermal or other special coordinates as in other works ([22], [16]).

On the second part of the thesis, we introduce the concept of critical point of a smooth map between submanifolds. Classically the focal set of a differential submanifold is given through the analysis of the singularities of the family of distance squared functions over the submanifold, see [25]. J. Montaldi characterized in [22] the singularities of corank 2 of distance squared functions on surfaces immersed in $\mathbb{R}^4$ as semiumbilic points.

Also, if we consider a differentiable k-dimensional manifold M immersed in $\mathbb{R}^{k+n}$, we know that its focal set can also be interpreted as the image of the critical points of the *normal map* $\nu(m,u) : NM \to \mathbb{R}^{k+n}$ defined by $\nu(m,u) = \pi_N(m,u) + u$, for $m \in M$ and $u \in N_m M$, where $\pi_N : NM \to M$ denotes the normal bundle.

In the same way, the parabolic set of a differential submanifold is given through the analysis of the singularities of the height functions over the submanifold. If we consider a differentiable k-dimensional manifold M immersed in $\mathbb{R}^{k+n}$, we know that its parabolic set can also be interpreted as the image of the critical points of the *generalized Gauss map* $\psi(m,u) : NM \to \mathbb{R}^{k+n}$ defined by $\psi(m,u) = u$, for $u \in N_m M$.

Finally, we characterize the asymptotic directions as the tangent set of a k-dimensional manifold M immersed in $\mathbb{R}^{k+n}$ throughout the study of the singularities of the tangent map $\Omega(m, y) : TM \to \mathbb{R}^{k+n}$ defined by $\Omega(m, y) = \pi(m, y) + y$, for $y \in T_mM$, where $\pi : TM \to M$ denotes the tangent bundle.

On the other hand, the concept of curvature ellipse at a point of a surface M immersed in $\mathbb{R}^4$ was treated with full details in [12]. It is defined as the locus of all the end points of the curvature vectors of the normal sections along all the tangent directions to M at a point in it. This ellipse lies in the normal subspace at that point and it is completely determined by the second fundamental form. We call *Veronese of curvature* to the natural generalization of the curvature ellipse for higher dimensions of M.

We describe first the focal set and its geometrical relation to the Veronese of curvature for k-dimensional immersions in $\mathbb{R}^{k+n}$. Then we define the r-critical points of a differential map $f : H \to K$ between two differential manifolds and characterize the 2 and 3-critical points of the normal map and generalized Gauss map. The number of these critical points at $m \in M$ may depend on the degeneration of the curvature ellipse and we calculate those numbers in the particular case that M is an immersed surface in $\mathbb{R}^4$ for the normal map and $\mathbb{R}^5$ for the generalized Gauss map.

Resumen

En general, el estudio del contacto con hiperplanos e hiperesferas se ha llevado a cabo usando la familia de funciones altura y la función distancia al cuadrado ([25],[21]). En la primera parte de la tesis analizamos el desarrollo de Taylor de la aplicación exponencial hasta orden 3 de una subvariedad M inmersa en $\mathbb{R}^n$. Nuestro principal objetivo es mostrar su utilidad en el estudio de contactos especiales de subvariedades con modelos geométricos.

A medida que analizamos los contactos de orden mayor, la complejidad de las cuentas aumenta. En este trabajo, a través del desarrollo de Taylor de la aplicación exponencial, caracterizamos la geometría de orden mayor que 3 en términos de invariantes geométricos de la inmersión, por lo que el trabajo con las cuentas en casos especiales se convierte en más manejable. Esto nos permite también obtener nuevos resultados geométricos.

Sea M una superficie regular inmersa en $\mathbb{R}^n$ y $\gamma : I \to \mathbb{R}$ una geodésica definida en M con condición inicial $\gamma'(0) = v \in T_mM$. Sea $h : I \to \mathbb{R}^n$ la geodésica definida en $\mathbb{R}^n$ con la misma velocidad inicial, esto es $h(t) = m + vt$. La diferencia $\gamma - h$ da la desviación geodésica de la inmersión para la condición inicial v. El desarrollo de Taylor de $\gamma(t) - h(t)$ comienza con el término de segundo orden el cual es proporcional a la segunda forma fundamental de M en m actuando sobre v, es decir, $\alpha(v, v)$. Esto es ortogonal a T_mM y su significado es bien conocido.

El tercer término tiene en general componentes ortogonales y tangentes que no se anulan con respecto a T_mM. La componente ortogonal depende esencialmente de la geometría de tercer orden de la superficie, esto es de la derivada covariante de la segunda forma fundamental.

La componente tangente depende sólo de la segunda forma fundamental en m y se puede descomponer de forma natural en dos componentes una tangente a v y la otra ortogonal a ella. Llamamos a la primera *desviación frontal*, y a la segunda *desviación lateral*. Distinguiremos las direcciones $v \in T_m M$ donde la desviación frontal (resp. desviación lateral) es extremal. En el caso de que M sea una superficie, existen en general como mucho cuatro direcciones de cada una de estas clases. También demostraremos que las direcciones donde la desviación lateral se anula corresponden a las direcciones de mayor contacto de la geodésica con la sección normal de la superficie.

Por otro lado, obtenemos una expresión para la torsión normal en términos de invariantes relacionados con la segunda forma fundamental y su derivada covariante. Finalmente, usando el desarrollo de Taylor de la exponencial normal, calculamos las direcciones de mayor contacto con hiperesferas en el caso de una superficie inmersa en $\mathbb{R}^4$, definidas por J. Montaldi en [22] y caracterizamos los centros de estas esferas a través de la curvatura normal y la torsión normal. También caracterizamos las direcciones asintóticas de una superficie inmersa en $\mathbb{R}^5$. En ambos casos, los resultados se dan en términos de invariantes de la inmersión, de modo que el cálculo numérico o simbólico de las direcciones se convierte en más asequible en comparación con el uso de cartas de Monge, coordenadas isotermas u otras coordenadas especiales como en otros trabajos ([22], [16]).

En la segunda parte de la tesis se introduce el concepto de punto crítico de una aplicación regular entre subvariedades. Clásicamente, el conjunto focal de una subvariedad diferencial viene dado a través del estudio de las singularidades de la función distancia al cuadrado sobre la subvariedad, [25]. J. Montaldi caracterizó en [22] las singularidades de corrango 2 de la función distancia al cuadrado para superficies inmersas en $\mathbb{R}^4$ como puntos semiumbílicos. Además, si consideramos una variedad diferenciable M de dimensión k e inmersa en $\mathbb{R}^{k+n}$, sabemos que su conjunto focal puede ser interpretado como la imagen de los puntos críticos de la *aplicación normal* $\nu(m, u) : NM \to \mathbb{R}^{k+n}$ definida por $\nu(m, u) = \pi_N(m, u) + u$, para $m \in M$ y $u \in N_m M$, donde $\pi_N : NM \to M$ denota el fibrado normal.

De la misma manera, el conjunto parabólico de una subvariedad diferencial viene dado por el análisis de las singularidades de la función altura sobre la subvariedad. Si consideramos una subvariedad M de dimensión k e inmersa en $\mathbb{R}^{k+n}$, sabemos que su conjunto parabólico puede ser interpretado como la imagen de los puntos críticos de la *aplicación generalizada de Gauss* $\psi(m, u) : NM \to \mathbb{R}^{k+n}$ definida por $\psi(m, u) = u$, donde $u \in N_m M$.

Finalmente, caracterizamos las direcciones asintóticas como el conjunto de direcciones del tangente de una subvariedad M de dimensión k e inmersa en $\mathbb{R}^{k+n}$ a través del estudio de las singularidades de la aplicación tangente $\Omega(m, y) : TM \to \mathbb{R}^{k+n}$ definida por $\Omega(m, y) = \pi(m, y) + y$, para $y \in T_m M$, donde $\pi : TM \to M$ denota el fibrado tangente.

Por otro lado, el concepto de elipse de curvatura en un punto de una superficie M inmersa en $\mathbb{R}^4$ es tratado con detalle en [12]. La elipse de curvatura es definida como el lugar geométrico que describen los vectores de curvatura de la sección normal a lo largo de todas las direcciones del tangente de M en un punto. Esta elipse está en el subespacio normal al punto y está completamente determinada por la segunda forma fundamental. Llamamos *Veronese of curvatura* a la generalización natural de la elipse de curvatura para mayores dimensiones de M.

Describimos primero el conjunto focal y su relación geométrica con la Veronese de curvatura para una variedad k dimensional inmersa en $\mathbb{R}^{k+n}$. Entonces, definimos los puntos r-críticos de una aplicación $f : H \to K$ entre dos subvariedades y caracterizamos los puntos 2 y 3 críticos de la aplicación normal y la aplicación generalizada de Gauss. El número de estos puntos críticos en $m \in M$ depende de la degeneración de la elipse de curvatura y calculamos ese número en el caso particular de una superficie inmersa en $\mathbb{R}^4$ para la aplicación normal y $\mathbb{R}^5$ para la aplicación generalizada de Gauss.

Resum

En general, l'estudi del contacte amb hiperplans i hiperesferes s'ha dut a terme utilitzant la família de funcions altura i la funció distància al quadrat ([25], [21]). A la primera part de la tesi analitzem el desenvolupament de Taylor de l'aplicació exponencial fins a ordre 3 d'una subvarietat M immersa en $\mathbb{R}^n$. El nostre principal objectiu és mostrar la seua utilitat en l'estudi de contactes especials de subvarietats amb models geomètrics.

A mesura que analitzem els contactes d'ordre major, la complexitat dels comptes augmenta. En aquest treball, a través del desenvolupament de Taylor de l'aplicació exponencial, caracteritzem la geometria d'ordre major que 3 en termes d'invariants geomètrics de la immersió, de manera que el treball amb els comptes en casos especials es converteix en més manejable. Això ens permet també obtenir nous resultats geomètrics.

Siga M una superfície regular immersa en $\mathbb{R}^n$ i $\gamma : I \to \mathbb{R}$ una geodèsica definida en M amb condició inicial $\gamma'(0) = v \in T_mM$. Siga $h : I \to \mathbb{R}^n$ la geodèsica definida en $\mathbb{R}^n$ amb la mateixa velocitat inicial, és a dir $h(t) = m + Vt$. La diferència $\gamma - h$ dóna la desviació geodèsica de la immersió per la condició inicial v. El desenvolupament de Taylor de $\gamma(t) - h(t)$ comença amb el terme de segon ordre el qual és proporcional a la segona forma fonamental de M a m actuant sobre v, és a dir, $\alpha(v, v)$. Això és ortogonal a T_mM i el seu significat és ben conegut.

El tercer terme té en general components ortogonals i tangents que no s'anul·len pel que fa a T_mM. La component ortogonal depèn essencialment de la geometria de tercer ordre de la superfície, és a dir de la derivada covariant de la segona forma fonamental.

La component tangent depèn només de la segona forma fonamental en m i es pot descompondre de forma natural en dues components, una tangent a v i l'altra ortogonal a ella. A la primera li diguem *desviació frontal*, i a la segona *desviació lateral*. Distingirem les direccions $v \in T_mM$ on la desviació frontal (resp. desviació lateral) és extremal. En el cas que M siga una superfície, hi ha en general com a molt 4 direccions de cadascuna d'aquestes classes. També demostrarem que les direccions on la desviació lateral s'anul·la corresponen a les direccions de major contacte de la geodèsica amb la secció normal de la superfície.

D'altra banda, obtenim una expressió per la torsió normal en termes d'invariants relacionats amb la segona forma fonamental i la seva derivada covariant. Finalment, usant el desenvolupament de Taylor de l'exponencial normal, calculem les direccions de més contacte amb hiperesferes en el cas d'una superfície immersa en $\mathbb{R}^4$, definides per J. Montaldi en [22] i caracteritzem els centres d'aquestes esferes a través de la curvatura normal i la torsió normal. També caracteritzem les direccions asimptòtiques d'una superfície immersa en $\mathbb{R}^5$. A ambdós casos, els resultats es donen en termes d'invariants de la immersió, de manera que el càlcul numèric o simbòlic de les direccions esdevé més assequible en comparació amb l'ús de cartes de Monge, coordenades isotermes o altres coordenades especials com en altres treballs ([22], [16]).

A la segona part de la tesi s'introdueix el concepte de punt crític d'una aplicació regular entre subvarietats. Clàssicament, el conjunt focal d'una subvarietat diferencial ve donat a través de l'estudi de les singularitats de la funció distància al quadrat sobre la subvarietat, [25]. J. Montaldi va caracteritzar en [22] les singularitats de corrango 2 de la funció distància al quadrat per a superfícies immerses en $\mathbb{R}^4$ com punts semiumbílics. A més, si considerem una varietat diferenciable M de dimensió k i immersa en $\mathbb{R}^{k+n}$, sabem que el seu conjunt focal pot ser interpretat com la imatge dels punts crítics de la *aplicació normal* $\nu(m,u) : NM \to \mathbb{R}^{k+n}$ definida per $\nu(m,u) = \pi_N(m,u) + o$, per $m \in M$ i $u \in N_mM$, on $\pi_N : NM \to M$ denota el fibrat normal.

De la mateixa manera, el conjunt parabòlic d'una subvarietat diferencial ve donat per l'anàlisi de les singularitats de la funció altura sobre la subvarietat. Si considerem una subvarietat M de dimensió k i immersa en $\mathbb{R}^{k+n}$, sabem que el seu conjunt parabòlic pot ser interpretat com la imatge dels punts crítics de la *aplicació generalitzada de Gauss* $\psi(m,u) : NM \to \mathbb{R}^{k+n}$ definida per $\psi(m,u) = u$, on $u \in N_mM$.

Finalment, caracteritzem les direccions asimptòtiques com el conjunt de direccions del tangent d'una subvarietat M de dimensió k i immersa en $\mathbb{R}^{k+n}$ a través de l'estudi de les singularitats de l'aplicació tangent $\Omega(m,y) : TM \to \mathbb{R}^{k+n}$ definida per $\Omega(m,y) = \pi(m,y) + y$, per $y \in T_mM$, on $\pi : TM \to M$ denota el fibrat tangent.

D'altra banda, el concepte d'el·lipse de curvatura en un punt d'una superfície M immersa en $\mathbb{R}^4$ és tractat amb detall en [12]. L'el·lipse de curvatura és definida com el lloc geomètric que descriuen els vectors de curvatura de la secció normal al llarg de totes les direccions del tangent de M en un punt. Aquesta el·lipse està en el subespai normal al punt i està completament determinada per la segona forma fonamental. Diguem *Veronese of curvatura* a la generalització natural de l'el·lipse de curvatura per a majors dimensions de M.

Descrivim primer el conjunt focal i la seva relació geomètrica amb la Veronese de curvatura per a una varietat k dimensional immersa en $\mathbb{R}^{k+n}$. Llavors, definim els punts r-crítics d'una aplicació $f : H \to K$ entre dues subvarietats i caracteritzem els punts 2 i 3 crítics de l'aplicació normal i l'aplicació generalitzada de Gauss. El nombre d'aquests punts crítics en $m \in M$ depèn de la degeneració de l'el·lipse de curvatura i calculem aquest nombre en el cas particular d'una superfície immersa en $\mathbb{R}^4$ per a l'aplicació normal i $\mathbb{R}^5$ per a l'aplicació generalitzada de Gauss.

Chapter 2

Preliminaries

In this chapter, we introduce the basic concepts that we shall use along the thesis.

2.1 Differential geometry

Let M be a k-dimensional differentiable manifold immersed in $\mathbb{R}^{k+n}$. Since all of our study will be local, we gain in brevity by assuming that it is a regular submanifold. For each $m \in M$ we consider the decomposition $T_m\mathbb{R}^{k+n} = T_mM \oplus N_mM$, where T_mM denotes the tangent space to M at $m \in M$, and N_mM denotes the normal subspace to M at m, i.e. $N_mM = (T_mM)^{\perp}$. Given $X \in T_m\mathbb{R}^{k+n}$, that decomposition will be written as $X_m = X_m^{\top} + X_m^{\perp}$ where $X^{\top} \in T_mM$, $X^{\perp} \in N_mM$.

Let $\pi : TM \to M$ and $\pi_N : NM \to M$ denote the tangent and normal bundles respectively. If E is the total space of a smooth bundle we will denote by $\Gamma(E)$ the space of its smooth sections. If $s \in \Gamma(E)$ then s_m shall denote usually the value of s at m. For the particular case of TM, we will put $\mathfrak{X}(M)$ instead of $\Gamma(TM)$. We define the linear connection $\nabla^\top$ for π by $\nabla^\top_X Y = (D_X Y)^\top$, $X, Y \in \mathfrak{X}(M)$, where D is the Levi-Civita connection in $\mathbb{R}^{k+n}$ which in fact is equal to the directional derivative. For π_N we define the connection $\nabla^\perp$ by $\nabla^\perp_X u = (D_X u)^\perp$, $u \in \Gamma(NM)$. These connections define a new connection ∇ in $\Gamma(T^{(r,s)}M \otimes N^{(p,q)}M)$ such that if, for example, we have $w = u \otimes Y \otimes \gamma$, where $u \in \Gamma(NM)$, $Y \in \mathfrak{X}(M)$, $\gamma \in \Gamma(T^*M)$, then:

$$\nabla_X w = (\nabla^\perp_X u) \otimes Y \otimes \gamma + u \otimes (\nabla^\top_X Y) \otimes \gamma + u \otimes Y \otimes (\nabla^\top_X \gamma).$$

This connection preserves the inner product. Sometimes we shall use here a notation more or less usual, with which from an 1-form $\mu \in T^*_m M$ we build the vector $\mu^\sharp \in T_m M$ such that $\mu^\sharp \cdot x = \mu(x)$, $\forall x \in T_m M$.

Suppose now that we have a linear connection with covariant derivative ∇ in a manifold M. Then we can define the operator:

$$\nabla : \mathfrak{X}(M) \times \Gamma^{(r,s)} M \to \Gamma^{(r,s)} M,$$

where $\Gamma^{(r,s)}M$ denotes the C^∞-module of differentiable sections of $T^{(r,s)}M$, in other words the tensor fields of type (r, s) of M. This operator acts as $\nabla(X, K) = \nabla_X K$, where $K \in \Gamma^{(r,s)}M$. We know that

$$\nabla(fX, K) = f\nabla(X, K), \qquad \nabla(X + Y, K) = \nabla(X, K) + \nabla(Y, K),$$

where $f : M \to \mathbb{R}$ is an arbitrary smooth function. Then, the operator is $C^\infty(M)$-linear in the first argument, but not in the second one.

Now for any $r > 1$ we can define the operator ∇^r which acts over any r differentiable vector fields $X_1, ..., X_r \in \mathfrak{X}(M)$ and a tensor field

K of type (r,s). This operator gives another tensor field of the same type (r,s) :

$$\nabla^r(X_1,\dots,X_r;K) = \nabla_{X_1}\big(\nabla^{r-1}(X_2,\dots,X_r;K)\big) - \sum_{i=2}^{r} \nabla^{r-1}(X_2,\dots,\nabla_{X_1}X_i,\dots,X_r;K).$$

Let us see that this operator is $C^\infty(M)$-linear in the first r arguments. This is clear for the sum. Let $\{f_i\}_{i=1}^r$ be a family of arbitrary smooth functions, $f_i : M \to \mathbb{R}$. For $r = 1$ we know that $\nabla(f_1X,K) = f_1\nabla(X,K)$. Let us see the general case:

$$\begin{aligned}
\nabla^r(f_1X_1, f_2X_2,\dots,f_rX_r;K) &= \nabla_{f_1X_1}(\nabla^{r-1}(f_2X_2,\dots,f_rX_r;K)) \\
&\quad - \sum_{i=2}^{r} \nabla^{r-1}(f_2X_2,\dots,\nabla_{f_1X_1}(f_iX_i),\dots,f_rX_r;K) \\
&= \nabla_{f_1X_1}(f_2\dots f_r\nabla^{r-1}(X_2,\dots,X_r;K)) \\
&\quad - \sum_{i=2}^{r} \nabla^{r-1}(f_2X_2,\dots,\nabla_{f_1X_1}(f_iX_i),\dots,f_rX_r;K)) \\
&= f_1\sum_{i=2}^{r} f_2\dots X_1(f_i)\dots f_r\nabla^{r-1}(X_2,\dots,X_r;K) \\
&\quad + f_1f_2\dots f_r\nabla_{X_1}(\nabla^{r-1}(X_2,\dots,X_r;K)) \\
&\quad - f_1\sum_{i=2}^{r} f_2\dots X_1(f_i)\dots f_r\nabla^{r-1}(X_2,\dots,X_r;K) \\
&\quad - f_1f_2\dots f_r\sum_{i=2}^{r} \nabla^{r-1}(X_2,\dots,\nabla_{X_1}X_i,\dots,X_r;K) \\
&= f_1f_2\dots f_r[\nabla_{X_1}(\nabla^{r-1}(X_2,\dots,X_r;K)) \\
&\quad - \sum_{i=2}^{r} \nabla^{r-1}(X_2,\dots,\nabla_{X_1}X_i,\dots,X_r;K)] \\
&= f_1f_2\dots f_r\nabla^r(X_1,X_2,\dots,X_r;K).
\end{aligned}$$

2.1.1 First and second fundamental forms

Definition 2.1.1 *The **first fundamental form** over M is the field g of positive definite symmetric bilinear forms induced over M by the Euclidean inner product in $\mathbb{R}^{k+n}$. Thus, if $X_m, Y_m \in T_mM$, then*

$$g_m(X_m, Y_m) = X_m \cdot Y_m.$$

Let $\phi : B \subset \mathbb{R}^k \to \mathbb{R}^{k+n}$ be a local parameterization of M. Thus $T_mM = d\phi_q(\mathbb{R}^k)$, where $q \in B$ and $m = \phi(q)$.

Definition 2.1.2 *A vector field X tangent to M along ϕ is a map $X : B \to \mathbb{R}^{k+n}$ such that for any $q \in B$, $X_m \in T_mM$.*

The maps $\phi_i : B \to TM$ given by $\phi_i(q) = (d\phi)_q(e_i)$, where (e_i), $i = 1, \dots, k$, is the canonical basis of $\mathbb{R}^k$, are tangent vector fields to M along ϕ. Since $(d\phi)_q$ is one-to-one, $\big(\phi_1(q), \dots, \phi_k(q)\big)$ is a basis of T_mM, where $m = \phi(q)$. Each vector field X tangent to M along ϕ can be written as $X = \sum_i X^i \phi_i$, where the smooth functions $X^i : B \to \mathbb{R}$ are called *components of X in the chart ϕ*. By applying a Gram-Schmidt process to the values of the vector fields ϕ_i at each point of B we may obtain a local reference of orthonormal tangent vector fields in $\phi(B)$.

Definition 2.1.3 *The **second fundamental form** α in M is defined as the section $\alpha \in \Gamma(NM \otimes T^*M \otimes T^*M)$ such that, if $X, Y \in \mathfrak{X}(M)$ then*

$$\alpha(X, Y) = (D_X Y)^{\perp}.$$

This is well defined, because α is evidently bilinear for the sum, and if $f, l : M \to \mathbb{R}$ are arbitrary smooth functions we have

$$\alpha(fX, lY) = (flD_X Y + fX(l)Y)^\perp = fl(D_X Y)^\perp.$$

Also, it is symmetric because $[X, Y] \in \mathfrak{X}(M)$ whence

$$\alpha(X, Y) - \alpha(Y, X) = (D_X Y - D_Y X)^\perp = [X, Y]^\perp = 0.$$

Another manner of looking at the second fundamental form is to consider for each $u \in N_m M$ the endomorphism

$$\mathcal{A}_m(u, .) : T_m M \to T_m M$$

that sends any vector $X_m \in T_m M$ to the vector $\mathcal{A}_m(u, X) \in T_m M$ whose inner product with any vector $Y_m \in T_m M$ is:

$$Y_m \cdot \mathcal{A}_m(u, X_m) = Y_m \cdot D_X \tilde{u} = -u \cdot \alpha_m(X_m, Y_m) = X_m \cdot \mathcal{A}_m(u, Y_m),$$

where $\tilde{u}$ is any normal vector field whose value at m be u. This defines a smooth tensor field $\mathcal{A}$ on M.

Definition 2.1.4 *Let $PT_m M$ denote the projective space of directions in $T_m M$. The* ***Veronese of curvature*** *of M at m is the map (or its image) $\eta_m : PT_m M \to N_m M$ given by*

$$\eta_m([t]) = \frac{\alpha_m(t, t)}{t \cdot t}, \quad \forall t \in T_m M \backslash \{0\}.$$

If $0 \neq t \in T_m M$, we shall put $\eta_m(t) = \eta_m([t])$ by a small abuse of notation. We note that the Veronese of curvature may also be defined as the restriction of η_m to the unit sphere in $T_m M$.

2.1.2 The case of surfaces. Curvature ellipse

The second fundamental form at a point $m \in M$ describes the geometry of the immersion M at m to the second order of approximation.

Thus in this subsection we assume that M is a surface, so that $k = 2$. Let $m \in M$, and let (t_1, t_2) be an orthonormal basis of T_mM. We denote by $S(T_mM)$ the circle of unit vectors of T_mM that is

$$S(T_mM) = \{t \in T_mM : t \cdot t = 1\}.$$

We can parameterize $S(T_mM)$ by the angle θ with respect to t_1. Thus any point of $S(T_mM)$ may be written in the form $t(\theta) = t_1 \cos\theta + t_2 \sin\theta$. Therefore, the Veronese of curvature at m may be written as

$$\begin{aligned} \eta_m(t(\theta)) =& \alpha_m(t_1 \cos\theta + t_2 \sin\theta, t_1 \cos\theta + t_2 \sin\theta) \\ =& \alpha_m(t_1, t_1)\cos^2\theta + \alpha_m(t_2, t_2)\sin^2\theta + 2\sin\theta\cos\theta\alpha_m(t_1, t_2). \end{aligned}$$

Then, by using $\cos^2\theta + \sin^2\theta = 1$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$, we have:

$$\eta_m(t(\theta)) = H_m + B_m \cos 2\theta + C_m \sin 2\theta,$$

where

$$\begin{aligned} H_m &= \frac{1}{2}(\alpha_m(t_1, t_1) + \alpha_m(t_2, t_2)), \\ B_m &= \frac{1}{2}(\alpha_m(t_1, t_1) - \alpha_m(t_2, t_2)), \end{aligned}$$

and

$$C_m = \alpha_m(t_1, t_2).$$

Putting $b_1 = \alpha_m(t_1, t_1)$, $b_2 = \alpha_m(t_2, t_2)$ and $b_3 = \alpha_m(t_1, t_2)$ we have that:

$$H_m = \frac{1}{2}(b_1 + b_2), \quad B_m = \frac{1}{2}(b_1 - b_2), \quad C_m = b_3,$$

Consider the affine map $\xi : TM \to NM$ given by $\xi(a_1t_1 + a_2t_2) = H + a_1B + a_2C$, where $a_1, a_2 \in \mathbb{R}$, and the map $\delta : PTM \to S(TM)$ given by $\delta([t(\theta)]) = t(2\theta)$. The map δ is well defined because

$$\delta([t(\theta + p\pi)]) = t(2\theta + 2p\pi) = t(2\theta), \quad \forall p \in \mathbb{Z}.$$

Since $\eta_m = \xi \circ \delta$, we see that the image of η_m is the image, by the affine map ξ, of a circle of T_mM and therefore, it is an ellipse, that may degenerate to a segment or to a point. It is called the **curvature ellipse** of M at m.

The center of that ellipse, denoted H_m, is called the *mean curvature vector*. The vectors B_m and C_m generate an affine subspace $\mathcal{S}_m$ of N_mM, passing by H_m. It is the affine span of the curvature ellipse. In general, it is an affine plane in N_mM. The vector subspace of N_mM of same dimension as $\mathcal{S}_m$ and parallel to it, that is the plane generated by B_m and C_m shall be denoted by ϵ_m. While B_m and C_m depend on the choice of the orthonormal basis (t_1, t_2), $H_m, \mathcal{S}_m$ and ϵ_m do not.

Except the last one, the following definitions were introduced by Little, see [12]. We say that m is:

- **Pseudo-umbilic**, when $\mathcal{S}_m$ is orthogonal to H_m.
- **Semiumbilic**, when B_m and C_m are linearly dependent. In this case the curvature ellipse degenerates into a segment or into a point.
- **Inflection point**, when there is a vector line that contains the curvature ellipse. In other words, m is semumbilic and there is an affine line by the origin that contains the ellipse.

- **Umbilic**, when the ellipse degenerates to a point.
- **Minimal**, when $H_m = 0$.
- **H-Umbilical,** when the curvature ellipse is a circle and the orthogonal projection of the origin into the plane which contains the ellipse coincides with the center of the circle. In this case, $H \cdot \eta$ is constant.

In the particular case that M is a surface immersed in $\mathbb{R}^{2+n}$, $n = 1, 2$, we say that m is:

- **Hyperbolic**, when the origin is outside the ellipse.
- **Elliptic**, when the origin is inside the ellipse.
- **Parabolic**, when the origin belongs to the ellipse.

Let (t_1', t_2') be another orthonormal basis of $T_m M$ given by $t_1' = t_1 \cos\beta + t_2 \sin\beta$, $t_2' = -t_1 \sin\beta + t_2 \cos\beta$ where β is the angle with respecto to t_1' and let us denote by B_m', C_m' the new vectors generating ϵ_m. Then a direct computation gives

$$B_m' = B_m \cos 2\beta + C_m \sin 2\beta, \quad C_m' = -B_m \sin 2\beta + C_m \cos 2\beta.$$

This could be expected beforehand because $\eta_m = \xi \circ \delta$. One may always choose β in order to have

$$|B_m| \geq |C_m|, \quad B_m \cdot C_m = 0,$$

where B_m and C_m determine the major and minor axis of the ellipse respectively, ([9]).

2.2 Contact theory

Let $f : M \to P$ be a smooth map between manifolds of dimension k and q respectively. Suppose given a point $m \in M$ and consider the set of all smooth mappings $U \to P$ whose domain U is a neighborhood of m in M. On this set we introduce an equivalence relation $\sim$.

Definition 2.2.1 *Given two such mappings $f_1 : U_1 \to P$, $f_2 : U_2 \to P$, we write $f_1 \sim f_2$ when there exists a neighborhood $U \subseteq U_1 \cap U_2$ of m in M for which the restrictions $f_1|_U, f_2|_U$ coincide. The equivalence classes under this relation are called smooth map-germs of mappings $M \to P$ at m, and the elements of the equivalence class are called representatives of the germ.*

Notice that if f_1, f_2 are representatives of the same map-germ then $f_1(m) = f_2(m)$, so all the representatives of the map-germ take the same value p, say at m. In view of this fact, it is usual to adopt the notation $f : (M, m) \to (P, p)$ for the map-germ and to call m, p the source and target of the map-germ, respectively. In the particular case when $M = P$, we use the notation $1_M : (M, m) \to (M, m)$ for the map-germ at m of the identity mapping $M \to M$.

Definition 2.2.2 *A map-germ $f : (M, m) \to (P, p)$ is invertible when there exists a map-germ $g : (P, p) \to (M, m)$ for which $f \circ g = 1_P$ and $g \circ f = 1_M$. In this case g is called the inverse of f.*

Further, to a map-germ $f : (M, m) \to (P, p)$ we associate a differential, denoted by $df_m : T_mM \to T_pP$ and defined to be the differential at m

of any representative. Notice that this definition does not depend on the choice of the representative and that a map-germ is invertible if and only if its differential is invertible.

Finally, the rank of the map-germ f is defined to be that of its differential, when the rank is equal to $\dim M$ the map-germ is immersive, and when its equal to $\dim P$ it is submersive. Thus a map-germ will be invertible if and only if it is both immersive and submersive. A map-germ which is neither immersive nor submersive is called singular.

Then, we have the following definition.

Definition 2.2.3 *We call two map-germ f_1, f_2, $\mathcal{A}$-equivalent when there exists invertible maps h, s for which the following diagrame commutes.*

$$\begin{array}{ccc} (M_1, m_1) & \xrightarrow{f_1} & (P_1, p_1) \\ \uparrow h & & \uparrow s \\ (M_2, m_2) & \xrightarrow{f_2} & (P_2, p_2). \end{array}$$

It is maybe worthwhile spelling out one simple consequence of the above definition, namely that any map-germ $f : (M, m) \to (P, p)$ is equivalent to some map-germ $(\mathbb{R}^k, 0) \to (\mathbb{R}^q, 0)$.

Definition 2.2.4 *Let M_i, N_i, $i = 1, 2$ be submanifolds of $\mathbb{R}^c$ with $\dim M_1 = \dim M_2$ and $\dim N_1 = \dim N_2$. We say that the contact of M_1 and N_1 at q_1 is of the same type as the contact of M_2 and N_2 at q_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^c, q_1) \to (\mathbb{R}^c, q_2)$ such that $\Phi(M_1) = M_2$ and $\Phi(N_1) = N_2$. In this case we write $K(M_1, N_1; q_1) = K(M_2, N_2; q_2)$.*

Definition 2.2.5 *We say the two germs $f, g : (\mathbb{R}^k, 0) \to (\mathbb{R}^{n+1}, 0)$ are $\mathcal{K}$-equivalents, and we shall denote it by $f \overset{\mathcal{K}}{\sim} g$, when there exists:*

(1) a germ of diffeomorfism $h : (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)$ *between the domains of the germs,*

(2) and a germ of diffeomorfism $H : (\mathbb{R}^k \times \mathbb{R}^{n+1}, (0,0)) \to (\mathbb{R}^k \times \mathbb{R}^{n+1}, (0,0))$ *between their graphics, such that* $H(x, 0) = (h(x), 0)$.

Then the following diagram is commutative:

$$\begin{array}{cccc} & (\mathbb{R}^k, 0) & \overset{(1_{\mathbb{R}^k}, f)}{\longrightarrow} & (\mathbb{R}^k \times \mathbb{R}^{n+1}, (0,0)) & \\ h & \downarrow & & \downarrow & H \\ & (\mathbb{R}^k, 0) & \underset{(1_{\mathbb{R}^k}, g)}{\longrightarrow} & (\mathbb{R}^k \times \mathbb{R}^{n+1}, (0,0)) & \end{array}$$

In other words $H(x, f(x)) = (h(x), (g \circ h)(x)),\ \forall x \in X$.

J. A. Montaldi gives in [22] the following characterization of the notion of contact by using the terminology of singularity theory:

Theorem 2.2.6 *([22]) Let* M_i, N_i $(i = 1, 2)$ *be submanifolds of* $\mathbb{R}^c$ *with* $\dim M_1 = \dim M_2$ *and* $\dim N_1 = \dim N_2$. *Let* $f_i : (M_i, p_i) \to (I\!R^c, q_i)$ *be immersion germs and* $g_i : (\mathbb{R}^c, q_i) \to (\mathbb{R}^c, 0)$ *be submersion germs with* $(N_i, q_i) = (g_i^{-1}(0), q_i)$. *In this case* $K(M_1, N_1; q_1) = K(M_2, N_2; q_2)$ *if and only if the germ* $(g_1 \circ f_1, p_1)$ *is* $\mathcal{K}$*-equivalent to the germ* $(g_2 \circ f_2, p_2)$.

Therefore, given two submanifolds M and N of $\mathbb{R}^c$, with a common point q, an immersion germ $f : (M, p) \to (\mathbb{R}^c, q)$ and a submersion germ $g : (\mathbb{R}^c, q) \to (\mathbb{R}^r, 0)$, such that $N = g^{-1}(0)$, the contact of $M \equiv f(M)$ and N at q is completely determined by the $\mathcal{K}$-singularity type of the germ $(g \circ f, p)$ (see [10] for details on $\mathcal{K}$-equivalence).

When N is a hypersurface, we have $r = 1$, and the function germ $(g \circ f, p)$ has a degenerate singularity if and only if its Hessian, $\mathcal{H}(g \circ f)(p)$, is a degenerate quadratic form. In such case, the tangent directions lying in the kernel of this quadratic form are called *contact directions* for M and N at q.

We shall apply this theory to the contact of surfaces with hyperplanes and hyperspheres in $\mathbb{R}^c$. In the following $\phi : U \subset \mathbb{R}^2 \to \mathbb{R}^c$ will be an immersed surface, where $M = \phi(U)$.

Definition 2.2.7 ***The family of*** *height functions on* M, $\lambda : M \times S^{c-1} \to \mathbb{R}$ *is defined as* $\lambda_u(m) = \lambda(m, u) = m \cdot u$, $u \in S^{c-1}$ *where* S^{c-1} *is the unit sphere in* $\mathbb{R}^c$ *centered at the origin.*

Varying u we obtain a family of functions λ_u on M that describes all the possible contacts of M with the hyperplanes on $\mathbb{R}^c$ ([15], [16]). The function λ_u has a singularity at $m = \phi(x_0, y_0) \in M$ if and only if

$$d_m\lambda_u = \left(\frac{\partial \phi}{\partial x}(x_0, y_0) \cdot u, \ \frac{\partial \phi}{\partial y}(x_0, y_0) \cdot u \right) = (0, 0),$$

which is equivalent to saying that $u \in N_m M$.

Let $D\lambda : M \times S^{c-1} \to S^{c-1} \times \mathbb{R}$ be the unfolding associated to the family λ. The singular set of $D\lambda$, given by

$$\Sigma(D\lambda) = \{(m, u) \in M \times S^{c-1} : d_m\phi \cdot u = 0\}$$

can clearly be identified with a canal hypersurface, CM, of M in $\mathbb{R}^c$. Moreover, the restriction of the natural projection $\pi : M \times S^{c-1} \to S^{c-1}$ to the submanifold $\Sigma(D\lambda) \equiv CM$ can be viewed as the normal Gauss map, $\Gamma : CM \to S^{c-1}$ on the hypersurface CM. It is not difficult to verify that m is a degenerate singularity of λ_u if and only if (m, u) is a singular point of Γ. This occurs when $\mathcal{K}(m, u) = 0$, where $\mathcal{K}$ denotes the gaussian curvature function on CM, i.e. $\mathcal{K} = det(d\Gamma)$, where $d\Gamma : T(CM) \to TS^{k+c-1}$.

Definition 2.2.8 *The family of* ***squared distance functions*** *over* M, $d^2 : M \times \mathbb{R}^c \to \mathbb{R}$, *is defined by* $d^2(m, a) = d_a^2(m) = \|m - a\|^2$.

The singularities of this family give a measure of the contacts of the immersion with the family of hyperspheres of $\mathbb{R}^c$ ([21], [25]). Then, we observe that the function d_a^2 has a singularity in a point $\phi(x_0, y_0) = m \in M$ iff

$$\frac{\partial \phi}{\partial x}(x_0, y_0) \cdot \big(\phi(x_0, y_0) - a\big) = 0, \quad \frac{\partial \phi}{\partial y}(x_0, y_0) \cdot \big(\phi(x_0, y_0) - a\big) = 0,$$

which is equivalent to saying that the point a lies in the normal subspace to M at m.

Definition 2.2.9 *Given a surface* M *immersed in* $\mathbb{R}^c$, *if the squared distance function* d_a^2 *has a degenerate singularity at* m *then we say that the point* $a \in \mathbb{R}^c$ *is a* ***focal center*** *at* $m \in M$. *The subset of* $\mathbb{R}^c$ *made of all the focal centers for all the points of* M *is called* ***focal set*** *of* M *in* $\mathbb{R}$. *A hypersphere tangent to* M *at* m *whose center lies in the focal set of* M *at* m *is said to be a* ***focal hypersphere*** *of* M *at* m.

In this case we say that the distance square function has a singularity of type A_2 at m.

The focal set is classically known as the singular set of the normal exponential map $\exp_M : NM \to \mathbb{R}^c$ ([25], [18]). It is easy to see that the directions of higher contacts of M with the focal hyperspheres are those contained in the kernel of the quadratic form

$$\frac{1}{2}\,\mathrm{Hess}(d_a^2) = g_m - (m-a)\cdot\alpha_m,$$

where g_m and α_m are the first and second fundamental forms at m, respectively.

In the remainder of this section, we will assume that $c = 4$. It follows from a general result of Montaldi [22] (and also Looijenga's Theorem in [13]) that for a residual set of immersions $\phi : M \to \mathbb{R}^4$, the family d^2 is a generic family of mappings. (The notion of a generic family is defined in terms of transversality to submanifolds of multi-jet spaces, see for example [10].) We call these immersions, generic immersions.

Definition 2.2.10 *If m is a degenerate singularity (non Morse) of λ_u, we say that u defines a* ***binormal*** *direction for M at m. The vector $v \in T_mM$ is an* ***asymptotic direction*** *at m if and only if v lies in the kernel of the hessian of some height function λ_u at m. In this case we say that v is an asymptotic direction associated to the binormal direction u at m.*

These directions were introduced in [15], where their existence and distribution over the generic submanifolds was analyzed.

Among all the focal hyperspheres which lie in the singular subset of the focal set of M, we have some special ones corresponding to distance-squared functions (from their centers) having singularities of type A_k, with $k \geq 3$, in this case corank is 1. Here, we remind that an

A_k singularity is a germ of function $\mathbb{R}^2 \to \mathbb{R}$ which can be transformed by a local change of coordinates in $\mathbb{R}^2$ to the germ of $x_1^2 \pm x_2^{k+1}$, [1].

Definition 2.2.11 *The centers of the focal hyperspheres of M which have contact of type A_k, $k \geq 3$ are called **(k-order) ribs** and they determine normal directions called **rib directions**. The corresponding points in M are known as **(k-order) ridges** and the corresponding directions are called **strong principal directions**.*

The k-order ridges with $k \geq 4$ (i.e. the A_k singularities of squared distance functions with $k \geq 4$) are the singular points of the ridges set. For a generic immersion, the ribs form a stratified subset of codimension one in the focal set and the k-order ridges, $k \geq 4$, form curves with the 5-order ridges as isolated points, [21]. Other special kind of focal hyperspheres is made by those corresponding to squared distance functions that have corank 2 singularities. In this case, all the coefficients of the quadratic form $\text{Hess}(d_u^2)$ vanish at the point m.

Definition 2.2.12 ([20]) *A focal center of M at a point m is said to be an **umbilical focus** provided the corresponding squared distance function has a singularity of corank 2 at m. A tangent 3-sphere centered at an umbilical focus is called **umbilical focal hypersphere**.*

Montaldi proved in [21] the following relation between the (non-radial) semiumbilic points and umbilical focal hyperspheres: *A point $m \in M$ is a (non-radial) semiumbilic if and only if it is a singularity of corank 2 of some distance squared function on M, in other words, it is a contact point of M with some umbilical focal hypersphere at m.*

The corank 2 singularities of distance-squared functions on generically immersed surfaces in $\mathbb{R}^4$ belong to the series $D_k^{\pm}$ (see [1]). Moreover, on a generic surface, there are only $D_4^{\pm}$ singularities along regular curves with isolated D_5.

Chapter 3

The Taylor expansion of the exponential map

In this chapter we analyze the Taylor expansion of the exponential map up to order three of a submanifold M immersed in $\mathbb{R}^{k+n}$ in order to introduce the concepts of *lateral and frontal deviation*. We compute the directions of extremal lateral and frontal deviation for surfaces in $\mathbb{R}^3$. Also we compute, by using the Taylor expansion, the directions of high contact with hyperspheres of a surface immersed in $\mathbb{R}^4$ and the asymptotic directions of a surface immersed in $\mathbb{R}^{2+n}$.

3.1 The Taylor expansion of the exponential map

Let M be an immersed submanifold in $\mathbb{R}^{k+n}$ and $m \in M$. We know that there is an open neighborhood $\mathcal{U}$ of $0 \in T_mM$ such that the exponential map $\exp_m : \mathcal{U} \to \mathbb{R}^{k+n}$ is an one-to-one immersion. We recall also that $\exp_m(x) = \gamma_x(1)$, where $\gamma_x : [0,1] \to \mathbb{R}^{k+n}$ is the geodesic in M with initial condition $\gamma_x(0) = m$, $\gamma_x'(0) = x \in \mathcal{U}$. We shall consider the Taylor expansion of $\exp_m$ around the origin of $T_m M$. It will be written as

$$\exp_m(x) = m + I_m(x) + \frac{1}{2}Q_m(x) + \frac{1}{6}K_m(x) + \dots,$$

where I_m, Q_m, K_m are respectively linear, quadratic and cubic forms in $T_m M$ with values in $\mathbb{R}^{k+n}$.

Our purpose is to write these forms in terms more familiar with the usual techniques of differential geometry. Let $x \in \mathcal{U}$ and put $x = tv$, where $t \in \mathbb{R}$ and $v \in T_m M$ is a unit vector. Then, as it is well known, $\exp_m(x) = \exp_m(tv) = \gamma_v(t)$. Therefore

$$\gamma_v(t) = m + I_m(v)t + \frac{1}{2}Q_m(v)t^2 + \frac{1}{6}K_m(v)t^3 + O(t^4).$$

Hence, $\gamma_v'(0) = v = I_m(v)$, so that $I_m : T_m M \to \mathbb{R}^{k+n}$ is the inclusion. We also have $\gamma_v''(0) = Q_m(v)$ and $\gamma_v'''(0) = K_m(v)$.

Now, γ_v is a geodesic in M and this implies that at every t we have $\gamma_v''(t) \in N_{\gamma_v(t)}M$. In fact, we have then $\gamma_v''(t) = \alpha_{\gamma_v(t)}(\gamma'(t), \gamma'(t))$. Hence,

$$Q_m(v) = \gamma_v''(0) = \alpha_m(v, v).$$

Thus, it is clear that the second order geometry of M around m is determined by the value at m of the second fundamental form of M. Let us study the third order geometry.

Let $\xi \in T_m M$. We may make the parallel transport of ξ along the geodesic γ_v in order to have a parallel vector field $X(t)$ along that geodesic. This means that $X(0) = \xi$, $X(t) \in T_{\gamma_v(t)}M$ and $X'(t) \in N_{\gamma_v(t)}M$. Then, we will have $X \cdot \gamma_v'' = 0$. Differentiating, we get

$$\begin{aligned} X \cdot \gamma_v''' &= -X' \cdot \gamma_v'' = -X' \cdot \alpha(\gamma_v', \gamma_v') = -(D_{\gamma_v'} X) \cdot \alpha(\gamma_v', \gamma_v') \\ &= -\alpha(X, \gamma_v') \cdot \alpha(\gamma_v', \gamma_v'). \end{aligned}$$

Hence, by evaluation at $t = 0$ we have

$$\xi \cdot K_m(v) = \xi \cdot \gamma_v'''(0) = -\alpha_m(\xi, v) \cdot \alpha_m(v, v).$$

We observe thus that the tangential part of the third order geometry at m depends only on the second order geometry at m. Now, let $\mathbb{Z} \in N_mM$. As before, we define the vector field $Z(t)$ along the curve γ_v as the parallel transport of $\mathbb{Z}$. Thus, for any t we will have $Z(t) \in N_{\gamma_v(t)}M$ and $Z'(t) \in T_{\gamma_v(t)}M$. Hence $Z' \cdot \gamma_v'' = 0$. Thus

$$Z \cdot \gamma_v''' = (Z \cdot \gamma_v'')' = (Z \cdot \alpha(\gamma_v', \gamma_v'))' = Z \cdot \big(\nabla_{\gamma_v'} \alpha\big)(\gamma_v', \gamma_v'),$$

because Z' and γ_v' are parallel along γ_v and $(Z \cdot \alpha(\gamma_v', \gamma_v'))' = D_{\gamma_v'}(Z \cdot \alpha(\gamma_v', \gamma_v'))$.

We have thus that $\mathbb{Z} \cdot K_m(v) = \mathbb{Z} \cdot \big(\nabla_v \alpha\big)(v, v)$. We conclude that, for any $u \in \mathbb{R}^{k+n}$ and for any $x \in \mathcal{U}$, we have

$$\begin{aligned} u \cdot \exp_m(x) =& u \cdot m + u \cdot x + \frac{1}{2} u \cdot \alpha_m(x, x) \\ & - \frac{1}{6} \alpha_m(u^\top, x) \cdot \alpha_m(x, x) + \frac{1}{6} u \cdot (\nabla_x \alpha)(x, x) + O(|x|^4). \end{aligned}$$

Let us put $\alpha_m^\sharp = \sum_i t_i \otimes \alpha_m(t_i, \cdot)$, where $(t_1, \ldots, t_k)$ is an orthonormal basis of T_mM, and take the convention that if $z \in \mathbb{R}^{k+n}$ and $X \in T_mM$ then

$$z \cdot \alpha_m^\sharp(X) = \sum_i (z \cdot t_i) \alpha_m(t_i, X) = \alpha_m(z^\top, X)$$

and

$$\alpha^\sharp_m(X) \cdot z = \sum_i t_i\big(\alpha_m(t_i, X) \cdot z\big).$$

Then

$$\begin{aligned}\exp_m(x) =& m + x + \frac{1}{2}\alpha_m(x,x) - \frac{1}{6}\alpha^\sharp_m(x) \cdot \alpha_m(x,x) + \frac{1}{6}(\nabla_x\alpha)(x,x) \\ &+ O(|x|^4), \\ \gamma_v(t) =& m + vt + \frac{1}{2}\alpha_m(v,v)t^2 \\ &+ \frac{1}{6}\left((\nabla_v\alpha)(v,v) - \alpha^\sharp_m(v) \cdot \alpha_m(v,v)\right)t^3 + O(t^4).\end{aligned}$$

By using the same technique, it is easy to compute higher order terms of these Taylor expansions, but we shall not use them here.

Definition 3.1.1 *we define the* ***geodesic deviation*** $\Delta_v(t)$ *defined by the unit vector* $v \in T_mM$ *as*

$$\begin{aligned}\Delta_v(t) =& \gamma_v(t) - (m + vt) = \frac{1}{2}\alpha_m(v,v)t^2 \\ &+ \frac{1}{6}((\nabla_v\alpha)(v,v) - \alpha^\sharp(v) \cdot \alpha_m(v,v))t^3 + O(t^4).\end{aligned}$$

The tangential and normal components of the geodesic deviation are given by

$$\begin{aligned}u^\top \cdot \Delta_v(t) &= -\frac{1}{6}\alpha_m(u^\top, v) \cdot \alpha_m(v,v)t^3 + O(t^4), \\ u^\perp \cdot \Delta_v(t) &= \frac{1}{2}u^\perp \cdot \alpha_m(v,v)t^2 + \frac{1}{6}u^\perp \cdot (\nabla_v\alpha)(v,v)t^3 + O(t^4),\end{aligned}$$

where $u \in \mathbb{R}^{k+n}$. We see that the term of second order of the normal deviation is $\frac{1}{2}\alpha_m(v,v)t^2$, and this gives a geometric interpretation of the second fundamental form. We will call its coefficient in t^2 the *normal deviation* of M in the direction $v \in T_m M$. In the following we will give geometric interpretations to the terms of third order.

3.2 Applications to geometry of surfaces

In this section, M will be a regular surface immersed in $\mathbb{R}^{2+n}$. Since the study is local we may assume that M is orientable, so that there is a well defined rotation of 90 degrees in $T_m M$ for each $m \in M$. It will be given by the tensor field J. In the following we shall suppress the subindex m which means the evaluation at m.

We will focus here in the principal term of the tangential part of the geodesic deviation which is

$$-\frac{1}{6}\alpha^{\sharp}(v) \cdot \alpha(v,v)t^3.$$

We decompose it into two components, one in the direction of v and the other one in the direction of Jv.

Definition 3.2.1 *We define the* ***frontal (geodesic) deviation*** *of M in the (unit) direction v by*

$$-\frac{1}{6}\alpha(v,v) \cdot \alpha(v,v).$$

The other component of this deviation, called ***lateral (geodesic) deviation*** *of M in the (unit) direction v, is given by*

$$-\frac{1}{6}\alpha(Jv,v) \cdot \alpha(v,v).$$

3.2.1 Lateral geodesic deviation of a surface in one direction

Now we are going to give an additional interpretation to the lateral deviation. Suppose that $\gamma_v''(0) \neq 0$. We consider the curve $\overline{\gamma}(t)$ obtained by the orthogonal projection of $\gamma_v(t)$ over the affine subspace by m generated by the orthonormal vectors $e_1 = \gamma_v'(0)$, $e_2 = \dfrac{\gamma_v''(0)}{\|\gamma_v''(0)\|}$ and $e_3 = J\gamma_v'(0)$. This affine subspace is the subspace generated by the tangent space to the surface at m and the vector $\alpha(v,v)$. As a consequence, the orthogonal projection of the geodesic into that subspace gives a curve whose osculating plane at m is generated by e_1 and e_2 and whose binormal at m is e_3. Obviously, when $n = 1$ the 3-euclidean space is the ambient space.

The projection of the geodesic will be given, in the affine frame $(m; e_1, e_2, e_3)$, by:

$$\overline{\gamma}(t) = ((\gamma_v(t) - m) \cdot e_1)e_1 + ((\gamma_v(t) - m) \cdot e_2)e_2 + ((\gamma_v(t) - m) \cdot e_3)e_3.$$

Proposition 3.2.2 *The lateral (geodesic) deviation can be written as follows:*

$$-\frac{1}{6}\alpha(Jv, v) \cdot \alpha(v, v) = \frac{1}{6}\overline{\kappa}(0)\,\overline{\tau}(0),$$

where $\overline{\kappa}(0)$ and $\overline{\tau}(0)$ are the curvature and the torsion of $\overline{\gamma}(t)$ at 0*, respectively.*

Proof. Evaluating the derivatives of $\overline{\gamma}(t)$ at 0, we have that $\overline{\gamma}'(0) =$

$v = e_1$, $\overline{\gamma}''(0) = \gamma_v''(0) = \|\alpha(v,v)\|e_2$, and

$$\begin{aligned}
\overline{\gamma}'''(0) \cdot e_1 &= \gamma_v'''(0) \cdot e_1 = -\|\alpha_m(v,v)\|^2, \\
\overline{\gamma}'''(0) \cdot e_2 &= \frac{(\nabla_v \alpha)(v,v) \cdot \alpha(v,v)}{\|\alpha(v,v)\|}, \\
\overline{\gamma}'''(0) \cdot e_3 &= -\alpha(Jv,v) \cdot \alpha(v,v).
\end{aligned}$$

Therefore $\overline{\gamma}'(0) \times \overline{\gamma}''(0) = \|\alpha(v,v)\|e_3$, hence the torsion of $\overline{\gamma}$ at $t = 0$ is given by:

$$\begin{aligned}
\overline{\tau} =& \frac{(\overline{\gamma}'(0) \times \overline{\gamma}''(0)) \cdot \overline{\gamma}'''(0)}{|\overline{\gamma}'(0) \times \overline{\gamma}''(0)|^2} \\
=& -\frac{\alpha(Jv,v) \cdot \alpha(v,v)}{\|\alpha(v,v)\|}.
\end{aligned}$$

Now, the curvature of $\overline{\gamma}(t)$ is given by $\overline{\kappa}(0) = \|\overline{\gamma}''(0)\| = \|\alpha(v,v)\|$. Then, the lateral deviation of M in the direction of the unit vector $v \in T_mM$ is

$$-\frac{1}{6}\alpha(Jv,v) \cdot \alpha(v,v) = \frac{1}{6}\overline{\kappa}(0)\,\overline{\tau}(0).$$

□

We can see that the lateral (geodesic) deviation generalize the following concept for surfaces in $\mathbb{R}^3$: The geometry of third order of the separation of the geodesic with respect to its initial osculating plane.

Finally, we know that if $\kappa_v(t)$ denotes the curvature of the geodesic $\gamma_v(t)$ then $\kappa_v(t)^2 = \|\alpha_{\gamma_v(t)}(\gamma_v'(t), \gamma_v'(t))\|^2$. Therefore,

$$\begin{aligned}
\kappa_v(t)\kappa_v'(t) &= \alpha_{\gamma_v(t)}(\gamma_v'(t), \gamma_v'(t)) \cdot \nabla_{\gamma_v'}\big(\alpha_{\gamma_v}(\gamma_v', \gamma_v')\big)(t) \\
=& \alpha_{\gamma_v(t)}(\gamma_v'(t), \gamma_v'(t)) \cdot (\nabla_{\gamma_v'}\alpha)(\gamma_v', \gamma_v')(t).
\end{aligned}$$

Evaluating at $t = 0$ we get

$$\kappa_v(0)\kappa_v'(0) = \alpha(v,v) \cdot (\nabla_v \alpha)(v,v),$$

so that if we denote $\kappa_v = \kappa_v(0)$ and $\kappa'_v = \kappa'_v(0)$, we have

$$\kappa_v \kappa'_v = \alpha(v,v) \cdot (\nabla_v \alpha)(v,v).$$

and it measures the geodesic ratio of change of the normal curvature in the direction v.

3.2.2 Retard of the geodesic with respect to the tangent vector

In this section we give an interpretation to the frontal (geodesic) deviation.

Let $t \mapsto m + tv$ be the geodesic in $\mathbb{R}^{1+n}$ with same initial condition as γ. We can approximate γ_v to order two by a curve β that describes, with velocity v, a circle of radius $R = \frac{1}{\gamma''_v(0)} = \frac{1}{\|\alpha(v,v)\|}$ that lies on the affine plane by m generated by $\gamma'(0)$ and $e_2 = \frac{\gamma''_v(0)}{\|\alpha(v,v)\|}$. The equation of this curve is

$$\beta(t) = m + R \sin \frac{t}{R}\, e_1 + (R - R \cos \frac{t}{R}) e_2.$$

The delay of the projection of $\beta(t)$ on the tangent plane with respect to the curve $m + tv$ is given by:

$$\begin{aligned}(R \sin \frac{t}{R} - t)v + \ldots &= -\frac{1}{6}\frac{t^3}{R^2} + \ldots \\ &= -\frac{1}{6}\alpha(v,v) \cdot \alpha(v,v) t^3 + \ldots\end{aligned}$$

This explains why the frontal deviation depends only on the second order geometry: it is a consequence of the curvature of γ_v together with the fact that it is parameterized by arc-length.

3.2.3 Extremal directions of the frontal geodesic deviation

Proposition 3.2.3 *Let M be a surface immersed in $\mathbb{R}^{2+n}$ and $m \in M$. At each point $m \in M$ there exist at most four extremal directions for the frontal geodesic deviation*

Proof. The frontal geodesic deviation of M in the direction v depends essentially on the norm of the second fundamental form. Hence, the extremal directions of this deviation are the directions where its derivative vanishes. We are going to find these directions when M is a surface. To simplify calculations, we differentiate the squared norm instead of the norm itself.

We know that

$$\eta(\theta) = \alpha(v, v) = H + B\cos 2\theta + C\sin 2\theta,$$

where $v = \cos\theta\, t_1 + \sin\theta\, t_2$. The derivative of the squared norm of $\eta(\theta)$ vanishes iff:

$$(H + B\cos 2\theta + C\sin 2\theta) \cdot (-B\sin 2\theta + C\cos 2\theta) = 0.$$

And this is equivalent to:

$$-hb\sin 2\theta + hc\cos 2\theta + (cc - bb)\sin 2\theta\cos 2\theta + bc\cos^2 2\theta - bc\sin^2 2\theta = 0,$$

where we have put $hb = H \cdot B$, $bb = B \cdot B$, etc. Now, putting $p = \tan\theta$, the extremal directions of the frontal deviation are given by the solutions of the following equation:

$$\begin{aligned} p^4(-hc + bc) + p^3(-2cc + 2bb - 2hb) + p^2(-2bc - 4bc) & \\ +p(-2hb + 2cc - 2bb) + bc + hc &= 0 \end{aligned}$$

whose discriminant is given by:

$$
\begin{aligned}
&256(bb^6 + 64bc^6 - 6bb^5cc + cc^6 - 3cc^4hb^2 + 3cc^2hb^4 - hb^6 - 3cc^4hc^2 \\
&- 21cc^2hb^2hc^2 - 3hb^4hc^2 + 3cc^2hc^4 - 3hb^2hc^4 - hc^6 + 48bc^4 \\
&(cc^2 - hb^2 - hc^2) + 3bb^4(4bc^2 + 5cc^2 - hb^2 - hc^2) \\
&- 54bc\; cc\; hb\; hc(hb^2 - hc^2) - 4bb^3cc(12bc^2 + 5cc^2 - 3(hb^2 + hc^2)) + \\
&3bc^2(4cc^4 - 5hb^4 + 26hb^2hc^2 - 5hc^4 - 8cc^2(hb^2 + hc^2)) + \\
&3bb^2(16bc^4 + 5cc^4 + hb^4 - 7hb^2hc^2 + hc^4 + 8bc^2(3cc^2 - hb^2 - hc^2) \\
&- 6cc^2(hb^2 + hc^2)) - 6bb(16bc^4cc + 8bc^2cc(cc^2 - hb^2 - hc^2) + \\
&bc(-9hb^3hc + 9hbhc^3) + cc(cc^4 + hb^4 - 7hb^2hc^2 + hc^4 \\
&- 2cc^2(hb^2 + hc^2)))).
\end{aligned}
$$

This equation could serve for computing numerically those directions and the corresponding lines of extremal frontal deviation. □

Let us see an example

Example 3.2.4 *Let M be the surface immersed in $\mathbb{R}^4$ by the chart*

$$\vec{x}\,(u,v) = (uv, v^2, u+v, u-v), \quad -1 < u, v < 1$$

and $\alpha = \vec{x} \circ \beta$, $\beta = (u(t), v(t))$. In this case, the differential equation of the extremal frontal deviation lines is given by:

$$
\begin{aligned}
&2((2+v^2)^3 + u^2(4 + 14v^2 + 9v^4))u'v' \\
&- (12\sqrt{2}uv(1+v^2)u'^2v'^2)(2 + u^2 + 5v^2 + 2v^4)^{1/2} \\
&+ 6(1+v^2)(-2 + v^2 + v^4 - u^2(1+3v^2))u'v'(u'^2 - v'^2) = 0.
\end{aligned}
$$

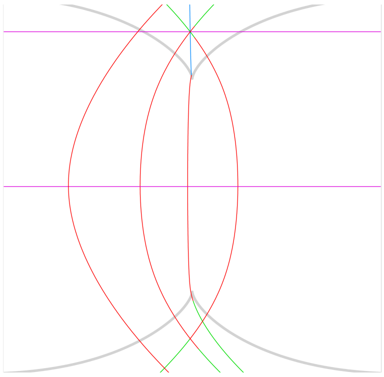


Figure 3.1: Extremal frontal deviation lines and the discriminant line of the surface $\vec{x}\,(u,v) = (uv, v^2, u+v, u-v)$

The thick gray line is the locus where the discriminant of the extremal frontal deviation lines vanishes. We have drawn also the extremal frontal deviation lines starting from two different points.

A notable thing in this example is that the discriminant curve seems not to separate regions with two or four directions, because at all points that are not in that curve there are four directions of extremal frontal deviation. However that fact could be due to a lack of resolution of the algorithm that computes that curve.

The figure 1, which has been made with the program [23] of A. Montesinos-Amilibia, shows some of the extremal frontal deviation lines.

At each point $m \in M$, there are two or four directions of extremal frontal deviation of geodesics. The program draws the lines that are at each point tangent to one of those directions. These curves are drawn in four colors: red indicates that at that point the curve is tangent to a maximal frontal deviation direction; fuchsia, to a minimal frontal deviation direction; green, for the greater intermediate extremal frontal deviation (if one); blue, for the lesser intermediate frontal deviation direction (if one). The gray thick lines represent the discriminant line. It separates the regions where there are four or two directions of extremal frontal deviation.

3.2.4 Relation between extremal frontal and lateral geodesic directions

Now we will find the directions θ of T_mM where the values of the frontal tangent deviation are extremal. These directions are the directions where $\alpha(Jv,v)\cdot\alpha(v,v)$ vanishes. In terms of H, B and C we have:

$$\begin{aligned}\alpha(Jv,v) &= \alpha(t_2\cos\theta - t_1\sin\theta, t_1\cos\theta + t_2\sin\theta)\\ &= -\frac{\sin 2\theta}{2}\,b_1 + \cos 2\theta\, b_3 + \frac{\sin 2\theta}{2}\,b_2.\end{aligned}$$

Since $b_1 = H + B$ and $b_2 = H - B$, we have

$$\alpha(Jv,v) = -B\sin 2\theta + C\cos 2\theta = \frac{1}{2}\eta(\theta)'.$$

Finally:

$$\begin{aligned}\alpha(Jv,v)\cdot\alpha(v,v) =&(-B\sin 2\theta + C\cos 2\theta)\cdot(H + B\cos 2\theta + C\sin 2\theta)\\ =&-hb\sin 2\theta + (cc - bb)\sin 2\theta\cos 2\theta - bc\sin^2 2\theta\\ &+ cb\cos^2 2\theta + ch\cos 2\theta.\end{aligned}$$

Note that $\alpha(Jv,v)\cdot\alpha(v,v) = \frac{1}{2}\eta(\theta)\cdot\eta(\theta)' = \frac{1}{4}\big(\eta(\theta)\cdot\eta(\theta)\big)'$. In other words, the lateral deviation is proportional to the derivative of the squared norm of the frontal deviation. With this, we have proved the following proposition.

Proposition 3.2.5 *Let M be a surface immersed in $\mathbb{R}^{2+n}$. The directions where the lateral deviation vanishes are the extremal directions of the frontal geodesic deviation.*

Proposition 3.2.6 *Let M be a surface immersed in $\mathbb{R}^{2+n}$ and $m \in M$. At each point $m \in M$ there exist at most four extremal directions for the lateral geodesic deviation*

Proof. Differentiating the expression $\alpha(Jv, v) \cdot \alpha(v, v)$, we get the equation for the directions of extremal lateral deviation:

$$-hb\cos 2\theta + (cc - bb)(\cos^2 2\theta - \sin^2 2\theta) - 4bc\cos 2\theta \sin 2\theta - ch\sin 2\theta = 0.$$

Using the same change as in the previous subsection, that equation is written as follows:

$$\begin{aligned} p^4(hb + cc - bb) + p^3(8bc - 2hc) + p^2(-6cc + 6bb) & \\ +p(-8bc - 2hc) + cc - bb - hb &= 0, \end{aligned}$$

whose discriminant is given by:

$$256(64bb^6 + 4096bc^6 - 384bb^5cc + 64cc^6 - 48cc^4hb^2 - 15cc^2hb^4 - hb^6 - 48cc^4hc^2 + 78cc^2hb^2hc^2 - 3hb^4hc^2 - 15cc^2hc^4 - 3hb^2hc^4 - hc^6 + 768bc^4(4cc^2 - hb^2 - hc^2) + 48bb^4(16bc^2 + 20cc^2 - hb^2 - hc^2) + 216bc\, cc\, hb\, hc(hb^2 - hc^2) - 64bb^3cc(48bc^2 + 20cc^2 - 3(hb^2 + hc^2)) + 3bb^2(1024bc^4 + 320cc^4 - 5hb^4 + 26hb^2hc^2 - 5hc^4 + 128bc^2(12cc^2 - hb^2 - hc^2) - 96cc^2(hb^2 + hc^2)) + 48bc^2(16cc^4 + hb^4 - 7hb^2hc^2 + hc^4 - 8cc^2(hb^2 + hc^2)) - 6bb(1024bc^4cc + 128bc^2cc(4cc^2 - hb^2 - hc^2) + 36bc\, hb\, hc(hb^2 - hc^2) + cc(64cc^4 - 5hb^4 + 26hb^2hc^2 - 5hc^4 - 32cc^2(hb^2 + hc^2)))).$$

□

Example 3.2.7 *Let M be the surface immersed in $\mathbb{R}^4$ by the chart*

$$\vec{x}\,(u,v) = (uv, v^2, u+v, u-v), \quad -1 < u, v < 1$$

and $\alpha = \vec{x} \circ \beta$, $\beta = (u(t), v(t))$.

The figure 2, that has also been made with the program [23] shows the lines that are at each point tangent to one of the two or four directions of extremal lateral deviation of geodesics. The thick gray line is the discriminant line separating regions where there are two from those where there are four extremal lateral deviation directions at each point. We have drawn the lines of extremal lateral deviation starting from a four-directions point and also from several two-directions points.

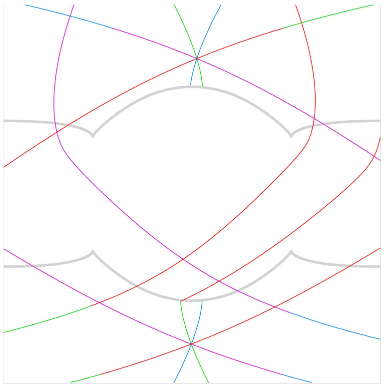


Figure 3.2: Extremal lateral deviation lines and the discriminant line of the surface $\vec{x}(u, v) = (uv, v^2, u + v, u - v)$

The differential equation of the extremal lateral deviation lines is given by:

$$\begin{aligned}
&-(6\sqrt{2}uv(1+v^2)u'v')(2+u^2+5v^2+2v^4)^{1/2}\\
&+((2+v^2)^3+u^2(4+14v^2+9v^4))(u'^2-v'^2)\\
&+24\sqrt{2}uv(1+v^2)u'v'(u'^2-v'^2)(2+u^2+5v^2+2v^4)^{1/2}\\
&+3(1+v^2)(-2+v^2+v^4-u^2(1+3v^2))(u'^2-2u'v'-v'^2)=0.
\end{aligned}$$

By using the expression $0 = \frac{1}{2}\eta(\theta) \cdot \eta(\theta)'$ we characterize the extremal frontal geodesic directions as the tangent directions where the distance of the ellipse to the origin is extremal. In other words, a tangent direction is an extremal frontal direction when the corresponding point of the ellipse belongs to a hypersphere of N_mM centered at the origin and tangent to the ellipse at that point. This guarantees the existence of at least 2 extremal directions.

With this, we have the following proposition.

Proposition 3.2.8 *Let M be a surface immersed in $\mathbb{R}^{2+n}$, $n \geq 1$ and $m \in M$. Depending on the degeneration of the point m, we have:*

i) If m is umbilic, the curvature ellipse is a point and every direction is extremal for the frontal deviation.

ii) If m is a pseudo-umbilic point, then all directions are extremal for the frontal deviation.

iii) If m is semiumbilic, the ellipse degenerates into a segment. In this case, if the orthogonal projection of the origin into the line which contains the ellipse is inside the segment, we have four extremal frontal directions. In the case that the projection is one of the extremes, three directions. Otherwise two directions.

iv) In the general case, we always have at least two extremal directions for the frontal deviation. In the case that $n = 2$, let $\mathcal{E}$ be the evolute of the ellipse of curvature at m and let $\mathcal{E}^0$ be equal to $\mathcal{E}$ minus its four singular points. Then the number of extremal directions of frontal deviation at m is four iff the origin of N_mM is in the interior of the bounded region whose frontier is $\mathcal{E}$, three iff it is on $\mathcal{E}^0$, or two otherwise.

Proof. By definition, it is clear for i), ii) and iii). Notice that in the case that the ellipse is a segment, and for a given direction $[t_0]$ of the projective space P_mM of tangent directions to M at m, the point $\eta([t_0])$ is not one of the two ends of that segment, then there is another direction $[t_1] \in P_mM$ such that $[t_1] \neq [t_0]$ and $\eta([t_1]) = \eta([t_0])$. This implies that, in the case that the orthogonal projection of the origin into the line which contains the ellipse is inside the segment, we have four directions of extremal frontal deviation, not three.

In the general case, the distance of the points of the ellipse to the origin is not constant. Therefore, we shall always have at least two directions, the directions where the distance to the origin is maximum or minimum.

Suppose now that $n = 2$ and that the ellipse of curvature at m is a non-degenerate ellipse in $N_m M$. In this case, the problem is equivalent to determine how many normals to the ellipse may pass by a given point in the plane. As far as we know, this problem was solved by Apollonius. A modern proof may be seen in [11]. □

Definition 3.2.9 *([6]) Suppose that M is a surface immersed in $\mathbb{R}^4$. The unit directions $v \in T_m M$ at which $||\eta(v) - H||^2$ is extremal define the lines of axial curvature.*

Corollary 3.2.10 *Suppose that M is a minimal surface immersed in $\mathbb{R}^4$, i.e. $H = 0$. Then the extremal frontal deviation lines coincide with the lines of axial curvature.*

Let M be a smooth compact surface (possibly with boundary), and suppose that V is a smooth vector field on M with finitely many critical points, all contained in the interior of M, and finitely inner or outer contact point respectively (i.e. boundary points of M such that the trajectory of V through them is internally or externally tangent to the boundary of M at the consider point).

Consider a continuous vector field V and a closed curve γ. Suppose that there are no critical points of V on γ. Let us move a point

P along the curve in the counterclockwise direction. The vector $V(P)$ will rotate during the motion. When P returns to its starting place after one revolution along the curve, $V(P)$ also returns to its original position. During the journey $V(P)$ will make some whole number of revolutions. Counting these revolutions positively if they are counter-clockwise, negatively if they are clockwise, the resulting algebraic sum of the number of revolutions is called the *winding number* of V on γ.

Definition 3.2.11 *The* index of an isolated critical point *of a vector field is the winding number of a small counterclockwise oriented circle with center at that point.*

The sum $Ind(V)$, of the indices at the critical points of V is given by the formula:

$$Ind(V) = \chi(M) + \frac{s-n}{2},$$

where s and n denote the number of inner or outer contact point respectively and $\chi(M)$ represents the Euler number of M.

Observe that if $\partial M = 0$ then $s = n = 0$ and this becomes the classical Poincaré-Hopf Formula. An immediate consequence of this formula is that given any line field on a surface M with non-vanishing Euler number, then there is at least one singularity of the field on M.

With this we have the following Theorem.

Theorem 3.2.12 *Any compact surface with non-vanishing Euler number immersed in $\mathbb{R}^{2+n}$ must have at least either an umbilic point or a pseudo-minimal point.*

Note that in general will be difficult to distinguish globally a line field among the two, three or four directions that may be present at each point. This is due to the fact that along the same integral curve of extremal frontal deviation, the direction defined by the tangent to that curve may pass from being that which gives the maximum frontal deviation at that point to that giving only an intermediate value, and so on. Therefore, this trick of selecting the direction that gives the maximum or minimum of some function, which works well for instance in case of lines of curvature for surfaces in $\mathbb{R}^3$, does not work here. For giving weight to this, the program [23] draws those curves on different colors, as we have said above.

3.2.5 Extremal directions in $\mathbb{R}^3$

Now suppose that M be a surface immersed in $\mathbb{R}^3$. In this case the curvature ellipse is reduced to a segment. Then there exist $h, b, c \in \mathbb{R}$ such that $H = hN$, $B = bN$ and $C = cN$, where N is the unit normal of M. Hence:

$$\alpha(t,t) = (h + b\cos 2\theta + c\sin 2\theta)N,$$

where $t = t_1 \cos\theta + t_2 \sin\theta$.

In the case of frontal deviation, the extremal directions are those that make the derivative of the squared norm of the second fundamental form to vanish. Then, in this case the expression $\alpha(v,v)\cdot\alpha(v,Jv) = 0$ gives:

$$4(h + b\cos 2\theta + c\sin 2\theta)(-b\sin 2\theta + c\cos 2\theta) = 0.$$

We have two possibilities:

1) $h + b\cos 2\theta + c\sin 2\theta = 0$, then θ is a asymptotic direction.

2) $-b\sin 2\theta + c\cos 2\theta = 0$, then θ is a principal direction.

In the case of the lateral deviation, the extremal directions are given by the equation:

$$-hb\cos 2\theta + (cc - bb)(\cos^2 2\theta - \sin^2 2\theta) - 4bc\cos 2\theta \sin 2\theta - hc\sin 2\theta = 0.$$

Another expression of this can be obtained as follows. Let k_1, k_2 be the principal curvatures. The Euler formula says that the normal curvature of M at m in the direction determined by θ is given by $k_n(\theta) = k_1\cos^2\theta + k_2\sin^2\theta$. Then $k_n'(\theta) = (k_2 - k_1)\sin 2\theta$.

In this case, we study the directions where the derivative of $k_n\, k_n' = 0$ vanishes. We know that $k_n\, k_n' = (k_1\cos^2\theta + k_2\sin^2\theta)(k_2 - k_1)\sin 2\theta$. Differentiating this equation we have:

$$(k_2 - k_1)(2k_1\cos 2\theta\cos^2\theta + 2k_2\cos 2\theta\sin^2\theta + (k_2 - k_1)\sin^2 2\theta) = 0.$$

Now, putting $p = \tan\theta$, this is

$$(k_2 - k_1)(-k_2p^4 + 3p^2(k_2 - k_1) + k_1) = 0.$$

Solving this equation at a non-umbilic point, the normal curvatures of the extremal directions of the lateral deviation are given by:

$$k_n = k_1\cos^2\theta + k_2\sin^2\theta = \frac{k_2k_1 - 3k_2^2 \pm k_2\sqrt{9(k_2^2 + k_1^2) - 14k_1k_2}}{3k_1 - 5k_2 \pm \sqrt{9(k_2^2 + k_1^2) - 14k_1k_2}},$$

where

$$p^2 = \frac{3(k_1 - k_2) \pm \sqrt{9(k_2^2 + k_1^2) - 14k_1k_2}}{-2k_2}$$

and the discriminant is given by:

$$-16k_1k_2(9k_1^2 - 14k_1k_2 + 9k_2^2)^2$$

One may verify from these values that the extremal lateral deviation directions are different from all of the special directions on surfaces that we know of, namely asymptotic, principal, arithmetic and geometric mean [7] or characteristic (harmonic mean) [30].

Example 3.2.13 *Let us see some examples for surfaces immersed in* $\mathbb{R}^3$.

If M is a sphere, then the geodesic curves are the maximum circles and its projection over the tangent space is in the line defined by the direction t. So, in this case the lateral deviation vanishes and the frontal deviation is extremal.

If M is a developable surface, i.e. it is a surface that can be flattened onto a plane without distortion, we know that $k_1 \cdot k_2 = 0$. Suppose that $k_1 = 0$, then $K_n = \alpha(v, v) = k_2 \sin^2\theta$. In one hand, the extremal directions of the frontal deviation are given by $\alpha \cdot \alpha' = 0$ i.e. $2k_2^2 \sin^3\theta\cos\theta = 0$ then $\theta = 0,\ \pi/2$, which are the principal directions. On the other hand, the extremal directions of the lateral deviation are given by $\theta = 0,\ \pi/3$, where one of this directions is a principal direction.

3.2.6 Normal curvature and torsion

In this section, we will show how the Taylor expansion of the exponential map allows us to obtain easily an intrinsic expression for the normal torsion of a surface in $\mathbb{R}^4$ in a tangent direction.

The definition of normal torsion at a point along one direction was given by W. Fessler in [5].

Definition 3.2.14 *Let M be a surface immersed in $\mathbb{R}^4$, $m \in M$, and $0 \neq v \in T_mM$. Consider the affine subspace of $\mathbb{R}^4$ which passes by m and is generated by v and N_mM. The intersection of this subspace with M is a curve that passes by m, called the* ***normal section*** *of M determined by v. The curvature and torsion of this curve, as a curve in that Euclidean affine 3-space, is the* ***normal curvature*** *and* ***normal torsion*** *of the surface M in the direction v, respectively.*

Proposition 3.2.15 *Let M be a surface immersed in $\mathbb{R}^4$, $m \in M$. The normal torsion of M at m in the unit direction $v \in T_mM$ is given by:*

$$\tau_v = \frac{J\alpha(v,v)\cdot(\nabla_v\alpha)(v,v)}{\alpha(v,v)\cdot\alpha(v,v)} = \frac{J\gamma_v''\cdot\gamma_v'''}{\gamma_v''\cdot\gamma_v''}(0),$$

where γ_v is the geodesic with initial condition v.

Proof. The inverse image by $\exp_m$ of the normal section of M in the direction given by the unit vector $v \in T_mM$ is a curve in T_mM whose Taylor expansion may be written as $\beta(t) = v\,t + \frac{1}{2}a\,Jvt^2 + \frac{1}{6}c\,t^3 + \ldots$, where $a \in \mathbb{R}$, $c \in T_mM$, and $v \cdot v = 1$.

We have:

$$\begin{aligned}(\exp(\beta(t)) - m)\cdot Jv &= Jv\cdot\beta(t) - \frac{1}{6}\alpha(Jv,\beta(t))\cdot\alpha(\beta(t),\beta(t)) + \ldots \\ &= \frac{1}{2}at^2 + \frac{1}{6}(Jv\cdot c)t^3 - \frac{1}{6}\alpha_m(Jv,v)\cdot\alpha_m(v,v)t^3 + O(t^4).\end{aligned}$$

Now since $\exp(\beta(t))$ is a normal section, $\exp(\beta(t)) - m$ will belong to the subspace generated by v and $N_m M$. Hence $(\exp(\beta(t)) - m) \cdot Jv = 0$ and this implies that $a = 0$ and $Jv \cdot c = \alpha_m(Jv, v) \cdot \alpha_m(v, v)$. Therefore

$$\beta(t) = vt + \frac{1}{6}\big((c \cdot v)v + \alpha_m(Jv, v) \cdot \alpha_m(v, v)Jv\big)t^3 + O(t^4).$$

We put $\mu(t)$ to denote the terms up to the third order in t of $\exp(\beta(t)) - m$. We compute the component of $\mu(t)$ along v

$$\begin{aligned} v \cdot \mu(t) =& v \cdot \beta(t) - \frac{1}{6}\alpha_m(v, \beta(t)) \cdot \alpha_m(\beta(t), \beta(t)) \\ =& t + \frac{1}{6}\big(v \cdot c - \alpha_m(v, v) \cdot \alpha_m(v, v)\big)t^3 \end{aligned}$$

As for the normal component of $\mu(t)$, it is given by

$$\mu(t)^{\perp} = \frac{1}{2}\alpha_m(v, v)t^2 + \frac{1}{6}(\nabla_v \alpha)(v, v)t^3.$$

In the following, the formulas for μ and its derivatives will have two components; the first one is the tangential component in the direction v (the tangential component in the direction Jv is zero); the second is the normal part which belongs to $N_m M$.

$$\begin{aligned} \mu(t) &= \Big(t + \frac{1}{6}\big(v \cdot c - \|\alpha_m(v, v)\|^2\big)t^3 \ , \ \frac{1}{2}\alpha_m(v, v)t^2 + \frac{1}{6}(\nabla_v \alpha)(v, v)t^3\Big), \\ \mu'(t) &= \Big(1 + \frac{1}{2}\big(v \cdot c - \|\alpha_m(v, v)\|^2\big)t^2 \ , \ \alpha_m(v, v)t + \frac{1}{2}(\nabla_v \alpha)(v, v)t^2\Big), \\ \mu''(t) &= \big((v \cdot c - \|\alpha_m(v, v)\|^2)t \ , \ \ \alpha_m(v, v) + (\nabla_v \alpha)(v, v)t\big), \\ \mu'''(t) &= \big(v \cdot c - \|\alpha_m(v, v)\|^2 \ , \ \ (\nabla_v \alpha)(v, v)\big). \end{aligned}$$

We evaluate the last three formulas at $t = 0$, and get

$$\begin{aligned} &\mu'(0) = (1, 0), \\ &\mu''(0) = (0, \alpha_m(v, v)), \\ &\mu'''(0) = \big(v \cdot c - \|\alpha_m(v, v)\|^2, \ (\nabla_v \alpha)(v, v)\big). \end{aligned}$$

Now it is easy to show that $\mu'(0) \times \mu''(0) = J\alpha(v,v)$ from which we have

$$(\mu'(0) \times \mu''(0)) \cdot \mu'''(0) = J\alpha(v,v) \cdot (\nabla_v \alpha)(v,v).$$

Therefore the normal torsion of M at m in the direction $v \in T_mM$ is given by:

$$\tau_v = \frac{J\alpha(v,v) \cdot (\nabla_v \alpha)(v,v)}{\alpha(v,v) \cdot \alpha(v,v)} = \frac{J\gamma_v'' \cdot \gamma_v'''}{\gamma_v'' \cdot \gamma_v''}(0).$$

The normal curvature in the same direction is $\kappa_v = \|\mu''(0)\| = \|\alpha_m(v,v)\|$. □

Proposition 3.2.16 *Let M be a k-dimensional submanifold in $\mathbb{R}^{k+n}$. For a given point $m \in M$ and a given unit vector $v \in T_mM$ there exists a unique geodesic $\gamma : I \to M$ with $\gamma(0) = m$ and $\gamma'(0) = v$ and a unique normal section $\beta : I \to M$ associated to m and v. Then the contact between γ and β is at least of order 3 on M if and only if v is an extremal frontal deviation direction.*

Proof. It is clear that $\gamma'(0) = \beta'(0) = v$ and we know that $\gamma''(0) = \beta''(0) = \alpha(v,v)$. On the other hand, we say that two regular curves γ, β with a point in common $\gamma(t_0) = \beta(t_0) = m$, have a contact of order **k** in m iff there exists a parametrization of the curves where the first $k-1$ derivatives coincides at that point, that is:

$$\begin{array}{rcl} \gamma^{(i)}(t_0) & = & \beta^{(i)}(t_0), \quad i = 1, \ldots, k-1, \\ \gamma^{(k)}(t_0) & \neq & \beta^{(k)}(t_0). \end{array}$$

Then we observe that the contact between the geodesic γ and the normal section β is at least of order 2. In ([2]) it is proved that the contact between γ and β is at least of order 3, that is, $\gamma'''(0) = \beta'''(0)$ if and only if $\alpha(v, Jv) \cdot \alpha(v,v) = 0$. □

3.3 Application to contact theory

3.3.1 Directions of high contact with 3-spheres in $\mathbb{R}^4$

Let M be a surface immersed in $\mathbb{R}^4$, $m \in M$ and $0 \neq u \in \mathbb{R}^4$. We will denote by $d_{3,u}$ the third order approximation of the function $f : T_mM \to \mathbb{R}$ defined as $f(x) = h(x) - h(0)$, where $h(x)$ is the composition of the exponential map with the distance squared function, that is $h(x) = d^2(\exp_m(x), m + u)$, where

$$\begin{aligned} d_{3,u}(x) = & -2u \cdot x + x \cdot x - u \cdot \alpha(x,x) + \frac{1}{3}\alpha(u^\top, x) \cdot \alpha(x,x) \\ & - \frac{1}{3}u^\perp \cdot (\nabla_x \alpha)(x,x). \end{aligned}$$

From definition 2.2.11 it is known that u determines a *rib direction* at m if and only if the following conditions are true:

(i) $u \in N_mM$.

(ii) There is some $x \in T_mM$, $x \neq 0$, such that $g(x,\cdot) - u \cdot \alpha(x,\cdot) = 0$.

(iii) $d_{3,u}(x) = 0$.

This vector x defines a *strong principal direction* at m i.e. a direction of at least A_k contact, $k \geq 3$, with the corresponding focal hypersphere, [25].

Theorem 3.3.1 *If a vector $0 \neq x \in T_mM$ defines a strong principal direction then it satisfies the following conditions:*

1. $\alpha(x,x) \neq 0$.

2. $J\alpha(x,x) \cdot \alpha(x,Jx) \neq 0 \quad$ *or* $\quad \alpha(x,Jx) = 0$.

3. $\det\big(\alpha(x,Jx), (\nabla_x \alpha)(x,x)\big) = 0$,

where the determinant is meaningful because both vectors belong to $N_m M$*, whose dimension is two.*

Proof. Assume that $0 \neq x \in T_m M$ defines a strong principal direction. Then there exists a rib direction $u \in \mathbb{R}^4$ satisfying properties (i)-(iii). Condition (i) says that $u^\top = 0$. Since (x, Jx) is a basis of $T_m M$ condition (ii) is equivalent to the following two conditions

$$x \cdot x = u \cdot \alpha(x,x), \qquad u \cdot \alpha(x,Jx) = 0.$$

Since $x \neq 0$, the first one requires that $\alpha(x,x) \neq 0$. Therefore we can put

$$u = p\alpha(x,x) + qJ\alpha(x,x)$$

for some $p, q \in \mathbb{R}$. Then $u \cdot \alpha(x,x) = x \cdot x = p\|\alpha(x,x)\|^2$, that is

$$p = \frac{x \cdot x}{\|\alpha(x,x)\|^2},$$

and

$$u \cdot \alpha(x,Jx) = 0 = \frac{(x \cdot x)\alpha(x,x) \cdot \alpha(x,Jx)}{\|\alpha(x,x)\|^2} + qJ\alpha(x,x) \cdot \alpha(x,Jx).$$

Hence, if $J\alpha(x,x) \cdot \alpha(x,Jx) \neq 0$ we can solve this for q. Otherwise we must have

$$J\alpha(x,x) \cdot \alpha(x,Jx) = \alpha(x,x) \cdot \alpha(x,Jx) = 0,$$

but since $\alpha(x,x) \neq 0$ and $J\alpha(x,x) \neq 0$ we conclude that $\alpha(x,Jx) = 0$. So, in any case condition *2* is satisfied.

Also, if (i) and (ii) are satisfied, then $d_{3,u}(x) = -\frac{1}{3}u \cdot (\nabla_x\alpha)(x,x)$ and this must be zero. Therefore, the non-zero vector $u \in N_mM$ must be orthogonal to $\alpha(x,Jx)$ and $(\nabla_x\alpha)(x,x) \in N_mM$. Since $\dim N_mM = 2$, we conclude that these two vectors must be linearly dependent, i.e.

$$\det\big(\alpha(x,Jx), (\nabla_x\alpha)(x,x)\big) = 0.$$

and this is condition *3.* □

Conversely to the previous theorem, we have

Theorem 3.3.2 *If a vector $x \in T_mM$ satisfies the following conditions:*

1. $\alpha(x,x) \neq 0, \quad J\alpha(x,x) \cdot \alpha(x,Jx) \neq 0,$
   $\det\big(\alpha(x,Jx), (\nabla_x\alpha)(x,x)\big) = 0,$

   or

2. $\alpha(x,x) \neq 0, \quad \alpha(x,Jx) = 0, \quad \{\alpha(x,x) \cdot (\nabla_x\alpha)(x,x) = 0$ *or* $J\alpha(x,x) \cdot (\nabla_x\alpha)(x,x) \neq 0\}.$

Then it defines a strong principal direction.

Proof. Suppose that x satisfies *1.* Then, as we have seen, there is a non-vanishing vector u that satisfies (i) and (ii). But then $u \cdot \alpha(x,Jx) = 0$, from which we conclude that u is orthogonal to $(\nabla_x\alpha)(x,x)$, because by the second and third conditions this vector is a multiple of $\alpha(x,Jx) \neq 0$ and this leads to (iii).

Now, suppose that x satisfies *2.* Then, for any value of $r \in \mathbb{R}$ we have that

$$u = \frac{x \cdot x}{\|\alpha(x,x)\|^2}\alpha(x,x) + r\, J\alpha(x,x)$$

satisfies (i) and (ii). The condition (iii) is now

$$\left(\frac{x \cdot x}{\|\alpha(x,x)\|^2}\alpha(x,x) + r\, J\alpha(x,x)\right) \cdot (\nabla_x \alpha)(x,x) = 0$$

If $\alpha(x,x) \cdot (\nabla_x \alpha)(x,x) = 0$ then the choice $r = 0$ solves the existence of the needed vector u. If $J\alpha(x,x) \cdot (\nabla_x \alpha)(x,x) \neq 0$, then we can solve the equation for r and find again the vector u. □

Condition *3* of Theorem 3.3.1 is an homogeneous polynomial equation of fifth degree in the components of x which generically gives at most 5 strong principal directions. That equation was first obtained by J. Montaldi in [22], by using a Monge chart. It was opaque in the sense that it is not given in terms geometrically recognizable, as our is. Another advantage of our equation is that it can be computed explicitly symbolically by means of a convenient package, or numerically.

Example 3.3.3 *Let M be a surface immersed in $\mathbb{R}^4$ by the chart:*

$$\vec{x}\,(u,v) = (u^3 v, v^3, u+v, u-v), \quad 0 < u, v < 1$$

and $\alpha = \vec{x} \circ \beta,\ \beta = (u(t), v(t))$. The differential equation that defines these lines is given by:

$$-\,72v^2(-2+45u^4v^2)u'^4 v + 216uv(2-23u^4v^2)u'^3 v'^2 - 36u^2(2+93u^4v^2)u'^2 v'^3$$
$$-\,72uv(-2+8u^6+45v^4)u' v'^4 + 36u^2(2+u^6-45v^4)v'^5 = 0.$$

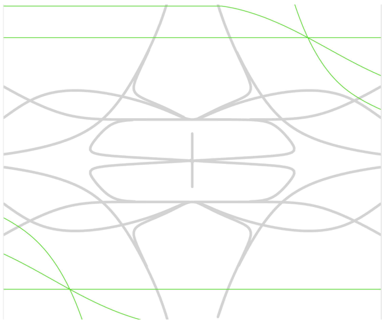


Figure 3.3: Strong principal directions of the surface $\vec{x}\,(u,v) = (u^3v, v^3, u+v, u-v)$

The figure 3 shows some strong principal lines of M. The thick gray line is the discriminant of the strong principal lines. We have drawn also the strong principal lines starting from three points of different regions.

Let $x \neq 0$ be a unit vector obtained by solving the fifth degree equation and put $b = b_x = \alpha(x,x)$ and $c = \alpha(x, Jx)$. Let us suppose in addition that $b \neq 0$ and $Jb \cdot c \neq 0$. Then, the conditions of the previous Theorem are satisfied and we will have that the corresponding rib is determined by

$$u = \frac{b}{||b||^2} - \frac{b \cdot c}{||b||^2 Jb \cdot c} Jb.$$

If $b \neq 0$ and $c = 0$ then

$$u = \frac{b}{||b||^2}.$$

In the first case, suppose that $\kappa' = \kappa'_x \neq 0$. Then c is a multiple of

$n = (\nabla_x \alpha)(x, x)$, so that if $\kappa = \kappa_x$ we may write

$$\begin{aligned} u =& \frac{b}{\kappa^2} - \frac{b \cdot n}{\kappa^2 Jb \cdot n} Jb \\ =& \frac{b}{\kappa^2} - \frac{\kappa'}{\kappa^3 \tau} Jb, \end{aligned}$$

where τ is the normal torsion of M at m in the direction x.

From definition 2.2.12 it is known that u determines an *umbilic direction* at m if and only if the following conditions are true:

(i) $u \in N_m M$.

(ii) $g(x, y) - u \cdot \alpha(x, y) = 0$, for any $x, y \in T_m M$.

In this case we have a singularity of corank 2 of the distance squared function on M at m, i.e. a contact of type D_k, $k \geq 4$, with the corresponding umbilic focal hypersphere. If m is umbilic then there is some vector $b \in N_m M$ such that we have at m that $\alpha = b \otimes g$. If $b = 0$, there are no umbilic directions at m. Otherwise, all vectors $u \in N_m M$ in the affine line given by the equation $u \cdot b = 1$ determine umbilic directions. The remaining cases are comprised in the following result, where we have reworded the theorem given in [28].

Theorem 3.3.4 *Let $m \in M$ be a non umbilic point. There is a vector $u \in N_m M$ determining an umbilic direction at $m \in M$ if and only if m is a semiumbilic non-inflection point.*

Proof. Assume that $u \in N_m M$ determines an umbilic direction. Let (t_1, t_2) be an orthonormal basis of $T_m M$ such that $B \cdot C = 0$

and $|B| \geq |C|$. Condition (ii) is then equivalent to the following three conditions

$$1 = u\cdot\alpha(t_1,t_1) = u\cdot b_1, \quad 1 = u\cdot\alpha(t_2,t_2) = u\cdot b_2, \quad 0 = u\cdot\alpha(t_1,t_2) = u\cdot b_3.$$

Therefore $u \neq 0,\ b_1 = \alpha(t_1,t_1) \neq 0$ and $b_2 = \alpha(t_2,t_2) \neq 0$. Also

$$\frac{1}{2}u \cdot (b_1 - b_2) = u \cdot B = 0.$$

Since B and C are orthogonal to the non-zero vector u, orthogonal to each other, and $|B| \geq |C|$ we conclude that $C = 0$. Then the curvature ellipse is a segment and m is semiumbilic. If b_1 and b_2 were linearly dependent, then both must be equal because $u \cdot b_1 = u \cdot b_2 = 1$. Thus, if $b_1 = rb_2,\ r \in \mathbb{R}$, we conclude that $u\cdot(b_1 - b_2) = (r-1)u\cdot b_2 = r-1 = 0$. Hence, $r = 1$ and $B = 0$. But then m would be umbilic against the hypothesis. If b_1 and b_2 are linearly independent, then m is not an inflection point, and it is easy to see that

$$u = \frac{JB}{H \cdot JB}.$$

Conversely, let m be a semiumbilic point that is not an inflection point. Then it is not umbilic. We can choose then the orthonormal basis (t_1, t_2) so that $C = 0$. It is easy to see that then $H \cdot JB \neq 0$, so that we can define a vector $u \in N_m M$ by the preceding formula and verify directly that it satisfies condition (ii). □

3.3.2 Application to the asymptotic directions for a surface in $\mathbb{R}^{2+n}$

Let M be a surface immersed in $\mathbb{R}^5$. We denote by $f_{3,u}$ the third order approximation of the height function composed with the exponential map as $x \in T_mM \mapsto u \cdot (\exp_m(x) - m)$, that is

$$f_{3,u}(x) = u{\cdot}x + \frac{1}{2}u{\cdot}\alpha_m(x,x) - \frac{1}{6}\alpha_m(u^\top, x){\cdot}\alpha_m(x,x) + \frac{1}{6}u^\perp{\cdot}(\nabla_x\alpha)(x,x).$$

In this section, we reword the characterization of asymptotic directions studied in [16] and [29] in the general case.

Definition 3.3.5 *Let $0 \neq u \in \mathbb{R}^5$. Then, u determines a* binormal direction *at m iff the following conditions are true:*

(i) $u \in N_mM$;

(ii) there is a non-vanishing vector $x \in T_mM$ such that $u{\cdot}\alpha_m(x,y) = 0$ for any $y \in T_mM$ and such that $f_{3,u}(x) = 0$. We say that such a vector x defines an asymptotic direction at m.

Notice that definition is a generalization of the definition of asymptotic direction for surfaces in $\mathbb{R}^4$.

Theorem 3.3.6 *A vector $0 \neq x \in T_mM$ defines an asymptotic direction at $m \in \mathbb{R}^5$ if and only if*

$$\det\big(\alpha_m(x,t_1), \alpha_m(x,t_2), (\nabla_x\alpha)(x,x)\big) = 0,$$

where $\{t_1, t_2\}$ is an orthonormal frame of T_mM.

Proof. Assume that $0 \neq x \in T_mM$ defines an asymptotic direction. Then there exists $u \in \mathbb{R}^5$ with the two properties of the above definition. These are equivalent clearly to the requirements that $u \in N_mM$, that $u \cdot \alpha_m(x,.) = 0$ and that $u \cdot (\nabla_x\alpha)(x,x) =$

0. Now, let t_1, t_2 be any basis of T_mM. Then the three vectors $\alpha_m(x, t_1)$, $\alpha_m(x, t_2)$, $(\nabla_x \alpha)(x, x) \in N_mM$ must have a non-vanishing vector $u \in N_mM$ orthogonal to them all. Since $\dim N_mM = 3$, we conclude that the necessary and sufficient condition for x being an asymptotic direction is that those three vectors be linearly dependent, that is

$$\det\big(\alpha_m(x, t_1), \alpha_m(x, t_2), (\nabla_x \alpha)(x, x)\big) = 0.$$

□

We have obtained thus a characterization of those asymptotic directions in terms of geometric invariants of the surface. The corresponding equation for the angle determining those directions can now be computed with the technique used for the strong principal directions. The program [24] draws the asymptotic lines, that is those whose tangent is an asymptotic direction at each point. The essential computations in that program are based in the preceding results.

Example 3.3.7 *Let M be a surface immersed in $\mathbb{R}^5$ given by the*

$$\vec{x}\,(u, v) = (u^3 + v^3, uv^3, u + v^3, u^2 - v^2, 3(u^2 + v^2))$$

and $\alpha = \vec{x} \circ \beta$, $\beta = (u(t), v(t))$. The differential equation that defines these lines is given by:

$$\begin{aligned} &432v^3u'^5 + 432(uv^2 - v^5)u'^4v' + 1296uv^4u'^3v'^2 \\ &+ (432v^3 + 1296u^2v^3 + 3(-432v^3 - 1296u^2v^3))u'^2v'^3 = 0. \end{aligned}$$

The figure 4, which have been made with the program [24] shows the asymptotic lines of a surface $M \in \mathbb{R}^5$. We have drawn the asymp-

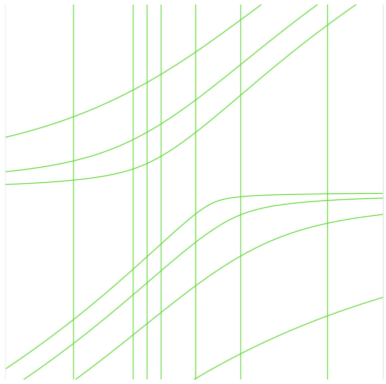


Figure 3.4: Asymptotic directions of the surface $\vec{x}\,(u,v) = (u^3 + v^3, uv^3, u + v^3, u^2 - v^2, 3(u^2 + v^2))$

totic lines starting from points where there are one or three asymptotic directions.

The program does not compute the discriminant curve, because the discriminant function for an equation of degree five is a homogeneous polynomial of degree eight with 59 terms and coefficients that reach the value 2500, in the coefficients of the polynomial equation. For simple examples of surfaces, the function and its derivatives take values so enormous (or so enormously small) that the zero locus cannot be computed accurately using machine precision, as does the program.

Let us see the general case. If $n > 5$, we can extend the previous definition as following.

Definition 3.3.8 *Let* $0 \neq u \in \mathbb{R}^n$*. we say that* u *is a* ***supernormal direction*** at m when the following conditions are satisfied:

(i) $u \in N_m M$;

(ii) there is a non-vanishing vector $x \in T_mM$ such that $u \cdot \alpha(x, \cdot) = 0$ and $f_{j,u}(x) = 0$, for $j = 3, \ldots, n-2$. Here $f_{j,u}$ denotes de j-th order approximation of the function $x \mapsto u \cdot (\exp_m(x) - m)$.

We say that such a vector x defines an *asymptotic direction* at m.

In the same way that in the previous case, we shall obtain an equation of degree n in $\tan\theta$, which will give us $n, n-2, \ldots$ of different asymptotic directions (note that an angle θ and its opposite have the same value for $\tan\theta$ but both angles define the same direction in T_mM).

Chapter 4

Critical points for smooth maps

In this chapter we introduce the concept of 1-critical point and give a geometric interpretation of these points in the case of the normal map, the generalized Gauss map and the exponential tangent map. Finally, we generalize the definition to introduce the concept of r-critical point.

4.1 The focal set as 1-critical points of the normal map

We will describe in this section and the next the relations between the Veronese of curvature and the focal set of an immersion M in $\mathbb{R}^{k+n}$. Let ν be the normal map of M given by $\nu : NM \to \mathbb{R}^{k+n}$ that sends an element $U = (m; U_m) \in NM$ to the point $m + U_m$, where $m \in M$ and $U_m \in N_m M$. Notice that both vectors are taken as vectors in $\mathbb{R}^{k+n}$. In the following, we shall consider that $U_m \neq 0$.

Definition 4.1.1 *We say that a vector $X \in T(NM)$ is a* ***1-critical point of ν*** *if $X \neq 0$ and $d\nu(X) = 0$. And we say that $U \in NM$ admits a 1-critical point of ν if there is a 1-critical point $X \in T_U(NM)$.*

This is the diagram of those maps:

$$\begin{array}{ccc} T(NM) & \xrightarrow{d\nu} & T\mathbb{R}^{k+n} \\ \downarrow{\scriptstyle \pi} & & \\ NM & \xrightarrow{\nu} & \mathbb{R}^{k+n} \\ \downarrow{\scriptstyle \pi_N} & & \\ M & & \end{array}$$

Definition 4.1.2 *The* ***focal set*** *of M, denoted here by $\mathcal{F}(M)$, is the set of elements $U \in NM$ that admit a 1-critical point of ν. Since there will be little risk of confusion, the same name will be used for the image of $\mathcal{F}(M)$ by ν.*

The next result is well-known. However we offer a proof in order to introduce the technique that we shall use throughout.

Proposition 4.1.3 *Let $\mathcal{F}_m(M) = \mathcal{F}(M) \cap N_mM$. The focal set at $m \in M$ is given by*

$$\mathcal{F}_m(M) = \{U_m \in N_mM \ : \ \det(g_m - U_m \cdot \alpha_m) = 0\},$$

where the determinant can be computed by means of any basis of T_mM.

Proof. Let $X \in T_{(m;U_m)}(NM)$ be a 1-critical point of ν. Consider a smooth curve $\gamma : I \to NM$ given by $\gamma(t) = (\beta(t); u(t))$, where I is an open neighborhood of $0 \in \mathbb{R}$, $\beta := \pi_N \circ \gamma$ and $u(t) \in N_{\beta(t)}M$, $\forall t \in$

I. Suppose that $(m; U_m) = \gamma(0) = (\beta(0); u(0))$ and $X = \gamma'(0) = (\beta'(0); u'(0))$. Let us put $v = \beta'(0)$. Since X is a 1-critical point of ν, we have:

$$d\nu(X) = (\nu \circ \gamma)'(0) = \beta'(0) + u'(0) = v + u'(0) = 0.$$

If $v = 0$, then we should have $u'(0) = 0$, that is $X = 0$, against the hypothesis. Therefore $v \neq 0$ and $u'(0) = -v \in T_m M$. Thus, we must find in what conditions upon v and U_m we may have $u'(0) = -v$.

The fact that $\beta'(0) \neq 0$ implies that β is an immersion in a neighborhood of 0. Then there exists a neighborhood $\mathcal{U}$ of m and a section $\tilde{u} : \mathcal{U} \subset M \to NM$ such that $\tilde{u} \circ \beta = \gamma$ and therefore $u(0) = \tilde{u}_m$ and $u'(0) = D_v \tilde{u}$. Hence, taking the normal and tangent components, we conclude that $d\nu(X) = v + D_v \tilde{u} = 0$ iff $(D_v \tilde{u})^\perp = \nabla_v^\perp \tilde{u} = 0$ and $v + \mathcal{A}_m(U_m, v) = 0$, or, equivalently $\nabla_v^\perp \tilde{u} = 0$ and $v \cdot x - \tilde{u}_m \cdot \alpha_m(v, x) = 0$, $\forall x \in T_m M$. The last equation says that if U_m admits a 1-critical point, then $\det(g_m - U_m \cdot \alpha_m) = 0$ and the nonzero vector $v \in T_m M$ satisfies $g_m(v, \cdot) = U_m \cdot \alpha_m(v, \cdot)$, where $d\pi_N(X) = v$.

Suppose that these conditions are satisfied. Then, let $\beta : I \to M$ be such that $\beta(0) = m$, $\beta'(0) = v$, and let $u : I \to \mathbb{R}^{k+n}$ be such that $u(t) \in N_{\beta(t)} M$ be the parallel transport of U_m along β. Then

$$u'(0) = D_v u = (D_v u)^\top = \mathcal{A}_m(U_m, v) = -v.$$

Therefore the tangent of the curve $(\beta(t); u(t))$ at $t = 0$ is equal to $X = (v; -v)$, so that $X \neq 0$ and $d\nu(X) = 0$. Hence X is a 1-critical point of ν. $\square$

Let $\{t_i\}_{i=1}^k$ be an orthonormal reference of T_mM at $m \in \mathcal{U} \subset M$ and $\{w_i\}_{i=1}^n$ be an orthonormal reference of N_mM in $\mathcal{U}$. Then, by using the previous notation, the matrix associated to $d(\nu \circ \tilde{u})_m$ is given by:

$$\left(\begin{array}{cccc|c} t_1 \cdot t_1 - U_m \cdot \alpha_m(t_1,t_1) & . & . & t_1 \cdot t_k - U_m \cdot \alpha_m(t_1,t_k) & \\ . & . & & . & \\ . & & . & . & 0_{k\times n} \\ . & & . & . & \\ t_k \cdot t_1 - U_m \cdot \alpha_m(t_k,t_1) & . & . & t_k \cdot t_k - U_m \cdot \alpha_m(t_k,t_k) & \\ \hline & & 0_{n\times k} & & I_{n\times n} \end{array}\right).$$

Then $U_m \in N_mM$ admits 1-critical point of ν iff the following determinant vanishes:

$$\left|\begin{array}{cccc} t_1 \cdot t_1 - U_m \cdot \alpha_m(t_1,t_1) & . & . & t_1 \cdot t_k - U_m \cdot \alpha_m(t_1,t_k) \\ . & . & & . \\ . & & . & . \\ . & & . & . \\ t_k \cdot t_1 - U_m \cdot \alpha_m(t_k,t_1) & . & . & t_k \cdot t_k - U_m \cdot \alpha_m(t_k,t_k) \end{array}\right| = 0.$$

Proposition 4.1.4 *Let $\mathcal{F}_m(M) = \mathcal{F}(M) \cap N_mM$ and $\eta_m : T_mM \backslash \{0\} \to N_mM$ be the Veronese of curvature given by $\eta_m(t) = \frac{\alpha_m(t,t)}{t \cdot t}$. Then the following properties are satisfied:*

1. *If $\tilde{u} : \mathcal{U} \to NM$ is a local section in a neighborhood of $m \in M$, then $\det(g_m - U_m \cdot \alpha_m) = \det(d(\nu \circ \tilde{u})_m^\top)$, where $U_m = \tilde{u}_m$.*

2. *If $U_m \in \mathcal{F}_m(M)$, then there exists $t \in T_mM \backslash \{0\}$, such that $g_m(t) = U_m \cdot \alpha_m(t) \in T_m^*M$. In the following items, t and U_m satisfy that property.*

3. *$U_m \cdot \eta_m(t) = 1$. In particular, $U_m \neq 0$, $\eta_m(t) \neq 0$.*

4. $\eta_m(t) \notin (d\eta_m)(T_tT_mM) \subset T_{\eta_m(t)}N_mM$, *under the usual identification of* N_mM *with* $T_{\eta_m(t)}N_mM$.

Proof. 1) and 2) Have been proved along the proof of the above proposition.

3) If the 1-form $(g_m - U_m \cdot \alpha_m)(t)$ acts upon the vector t itself, we get $t \cdot t - U_m \cdot \alpha_m(t,t) = 0$, whence, by dividing by $t \cdot t \neq 0$, we obtain the claim.

4) For making the calculations easier we can assume that $t \cdot t = 1$. Then, if $X \in T_tT_mM$, we have

$$\begin{aligned} d\eta_m(X) =& \frac{2}{(t\cdot t)^2}((t\cdot t)\alpha_m(t,X) - (t\cdot X)\alpha_m(t,t)) \\ =& 2(\alpha_m(t,X) - (t\cdot X)\eta_m(t)). \end{aligned}$$

Suppose that this is equal to $\eta_m(t)$. By inner multiplication of this with U_m we get $U_m \cdot \eta_m(t) = t \cdot t = 1$, while the same multiplication with $d\eta_m(X)$ yields $2(U_m \cdot \alpha_m(t) - g_m(t))(X) = 0$, which is absurd. □

In general, $\mathcal{F}(M)$ will be a hypersurface of NM, possibly with singularities, whose intersection with each fiber N_mM will be an algebraic hypersurface of degree k. Thus, in the case of a surface $M \subset \mathbb{R}^{2+n}$, $\mathcal{F}_m(M)$ is an algebraic hypersurface of degree 2 given by:

$$\begin{vmatrix} 1-U_m\cdot\alpha_m(t_1,t_1) & -U_m\cdot\alpha_m(t_1,t_2) \\ -U_m\cdot\alpha_m(t_2,t_1) & 1-U_m\cdot\alpha_m(t_2,t_2) \end{vmatrix} = 0.$$

This gives an equation of the form:

$$\sum_{i,j=1}^{n} a_{ij}u_iu_j + \sum_{i=1}^{n} b_iu_i + 1 = 0, \tag{4.1.1}$$

where $U_m = \sum_{i=1}^{n} u_iw_i$ and $a_{ij}, b_i \in \mathbb{R}$. In this case $\mathcal{F}_m(M)$ is a quadric.

4.1.1 The inverted pedal

Definition 4.1.5 *Let P be a smooth manifold and $\mu : P \to \mathbb{R}^d$ a smooth map. For each $p \in P$, let $\mathrm{ped}_\mu(p)$ be the nearest point to the origin among those of the affine subspace tangent to $\mu(P)$ at $\mu(p)$, i.e. $\{\mu(p) + d\mu(X) \ : \ X \in T_pP\}$. The resulting map $\mathrm{ped}_\mu : P \to \mathbb{R}^d$ is called the pedal map of μ. Let $\tilde{P}_\mu = \{p \in P \ : \ \mathrm{ped}_\mu(p) \neq 0\}$. If $R : \mathbb{R}^d \backslash \{0\} \to \mathbb{R}^d \backslash \{0\}$ is the inversion with respect to the hypersphere with center 0 and unit radius, the composition $R \circ \mathrm{ped}_\mu : \tilde{P} \to \mathbb{R}^d \backslash \{0\}$ (and sometimes, also its image) will be called the* ***inverted pedal of*** *μ.*

Let us show the relation between the focal set and the inverted pedal of $\eta_m : T_mM \backslash \{0\} \to N_mM$ at $m \in M$. We are here interested solely in the study of $\mathcal{F}_m(M)$. This justifies the use of the following simplified notation in this subsection:

$$T = T_mM \backslash \{0\}, \ N = N_mM, \ \alpha = \alpha_m, \ \eta = \eta_m, \ \mathcal{F} = \mathcal{F}_m(M), \ g = g_m.$$

Proposition 4.1.6 *Let $R(z) \in N$ be a point in the inverted pedal of η. Then $R(z) \in \mathcal{F}$.*

Proof. Let $t \in T$ and let $z = \eta(t) + d\eta(X)$, with $X \in T_tT$, be a non-zero point in the pedal of η so that $R(z)$ belongs to the inverted pedal of η. We must have $z \cdot d\eta(T_tT) = d(z \cdot \eta)(T_tT) = 0$ because z is the nearest point to the subspace $\{\eta(t) + d\eta(Y)\,;\, Y \in T\}$. In particular, $z \cdot d\eta(X) = 0$, whence $z \cdot z = z \cdot \eta(t) \neq 0$. Also, $d(z \cdot \eta)_t = 0$. Hence t is a critical point of the map $t \mapsto z \cdot \eta(t)$. But one sees easily that this entails

$$g(t,t)z \cdot \alpha(t,X) - (t \cdot X)z \cdot \alpha(t,t) = 0$$

for any $X \in T_tT$, i.e. $(z \cdot \alpha - z \cdot \eta(t)g)(t) = 0$, and this requires the vanishing of $\det(z \cdot \alpha - z \cdot \eta(t)g)$. By dividing that determinant by $(-z \cdot \eta(t))^k$, we conclude that

$$\det(g - \frac{z}{z \cdot \eta(t)} \cdot \alpha) = 0,$$

that is

$$\frac{z}{z \cdot \eta(t)} = \frac{z}{z \cdot z} = R(z) \in \mathcal{F}.$$

□

Let us see whether there is some form of converse of this.

Let $Q : T \to N$ be the map given by $Q(t) = \alpha(t,t),\ t \in T$. Notice that $\alpha(t,t) = |t|^2\eta(t)$, then the image of T by Q is a cone centered at the origin and based on the Veronese of curvature. Hence, the map $(dQ)_t : T_tT \to T_{Q(t)}N$ gives $(dQ)_t(X) = 2\alpha(t,X),\ X \in T_tT$. In other words, the vector space $\alpha(t,T)$ is equivalent to the image of T by $(dQ)_t,\ t \in T$. Then, $\alpha(t,T)$ is the vector space generated by $\eta(t)$ and the tangent space of the Veronese of curvature at $\eta(t)$.

We put $B_t = R(\mathrm{ped}_\eta(t)) + \perp\alpha(t,T)$, where $\perp\alpha(t,T)$ denotes the orthogonal subspace of $\alpha(t,T)$ in N. Thus, B_t is an affine subspace of N passing by $R(\mathrm{ped}_\eta(t))$.

Let $\tilde{T} = \{t \in T : \mathrm{ped}_\eta(t) \neq 0\}$.

Theorem 4.1.7 *$\mathcal{F}$ is the union of the inverted pedal of η with $\cup_{t\in\tilde{T}}B_t$.*

Proof. Let $x \in N$ be a point in $\mathcal{F}$. Then $\det(g - x \cdot \alpha) = 0$. Let $t \in T,\ t \cdot t = 1$, be such that $g(t) = x \cdot \alpha(t)$. We know that then

$x \cdot \eta(t) = 1$ and $\eta(t) \notin (d\eta)(T_tT)$. Let $z = \text{ped}_\eta(t)$; if $z = 0$ we would have $\eta(t) \in (d\eta)(T_tT)$, which is absurd. As we have seen before we will have $(z \cdot z)g(t) = z \cdot \alpha(t)$, from which we obtain $g(t) = R(z) \cdot \alpha(t)$. Therefore $(R(z) - x) \cdot \alpha(t) = 0$, that is $R(z) - x \in B_t$. Hence we can write this in the form

$$x = R(z) + u, \ u \in \perp\alpha(t, T).$$

□

This describes completely $\mathcal{F}$. Note that dim $\alpha(t, T) \leq k$. Hence, if for example $k = 2$ (M is thus a surface) and $n = 2$ then generically the dimension of $\mathcal{F}$ will be that of $\eta(PT)$, that is one; thus, $\mathcal{F}$ will be a conic. If $n = 3$, it will be generally a ruled quadric surface.

Let us see some examples.

Example 4.1.8 *Suppose that $\beta : I \to \mathbb{R}^{1+n}$ is a curve immersed in $\mathbb{R}^{1+n}$, where I is an open neighborhood of $0 \in \mathbb{R}$ and $\beta(0) = m$. If $\alpha(t, t) \neq 0, \ t \in T_m\beta$, $\mathcal{F}_m(\beta)$ is given by $R(\text{ped}_\eta(t)) + \perp\alpha(t, t)$, which gives $\frac{\alpha(t,t)}{\alpha(t,t)\cdot\alpha(t,t)} + \perp\alpha(t, t)$. Otherwise, $\text{ped}_\eta(t) = 0$ and $\mathcal{F}_m(\beta)$ is empty.*

Notice that in the case of $n = 2$, $\mathcal{F}_m(\beta)$ is the affine line defined by the binormal vector at m and passing through $R(\text{ped}_\eta(t))$.

Let us see the case of a surface $M \subset \mathbb{R}^{2+n}$ in detail. The image of T by η is an ellipse in N called *the curvature ellipse* and the affine tangent subspace to the ellipse at the point $\eta(t)$ is generally an affine line given by $\eta(t) + (d\eta)_t(T_tT)$. We shall distinguish different cases depending on the degeneration of the curvature ellipse at the point $m \in M$.

Notice that if $t \in T$ is such that $\text{ped}_\eta(t) = \eta(t)$, then $\eta(t) \cdot d\eta(X) = 0, \ X \in T_tT$. These tangent directions coincide with the *extremal* *frontal geodesic directions* defined in Chapter 2. We characterized these directions as the tangent directions where the distance of the ellipse to the origin is extremal. In other words, a tangent direction is an extremal frontal direction when the corresponding point of the ellipse belongs to a hypersphere of N_mM centered at the origin and tangent to the ellipse at that point.

Proposition 4.1.9 *Let M be a surface immersed in $\mathbb{R}^3$, where $m \in M$. Since $dimN_mM = 1$, the curvature ellipse degenerates into a segment or a point. Then, if $\eta(t_1)$ and $\eta(t_2)$ are the extremes of the ellipse, we have the following cases:*

i) If the ellipse degenerates into a segment and m is not parabolic, hence, $\mathcal{F}$ is the union of $R(\eta(t_1))$ with $R(\eta(t_2))$.

ii) If m is parabolic, $\mathcal{F}$ is $R(\eta(t_1))$, $\eta(t_1) \neq 0$.

iii) If m is umbilic not plane, $\mathcal{F}$ is reduced to $R(\eta(t))$.

iv) If m is umbilic plane, $\mathcal{F}$ is empty.

Proof. Suppose that the ellipse degenerates into a segment and that m is not parabolic. If $t \in T$ is such that $\eta(t)$ is not an extreme of the ellipse, then $\mathrm{ped}_\eta(t) = 0$ because of its affine tangent subspace is the line which contains the ellipse and the origin. If $t_1 \in T$ is such that $\eta(t_1)$ is an extreme of the ellipse, its affine tangent subspace is the same point and $\mathrm{ped}_\eta(t_1) = \eta(t_1)$. Hence, $\mathcal{F}$ is the union of $R(\eta(t_1))$ with $R(\eta(t_2))$, which gives $\frac{\alpha(t_1,t_1)}{\alpha(t_1,t_1)\cdot\alpha(t_1,t_1)}$ and $\frac{\alpha(t_2,t_2)}{\alpha(t_2,t_2)\cdot\alpha(t_2,t_2)}$, where $\eta(t_1)$ and $\eta(t_2)$ are the principal curvatures at m and $t_1, t_2 \in T$ are two extremal directions of frontal deviation which coincides with the principal directions. If m is parabolic, i.e. the origin coincides with one of the extremes of the segment, for example with $\eta(t_2)$, $\mathcal{F}$ is reduced to the point $R(\eta(t_1))$.

If the ellipse degenerates into a point $\eta(t)$ different from the origin, i.e. m is umbilic not plane, then $\mathrm{ped}_\eta(t) = \eta(t)$ and every direction $t \in T$ is extremal for the frontal deviation. In this case $\mathcal{F}$ is reduced to $R(\eta(t))$.

Finally, if the curvature ellipse degenerates into a point which coincides with the origin, i.e. m is umbilic plane, $\mathcal{F}$ is empty. □

Let us see the case of a surface in $\mathbb{R}^4$.

Proposition 4.1.10 *Let M be a surface immersed in $\mathbb{R}^4$, where $m \in M$. Then, depending on the degeneration of the curvature ellipse at $m \in M$, we have the following cases:*

i) If m is elliptic, $\mathcal{F}$ is an ellipse.

ii) If m is hyperbolic, $\mathcal{F}$ is an hyperbola.

iii) If m parabolic, $\mathcal{F}$ is a parabola.

iv) If m is semiumbilic but not an inflection point, $\mathcal{F}$ is the union of $R(\eta(t_1)) + \perp L_1$ with $R(\eta(t_2)) + \perp L_2$ whose intersection point is $R(\mathrm{ped}_\eta(t))$. The directions t_1, t_2 are the asymptotic directions.

v) If m is an inflection point, $\mathcal{F}$ is the union of $R(\eta(t_1)) + \perp L$ with $R(\eta(t_2)) + \perp L$, where $\eta(t_1)$ and $\eta(t_2)$ are the extremes of the ellipse and t_1, t_2 are asymptotic directions.

vi) If m is umbilic not plane, then $\mathcal{F}$ is reduced to $R(\eta(t)) + \perp L$, where L is the line from the origin which contains $\eta(t)$ and every direction $t \in T$ is an asymptotic direction.

vii) If m is umbilic plane, then $\mathcal{F}$ is empty.

Proof. If m is elliptic, the tangent line at $\eta(t)$ does not contain the origin and hence every $t \in T$ provides a point of $R(\text{ped}_\eta(t))$ and therefore a point of $\mathcal{F}$. In this case, $\mathcal{F}$ is equal to the inverted pedal of η and the equation 4.1.1 gives an ellipse.

If m is hyperbolic, there exists $t_1, t_2 \in T$ such that $\eta(t_i)$ and $(d\eta)_{t_i}$ are parallel, where $i = 1, 2$. Then $\text{ped}_\eta(t_1) = \text{ped}_\eta(t_2) = 0$. Notice that as $t \in T$ approaches to one of these directions, the tangent line to the ellipse at $\eta(t)$ approaches to the origin and thus $R(\text{ped}_\eta(t))$ is increasingly further moving away. The directions t_1, t_2 are the asymptotic directions defined in [14] and $\mathcal{F}$ is a not bounded conic, it is a hyperbola.

Finally, if m is parabolic, there exists only one $t_1 \in T$ such that $\text{ped}_\eta(t_1) = \eta(t_1) = 0$. In this case $\mathcal{F}$ is a parabola and t_1 is an asymptotic direction which is also an extremal direction of frontal deviation.

Suppose now that the ellipse degenerates into a segment. In this case, as the following case where the ellipse degenerates into a point, the extremal directions of frontal deviation coincides with the asymptotic directions. Suppose that the line which contains the ellipse does not contain the origin, i.e. m is semiumbilic but not an inflection point. In this case, if $t \in T$ is such that $\eta(t)$ is not an extreme of the ellipse, the affine tangent subspace to the ellipse at $\eta(t)$ is the line which contains the ellipse. So, all these points gives just one point of $R(\text{ped}_\eta(t))$. If $t_1 \in T$ is such that $\eta(t_1)$ is an extreme of the ellipse, the affine tangent subspace to the ellipse at $\eta(t_1)$ is the same point $\eta(t_1)$ and then $R(\text{ped}_\eta(t_1))$ is well defined and coincides with $R(\eta(t_1))$. In this case $\alpha(t_1, T)$ is the line L_1 from the origin which contains $\eta(t_1)$. In the same way, the other extreme of the ellipse $\eta(t_2),\ t_2 \in T$, generates the point $R(\eta(t_2))$ and the line L_2. In this case $\mathcal{F}$ is the union of $R(\eta(t_1)) + \perp L_1$ with $R(\eta(t_2)) + \perp L_2$ whose intersection point is $R(\text{ped}_\eta(t))$. The directions t_1, t_2 are the asymptotic directions.

Suppose now that the ellipse degenerates into a segment and that the line, L, which contains the ellipse contains also the origin, i.e. m is an inflection point. In this case, if $t \in T$ is such that $\eta(t)$ is not an extreme of the ellipse, then $\text{ped}_\eta(t) = 0$. Otherwise, if $t_1 \in T$ is such that $\eta(t_1)$ is an extreme of the ellipse, $\text{ped}_\eta(t_1) = \eta(t_1)$. Hence, $\mathcal{F}$ is the union of $R(\eta(t_1)) + \perp L$ with $R(\eta(t_2)) + \perp L$, where $\eta(t_1)$ and $\eta(t_2)$ are the extremes of the ellipse and t_1, t_2 are asymptotic directions. In the case that one of the extremes coincides with the origin, for example $\eta(t_2)$, then $\mathcal{F}$ is $R(\eta(t_1)) + \perp L$.

If m is umbilic not plane, then $\mathcal{F}$ is reduced to $R(\eta(t)) + \perp L$, where L is the line from the origin which contains $\eta(t)$ and every direction $t \in T$ is an asymptotic direction.

Finally, if m is umbilic plane, then $\mathcal{F}$ is empty. $\square$

Let us see the case of a surface M immersed in $\mathbb{R}^{2+n},\ n > 2$.

Proposition 4.1.11 *Let M be a surface immersed in $\mathbb{R}^{2+n}$, $n > 2$. Then:*

i) If the ellipse is not degenerated, the origin is inside the plane *containing the ellipse and the origin is inside the ellipse,* $\mathcal{F}$ *is the union of the inverted pedal of* η *with* $\cup_{t\in T} B_t$.

ii) *If the ellipse is not degenerated and the origin is outside the ellipse, in this case* $\mathcal{F}$ *is the union of the inverted pedal of* η *with* $\cup_{t\in\tilde{T}} B_t$*, where* $\tilde{T} = T\backslash\{t_1, t_2\}$.

iii) *If the ellipse is not degenerated and the origin belongs to the ellipse,* $\mathcal{F}$ *is the union of the inverted pedal of* η *with* $\cup_{t\neq t_1\in T} B_t$*, where* t_1 *is an extremal direction for the frontal deviation.*

iv) *If the origin does not belong to the plane containing the ellipse,* $\mathcal{F}$ *is the union of the inverted pedal of* η *with* $\cup_{t\in T} B_t$.

v) *If* m *is semiumbilic but not an inflection point, In this case* $\mathcal{F}$ *is the union of* $R(\eta(t_1)) + \perp L_1$ *with* $R(\eta(t_2)) + \perp L_2$ *and* $R(\mathrm{ped}_\eta(t)) + \perp\alpha(t, T)$*, where* t_1, t_2 *are extremal directions for the frontal deviation.*

vi) *If* m *is an inflection point,* $\mathcal{F}$ *is the union of* $R(\eta(t_1)) + \perp L$ *with* $R(\eta(t_2)) + \perp L$*, where* $\eta(t_1)$ *and* $\eta(t_2)$ *are the extremes of the ellipse,* $t_1, t_2 \in T$ *are extremal directions for the frontal deviation and* L *is the line from the origin which contains* $\eta(t)$.

vii) *If* m *is umbilic not plane, then* $\mathcal{F}$ *is reduced to* $R(\eta(t)) + \perp L$*, where* L *is the line from the origin which contains* $\eta(t)$*,* $\forall t \in T$*. In this case every direction is extremal for the frontal deviation.*

viii) *Finally, if* m *is umbilic plane,* $\mathcal{F}$ *is empty.*

Proof. Consider the general case, where the ellipse is not degenerated and suppose that the origin is inside the plane containing the ellipse. If the origin is inside the ellipse, the tangent line at $\eta(t)$ does not pass through the origin and hence every $t \in T$ provides a point of $R(\mathrm{ped}_\eta(t))$ and therefore a point of $\mathcal{F}$. In this case, $\mathcal{F}$ is the union of the inverted pedal of η with $\cup_{t\in T} B_t$, where B_t is an affine subspace of N with dimension $n-2$ and passing through $R(\mathrm{ped}_\eta(t))$. If the origin is outside the ellipse, there exists $t_1, t_2 \in T$ such that $\mathrm{ped}_\eta(t_1) = \mathrm{ped}_\eta(t_2) = 0$. In this case $\mathcal{F}$ is the union of the inverted pedal of η with $\cup_{t\in\tilde{T}} B_t$, where $\tilde{T} = T\backslash\{t_1, t_2\}$ and dim $B_t = n-2$. Finally, if the origin belongs to the ellipse, there exists only one $t_1 \in T$ such that $\eta(t_1) = 0$. In this case $\mathcal{F}$ is the union of the inverted pedal of η with $\cup_{t\neq t_1\in T} B_t$, where t_1 is an extremal direction for the frontal deviation and dim $B_t = n-2$.

Suppose now that the origin does not belong to the plane containing the ellipse. In this case the tangent line at $\eta(t)$ does not pass through the origin and hence every $t \in T$ provides a point of $R(\mathrm{ped}_\eta(t))$ and therefore a point of $\mathcal{F}$. In this case, $\mathcal{F}$ is the union of the inverted pedal of η with $\cup_{t\in T} B$, where dim $B_t = n-2$.

Suppose now that m is semiumbilic but not an inflection point. In this case, if $t \in T$ is such that $\eta(t)$ is not an extreme of the ellipse, the affine tangent subspace to the ellipse at $\eta(t)$ is the line which contains the ellipse. So, all these points gives just one point of $R(\mathrm{ped}_\eta(t))$. If $t_1 \in T$ is such that $\eta(t_1)$ is an extreme of the ellipse, the affine tangent subspace to the ellipse at $\eta(t_1)$ is the same point $\eta(t_1)$ and then $R(\mathrm{ped}_\eta(t_1))$ is well defined and coincides with $R(\eta(t_1))$. In this case

$\alpha(t_1, T)$ is reduced to the line L_1 from the origin which contains $\eta(t_1)$. In the same way, the other extreme of the ellipse $\eta(t_2)$ generates the point $R(\eta(t_2))$ and the line L_2. In this case $\mathcal{F}$ is the union of $R(\eta(t_1))+\perp L_1$ with $R(\eta(t_2)) + \perp L_2$ and $R(\text{ped}_\eta(t)) + \perp\alpha(t, T)$, where t_1, t_2 are extremal directions for the frontal deviation and $\dim \perp\alpha(t, T) = n-2$.

Suppose now that m is an inflection point. In this case, if $t \in T$ is such that $\eta(t)$ is not an extreme of the ellipse, then $\text{ped}_\eta(t) = 0$. Otherwise, if t_1 is such that $\eta(t_1)$ is an extreme of the ellipse, $\text{ped}_\eta(t_1) = \eta(t_1)$. Hence, $\mathcal{F}$ is the union of $R(\eta(t_1))+\perp L$ with $R(\eta(t_2))+\perp L$, where $\eta(t_1)$ and $\eta(t_2)$ are the extremes of the ellipse, $t_1, t_2 \in T$ are extremal directions for the frontal deviation and L is the line from the origin which contains $\eta(t)$. In the case that one of the extremes coincides with the origin, for example $\eta(t_2)$, then $\mathcal{F}$ is given by $R(\eta(t_1)) + \perp L$.

If m is umbilic not plane, then $\mathcal{F}$ is reduced to $R(\eta(t))+\perp L$, where L is the line from the origin which contains $\eta(t)$, $\forall t \in T$. In this case every direction is extremal for the frontal deviation.

Finally, if m is umbilic plane, $\mathcal{F}$ is empty. □

4.2 The parabolic set as 1-critical points of the generalized Gauss map

Let

$$CM = \{U \in PNM\,;\, U \cdot U = 1\} \subset NM$$

and let $\psi : CM \to \mathbb{R}^{k+n}$ be the map given by $\psi(m; U_m) = U_m$, $U_m \in C_mM$, where $C_mM = CM \cap PN_mM$. This map is known as the *gen-*

eralized Gauss map. Of course, its image lies in the unit sphere of N_mM.

Definition 4.2.1 *A vector $X \in T(CM)$ is a **1-critical point of ψ** iff it is non-zero and $d\psi(X) = 0$. And we say that $U \in CM$ admits a 1-critical point of ψ if there is a 1-critical point $X \in T_U(CM)$.*

Let us see a diagram of those maps:

$$\begin{array}{ccc} T(CM) & \xrightarrow{d\psi} & T\mathbb{R}^{k+n} \\ \downarrow{\scriptstyle \pi} & & \\ CM & \xrightarrow{\psi} & \mathbb{R}^{k+n} \\ \downarrow{\scriptstyle \pi_N} & & \\ M & & \end{array}$$

Definition 4.2.2 *If $U \in CM$ is such that admits a 1-critical point of ψ, we say that it belongs to the **parabolic set** of M. We shall denote by $\mathcal{P}(M)$ the set of those points.*

A computation almost equal to that made for the normal map leads immediately to the following result:

Proposition 4.2.3 *The point $U_m \in C_mM$ admits a 1-critical point $X \in T_{(m;U_m)}(CM)$ of ψ iff $\det(U_m \cdot \alpha_m(\cdot,\cdot)) = 0$, and in such case $X = (v; 0)$, where $0 \neq v = d\pi_N(X) \in T_mM$ satisfies $U_m \cdot \alpha_m(v, \cdot) = 0$.*

By using the notation of the previous section, the matrix associated to $d(\psi \circ \tilde{u})_m$ is given by:

$$\left(\begin{array}{cccc|c} -U_m\cdot\alpha_m(t_1,t_1) & \cdot & \cdot & \cdot\ -U_m\cdot\alpha_m(t_1,t_k) & \\ \cdot & \cdot & & \cdot & \\ \cdot & & \cdot & \cdot & 0_{k\times n} \\ \cdot & & & \cdot\ \ \cdot & \\ -U_m\cdot\alpha_m(t_k,t_1) & \cdot & \cdot & \cdot\ -U_m\cdot\alpha_m(t_k,t_k) & \\ \hline & & 0_{n\times k} & & I_{n\times n} \end{array}\right),$$

where $U_m = \tilde{u}_m$. Then $U_m \in C_m M$ admits a 1-critical point of ψ iff the following determinant vanishes:

$$\left|\begin{array}{cccc} U_m\cdot\alpha_m(t_1,t_1) & \cdot & \cdot & \cdot\ U_m\cdot\alpha_m(t_1,t_k) \\ \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & & \cdot\ \ \cdot \\ U_m\cdot\alpha_m(t_k,t_1) & \cdot & \cdot & \cdot\ U_m\cdot\alpha_m(t_k,t_k) \end{array}\right| = 0.$$

Proposition 4.2.4 *Let $\mathcal{P}_m(M) = \mathcal{P}(M) \cap C_m M$. The following properties are satisfied:*

1. *If $\tilde{u} : \mathcal{U} \to CM$ is a local section in a neighborhood of $m \in M$, then $\det(U_m \cdot \alpha_m) = \det(d(\psi \circ \tilde{u})^{\top}_m)$, where $U_m = \tilde{u}_m$.*

2. *If $U_m \in \mathcal{P}_m(M)$, then there exists $t \in T_m M \backslash \{0\}$, such that $0 = U_m \cdot \alpha_m(t) \in T^*_m M$. In the following item, t and U_m satisfy that property.*

3. *$U_m \cdot d\eta_m(X) = 0$, for all $X \in T_t(T_m M \backslash \{0\})$ such that $X \cdot t = 0$.*

Proof. 1) and 2) are clear by the above proposition.

3) Note that here η_m is taken as a map from $T_mM\backslash\{0\}$ to N_mM. For making the calculations easier we can assume that $t \cdot t = 1$, i.e. $t \in S(T_mM)$. Then, if $X \in T_t(T_mM\backslash\{0\})$, $X \cdot t = 0$ and

$$d\eta_m(X) = 2\alpha_m(t, X),$$

by inner multiplication of this with U_m we get that $U_m \cdot d\eta_m(X) = 0$. Notice that $d\eta_m(T_tS(T_mM))$ is the tangent space to the Veronese of curvature at $\eta_m(t)$, whose dimension is in general $min(k-1, n)$. □

In general, $\mathcal{P}(M)$ will be a hypersurface of CM, with singularities, whose intersection with each fiber C_mM will be an algebraic hypersurface of degree k.

Example 4.2.5 *Let $\beta : I \to \mathbb{R}^{1+n}$ be a curve where $\beta(0) = m$. Since dim $T_m\beta = 1$, then $\mathcal{P}_m(\beta)$ is given by the union of the unit vectors $U_m \in C_mM$ such that $U_m \cdot \alpha_m(t,t) = 0$. Notice that if $\alpha_m(t,t) = 0$, i.e. m is plane, then $\mathcal{P}_m(\beta)$ is C_mM.*

Let us see the case of a surface in detail. As in the previous section, we shall distinguish different cases depending on the degeneration of the curvature ellipse at the point $m \in M$. We are here interested solely in the study of $\mathcal{P}_m(M)$. This justifies the use of the following simplified notation in these examples:

$$T = T_mM\backslash\{0\},\ C = C_mM,\ N = N_mM,\ \alpha = \alpha_m,\ \eta = \eta_m$$

and

$$\mathcal{P} = \mathcal{P}_m(M),\ (d\eta)_t = (d\eta_m)_t(T_t(T_mM\backslash\{0\})).$$

Example 4.2.6 *Let M be a surface immersed in $\mathbb{R}^3$, $m \in M$. If m is parabolic, there exists $t \in T$ such that $\alpha(t,\cdot) = 0$. Then $\mathcal{P}$ is the unit normal vector U_m and $t \in T$ is an asymptotic direction. If m is umbilic plane, then $\alpha(\cdot,\cdot) = 0$. In this case, $\mathcal{P}$ is the unit normal vector U_m. Otherwise, $\mathcal{P}$ is empty.*

Suppose now that M is a surface immersed in $\mathbb{R}^4$. In this case, the conditions of Proposition 4.2.4 imply that $\eta(t)$ and $(d\eta)_t$ are parallel, i.e. $\eta(t)||(d\eta)_t$, where $t \in T$, and perpendicular to $U_m \in \mathcal{P}$. These directions t_1, t_2 are the asymptotic directions and U_1, U_2 are the binormal directions defined in [14]. Let us see a Proposition whose proof is trivial.

Proposition 4.2.7 *Let M be a surface immersed in $\mathbb{R}^4$. Depending on the degeneration of the curvature ellipse at the point $m \in M$, we have:*

i) If m is elliptic, then $\mathcal{P}$ is empty.

ii) If m is hyperbolic, there exists $t_1, t_2 \in T$ such that $\eta(t_i)\ ||(d\eta)_{t_i},\ i = 1,2$. In this case $\mathcal{P}$ is the union of two vectors $U_1, U_2 \in C$ such that U_i is perpendicular to the line which contains the origin and the extreme of the segment $\eta(t_i)$, $i = 1,2$.

iii) If m is parabolic, there exists only one $t_1 \in T$ such that $\eta(t_1) = 0$. Then $\mathcal{P}$ is a unit vector $U_m \in C$ perpendicular to the tangent line at $\eta(t_1)$.

iv) If m semiumbilic but not inflection point, $\mathcal{P}$ is the union of two *vectors $U_1, U_2 \in C$ such that U_i is perpendicular to the line which contains the origin and the extreme of the ellipse $\eta(t_i)$, $i = 1, 2$.*

v) If m is an inflection point, $\mathcal{P}$ is a unit vector $U_m \in C$ orthogonal to the line which contains the ellipse.

vi) If m is umbilic not plane, $\eta(t)||(d\eta)_t$, $\forall t \in T$. In this case $\mathcal{P}$ is a vector $U_m \in C$ perpendicular to the line which contain the origin and $\eta(t)$.

vii) If m is plane then $\mathcal{P}$ is C.

Proposition 4.2.8 *Let M be a surface immersed in $\mathbb{R}^{2+n}$, $n > 2$ and $m \in M$. In this case, depending on the degeneration of the curvature ellipse at the point $m \in M$ we have:*

i) If the the ellipse is not degenerated and suppose that the origin is inside the plane containing the ellipse, then $\mathcal{P}$ is the orthogonal vector subspace in N to the plane that contains the ellipse. Otherwise, $\mathcal{P}$ is the union of the unit vectors U_m such that $U_m \cdot \alpha(t, T) = 0$, $\forall t \in T$.

ii) If m is semiumbilic but not an inflection point, $\mathcal{P}$ is the orthogonal vector subspace in N to the plane that contains the segment and the origin.

iii) If m is an inflection point or umbilic not plane, $\mathcal{P}$ is the orthogonal vector subspace in N to the line that contains the origin and $\eta(t)$.

iv) If m is umbilic plane, $\mathcal{P}$ is C.

In this case, $\mathcal{P}_m(M)$ will be, in general, an algebraic hypersurface of degree 2.

Proof. The proof of 1-4 is trivial.

In the case of a surface $M \subset \mathbb{R}^{2+n}$, $\mathcal{P}_m(M)$ will be, in general, an algebraic hypersurface of degree 2 given by:

$$\begin{vmatrix} U_m \cdot \alpha_m(t_1,t_1) & U_m \cdot \alpha_m(t_1,t_2) \\ U_m \cdot \alpha_m(t_2,t_1) & U_m \cdot \alpha_m(t_2,t_2) \end{vmatrix} = 0.$$

This gives an equation of the form:

$$\sum_{i,j=1}^{n} a_{ij} u_i u_j = 0, \tag{4.2.1}$$

where $U_m = \sum_{i=1}^{n} u_i w_i$ and $a_{ij} \in \mathbb{R}$. In this case $\mathcal{P}_m(M)$ is a cone centered at the origin. □

4.3 1-critical points of the exponential tangent map

Let

$$SM = \{Y \in PTM : Y \cdot Y = 1\} \subset PTM$$

and $S_m M = SM \cap PT_m M$. Let $\Omega : TM \to \mathbb{R}^{k+n}$ be the exponential tangent map given by $\Omega(m; Y_m) = m + Y_m$, where $m \in M$ and $Y_m \in S_m M$.

Definition 4.3.1 *A vector $X \in T(SM)$ is a **1-critical point of Ω** iff it is non-zero and $d\Omega(X) = 0$. And we say that $Y \in SM$ admits a 1-critical point of Ω if there is a 1-critical point $X \in T_Y(SM)$.*

This is the diagram of those maps:

$$\begin{array}{ccc} T(SM) & \xrightarrow{d\Omega} & T\mathbb{R}^{k+n} \\ \downarrow{\scriptstyle \pi} & & \\ SM & \xrightarrow{\Omega} & \mathbb{R}^{k+n} \\ \downarrow{\scriptstyle \pi_T} & & \\ M & & \end{array}$$

Definition 4.3.2 *If $Y \in SM$ is such that admits a 1-critical point of Ω, then we say that it belongs to the **tangent set** of M. We shall denote by $\mathcal{T}(M)$ the set of these points.*

Proposition 4.3.3 *The point $Y_m \in S_m M$ admits a 1-critical point $X \in T_{(m;Y_m)}(SM)$ of Ω iff there exists $0 \neq v = d\pi_T(X) \in T_m M$ and a smooth vector field $\tilde{y} : M \to \mathbb{R}^{k+n}$ tangent to M such that $Y_m = y(0)$ and:*

a) $\alpha_m(Y_m, v) = 0$,

b) $v + \nabla_v^\top \tilde{y} = 0$.

In such case, $X = (v; -v)$.

Proof. Let $X \in T_{(m;Y_m)}(SM)$ be a 1-critical point of Ω. Consider a smooth curve $\gamma : I \to SM$ given by $\gamma(t) = (\beta(t); y(t))$, where I is an open neighborhood of $0 \in \mathbb{R}$ and $y(t) \in T_{\beta(t)}M$, $\forall t \in \mathbb{R}$. Suppose that

$(m; Y_m) = \gamma(0) = (\beta(0); y(0))$ and $X = \gamma'(0) = (\beta'(0); y'(0))$. Let us put $v = \beta'(0)$. Since X is a 1-critical point of Ω, we have:

$$d\Omega(X) = (\Omega \circ \gamma)'(0) = \beta'(0) + y'(0) = v + y'(0) = 0.$$

If $v = 0$, then we should have $y'(0) = 0$, that is $X = 0$, against the hypothesis. Therefore $v \neq 0$ and $y'(0) = -v \in T_mM$. Thus, we must find in what conditions upon v and Y_m we may have $y'(0) = -v$.

The fact that $\beta'(0) \neq 0$ implies that β is an immersion in a neighborhood of 0. Then there exists a neighborhood $\mathcal{U}$ of m and a vector tangent field $\tilde{y} : \mathcal{U} \subset M \to SM$ such that $\tilde{y} \circ \beta = \gamma$ and therefore $y'(0) = D_v\tilde{y}$.

Hence, taking the tangent and normal components, we conclude that $d\Omega(X) = v + D_v\tilde{y} = 0$ iff $v + \nabla_v^\top y = 0$, where $(D_v\tilde{y})^\top = \nabla_v^\top \tilde{y}$, and $\alpha_m(Y_m, v) = 0$.

Suppose that these conditions are satisfied, where $d\pi(X) = v$. Then, let $\beta : I \to M$ be such that $\beta(0) = m$, $\beta'(0) = v$, and let $y : I \to \mathbb{R}^{k+n}$ a smooth tangent vector field along β. Then

$$y'(0) = (D_v\tilde{y})^\perp + (D_v\tilde{y})^\top = -v.$$

Therefore the tangent of the curve $(\beta(t); y(t))$ at $t = 0$ is equal to $X = (v; -v)$, so that $X \neq 0$ and $d\Omega(X) = 0$. Hence X is a 1-critical point of Ω. □

In [3] the author gives the following definition and Theorem.

Definition 4.3.4 *Let M be a k-dimensional submanifold immersed in $\mathbb{R}^{k+n}$. Two vectors $X_m, Y_m \in S_mM$ are called conjugate if $u \cdot$*

$\alpha_m(X_m, Y_m) = 0, \forall u \in N_m M$. *Define the set* CP *as the set* $CP = \{(m, X_m, Y_m) \in M \times S_m M \times S_m M : X_m, Y_m$ *are conjugate*$\}$.

Theorem 4.3.5 *([3]) For a generic immersion, the set* CP *is an embedded manifold in* $M \times TM \times TM$ *of dimension* $3k - n - 2$.

In general, if $n < 3k - 2$, then every tangent vector is conjugate to at least one other vector. If $n > k$, the set of conjugate vectors becomes more sparse, until we get to the point where conjugate vectors are isolated in general ($n = 3k - 2$), there are no conjugate directions in general ($n > 3k - 2$). For us, the most interesting structure for the conjugate vectors occurs when $k = n$.

In general, we have the following proposition.

Proposition 4.3.6 *([3]) Let* m *be a point on a* n*-dimensional manifold immersed in* $\mathbb{R}^{2n}$. *Let* $\{w_i\}_{i=1}^n$ *be a basis of* $N_m M$ *and* $\{t_i\}_{i=1}^k$ *a basis of* $T_m M$. *A vector* $Y_m \in S_m M$ *is a conjugate vector iff det* $(w_i \cdot \alpha_m(Y_m, t_i)) = 0$ *and iff there is a 1-critical point* $U_m \in C_m M$ *of* ψ *such that* $U_m \cdot \alpha_m(Y_m, \cdot) = 0$.

This means that, in the particular case of a surface immersed in $\mathbb{R}^4$ the directions of $S_m M$ that admits a 1-critical point of Ω are the asymptotic directions and the $U_m \in C_m M$ are the binormal directions defined in [14].

4.4 Definition of r-critical point

In this section we recall first the well known notion of r-tangent bundle and define r_δ-critical points.

Let M be a smooth manifold. The r-tangent bundle $\pi^r : T^rM \to M$, for any non-negative integer r, is defined recursively as follows: the 0-tangent bundle is $\mathrm{id} : M \to M$ and the 1-tangent bundle is the tangent bundle $\pi : TM \to M$. Suppose that we have defined $\pi^r : T^rM \to M$ and also a bundle $\pi^r_{r-1} : T^rM \to T^{r-1}M$ such that $\pi^r = \pi^{r-1} \circ \pi^r_{r-1}$, where $\pi^1_0 = \pi$. Then we define the next total space

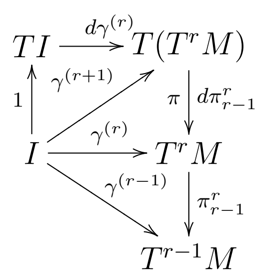
$$T^{r+1}M = \{X \in T(T^rM) \,:\, \pi(X) = d\pi^r_{r-1}(X)\}$$

and the maps $\pi^{r+1} = \pi^r \circ \pi$ and $\pi^{r+1}_r(X) = \pi(X) = d\pi^r_{r-1}(X)$.

Lemma 4.4.1 *T^rM is the bundle $J^r_0(\mathbb{R}, M)$ of r-jets of curves in M with source* 0.

Proof. Assume that for each integer $i = 1, \dots, r$ we have a smooth map $\gamma^{(i)}$ from an open neighborhood I of $0 \in \mathbb{R}$ to T^iM, and that for each $i = 2, \dots, r$ we have $\pi^i_{i-1} \circ \gamma^{(i)} = \gamma^{(i-1)}$.

Then, define $\gamma^{(r+1)} = \gamma^{(r)\prime} = d\gamma^{(r)} \circ \mathbf{1} : I \to T(T^rM)$, where $\mathbf{1} : \mathbb{R} \to T\mathbb{R}$ is the unit vector field ∂_t, where t is the canonical coordinate in $\mathbb{R}$. Then, the follows diagram is commutative:



Therefore

$$\pi(\gamma^{(r+1)}) = (\pi \circ d\gamma^{(r)} \circ \mathbf{1}) = \gamma^{(r)}$$
$$d\pi^r_{r-1}(\gamma^{(r+1)}) = d(\pi^r_{r-1} \circ \gamma^{(r)}) \circ \mathbf{1} = d\gamma^{(r-1)} \circ \mathbf{1} = \gamma^{(r-1)\prime} = \gamma^{(r)}.$$

Hence, $\gamma^{(r+1)} \in T^{r+1}M$.

In the other way, let $X^i := \pi_i^{r+1}(X^{r+1})$. By definition, there exists a curve $\beta^r : I \to T^rM$ such that $X^{r+1} = (\beta^r)'$. We know that $\pi(X^{r+1}) = X^r$, then, $X^r = d\pi^r_{r-1}((\beta^r)') = (\pi^r_{r-1} \circ \beta^r)'$. Let $\beta^{r-1} = \pi^r_{r-1} \circ \beta^r$, then $X^{r+1} = \beta^{(r+1)}$, where $\beta = \pi \circ \beta^1$.

Notice that $\dim T^rM = \dim J^r_0(\mathbb{R}, M) = (r+1)k$, see [8]. Given a chart $\varphi : U \to M$ of M, we can define the chart $\varphi^r : T^rU \to \mathbb{R}^{(r+1)k}$, as follows: let $\beta : I \to M$ whose components in the chart φ are $(\beta_1, \ldots, \beta_k)$. Then, $\varphi^r(\beta^{(r)})$ is given by:

$$\big(\beta_1(0), \ldots \beta_k(0);\ {\beta_1}'(0), \ldots {\beta_k}'(0); \ldots; {\beta_1}^{(r)}(0), \ldots, {\beta_k}^{(r)}(0)\big),$$

which are the components of the induced chart by φ for $J^r_0(\mathbb{R}, M)$. □

If $f : M \to P$ is a smooth map between manifolds and $\beta : I \to M$ is a smooth curve in M, then it defines a smooth map

$$f^{(r)} : T^rM \to T^rP,$$

where $f^{(r)}(j^r(\beta)) = j^r(f \circ \beta)$, and $j^r(\beta)$ is the r-jet of β in 0, i.e. we can write: $j^r(\beta) = (\beta(0), \beta^{(1)}(0), \ldots \beta^{(r)}(0))$. Also if $g : P \to Q$ is another smooth map, then $(g \circ f)^{(r)} = g^{(r)} \circ f^{(r)}$.

Definition 4.4.2 *We define*

$$V(T^rM) = \{X \in T(T^rM) \,:\, d\pi^r_{r-1}(X) = 0\}$$

as the ***vertical bundle*** *over T^rM, which is a vector bundle whose fiber upon $X^r \in T^rM$ is denoted $V_{X^r}(T^rM)$.*

If $A \in T^rM$, we shall denote by $T_A^{r+1}M$ the fiber of π_r^{r+1} over A.

Lemma 4.4.3 *If $A \in T^rM$, then $T_A^{r+1}M$ is a real affine space of dimension k whose associated vector space is $V_A(T^rM)$.*

Proof. Let $X, Y \in T_A^{r+1}M$, in this case, we have that

$$\begin{aligned} d\pi_{r-1}^r(X - Y) &= d\pi_{r-1}^r(X) - d\pi_{r-1}^r(Y) \\ &= \pi(X) - \pi(Y) = A - A \\ &= 0. \end{aligned}$$

Hence, $X - Y \in V_A T^rM$.

In the other way, let $V \in V_A(T^rM)$, then $d\pi_{r-1}^r(X+V) = d\pi_{r-1}^r(X) + d\pi_{r-1}^r(V) = A + 0 = A$. Hence, $X + V \in T_A^{r+1}M$.

Then, we can write $T_A^{r+1}M = X + V_A(T^rM)$. Therefore the fibre $T_A^{r+1}M$ is an affine space. Its dimension is that of $V_A(T^rM)$, that is k. □

If $0 = A \in T^rM$, then $\forall X^{r+1} \in T_A^{r+1}M$ we have that $d\pi_{r-1}^r(X^{r+1}) = \pi(X^{r+1}) = A = 0$. In other words, $T_0^{r+1}M = V_0(T^rM)$. Then, in this case $T_0^{r+1}M$ is a vector space of $T_0(T^rM)$. This implies that there exists a unique well-defined zero in $T_A^{r+1}M$ when $A = 0$. Let $f : M \to P$ be an affine map between manifolds and $A \in T^rM$, then the map $f^{(r+1)} : T_A^{r+1}M \to T_{f^{(r)}(A)}^{r+1}P$ is an smooth map. If $m \in M$, we shall denote by $c_m : \mathbb{R} \to M$ the constant curve, that is $c_m(t) = m$, for all $t \in \mathbb{R}$. Then we define $O^r : M \to T^rM$ as the smooth section defined by $O_m^r = c_m^{(r)}$. It is clear that $\pi_j^r \circ O^r = O^j$.

Definition 4.4.4 *Let $f : M \to P$ be a smooth map between manifolds. If $X^1 \in TM$ we say that X^1 is a* 1-critical point of *f if $X^1 \neq 0$ and $f^{(1)}(X^1) = O^1_{f(m)}$. If $X^r \in T^rM$ and $r > 1$ we say that X^r is an r*-critical point of *f if $\pi^r_1(X^r)$ is a 1-critical point of f and $f^{(r)}(X^r) = O^r_{f(m)}$, where $m = \pi^r_0(X^r)$. By the commutativity $\pi^r_j \circ f^{(r)} = f^{(j)} \circ \pi^r_j$, we see that if X^r is a r-critical point of f and $1 \leq j < r$, then $X^j := \pi^r_j(X^r)$ is a j-critical point of f.*

Notice that, if $\beta : I \to M$ is a smooth curve in M, where I is an open neighborhood of $0 \in \mathbb{R}$, such that $X^r = j^r(\beta)$ and X^r is an *r-critical point of* $f : M \to P$, then $(f \circ \beta)^{(i)}(0) = 0$, $\forall i = 1, 2, \ldots, r$.

4.4.1 Probes and r-critical points

The following definition is due to I. Porteous, ([26], [27]).

Definition 4.4.5 *Let $f : (\mathbb{R}^k, m) \to (\mathbb{R}^q, f(m))$ be a smooth map-germ. Then a d-probe of f or a probe of index d, is an immersive smooth map-germ $\sigma : (\mathbb{R}^d, 0) \to (\mathbb{R}^k, m)$ such that:*

$$(f \circ \sigma)_1(0) = d(f \circ \sigma)(0) = 0.$$

Asserting the existence of a probe of index d is equivalent to asserting that the dimension of the kernel of the first differential of f at m is greater than or equal to d. The map-germ is said to be Σ^d if it has a d-probe but no $(d+1)-$probe, where necessarily $d \leq k$. Clearly f is Σ^d if and only if the dimension of the kernel at df_m is equal to d. Moreover, the probe σ may be taken as the inclusion of the kernel of df_m in $\mathbb{R}^k$.

Suppose now that $\varphi : (\mathbb{R}^j, 0) \to (\mathbb{R}^d, 0)$ is a j-probe of $(f \circ \sigma)_1 : (\mathbb{R}^d, 0) \to (L(\mathbb{R}^d, \mathbb{R}^q), 0)$ such that

$$((f \circ \sigma)_1 \circ \varphi)_1(0) = 0,$$

where $L(\mathbb{R}^d, \mathbb{R}^q)$ denotes the set of linear maps from $\mathbb{R}^d$ to $\mathbb{R}^q$. Then (σ, φ) is said to be a (d, j)-probe of f, f being said to be $\Sigma^{d,j}$ if it has a (d, j)-probe, but no $(d + 1)$-probe and no $(d, j + 1)$-probe. When $f : (\mathbb{R}^k, m) \to (\mathbb{R}^q, f(m))$ is $\Sigma^{d,d}$, it has a d-probe $\sigma : (\mathbb{R}^d, 0) \to (\mathbb{R}^k, m)$ such that:

$$(f \circ \sigma)_2(0) := ((f \circ \sigma)_1 \circ Id)_1(0) = 0,$$

being Id the identity map-germ on $\mathbb{R}$ at 0.

This definitions extend indefinitely in the obvious way.

Let us see the equivalence between the $\Sigma^{d,j}$ and the critical points. Suppose that $f : (\mathbb{R}^k, m) \to (\mathbb{R}^q, f(m))$ is of type Σ^d. In this case we have seen that there exits a d-probe $\sigma : (\mathbb{R}^d, 0) \to (\mathbb{R}^k, m)$, such that $d(f \circ \sigma)(0) = 0$. Let $\beta : (\mathbb{R}, 0) \to (\mathbb{R}^d, 0)$ a curve defined in σ, we put $\gamma := \sigma \circ \beta$. Then:

$$d(f \circ \sigma)_{(0)}(\beta'(0)) = d(f \circ \sigma \circ \beta)_{(0)}(0) = d(f \circ \gamma)_{(m)}(0) = (f \circ \gamma)'(0) = 0.$$

Then f admits a 1-critical point and $\dim ker(df_m) = d$.

Suppose now that f admits a 1-critical point and $\dim ker(df_m) = d$. In this case, there exists a d-probe $\sigma : (\mathbb{R}^d, 0) \to (\mathbb{R}^k, m)$ which may be taken as the inclusion of the kernel of df_m in $\mathbb{R}^k$ and no $(d+1)$-probe such that:

$$(f \circ \sigma)_1(0) = d(f \circ \sigma)(0) = 0.$$

Suppose now that $f : (\mathbb{R}^k, m) \to (\mathbb{R}^q, f(m))$ is of type $\Sigma^{d,j}$. Then there exists a d-probe $\varphi : (\mathbb{R}^j, 0) \to (\mathbb{R}^d, 0)$ such that $d(d(f \circ \sigma) \circ \varphi)(0) = 0$. Let $\beta : (\mathbb{R}, 0) \to (\mathbb{R}^j, 0)$ such that $(d\sigma \circ \varphi \circ \beta)(0) \in T_m\mathbb{R}^k$. Then there exists $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^k, m)$ such that $(d\sigma \circ \varphi \circ \beta)(0) = \gamma'(0)$. Then:

$$d(d(f \circ \sigma) \circ \varphi)_{(0)}(\beta'(0)) = d(df \circ d\gamma)_{(m)}(0) = (f \circ \gamma)''(0) = 0.$$

Therefore, f admits a 2-critical point.

Suppose now that f admits a 2-critical point such that $\dim(kerdf_m) = d$ and $\dim\{v = \gamma'(0) \in T_m\mathbb{R}^k / (f \circ \gamma)''(0) = 0\} = j$, where $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^k, m)$. Then it is trivial the existence of a d-probe $\sigma : (\mathbb{R}^d, 0) \to (\mathbb{R}^k, a)$ such that $(f \circ \sigma)_1(0) = d(f \circ \sigma)(0) = 0$ and the existence of a j-probe $\varphi : (\mathbb{R}^j, 0) \to (\mathbb{R}^d, 0)$ such that $d(d(f \circ \sigma) \circ \varphi)(0) = 0$.

In [4] the authors prove the equivalence between the Boardman symbols and the probe structure for $\Sigma^{d,j}$.

Chapter 5

Analysis of the critical points

In the following chapter we analyze the critical points of the normal map, the generalized Gauss map.

In the next lemma, by *free* we shall mean that if $t \in I$, we may choose the field X or the field u, in such a way that the value at t of the object that is said to be free may have any value specified beforehand, of course in the respective subspace $T_m M$ or $N_m M$, where $m \in M$.

Lemma 5.0.6 *Let $\beta : I \to M$ be a smooth curve, where I is an open interval of $\mathbb{R}$ and M is a k dimensional submanifold immersed in $\mathbb{R}^{k+n}$, and suppose that $X \circ \beta : I \to TM$ and $u \circ \beta : I \to NM$ are smooth vector fields along β. The following conditions are satisfied:*

a) $\beta'^{\perp} = 0, \quad \beta'^{\top}$ *is free.*

b) $\beta''^{\perp} = \alpha\big(\beta', \beta'\big), \quad \beta''^{\top}$ *is free.*

c) $\beta'''^{\perp} = (\nabla_{\beta'}\alpha)\big(\beta', \beta'\big) + 3\alpha\big(\beta', \beta''^{\top}\big), \quad \beta'''^{\top}$ *is free.*

d) $u'^{\top} = \mathcal{A}(u, \beta'), \quad u'^{\perp}$ *is free.*

e) $u''^{\top} = (\nabla_{\beta'}\mathcal{A})(u, \beta') + 2\mathcal{A}(u'^{\perp}, \beta') + \mathcal{A}(u, \beta''^{\top}), \quad u''^{\perp}$ *is free.*

f) $u'''^{\perp}$ *is free,*

$$\begin{aligned} u'''^{\top} =& (\nabla^2_{\beta'}\mathcal{A})(\beta', u, \beta') + (\nabla_{\beta''^{\top}}\mathcal{A})(u, \beta') + 3(\nabla_{\beta'}\mathcal{A})(u'^{\perp}, \beta') \\ &+ 2(\nabla_{\beta'}\mathcal{A})(u, \beta''^{\top}) + \mathcal{A}(u''^{\perp}, \beta') + 2\mathcal{A}(\nabla_{\beta'}(u'^{\perp}), \beta) \\ &+ 3\mathcal{A}(u'^{\perp}, \beta''^{\top}) + \mathcal{A}(u, \nabla_{\beta'}(\beta''^{\top})). \end{aligned}$$

g) $X'^{\perp} = \alpha(\beta', X), \quad X'^{\top}$ *is free.*

h) $X''^{\perp} = (\nabla_{\beta'}\alpha)(\beta', X) + \alpha(\beta''^{\top}, X) + 2\alpha(\beta', X'^{\top}), \quad X''^{\top}$ *is free,*

i) $X'''^{\top}$ *is free,*

$$\begin{aligned} X'''^{\perp} =& (\nabla^2_{\beta'}\alpha)(\beta', \beta', X) + (\nabla_{\beta''^{\top}}\alpha)(\beta', X) \\ &+ 2(\nabla_{\beta'}\alpha)(\nabla_{\beta'}\beta', X) + 3(\nabla_{\beta'}\alpha)(\beta', \nabla_{\beta'}X) \\ &+ \alpha(\nabla_{\beta'}(\beta''^{\top}), X) + 3\alpha(\nabla_{\beta'}\beta', \nabla_{\beta'}X) + 2\alpha(\beta', \nabla_{\beta'}(X'^{\top})) \\ &+ \alpha(\beta', X''^{\top}). \end{aligned}$$

where, for brevity we write α *instead of* $\alpha \circ \beta$ *and* $\mathcal{A}$ *instead of* $\mathcal{A} \circ \beta$ *which means the evaluation of* α *and* $\mathcal{A}$ *at* $\beta(t)$.

Proof. Obviously $\beta'(t)^{\perp} = 0$ and $\beta'(t)^{\top} = \beta'(t)$ is free because TM is generated by the tangents to smooth curves passing by $\beta(t)$, and this is a). Also $u'^{\top} = (D_{\beta'}u)^{\top} = \mathcal{A}(u, \beta')$ by definition of $\mathcal{A}$ and this is the first part of d). Also $X'^{\perp} = (D_{\beta'}X)^{\perp} = \alpha(\beta', X)$, by definition, and this is the first part of g). So, we pass to the next derivative. We have:

$$u \cdot \beta'' = u \cdot D_{\beta'}\beta' = u \cdot \alpha(\beta', \beta').$$

Hence b) is:

$$\beta''^{\perp} = \alpha\big(\beta'^{\top}, \beta'^{\top}\big).$$

Suppose now that u is parallel, that is $u'^{\perp} = 0$. We take the derivative of the left-hand side of the equation $u \cdot \beta'' = u \cdot \alpha(\beta', \beta')$ and get:

$$(u \cdot \beta'')' = u' \cdot \beta''^{\top} + u \cdot \beta''' = -u \cdot \alpha(\beta', \beta''^{\top}) + u \cdot \beta'''.$$

And now we take the derivative of the right-hand side of same equation, and get

$$(\nabla_{\beta'} u) \cdot \alpha\big(\beta', \beta'\big) + u \cdot (\nabla_{\beta'}\alpha)\big(\beta', \beta'\big) + 2u \cdot \alpha\big(\beta', \nabla_{\beta'}\beta'\big).$$

Then

$$\begin{aligned} -u \cdot \alpha(\beta', \beta''^{\top}) + u \cdot \beta''' =& (\nabla_{\beta'} u) \cdot \alpha\big(\beta', \beta'\big) + u \cdot (\nabla_{\beta'}\alpha)\big(\beta', \beta'\big) \\ &+ 2u \cdot \alpha\big(\beta', \nabla_{\beta'}\beta'\big). \end{aligned}$$

Since $u'^{\perp} = 0$ and $u \cdot \alpha\big(\beta', \nabla_{\beta'}\beta'\big) = u \cdot \alpha\big(\beta', \beta''^{\top}\big)$, we obtain finally

$$u \cdot \beta''' = u \cdot (\nabla_{\beta'}\alpha)\big(\beta', \beta'\big) + 3u \cdot \alpha\big(\beta', \beta''^{\top}\big).$$

In other words, c) is:

$$\beta'''^{\perp} = (\nabla_{\beta'}\alpha)\big(\beta', \beta'\big) + 3\alpha\big(\beta', \beta''^{\top}\big).$$

Let X be a parallel tangent vector field along β, so that $X'^{\top} = 0$. We have $X \cdot u' = X \cdot \mathcal{A}(u, \beta')$. Thus we take the derivative of both sides of this equation. The left-hand side gives:

$$\begin{aligned}(X \cdot u')' &= X' \cdot u' + X \cdot u'' = \alpha(\beta', X) \cdot u'^{\perp} + X \cdot u'' \\ &= -X \cdot \mathcal{A}(u'^{\perp}, \beta') + X \cdot u''.\end{aligned}$$

The right-hand side gives

$$X \cdot \Big((\nabla_{\beta'} \mathcal{A})(u, \beta') + \mathcal{A}(u'^{\perp}, \beta') + \mathcal{A}(u, \beta''^{\top}) \Big).$$

Hence, we have that

$$u''^{\top} = (\nabla_{\beta'} \mathcal{A})(u, \beta') + 2\mathcal{A}(u'^{\perp}, \beta') + \mathcal{A}(u, \beta''^{\top}).$$

Now, taking derivatives on both sides of the equation:

$$X \cdot u'' = X \cdot \left((\nabla_{\beta'} \mathcal{A})(u, \beta') + 2\mathcal{A}(\nabla_{\beta'} u, \beta') + \mathcal{A}(u, \nabla_{\beta'} \beta') \right),$$

the left-hand side gives:

$$\begin{aligned}(X \cdot u'')' =& X' \cdot u'' + X \cdot u''' = \alpha(X, \beta') \cdot \nabla_{\beta'} u' + X \cdot u''' \\ =& - X \cdot \mathcal{A}(u''^{\perp}, \beta') + X \cdot u'''.\end{aligned}$$

Considering that $(\nabla_{\beta'} \mathcal{A})(u, \beta') = (\nabla \mathcal{A})(\beta', u, \beta')$, the right-hand side gives:

$$\begin{aligned}X \cdot (&(\nabla^2_{\beta'} \mathcal{A})(\beta', u, \beta') + (\nabla \mathcal{A})(\nabla_{\beta'} \beta', u, \beta') + (\nabla \mathcal{A})(\beta', \nabla_{\beta'} u, \beta') \\ &+ (\nabla \mathcal{A})(\beta', u, \nabla_{\beta'} \beta') + 2(\nabla_{\beta'} \mathcal{A})(\nabla_{\beta'} u, \beta') + 2\mathcal{A}(\nabla_{\beta'}(u'^{\perp}), \beta') \\ &+ 2\mathcal{A}(\nabla_{\beta'} u, \nabla_{\beta'} \beta') + (\nabla_{\beta'} \mathcal{A})(u, \nabla_{\beta'} \beta') + \mathcal{A}(\nabla_{\beta'} u, \nabla_{\beta'} \beta') \\ &+ \mathcal{A}(u, \nabla_{\beta'}(\beta''^{\top}))).\end{aligned}$$

Finally, we obtain:

$$\begin{aligned} u'''^{\top} =&(\nabla^2_{\beta'}\mathcal{A})(\beta', u, \beta') + (\nabla_{\beta''^{\top}}\mathcal{A})(u, \beta') + 3(\nabla_{\beta'}\mathcal{A})(\nabla_{\beta'}u, \beta') \\ &+ 2(\nabla_{\beta'}\mathcal{A})(u, \nabla_{\beta'}\beta') + 2\mathcal{A}(\nabla_{\beta'}(u'^{\perp}), \beta') + \mathcal{A}(u''^{\perp}, \beta') \\ &+ 3\mathcal{A}(\nabla_{\beta'}u, \nabla_{\beta'}\beta') + \mathcal{A}(u, \nabla_{\beta'}(\beta''^{\top})) \end{aligned}$$

and $u'''^{\perp}$ is free.

Suppose now that u is parallel, then $u'^{\perp} = 0$. We know that $u \cdot X' = u \cdot \alpha(\beta', X)$. Then taking derivatives of the left-hand side, we get:

$$(u \cdot X')' = u' \cdot X'^{\top} + u \cdot X'' = -u \cdot \alpha(\beta', X'^{\top}) + u \cdot X''.$$

We take now the derivative of $u \cdot \alpha(\beta', X)$ and we obtain

$$u' \cdot \alpha(\beta', X) + u \cdot \Big((\nabla_{\beta'}\alpha)(\beta', X) + \alpha(\beta''^{\top}, X) + \alpha(\beta', X'^{\top})\Big).$$

Hence, since $u'^{\perp} = 0$ we get:

$$X''^{\perp} = (\nabla_{\beta'}\alpha)(\beta', X) + \alpha(\beta''^{\top}, X) + 2\alpha(\beta', X'^{\top}),$$

and $X''^{\top}$ is free.

Finally, taking derivatives in the equation

$$u \cdot X'' = u \cdot ((\nabla_{\beta'}\alpha)(\beta', X) + \alpha(\nabla_{\beta'}\beta', X) + 2\alpha(\beta', \nabla_{\beta'}X)),$$

the left-hand side gives:

$$\begin{aligned} (u \cdot X'')' =&u' \cdot X'' + u \cdot X''' = \mathcal{A}(u, \beta') \cdot \nabla_{\beta'}X' + u \cdot X''' \\ =& - u \cdot \alpha(\beta', X''^{\top}) + u \cdot X'''. \end{aligned}$$

And the right-hand side:

$$\begin{aligned}
&u \cdot ((\nabla^2_{\beta'}\alpha)(\beta',\beta',X) + (\nabla\alpha)(\nabla_{\beta'}\beta',\beta',X) + (\nabla\alpha)(\beta',\nabla_{\beta'}\beta',X) \\
&+ (\nabla\alpha)(\beta',\beta',\nabla_{\beta'}X) + (\nabla_{\beta'}\alpha)(\nabla_{\beta'}\beta',X) + \alpha(\nabla_{\beta'}(\beta''^{\top}),X) \\
&+ \alpha(\nabla_{\beta'}\beta',\nabla_{\beta'}X) + 2(\nabla_{\beta'}\alpha)(\beta',\nabla_{\beta'}X) + 2\alpha(\nabla_{\beta'}\beta',\nabla_{\beta'}X) \\
&+ 2\alpha(\beta',\nabla_{\beta'}(X'^{\top}))).
\end{aligned}$$

Finally:

$$\begin{aligned}
X'''^{\perp} =&(\nabla^2_{\beta'}\alpha)(\beta',\beta',X) + (\nabla_{\beta''^{\top}}\alpha)(\beta',X) + 2(\nabla_{\beta'}\alpha)(\nabla_{\beta'}\beta',X) \\
&+ 3(\nabla_{\beta'}\alpha)(\beta',\nabla_{\beta'}X) + \alpha(\nabla_{\beta'}(\beta''^{\top}),X) + 3\alpha(\nabla_{\beta'}\beta',\nabla_{\beta'}X) \\
&+ 2\alpha(\beta',\nabla_{\beta'}(X'^{\top})) + \alpha(\beta',X''^{\top}).
\end{aligned}$$

□

The conclusions about being free all tangent components of the derivatives of β are due to the following facts. The r-jet of the curve β defines, at each $t \in I$, a point in T^rM and each point in T^rM may be obtained in this manner. But if $\iota : M \to \mathbb{R}^{k+n}$ is the inclusion map, which is a regular immersion (at least locally), then it defines the inclusion maps $\iota^{(r)} : T^rM \to T^r\mathbb{R}^{k+n}$ that are locally regular immersions.

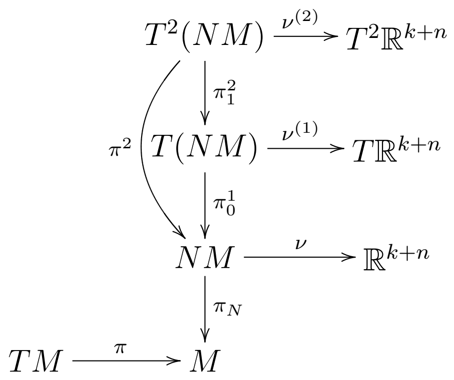
The dimension of T^rM is $(r+1)k$ and the dimension of $T^r\mathbb{R}^{k+n}$ is $(r+1)(k+n)$. Now, let $q^{(r-1)} \in T^{r-1}M$ be given by the jet $q^{(r-1)} = (\beta(0), \beta'(0), \dots, \beta^{(r-1)}(0))$. In order to obtain any point in the fiber in T^rM over $q^{(r-1)}$ we need to get a jet of the form $q^{(r)} = (\beta(0), \dots, \beta^{(r-1)}(0), \beta^{(r)}(0))$, that is by the choice of $\beta^{(r)}(0)$, that may be decomposed into its tangent and its normal part. Since the normal part of $\beta^{(r)}(0)$ is common to all the points on that fiber as we have proved, we shall obtain a fiber with the desired dimension, that is k, by determining freely the other summand of the decomposition, that is the tangent part of $\beta^{(r)}(0)$. The argument for u and X is similar.

In the following we shall suppress the subindex m, that means evaluation at $m \in M$.

5.1 Analysis of the critical points of ν

Now, we will study the 2-critical points of the normal map $\nu : NM \to \mathbb{R}^{k+n}$. We use the following notations for the different bundles that we consider: $\pi : TM \to M$, $\pi_N : NM \to M$, $\pi_0^1 : T(NM) \to NM$, $\pi_1^2 : T^2(NM) \to T(NM)$ and $\pi^2 : T^2(NM) \to NM$.

We may see the relations among those maps by means of the following commutative diagram:



Definition 5.1.1 *Let $v \in T_mM$ and $U \in N_mM$. In the following we will say that* there is a 2-critical point of ν over (v, U), or that (v, U) admits a 2-critical point of ν *if there is some 2-critical point $X^2 \in T^2(NM)$ of ν such that $U = \pi^2(X^2)$ and $v = (\pi_1^2 \circ \pi_N^{(2)})(X^2)$. In other terms, if $X^2 = \gamma''$, being $\gamma : I \to NM$ a smooth curve, then $v = (\pi_N \circ \gamma)'(0)$ and $U = \gamma(0)$.*

In the same manner we say that $v \in T_mM$ admits a 2-critical point of ν *if there is $U \in N_mM$ such that (v, U) admits a 2-critical point of ν.*

The following theorem characterizes the properties that must have a pair (v, U) as above in order to admit a 2-critical point of ν.

Theorem 5.1.2 *Let $m \in M$, $v \in T_mM$ and $U \in N_mM$. Then, (v, U) admits a 2-critical point iff the following conditions are satisfied:*

a) $v \neq 0$ and $g(v, .) - U \cdot \alpha(v, .) = 0$, i.e. U belongs to the focal set over v.

b) $U \cdot (\nabla_v \alpha)(v, v) = 0$.

c) Let $(t_1, \dots, t_k)$ be an orthonormal basis of T_mM such that t_1 and v are parallel. Then the following linear system, whose unknowns are the components of $x = x^2 t_2 + \cdots + x^k t_k$, has a solution:

$$g(x, t_j) - U \cdot \alpha(x, t_j) = U \cdot (\nabla_v \alpha)(v, t_j), \ j = 2, \dots, k.$$

Proof. Suppose that $X^2 \in T^2(NM)$ can be written as $X^2 = \gamma''(0) = (\beta''(0); u''(0))$, where $\gamma : I \to NM$ is a smooth curve given by $\gamma(t) = (\beta(t); u(t))$, where $\beta := \pi_N \circ \gamma$, I is an open neighborhood of $0 \in \mathbb{R}$ and $u(t) \in N_{\beta(t)}M \subset \mathbb{R}^{k+n}$, $\forall t \in I$.

We have that $\pi_1^2 \circ \gamma'' = \gamma'$ and $\pi_0^1 \circ \gamma' = \gamma$. Let $X := \pi_1^2(X^2)$, where we suppose can be written as

$$X = \gamma'(0) = (\beta'(0); u'(0)).$$

Then we shall denote for brevity:

$$m = \beta(0), \ v = \beta'(0), \ a = \beta''(0), \ U = u(0), \ V = u'(0), \ A = u''(0).$$

Since X^2 is a 2-critical point of ν, then X must be 1-critical, that is

$$\nu^{(1)}(X) = (\nu \circ \gamma)'(0) = (\beta + u)'(0) = v + V = 0.$$

In Proposition 4.1.3 we proved that, in this case there exists a neighborhood $\mathcal{U}$ of $m \in M$ and a section $u : \mathcal{U} \subset M \to NM$ such that $\tilde{u} \circ \beta = \gamma$ and therefore $u'(0) = D_v u$.

Hence, taking the normal and tangent components, we conclude that $\nu^{(1)}(X) = 0$ iff

1.1) $v + \mathcal{A}(U, v) = 0$,

or, equivalently $v \cdot x - U \cdot \alpha(v, x) = 0, \ \forall x \in T_m M$, which is condition a) and

1.2) $u'(0)^{\perp} = 0$, then $\nabla_v u = 0$.

Also, since X^2 must be 2-critical, we must have

$$\nu^{(2)}(X^2) = (\nu \circ \gamma)''(0) = \beta''(0) + u''(0) = a + A = 0.$$

According with the results of the Lemma 5.0.6 and considering that $\nabla_v u = 0$, the tangent part of $a + A = 0$ is:

$$a^{\top} + (\nabla_v \mathcal{A})(U, v) + \mathcal{A}(U, a^{\top}) = 0,$$

which may be written equivalently as

$$g(a^{\top}, \cdot) - U \cdot (\nabla_v \alpha)(v, \cdot) - U \cdot \alpha(a^{\top}, \cdot) = 0. \qquad (5.1.1)$$

Let us put $a^{\top} = pv + b$, with $b \cdot v = 0$ and $p \in \mathbb{R}$. Then (5.1.1) reads

$$g(b, .) - U \cdot \alpha(b, \cdot) - U \cdot (\nabla_v \alpha)(v, \cdot) = 0,$$

by condition a). If this 1-form acts over v, and we consider condition a), we obtain:

$$b) \qquad U \cdot (\nabla_v \alpha)(v,v) = 0.$$

And if that 1-form acts upon the vectors $t_j,\ j = 2, \ldots, k$, we get condition c).

Finally, we have that $a^\perp = \beta''^\perp(0) = \alpha(v,v)$, hence the normal part of $\beta''(0) + u''(0) = 0$ is:

$$2.2) \quad \alpha(v,v) + A^\perp = 0.$$

which may be satisfied always, because $A^\perp$ is free by Lemma 5.0.6. Hence, our claim is true.

Suppose now that the system a), b), c) is satisfied by v, U and $x = x^2 t_2 + \cdots + x^k t_k$. It is enough to prove that we may choose β and u such that $\beta'(0) = v,\ \beta''(0)^\top = x,\ u(0) = U$, and that 1.2) and 2.2) are satisfied. The conditions on β can always be satisfied because $v \in T_m M$ and the tangent part of $\beta''(0)$ may be arbitrary. As for u, we take first a parallel orthonormal reference of NM along β, $(u_1, \ldots, u_n)$. Thus we can write $u(t) = \sum_i p^i(t) u_i(t)$ with the conditions $\sum_i p^i(0) u_i(0) = u(0) = U$ and $p^{i\prime}(0) = 0, i = 1, \ldots n$. Then $u'(0) = \sum_i p^i(0) u_i'(0)$. Since the u_i are parallel, this implies that $u'(0)^\perp = 0$, which is condition 1.2).

Condition 2.2) can be written as $u''(0)^\perp = -\alpha(v,v)$ or equivalently as $u_i(0) \cdot u''(0) = -u_i(0) \cdot \alpha(v,v),\ i = 1, \ldots, n$. Since we have

$$u''(0) = \sum_i (p^{i\prime\prime}(0) u_i(0) + p^i(0) u_i''(0)),$$

this condition means that 2.2) is satisfied by choosing

$$p^{i''}(0) = -u_i(0) \cdot \alpha(v, v) - \sum_j p^j(0) u_i(0) \cdot u_j''(0).$$

Hence, our claim is true. □

In the following subsections we want to compute the equation that must satisfy a unit vector $v \in T_m M$ where there exists a normal vector $U \in N_m M$ such that (v, U) admits a 2-critical point of ν. Since it is obvious from the preceding Theorem that there is a 2-critical point over (v, U) with $v \neq 0$ iff there is one over $(v/|v|, U)$ in the following subsections we may assume that $g(v, v) = 1$.

5.1.1 2-Critical points of ν for curves immersed in $\mathbb{R}^{1+n}$

Let $\beta : I \to \mathbb{R}^{1+n}$ be a curve parameterized by arc-length, where $m = \beta(0)$ is not plane, $U \in N_m\beta$ and $v = \beta'(0) \in T_m\beta$. Now, (v, U) admits a 2-critical point when the following conditions are satisfied:

a) $g(v, \cdot) - U \cdot \alpha(v, \cdot) = 0$,

b) $U \cdot (\nabla_v \alpha)(v, v) = 0$.

Since $\dim(T_m\beta) = 1$, the first condition is equivalent to apply the expression over v and we obtain:

$$1 - U \cdot N_1 k_1 = 0 \to U = \frac{1}{k_1} N_1 + \sum_{i=2}^{n} \lambda_i N_i,$$

where k_1 is the curvature of β at m and $\{N_i\}_{i=1}^n$ is an orthonormal reference of $N_m\beta$, $\lambda_i \in \mathbb{R}$.

The condition b) implies that:

$$U \cdot (\nabla_v \alpha)(v, v) = 0 \to U \cdot \beta'''(0)^{\perp} = 0.$$

In this case $\beta'''(0)^\perp = (k_1N_1)'^\perp$. Then by using Frenet formulas, we obtain that the expression $U \cdot \beta'''(0)^\perp = 0$ is equivalent to the following:

$$\left(\frac{1}{k_1}N_1 + \sum_{i=2}^{n} \lambda_i N_i\right)(k_1'N_1 + k_1k_2N_2) = 0 \to \frac{k_1'}{k_1} + \lambda_2 k_1 k_2 = 0,$$

where k_2 is the second curvature (or torsion in the case of a curve in $\mathbb{R}^3$) of the curve at m. This implies that

$$\lambda_2 = -\frac{k_1'}{k_1^2 k_2}.$$

In the case that $n = 1$, we obtain that $k_1' = 0$.

5.1.2 2-Critical points of ν for surfaces immersed in $\mathbb{R}^3$ and $\mathbb{R}^4$

In this subsection we study the 2-critical points of the normal map for a surface $M \subseteq \mathbb{R}^{2+n}$, $m \in M$. We shall denote by $J : T_mM \to T_mM$ the rotation of $90°$.

We use the following orthonormal frame of T_mM which is parameterized by the chart $\overrightarrow{x}(x, y)$:

$$t_1 = \frac{\overrightarrow{x}_x}{\sqrt{E}}, \qquad t_2 = \frac{E\overrightarrow{x}_y - F\overrightarrow{x}_x}{\sqrt{E(EG - F^2)}},$$

where $E = \overrightarrow{x}_x \cdot \overrightarrow{x}_x$, $F = \overrightarrow{x}_x \cdot \overrightarrow{x}_y$ and $G = \overrightarrow{x}_y \cdot \overrightarrow{x}_y$. We shall consider $v = \cos\theta t_1 + \sin\theta t_2$ as a unit vector of T_mM, where $\theta \in [0, 2\pi[$ and $U \in N_mM$, where $b_1 = \alpha(t_1, t_1)$, $b_2 = \alpha(t_2, t_2)$ and $b_3 = \alpha(t_1, t_2)$. We call $v^\sharp = g(v, \cdot)$, for any $v \in T_mM$ and $Jv = -\sin\theta t_1 + \cos\theta t_2$.

Proposition 5.1.3 *Let M be a surface immersed in $\mathbb{R}^{2+n}$, $m \in M$. The pair (v, U), where $0 \neq v \in T_mM$ and $U \in N_mM$, admits a 2-critical point of ν iff the following conditions are satisfied:*

a) $U \cdot \alpha(v, v) = v \cdot v$, *and* $U \cdot \alpha(v, Jv) = 0$;

b) $U \cdot (\nabla_v \alpha)(v, v) = 0$;

c) $U \cdot \alpha(Jv, Jv) \neq v \cdot v$ *or* $U \cdot (\nabla_v \alpha)(v, Jv) = 0$.

Proof. A vector $x \in T_m M$ that is orthogonal to v may be written as qJv, for some $q \in \mathbb{R}$. Also J is an isometry, whence $g(Jv, Jv) = v \cdot v$. Thus, the conditions of Theorem 5.1.2 for (v, U) are now

a') $g(v, .) - U \cdot \alpha(v, .) = 0$;

b) $U \cdot (\nabla_v \alpha)(v, v) = 0$;

c') $q\big(v \cdot v - U \cdot \alpha(Jv, Jv)\big) = U \cdot (\nabla_v \alpha)(v, Jv)$ has a solution for $q \in \mathbb{R}$.

Then, a) is equivalent to a') because (v, Jv) is a basis of $T_m M$. Obviously c) is equivalent to c'). □

Lemma 5.1.4 *Let* M *be a surface immersed in* $\mathbb{R}^{2+n}$, $m \in M$, $v = \cos\theta t_1 + \sin\theta t_2 \in T_m M$, $\theta \in [0, 2\pi[$ *and* $U \in N_m M$. *Then the following conditions are satisfied:*

a) $g(v) - U \cdot \alpha(v) = 0$ *is equivalent to the following expressions:*

$((U \cdot b_1) - 1)\cos\theta + (U \cdot b_3)\sin\theta = 0,$

$((U \cdot b_2) - 1)\sin\theta + (U \cdot b_3)\cos\theta = 0.$

In the following items, the pair (v, U) *satisfies this property.*

b) $U \cdot (\nabla_v \alpha)(v, v) = 0$ *is equivalent to the following equation:*

$$U \cdot ((D_{t_1} b_1) \cos^3 \theta + (D_{t_2} b_1) \sin \theta \cos^2 \theta + 2(D_{t_1} b_3) \sin \theta \cos^2 \theta$$
$$+2(D_{t_2} b_3) \sin^2 \theta \cos \theta + (D_{t_1} b_2) \sin^2 \theta \cos \theta + (D_{t_2} b_2) \sin^3 \theta) = 0.$$

c) $U \cdot (\nabla_v \alpha)(v, Jv) = 0$ *is equivalent to the following:*

$$\begin{aligned}
& U \cdot ((D_{t_1} b_3) \cos^3 \theta - (D_{t_1} b_1) \sin \theta \cos^2 \theta + (D_{t_2} b_3) \sin \theta \cos^2 \theta \\
& +(D_{t_1} b_2) \sin \theta \cos^2 \theta - (D_{t_2} b_1) \sin^2 \theta \cos \theta - (D_{t_1} b_3) \sin^2 \theta \cos \theta \\
& +(D_{t_2} b_2) \sin^2 \theta \cos \theta - (D_{t_2} b_3) \sin^3 \theta)) \\
& +(t_1 \cdot D_{t_1} t_2)(-(U \cdot b_1)(\cos^3 \theta - \cos \theta \sin^2 \theta) - 4(U \cdot b_3) \sin \theta \cos^2 \theta \\
& +(U \cdot b_2)(\cos^3 \theta - \sin \theta \cos \theta)) \\
& +(t_1 \cdot D_{t_2} t_2)(-(U \cdot b_1)(\cos^2 \theta \sin \theta - \sin^3 \theta) - 4(U \cdot b_3) \sin^2 \theta \cos \theta \\
& +(U \cdot b_2)(\cos^2 \theta \sin \theta - \sin^2 \theta)) = 0.
\end{aligned}$$

Proof. In this case we have that:

$$\begin{aligned}
g(v) - U \cdot \alpha(v) &= g(\cos \theta t_1 + \sin \theta t_2) - U \cdot \alpha(\cos \theta t_1 + \sin \theta t_2) \\
&= \cos \theta t_1^\sharp + \sin \theta t_2^\sharp - \cos \theta U \cdot \alpha(t_1) - \sin \theta U \cdot \alpha(t_2) \\
&= \cos \theta (t_1^\sharp - U \cdot \alpha(t_1)) + \sin \theta (t_2^\sharp - U \cdot \alpha(t_2)) \\
&= 0.
\end{aligned}$$

If this expression vanishes, then it is equivalent to apply the equation over $\{t_1, t_2\}$, an orthonormal frame of $T_m M$ and the result vanishes. Then applying over these vectors, we have:

$$((U \cdot b_1) - 1) \cos \theta + (U \cdot b_3) \sin \theta = 0, \tag{5.1.2}$$
$$((U \cdot b_2) - 1) \sin \theta + (U \cdot b_3) \cos \theta = 0. \tag{5.1.3}$$

Let us see the expression $U \cdot (\nabla_v \alpha)(v,v) = 0$. We can write:

$$\alpha := b_1 \otimes t_1^\sharp \otimes t_1^\sharp + b_3 \otimes (t_1^\sharp \otimes t_2^\sharp + t_2^\sharp \otimes t_1^\sharp) + b_2 \otimes t_2^\sharp \otimes t_2^\sharp.$$

We have that $U \cdot \nabla_v b_i = U \cdot D_v b_i$, and

$$(\nabla_v t_i^\sharp)(v) = (\nabla_v g(t_i))(v) = (\nabla_v t_i) \cdot v = (D_v t_i) \cdot v.$$

Then

$$\begin{aligned}
&U \cdot (\nabla_v \alpha)(v,v) = \\
=&(U \cdot D_v b_1)\cos^2\theta + (U \cdot b_1)(t_2 \cdot D_v t_1)\sin 2\theta + (U \cdot D_v b_3)\sin 2\theta \\
&+ 2(U \cdot b_3)((t_2 \cdot D_v t_1)\sin^2\theta + (t_1 \cdot D_v t_2)\cos^2\theta) \\
&+ (U \cdot D_v b_2)\sin^2\theta + (U \cdot b_2)(t_1 \cdot D_v t_2)\sin 2\theta \\
=&(U \cdot D_v b_1)\cos^2\theta + (U \cdot D_v b_3)\sin 2\theta + (U \cdot D_v b_2)\sin^2\theta \\
&+ (t_1 \cdot D_v t_2)(-(U \cdot b_1)\sin 2\theta + 2(U \cdot b_3)\cos 2\theta + (U \cdot b_2)\sin 2\theta).
\end{aligned}$$

We compute now the second factor of the last part, having in mind condition a). This gives:

$$\begin{aligned}
&-2(U \cdot b_1)\sin\theta\cos\theta + 2(U \cdot b_3)(\cos^2\theta - \sin^2\theta) + 2(U \cdot b_2)\sin\theta\cos\theta \\
=&-2\sin\theta\cos\theta + 2(U \cdot b_3)\sin^2\theta + 2(U \cdot b_3)(\cos^2\theta - \sin^2\theta) \\
&+ 2\sin\theta\cos\theta - 2(U \cdot b_3)\cos^2\theta \\
=&0.
\end{aligned}$$

Using this in the previous equation, $U \cdot (\nabla_v \alpha)(v,v) = 0$ can be written as following:

$$\begin{aligned}
&U \cdot ((D_{t_1} b_1)\cos^3\theta + (D_{t_2} b_1)\sin\theta\cos^2\theta + 2(D_{t_1} b_3)\sin\theta\cos^2\theta \\
&\quad + 2(D_{t_2} b_3)\sin^2\theta\cos\theta + (D_{t_1} b_2)\sin^2\theta\cos\theta + (D_{t_2} b_2)\sin^3\theta) = 0.
\end{aligned}$$

Finally, condition c) can be proved in the same way. □

2-Critical points of ν for surfaces in $\mathbb{R}^3$

Let M be a surface immersed in $\mathbb{R}^3$, $m \in M$. By the previous Lemma, we know that the expression $g(v) - U \cdot \alpha(v) = 0$ is equivalent to the following:

$$((U \cdot b_1) - 1)\cos\theta + (U \cdot b_3)\sin\theta = 0, \tag{5.1.4}$$

$$((U \cdot b_2) - 1)\sin\theta + (U \cdot b_3)\cos\theta = 0. \tag{5.1.5}$$

According to the point m, we distinguish different cases. If m is **plane**, then $b_1 = b_2 = b_3 = 0$, and this implies that $\alpha = 0$ and $v = 0$ which is a contradiction. Suppose now that m is an **umbilic not plane** point, then $b_1 - b_2 = b_3 = 0$. In this case, the equations (5.1.4) and (5.1.5) are reduced to the following:

$$\cos\theta(1 - U \cdot b_1) = 0,$$
$$\sin\theta(1 - U \cdot b_1) = 0.$$

Since $\sin\theta$ and $\cos\theta$ cannot vanishes at the same time, this implies that $U \cdot b_1 = 1$, i.e.:

$$U = \frac{b_1}{b_1 \cdot b_1}$$

and we do not have any condition over θ.

If $b_1 - b_2 = b_3 = 0$, then $g - U \cdot \alpha = 0$ and the other conditions are equivalent to $U \cdot (\nabla_v \alpha)(v, v) = 0$. Then, *if m is umbilic not plane in general, there exists an odd number, less than or equal to 3 of directions of T_mM over each one there exists a 2-critical point of ν over m.*

Now, we consider the **general case**. We know that $g(v) - U \cdot \alpha(v) = 0$. Applying this equation over Jv we obtain that $U \cdot \alpha(v, Jv) = 0$ and therefore the directions $v \in T_mM$ that admits a 1-critical point of ν are the principal directions, see Subsection 3.2.5.

Since $\dim(N_mM) = 1$, then we can consider that $U = pN$, where $p \in \mathbb{R}$ and $N \in N_mM$ is the unit normal vector. Then, for each principal direction, $v_1,\ v_2 \in T_mM$ determined by $\theta_1,\ \theta_2$, we obtain that $\alpha(v_i, v_i) = k_iN$, where k_i is the principal curvature at m associated to $v_i,\ i = 1, 2$ and:

$$g(v_i, v_i) - U \cdot \alpha(v_i, v_i) = 1 - p_ik_i = 0 \to p_i = \frac{1}{k_i},\ i = 1, 2.$$

In this case, we can write

$$U_i = \frac{\alpha(v_i, v_i)}{||\alpha(v_i, v_i)||^2}, \quad i = 1, 2.$$

Finally, by the equation $U \cdot (\nabla_v\alpha)(v, v) = 0$, we obtain that:

$$\frac{\alpha(v_i, v_i)}{||\alpha(v_i, v_i)||^2} \cdot (\nabla_{v_i}\alpha)(v_i, v_i) = \frac{k_i'}{k_i^2} = 0.$$

Then $k_i' = 0$.

2-Critical points of ν for surfaces in $\mathbb{R}^4$

We start analyzing the equations which a unit vector $v \in T_mM$ and $U \in N_mM$ must satisfy in order to admit a 2-critical point of ν. As in the previous subsection, we know that the expression $g(v) - U \cdot \alpha(v) = 0$ which characterize the focal set is equivalent to the following:

$$((U \cdot b_1) - 1)\cos\theta + (U \cdot b_3)\sin\theta = 0 \tag{5.1.6}$$

$$((U \cdot b_2) - 1)\sin\theta + (U \cdot b_3)\cos\theta = 0. \tag{5.1.7}$$

Note that, for $M \in \mathbb{R}^4$ conditions a) and b) of Proposition 5.1.3 are the same that characterize the strong principal directions defined by Montaldi in [22]. Note also that c) is true generically, so that generically the question whether a direction v admits 2-critical points is answered by ascertaining that v satisfies the equation

$$\det(\alpha(v, Jv), (\nabla_v \alpha)(v, v)) = 0, \tag{5.1.8}$$

which leads to a polynomial equation of fifth degree. Also, conditions a) or a), mean that U is a focal point corresponding to v and this can be analyzed easily by using the description of the focal locus by means of the inverted pedal.

Now, we distinguish different cases depending on the degeneration of the curvature ellipse at the point m. Suppose that m is **plane**, i.e. $b_1 = b_2 = b_3 = 0$. Then $\alpha = 0$ and this implies that $v = 0$ but by hypothesis $v \neq 0$. Then *if m is plane it does not admit strong principal directions nor 2-critical points of the map ν.*

Suppose now that m is **umbilic not plane**, i.e. $b_1 - b_2 = b_3 = 0$ and $b_1 \neq 0$. The equations (5.1.6) and (5.1.7) can be written as $(U \cdot b_1 - 1)\cos\theta = (U \cdot b_1 - 1)\sin\theta = 0$. Then, we can write:

$$U = \frac{b_1}{b_1 \cdot b_1} + sJb_1, \quad s \in \mathbb{R}.$$

The condition b) of the theorem is now equivalent to the following:

$$U \cdot (\nabla_v \alpha)(v, v) = U \cdot (\nabla_v \alpha)(v, Jv) = 0.$$

Substituting the value of U, we obtain:

$$a(\theta) + sb(\theta) = 0, \quad c(\theta) + sd(\theta) = 0,$$

where a, b, c and d are homogeneous polynomials of degree 3 in $\sin\theta$ and $\cos\theta$. Let s_0, θ_0 be a solution of the system. Eliminating s_0 we obtain the equation $a(\theta)d(\theta) - c(\theta)b(\theta) = 0$ which is an homogeneous equation of degree 6 in $\sin\theta_0$ and $\cos\theta_0$. If θ_0 is a solution, we can obtain s_0 in a unique way when $b(\theta_0) \neq 0$ or $d(\theta_0) \neq 0$, which is the general case. Then, *if m is an umbilic not plane point, there exists in general an even number of directions $[v]$ of T_mM less than or equal to* 6, *over each one there exists a 2-critical point of ν over m.* In the case that $b(\theta_0) = 0$ and $d(\theta_0) = 0$, we can not determine s_0 and therefore there no exist 2-critical points.

Suppose now that m is **semiumbilic** but not umbilic. In this case, we can take an orthonormal basis (t_1, t_2) of T_mM in such away that $b_3 = 0$. Then, the equations (5.1.6) and (5.1.7) can be written as following:

$$((U \cdot b_1) - 1)\cos\theta = 0, \qquad ((U \cdot b_2) - 1)\sin\theta = 0.$$

The extremes of the curvature ellipse are when $\cos 2\theta = \pm 1$, i.e. when $\cos\theta = 0$ or $\sin\theta = 0$. Suppose that θ satisfies $\alpha(v, v)$ is not an extreme of the curvature ellipse. Then U must satisfy the equations $U \cdot b_1 = U \cdot b_2 = 1$. If b_1 and b_2 are linearly independent, as we proved in Theorem 3.3.4, this system has the unique solution:

$$U_0 = \frac{JB}{H \cdot JB}.$$

In this case $g - U \cdot \alpha = 0$, then the other conditions are equivalent to $U_0 \cdot (\nabla_v \alpha)(v, v) = U_0 \cdot (\nabla_v \alpha)(v, Jv) = 0$. In other words, θ must satisfy two homogeneous equations simultaneously of degree 3 in $\sin\theta$ and $\cos\theta$.

Then, in general *there no exist 2-critical points over the directions of T_mM which no correspond to the extremes of the curvature ellipse.*

If b_1 and b_2 are linearly dependent, then $b_1 = pb_2$, $p \in \mathbb{R}$ and $b_3 = 0$. In this case, $U \cdot (b_1 - b_2) = (p-1)U \cdot b_2 = p - 1 = 0$. Hence, $p = 1$ but then m would be umbilic. Finally, if $b_1 = 0$ then $U \cdot b_2 = 1$ which is analogous to the case of m umbilic not plane. And it is the same for $b_2 = 0$.

Suppose now that $\cos\theta = 0$, or $\sin\theta = 0$. Since both cases are similar we suppose that $\sin\theta = 0$, in other words, $v = t_1$.

The equations (5.1.6) and (5.1.7) can be written as $U \cdot b_1 - 1 = 0$. If $b_1 = 0$, i.e. $\alpha(v, v) = 0$, this equation has no solution. In other case, the equations (5.1.6) and (5.1.7) are satisfied for any real number λ,

$$U = \frac{b_1}{b_1 \cdot b_1} + \lambda J b_1,$$

which is an affine line R in N_mM. Since θ is now fixed, the solutions of the condition b) of Proposition 5.1.3 are the intersection of the affine line R with the orthogonal subspace to the vector $(\nabla_v \alpha)(v, v)$. In general, this intersection is a point, but occasionally can be an affine line or the empty set. In the last two cases there no exist 2-critical points. Suppose that

$$U_0 = \frac{b_1}{b_1 \cdot b_1} + \lambda_0 J b_1$$

satisfies $U_0 \cdot (\nabla_v \alpha)(v,v) = 0$. In this case, we have

$$\begin{aligned} g - U_0 \cdot \alpha &= g - \left(\frac{b_1}{b_1 \cdot b_1} + \lambda_0 J b_1 \right) \cdot (b_1 \otimes t_1^\sharp \otimes t_1^\sharp + b_2 \otimes t_2^\sharp \otimes t_2^\sharp) \\ &= g - t_1^\sharp \otimes t_1^\sharp - \left(\frac{b_1 \cdot b_2}{b_1 \cdot b_1} + \lambda_0 J b_1 \cdot b_2 \right) t_2^\sharp \otimes t_2^\sharp \\ &= \left(1 - \frac{b_1 \cdot b_2}{b_1 \cdot b_1} - \lambda_0 J b_1 \cdot b_2 \right) t_2^\sharp \otimes t_2^\sharp \\ &= (1 - U_0 \cdot b_2) t_2^\sharp \otimes t_2^\sharp . \end{aligned}$$

Let $b = cJv,\ c \in \mathbb{R}$. Therefore

$$g(b, Jv) - U_0 \cdot \alpha(b, Jv) = (1 - U_0 \cdot b_2)c.$$

Then, the condition c) of Proposition 5.1.3 is satisfied if we can find a real number $c \in \mathbb{R}$ where

$$(1 - U_0 \cdot b_2)c = U_0 \cdot (\nabla_v \alpha)(v, Jv).$$

In other words, if $\alpha(t_1, t_1) \neq 0$, over $v = t_1$ *there exist as many 2-critical points as points* U_0 *of* R *are orthogonal to* $(\nabla_v \alpha)(v,v)$ *and satisfy*

$$U_0 \cdot b_2 \neq 1$$

or $U_0 \cdot (\nabla_v \alpha)(v, Jv) = 0$. If $U_0 \cdot b_2 \neq 1$, in Theorem 3.3.2, we proved that in this case we may write

$$U = \frac{b_1}{\kappa^2} - \frac{\kappa'}{\kappa^3 \tau} J b_1,$$

where κ is the curvature at m and τ is the normal torsion of M at m in the direction t_1. If $U_0 \cdot b_2 = 1$, this coincides with the previous case and if $\alpha(t_1, t_1) = 0$, then *there no exist 2-critical points over* t_1. The same considerations are valid when $\cos\theta = 0$.

Let us see the **general case**, where m is not semiumbilic. Since we know, for any direction $v = \cos\theta t_1 + \sin\theta t_2$ of T_mM there exists an unique vector $U \in N_mM$ which satisfies the equations (5.1.6) and (5.1.7). It is possible an exception, when the tangent line to the curvature ellipse $\alpha(v, v)$ goes through the origin of N_mM. Then, there is no solution and there no exist 2-critical points of ν over v. If m is elliptic, the origin of N_mM is in a bounded open set limited by the curvature ellipse, then this exception does not exist. Let us see the equation that ν must satisfy in the other cases. For that, we compute U. We know that B and C are linearly independent because m is not semiumbilic. Therefore, we can write as a unique way $U = pB + qC$ and the the equations (5.1.6) and (5.1.7) can be written as following:

$$
\begin{aligned}
&(pB \cdot b_1 + qC \cdot b_1 - 1)\cos\theta + (pB \cdot b_3 + qC \cdot b_3)\sin\theta \\
&= p(B \cdot b_1 \cos\theta + B \cdot b_3 \sin\theta) + q(C \cdot b_1 \cos\theta + C \cdot b_3 \sin\theta) = \cos\theta, \\
&p(B \cdot b_2 \sin\theta + B \cdot b_3 \cos\theta) + q(C \cdot b_2 \sin\theta + C \cdot b_3 \cos\theta) = \sin\theta.
\end{aligned}
$$

Solving this system, we obtain that:

$$
\begin{aligned}
p = \frac{1}{det}&((C \cdot b_2 \sin\theta + C \cdot b_3 \cos\theta)\cos\theta \\
&- (C \cdot b_1 \cos\theta + C \cdot b_3 \sin\theta)\sin\theta), \\
q = \frac{1}{det}&((B \cdot b_1 \cos\theta + B \cdot b_3 \sin\theta)\sin\theta \\
&- (B \cdot b_2 \sin\theta + B \cdot b_3 \cos\theta)\cos\theta),
\end{aligned}
$$

where

$$\begin{aligned} det =&(B \cdot b_1 \cos\theta + B \cdot b_3 \sin\theta)(C \cdot b_2 \sin\theta + C \cdot b_3 \cos\theta) \\ &- (B \cdot b_2 \sin\theta + B \cdot b_3 \cos\theta)(C \cdot b_1 \cos\theta + C \cdot b_3 \sin\theta). \end{aligned}$$

We see that the numerator of the components of U are homogeneous polynomials of degree 2 in $\sin\theta$ and $\cos\theta$. This implies that the equation b) of Proposition 5.1.3 is a homogeneous polynomial of degree 5 in $\sin\theta$ and $\cos\theta$.

Finally, let us see the condition c), and which is that the rank of $g-U\cdot\alpha$ not vanishes. If this not happen, then $g-U\cdot\alpha=0$, but in this case, considering a unit vector v, that $U\cdot\alpha(v,v)=U\cdot\eta(v)=1$, i.t. the height of the points of the curvature ellipse $\eta(v)$ in m with respect to the orthogonal line to U which goes through the origin is constant and equal to 1. This only happen when the curvature ellipse degeneres in a segment or in a point. In that case m is a semiumbilic or an umbilic point, which is a contradiction. Then if *m is not a semiumbilic point, then there exists in general an odd number, less than or equal to* 5, *of strong principal directions $[v]\in T_mM$ for each one there exist an element $U\in N_mM$ which admits a 2-critical point of ν.* In Theorem 3.3.2, we proved that in this case we may write

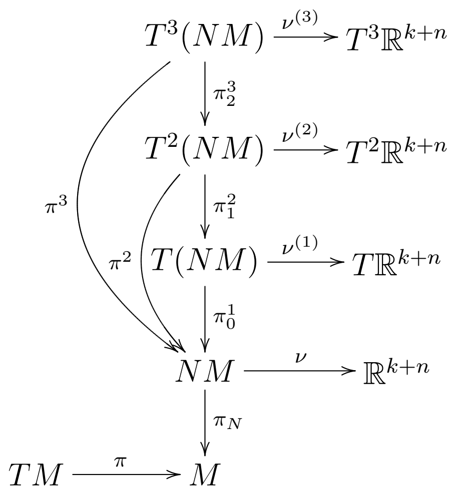
$$U = \frac{b}{\kappa^2} - \frac{\kappa'}{\kappa^3\tau} Jb,$$

where $b=\alpha(v,v)$, κ is the curvature at m and τ is the normal torsion of M at m in the direction v.

5.1.3 3-Critical points of ν for surfaces

Now, we will study the 3-critical points of the normal map. In addition to the notation of the preceding section, we will use $\pi^3 : T^3(NM) \to NM$ and $\nu^{(3)} : T^3(NM) \to T^3\mathbb{R}^{k+n}$.

We may see the relations among those maps by means of the following commutative diagram:



Definition 5.1.5 *Let $v \in T_mM$ and $U \in N_mM$. In the following we will say that ν admits a 3-critical point over (v, a, U), where $v \in T_mM$, $a \in T^2_{(m,v)}M$ and $U \in N_mM$ if there is a 3-critical point of ν, $X^3 \in T^3(NM)$, such that $(\pi_1^3 \circ \pi_N^{(3)})(X^3) = v$, $(\pi_2^3 \circ \pi_N^{(3)})(X^3) = a$ and $\pi_3(X^3) = U$.*

Theorem 5.1.6 *Let $m \in M$, $v \in T_mM$, $a \in T^2_mM$ and $U \in N_mM$. Let us choose any orthonormal basis of T_mM, $(t_1, \ldots, t_k)$ such that t_1 and v are parallel. Then, ν admits a 3-critical point X^3 over (v, a, U) iff the following conditions are satisfied:*

a) $v \neq 0$ *and* $g(v, .) - U \cdot \alpha(v, .) = 0$, *i.e.* U *belongs to the focal set over* v.

b) $U \cdot (\nabla_v \alpha)(v, v) = 0$.

c) $g(\cdot, a) - U \cdot \alpha(\cdot, a) = U \cdot (\nabla_v \alpha)(v, a)$.

d) $2U \cdot (\nabla_v \alpha)(v, a) + U \cdot (\nabla_v^2 \alpha)(v, v, v) + U \cdot (\nabla_{a^\top} \alpha)(v, v) = 0$.

e) The following linear system, whose unknowns are the components of $y = y^2 t_2 + \cdots + y^k t_k$ *has a solution:*

$$g(y, t_j) - U \cdot \alpha(y, t_j) = -\alpha(v, v) \cdot \alpha(v, t_j) + U \cdot (\nabla_v^2 \alpha)(v, v, t_j) \\ + U \cdot (\nabla_{a^\top} \alpha)(v, t_j) - U \cdot \alpha(\mathcal{A}_{\alpha(v,v)} v, t_j) + 2U \cdot (\nabla_v \alpha)(a^\top, t_j).$$

Proof. Note that a), b) and c) are the conditions of Theorem 5.1.2, where

$$m = \beta(0),\ v = \beta'(0),\ a = \beta''(0),\ U = u(0),\ V = u'(0),\ A = u''(0).$$

Thus, the condition that we must require in addition to a), b) and c) is

$$\nu^{(3)}(X^3) = \beta'''(0) + u'''(0) = b + B = 0.$$

Let us compute the normal part of $b + B = 0$. By Lemma 5.0.6, we know that:

$$b^\perp = (\nabla_v \alpha)(v, v) + 3\alpha(v, a^\top),$$

that is we get the equation $B^\perp = -(\nabla_v \alpha)(v, v) - 3\alpha(v, a)$. Now we need to prove that we can set the values of the $p^{i'''}(0)$ in order to satisfy this equation (see proof of Theorem 5.1.2), and this is trivial.

Let Y denote the parallel transport of $y \in T_mM$ along β. We have $a + A = a + D_v D_{\beta'} u = 0$. Therefore

$$(D_v Y) \cdot D_v D_v u = -(D_v Y) \cdot a = -(D_v Y) \cdot a^{\perp} = -\alpha(v,v) \cdot \alpha(v,y).$$

We compute separately the following, having in mind that $a + A = 0$ and $v + V = 0$:

$$\begin{aligned}\nabla_v((D_v u)^{\perp}) \cdot \alpha(y,v) &= D_v(D_v u - (D_v u)^{\top}) \cdot \alpha(y,v) \\ &= D_v(D_v u + (D_v v)^{\top}) \cdot \alpha(y,v) = A \cdot \alpha(y,v) - \alpha(y,v) \cdot \alpha(v,v) \\ &= -a \cdot \alpha(y,v) + \alpha(y,v) \cdot \alpha(v,v) = 0.\end{aligned}$$

Then, by Lemma 5.0.6, where we use that $-U \cdot (\nabla_v \alpha)(v,y) = y \cdot (\nabla_v \mathcal{A})(U,v)$, $-U \cdot \alpha(v,y) = y \cdot \mathcal{A}(U,v)$ and $-U \cdot (\nabla_v^2 \alpha)(y,v,v) = y \cdot (\nabla_v^2 \mathcal{A})(U,v,v)$, and recalling that $\nabla_v Y = 0$ and $V^{\perp} = 0$ we get

$$\begin{aligned}y \cdot u'''(0) =& - U \cdot (\nabla^2 \alpha)(v,v,v,y) - U \cdot (\nabla_{a^{\top}} \alpha)(y,v) \\ & - 2U \cdot (\nabla_v \alpha)(a,y) - U \cdot \alpha(y, \nabla_v(\beta''^{\top})) + \alpha(v,v) \cdot \alpha(v,y) \\ =& - U \cdot (\nabla^2 \alpha)(v,v,v,y) - U \cdot (\nabla_{a^{\top}} \alpha)(y,v) \\ & - 2U \cdot (\nabla_v \alpha)(a,y) - U \cdot \alpha(y, b - \mathcal{A}_{\alpha(v,v)} v) \\ & + \alpha(v,v) \cdot \alpha(v,y).\end{aligned}$$

Hence, from $b + B = 0$ we obtain

$$\begin{aligned}g(b,\cdot) - U \cdot \alpha(b,\cdot) =& -\alpha(v,v) \cdot \alpha(v,\cdot) + U \cdot (\nabla^2 \alpha)(v,v,v,\cdot) \\ & + U \cdot (\nabla_{a^{\top}} \alpha)(v,\cdot) + 2U \cdot (\nabla_v \alpha)(a,\cdot) - U \cdot \alpha(\mathcal{A}_{\alpha(v,v)} v, \cdot)\,.\end{aligned}$$

If this 1-form acts upon v we have, taking account of condition a):

d) $\quad 2U \cdot (\nabla_v \alpha)(v,a) + U \cdot (\nabla^2 \alpha)(v,v,v,v) + U \cdot (\nabla_{a^{\top}} \alpha)(v,v) = 0.$

Then we may find b such that $b + B = 0$ iff there is a vector $y = y^2 t_2 + \cdots + y^k t_k$ such that:

$$g(y, t_j) - U \cdot \alpha(y, t_j) = -\alpha(v, v) \cdot \alpha(v, t_j) + U \cdot (\nabla^2 \alpha)(v, v, v, t_j)$$
$$+ U \cdot (\nabla_{a^\top} \alpha)(v, t_j) - U \cdot \alpha(\mathcal{A}_{\alpha(v,v)} v, t_j) + 2U \cdot (\nabla_v \alpha)(a^\top, t_j),$$

which is condition e). □

Let us see the case of a surface. By using the same notation as in the previous section, we obtain the following Proposition, the proof is trivial.

Proposition 5.1.7 *Let M be a surface immersed in $\mathbb{R}^{2+n}$, $m \in M$. Then ν admits a 3-critical point over (v, a, U), where $0 \neq v \in T_m M$, $a \in T^2_{(m,v)} M$ and $U \in N_m M$ iff the following conditions are satisfied:*

a) $U \cdot \alpha(v, v) = v \cdot v$, *and* $U \cdot \alpha(v, Jv) = 0$;

b) $U \cdot (\nabla_v \alpha)(v, v) = 0$;

c) $U \cdot \alpha(Jv, Jv) \neq v \cdot v$ *or* $U \cdot (\nabla_v \alpha)(v, Jv) = 0$;

d) $2U \cdot (\nabla_v \alpha)(v, a) + U \cdot (\nabla^2_v \alpha)(v, v, v) + U \cdot (\nabla_{a^\top} \alpha)(v, v) = 0$;

e) $U \cdot \alpha(Jv, Jv) \neq v \cdot v$ *or*

$$- \alpha(v, v) \cdot \alpha(v, Jv) + U \cdot (\nabla^2 \alpha)(v, v, v, Jv)$$
$$+ U \cdot (\nabla_{a^\top} \alpha)(v, Jv) + 2U \cdot (\nabla_v \alpha)(a, Jv) - U \cdot \alpha(\mathcal{A}_{\alpha(v,v)} v, Jv) = 0\,,$$

where $J : T_m M \to T_m M$ denotes the rotation of $90°$.

5.2 Analysis of the critical points of ψ

Let M be a k-dimensional submanifold immersed in $\mathbb{R}^{k+n}$, $m \in M$. Now, we will study the 2-critical points of the generalized Gauss map $\psi : NM \to \mathbb{R}^{k+n}$, given by $\psi(m; U) = U \in C_m M$, where

$$CM = \{U \in PNM\,;\, U \cdot U = 1\} \subset NM$$

and $C_m M = CM \cap PN_m M$.

Definition 5.2.1 *Let $v \in T_m M$ and $U \in C_m M$. In the following we will say that* there is a 2-critical point of ψ over (v, U), or that (v, U) admits a 2-critical point of ψ *if there is some 2-critical point $X^2 \in T^2(CM)$ of ψ such that $U = \pi^2(X^2)$ and $v = (\pi_1^2 \circ \pi_N^{(2)})(X^2)$. In other terms, if $X^2 = \gamma''$, being $\gamma : I \to CM$ a smooth curve, then $v = (\pi_N \circ \gamma)'(0)$ and $U = \gamma(0)$.*

In the same manner we say that $v \in T_m M$ admits a 2-critical point of ψ *if there is $U \in C_m M$ such that (v, U) admits a 2-critical point of ψ.*

The following theorem characterizes the properties that must have a pair (v, U), $v \in T_m M$ and $U \in C_m M$ in order to admit a 2-critical point of ψ. A computation almost equal to that made for the normal map leads immediately to the following result:

Theorem 5.2.2 *Let $m \in M$, $v \in T_m M$ and $U \in C_m M$. Then (v, U) admits a 2-critical point of ψ iff the following conditions are satisfied:*

a) $v \neq 0$ *and* $U \cdot \alpha(v, .) = 0$.

b) $U \cdot (\nabla_v \alpha)(v, v) = 0$.

c) Let $(t_1, \ldots, t_k)$ *be an orthonormal basis of* T_mM *such that* t_1 *and* v *are parallel. Then the following linear system, whose unknowns are the components of* $x = x^2 t_2 + \cdots + x^k t_k$*, has a solution:*

$$U \cdot \alpha(x, t_j) = -U \cdot (\nabla_v \alpha)(v, t_j), \ j = 2, \ldots, k.$$

5.2.1 2-Critical points of ψ for surfaces in $\mathbb{R}^5$

In this subsection we shall study the 2-critical points of ψ for a surface immersed in $\mathbb{R}^5$. Let us see first a Proposition that characterizes the 2-critical points of ψ for a surface immersed in $\mathbb{R}^{2+k}$.

Proposition 5.2.3 *Let* M *be a surface immersed in* $\mathbb{R}^{2+n}$*. The pair* (v, U)*, where* $0 \neq v \in T_mM$ *and* $U \in C_mM$*, admits a 2-critical point of* ν *iff the following conditions are satisfied:*

a) $U \cdot \alpha(v, \cdot) = 0$;

b) $U \cdot (\nabla_v \alpha)(v, v) = 0$;

c) $U \cdot \alpha(Jv, Jv) \neq 0 \quad$ *or* $\quad U \cdot (\nabla_v \alpha)(v, Jv) = 0$,

where $J : T_mM \to T_mM$ *denotes the rotation of* $90°$.

Proof. A vector $x \in T_mM$ that is orthogonal to v may be written as qJv, for some $q \in \mathbb{R}$. Thus, the conditions of previous Theorem for (v, U) are now a') $U \cdot \alpha(v, v) = 0$ and $U \cdot \alpha(v, Jv) = 0$;

b) $U \cdot (\nabla_v \alpha)(v, v) = 0$;

c') $qU \cdot \alpha(Jv, Jv) = U \cdot (\nabla_v \alpha)(v, Jv)$ has a solution for $q \in \mathbb{R}$.

Then, a) is equivalent to a') because (v, Jv) is a basis of T_mM. Obviously c) is equivalent to c'). □

Suppose now that M is a surface immersed $\mathbb{R}^5$. We shall use the same notation as in the case of surfaces for ν. We shall suppose that $v = \cos\theta t_1 + \sin\theta t_2 \in T_mM$, where t_1, t_2 is an orthonormal basis of T_mM. Then, conditions a') and b) of the above Proposition are the same that characterize the asymptotic directions defined in Definition 3.3.5. Note also that c') is true generically, so that generically the question whether a direction v admits 2-critical points is answered by ascertaining that v satisfies the equation

$$\det(\alpha(v, t_1), \alpha(v, t_2), (\nabla_v \alpha)(v, v)) = 0, \qquad (1)$$

which leads to a polynomial equation of fifth degree.

Let us see different cases depending on the degeneration of the curvature ellipse at the point $m \in M$. Let m be an **umbilic point**. In this case $\alpha = 0$ and we do not have any condition over v and U.

Suppose now that m is **umbilic not plane**. In this case $b_1 - b_2 = b_3 = 0$ and $U \cdot \alpha = 0$. Then, the conditions b) and c) of the Proposition leads to the equation:

$$det(\alpha(t_1, t_1), (\nabla_v \alpha)(v, v), (\nabla_v \alpha)(v, Jv)) = 0.$$

Then, if *m is an umbilic not plane point, then there exists in general an even number, less than or equal to 6, of directions $[v] \in T_mM$ for* *each one there exists an element $U \in C_mM$ which admits a 2-critical point of ψ.*

Suppose now that m is a **semiumbilic** not umbilic point. The extremes of the curvature ellipse are when $\cos 2\theta = \pm 1$, i.e. when $\cos\theta = 0$ or $\sin\theta = 0$. If $v \in T_mM$ is such that $\eta(v)$ is not an extreme of the curvature ellipse, in this case U must satisfy the equations $U \cdot b_1 = U \cdot b_2 = 0$. If b_1 and b_2 are independent, then U is the orthogonal unit vector to the plane defined by b_1 and b_2. In this case $U \cdot \alpha = 0$ and the other conditions are equivalent to $U_0 \cdot (\nabla_v \alpha)(v, v) = U_0 \cdot (\nabla_v \alpha)(v, Jv) = 0$. In other words, θ must satisfy two homogeneous equations simultaneously of degree 3 in $\sin\theta$ and $\cos\theta$. Then, in general *there no exist 2-critical points over the directions of T_mM which no correspond to the extremes of the curvature ellipse.*

If b_1 and b_2 are dependant, i.e. m is an inflection point, this case coincides with the umbilic not plane.

Suppose now that $\cos\theta = 0$, or $\sin\theta = 0$. Since both cases are similar we suppose that $\sin\theta = 0$, in other words, $v = t_1$. In this case, $U \cdot b_1 = 0$. Since θ is now fixed, the solutions of the condition b) of Proposition 5.2.3 are the intersection of the orthogonal subspace to the vector b_1 and $(\nabla_v \alpha)(v, v)$. In general, this intersection is a line, but eventually can be a plane or empty. Suppose that $U_0 \in C_mM$ is the vector that defines the intersection line. Then, the condition c) of the Previous Theorem is satisfied if we can find a real number $c \in \mathbb{R}$ where

$$U_0 \cdot b_2 c = -U_0 \cdot (\nabla_v \alpha)(v, Jv).$$

In other words, if $\alpha(t_1, t_1) \neq 0$, over $v = t_1$ *there exist as many 2-critical points as vector* $U_0 \in C_m M$ *are orthogonal to* $(\nabla_v \alpha)(v, v)$ *and satisfy*

$$U_0 \cdot b_2 \neq 0$$

or $U_0 \cdot (\nabla_v \alpha)(v, Jv) = 0$. If $\alpha(t_1, t_1) = 0$, then *there no exist 2-critical points over* t_1. The same considerations are valid when $\cos\theta = 0$.

If $b_1 = 0$, then U is the orthogonal unit vector to $(\nabla_v \alpha)(v, v)$ and $(\nabla_v \alpha)(v, Jv) = 0$. If These vector are parallel, then U cannot be determined and therefore there no exists 2-critical point over the directions of $T_m M$.

Finally, we consider the general case where the curvature ellipse is not degenerated and the plane which contains the ellipse does not contain the origin. In this case v admits 2-critical points if the following equation is satisfied:

$$\det(\alpha(v, t_1), \alpha(v, t_2), (\nabla_v \alpha)(v, v)) = 0,$$

which leads to a polynomial equation of fifth degree. Then, if m *is no semiumbilic point, there exists in general an odd number, less than or equal to* 5, *of directions* $[v] \in T_m M$ *for each one there exist an element* $U \in C_m M$ *which admits a 2-critical point of* ψ.

Otherwise, if the curvature ellipse is not degenerated and the plane which contains the ellipse contains the origin, then this plane is defined by $\alpha(t_1, t_1)$ and $\alpha(t_1, t_2)$. In this case we obtain the following determinant:

$$\det(\alpha(t_1, t_1), \alpha(t_1, t_2), (\nabla_v \alpha)(v, v)) = 0,$$

which gives a polynomial equation of third degree. Then, if m *is not a semiumbilic point and the plane which contains the ellipse contains the origin, there exists in general an odd number, less than or equal to* 3, *of directions* $[v] \in T_mM$ *for each one there exist an element* $U \in C_mM$ *which admits a 2-critical point of* ψ.

5.2.2 3-Critical points of ψ for surfaces

Now, we will study the 3-critical points of the generalized Gauss map.

Definition 5.2.4 *Let* $v \in T_mM$ *and* $U \in C_mM$. *In the following we will say that* ψ *admits a 3-critical point over* (v, a, U), *where* $v \in T_mM$, $a \in T^2_mM$ *and* $U \in C_mM$ *if there is a 3-critical point of* ψ, $X^3 \in T^3(CM)$, *such that* $(\pi^3_1 \circ \pi^{(3)}_N)(X^3) = v$, $(\pi^3_2 \circ \pi^{(3)}_N)(X^3) = a$ *and* $\pi_3(X^3) = U$.

Theorem 5.2.5 *Let* $m \in M$, $v \in T_mM$, $a \in T^2_{(m,v)}M$ *and* $U \in C_mM$. *Let us choose any orthonormal basis of* T_mM, $(t_1, \ldots, t_k)$ *such that* t_1 *and* v *are parallel. Then,* ψ *admits a 3-critical point* X^3 *over* (v, a, U) *iff the following conditions are satisfied:*

a) $v \neq 0$ *and* $U \cdot \alpha(v, \cdot) = 0$.

b) $U \cdot (\nabla_v \alpha)(v, v) = 0$.

c) $-U \cdot \alpha(\cdot, a) = U \cdot (\nabla_v \alpha)(v, a)$.

d) $2U \cdot (\nabla_v \alpha)(v, a) + U \cdot (\nabla^2_v \alpha)(v, v, v) + U \cdot (\nabla_{a^\top} \alpha)(v, v) = 0$.

e) The following linear system, whose unknowns are the components of $y = y^2 t_2 + \cdots + y^k t_k$ has a solution:

$$\begin{aligned} -U \cdot \alpha(y, t_j) =& U \cdot (\nabla_v^2 \alpha)(v, v, t_j) + U \cdot (\nabla_{a^\top} \alpha)(v, t_j) \\ &- U \cdot \alpha(\mathcal{A}_{\alpha(v,v)} v, t_j) + 2U \cdot (\nabla_v \alpha)(a^\top, t_j). \end{aligned}$$

Proof. Note that a), b) and c) are the conditions of Theorem for 2-critical points of ψ, where

$$m = \beta(0),\ v = \beta'(0),\ a = \beta''(0),\ U = u(0),\ V = u'(0),\ A = u''(0).$$

Thus, the condition that we must require in addition to a), b) and c) is

$$\nu^{(3)}(X^3) = u'''(0) = B = 0.$$

Let Y denote the parallel transport of $y \in T_m M$ along β.

Considering that $u''(0) = 0$ and $\nabla_v((D_v u)^\perp) = 0$ then, by Lemma 5.0.6, where we use that $-U \cdot (\nabla_v \alpha)(v, y) = y \cdot (\nabla_v \mathcal{A})(U, v)$, $-U \cdot \alpha(v, y) = y \cdot \mathcal{A}(U, v)$ and $-U \cdot (\nabla_v^2 \alpha)(y, v, v) = y \cdot (\nabla_v^2 \mathcal{A})(U, v, v)$, and recalling that $\nabla_v Y = 0$ and $V^\perp = 0$ we get

$$\begin{aligned} y \cdot u'''(0) =& - U \cdot (\nabla^2 \alpha)(v, v, v, y) - U \cdot (\nabla_{a^\top} \alpha)(y, v) \\ &- 2U \cdot (\nabla_v \alpha)(a, y) - U \cdot \alpha(y, \nabla_v(\beta''^\top)) \\ =& - U \cdot (\nabla^2 \alpha)(v, v, v, y) - U \cdot (\nabla_{a^\top} \alpha)(y, v) \\ &- 2U \cdot (\nabla_v \alpha)(a, y) - U \cdot \alpha(y, b - \mathcal{A}_{\alpha(v,v)} v). \end{aligned}$$

Hence, from $B^\top = 0$ we obtain

$$\begin{aligned} -U \cdot \alpha(b, \cdot) =& U \cdot (\nabla^2 \alpha)(v, v, v, \cdot) \\ &+ U \cdot (\nabla_{a^\top} \alpha)(v, \cdot) + 2U \cdot (\nabla_v \alpha)(a, \cdot) - U \cdot \alpha(\mathcal{A}_{\alpha(v,v)} v, \cdot). \end{aligned}$$

If this 1-form acts upon v we have, taking account of condition a):

$$d)\quad 2U\cdot(\nabla_v\alpha)(v,a)+U\cdot(\nabla^2\alpha)(v,v,v,v)+U\cdot(\nabla_{a^\top}\alpha)(v,v)=0.$$

Finally, given a vector $y=y^2t_2+\cdots+y^kt_k$ we have that that:

$$\begin{aligned}-U\cdot\alpha(y,t_j)=&U\cdot(\nabla^2\alpha)(v,v,v,t_j)+U\cdot(\nabla_{a^\top}\alpha)(v,t_j)\\&-U\cdot\alpha(\mathcal{A}_{\alpha(v,v)}v,t_j)+2U\cdot(\nabla_v\alpha)(a^\top,t_j),\end{aligned}$$

which is condition e). Finally, $B^\perp=0$ can be always satisfied because it is free. □

Let us see the case of a surface. By using the same notation as in the previous section, we obtain the following Proposition, the proof is trivial.

Proposition 5.2.6 *Let M be a surface immersed in $\mathbb{R}^{2+n}$, $m\in M$. Then ψ admits a 3-critical point over (v,a,U), where $0\neq v\in T_mM$, $a\in T^2_{(m,v)}M$ and $U\in N_mM$ iff the following conditions are satisfied:*

a) $U\cdot\alpha(v,\cdot)=0$;

b) $U\cdot(\nabla_v\alpha)(v,v)=0$;

c) $U\cdot\alpha(Jv,Jv)\neq 0 \quad$ *or* $\quad U\cdot(\nabla_v\alpha)(v,Jv)=0$;

d) $2U\cdot(\nabla_v\alpha)(v,a)+U\cdot(\nabla^2_v\alpha)(v,v,v)+U\cdot(\nabla_{a^\top}\alpha)(v,v)=0$;

e) $U\cdot\alpha(Jv,Jv)\neq 0$ *or*

$$\begin{aligned}&+U\cdot(\nabla^2\alpha)(v,v,v,Jv)+U\cdot(\nabla_{a^\top}\alpha)(v,Jv)\\&+2U\cdot(\nabla_v\alpha)(a,Jv)-U\cdot\alpha(\mathcal{A}_{\alpha(v,v)}v,Jv)=0\,,\end{aligned}$$

where $J:T_mM\to T_mM$ denotes the rotation of $90°$.

Future work

Usually the computations of the study of the contact between a k-dimensional manifold M immersed in $\mathbb{R}^{k+n}$ and a hypersphere or hyperplane is based on the derivatives of a Monge chart defined in M. However, as the contact increases, the operations become less affordable. This is one of the reasons why there is no many results in the study of the contact between a manifold and a hypersphere or hyperplane for a large number n or k.

Although the results presented here have demonstrated the effectiveness of the critical points, it could be further developed in a number of ways.

In chapter 5 we give the theorems for 3-critical points of the normal map and the generalized Gauss map. In a future work, it could be studied the applications of these theorems in the contact between a surface immersed in $\mathbb{R}^{2+n}$ and hyperspheres $n \geq 3$ or hyperplanes for $n > 3$.

Finally, we could study the contact between 3-manifolds immersed in $\mathbb{R}^{3+n}$ and hyperspheres or hyperplanes by using the corresponding critical points.

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