The star partial order and the eigenprojection at 0 on EP matrices*

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Abstract

The space of $n \times n$ complex matrices with the star partial order is considered in the first part of this paper. The class of EP matrices is analyzed and several properties related to this order are given. In addition, some information about predecessors and successors of a given EP matrix is obtained. The second part is dedicated to the study of some properties that relate the eigenprojection at 0 with the star and sharp partial orders.

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1 Introduction and Preliminaries

Partial orders on matrices have been studied by several authors. They involve different generalized inverses in their definitions. Some results about matrices, generalized inverses and partial orders on matrices can be found, for instance, in [7, 8, 22, 24, 25, 27, 30, 31]. Different applications of partial orders in many areas, such as statistics, generalized inverses, electrical networks, etc. can be found in [3, 4, 24, 26, 28].

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A matrix A is called EP if the projectors $AA^\dagger$ and $A^\dagger A$, determined by its Moore-Penrose inverse, are equal. Several representations were given for these matrices. For example, Tian et al. summarize thirty five characterizations of EP matrices in [29]. Theorems 2.3 and 2.4 in the present paper are related to these characterizations and Theorems 2.1 and 2.2 state conditions under which an EP matrix commutes with an arbitrary matrix.

The star partial order, denoted by $\leq^*$, was introduced by Drazin in [13] and it has been widely studied in the literature. In [1], Baksalary et al. studied certain partial orders on complex matrices and some relationships with their powers. In particular, the authors showed that if A is an EP matrix and B is a successor of A under the star partial order then B^2 is a successor of A^2 as well. Moreover, in [2], under the same hypothesis it was proved that A and B commute. Related to its algebraic structure, Hartwig et al. established the fact that the set of complex $m \times n$ matrices is a lower semi-lattice with respect to star partial order in [17].

The minus partial order, denoted by $\leq^-$, was introduced by Hartwig and it has been analyzed by several authors [15, 18, 19]. For instance, some results about the minus partial order on a class of nonnegative matrices are found in [5, 6].

The sharp partial order, denoted by $\leq^\#$, was introduced by Mitra in [23] on the class of index 1 matrices. By using the singular value decomposition for these matrices, Groß showed in [14] that A is a predecessor of B with respect to the sharp partial order if and only if A and B have a special form. This result will be recovered in our Theorem 3.5 for the star partial order since Moore-Penrose inverse and group inverse coincide in the class of EP matrices. However, we present a direct proof based on a factorization of EP matrices.

Let $\mathbb{C}^{m\times n}$ denote the space of complex $m \times n$ matrices; in particular, I_n stands for the identity matrix of size $n \times n$. The symbols A^*, A^{-1}, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ will denote the conjugate transpose, the inverse $(m = n)$, the range and the null space of a matrix $A \in \mathbb{C}^{m\times n}$, respectively. The symbol $A \oplus B$ stands for the direct sum of two square matrices A and B. If S is a subspace of $\mathbb{C}^n$, then $S^\perp$ will denote its orthogonal complementary subspace and P_S will stand for the orthogonal projector onto S along $S^\perp$.

For every matrix $A \in \mathbb{C}^{m\times n}$ there is a unique matrix in $\mathbb{C}^{n\times m}$, called the Moore-Penrose inverse of A, denoted by $A^\dagger$, which satisfies the four conditions below:

(M-P1) $AA^\dagger A = A$

(M-P2) $A^\dagger AA^\dagger = A^\dagger$

(M-P3) $(AA^\dagger)^* = AA^\dagger$

(M-P4) $(A^\dagger A)^* = A^\dagger A$

The following property on Moore-Penrose inverses will often be used [8].

(M-P) If $W \in \mathbb{C}^{n\times n}$, $V \in \mathbb{C}^{s\times s}$ and $U, T \in \mathbb{C}^{(n+s)\times(n+s)}$, with U and T unitary matrices, then $[U(W \oplus V)T]^\dagger = T^*(W^\dagger \oplus V^\dagger)U^*$.

A similar property to (M-P) is also valid for group inverses setting $U = P$ and $T = P^{-1}$, where $P \in \mathbb{C}^{n\times n}$ is nonsingular.

A matrix $A \in \mathbb{C}^{n\times n}$ with rank a is called *range-Hermitian* or *EP* if one of the following equivalent conditions is satisfied:

(EP1) $\mathcal{R}(A) \perp \mathcal{N}(A)$ (i.e., $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are mutually orthogonal subspaces of $\mathbb{C}^n$)

(EP2) $\mathcal{R}(A) = \mathcal{R}(A^*)$

(EP3) $\mathcal{N}(A) = \mathcal{N}(A^*)$

(EP4) $A = O$ or there exist a unitary matrix $U_A \in \mathbb{C}^{n\times n}$ and a nonsingular matrix $C_A \in \mathbb{C}^{a\times a}$ such that $A = U_A(C_A \oplus O)U_A^*$

(EP5) $AA^\dagger = A^\dagger A$

As usual, the last null row and column blocks may be absent in (EP4).

Notice that a nonzero matrix $A \in \mathbb{C}^{n\times n}$ with rank a is *EP* if and only if there exist a unitary matrix $V_A \in \mathbb{C}^{n\times n}$ and a nonsingular matrix $C_A \in \mathbb{C}^{a\times a}$ such that $A = V_A(O \oplus C_A)V_A^*$.

For two given matrices $A, B \in \mathbb{C}^{m\times n}$ it is said that A is below B under the star partial order, and it is denoted by $A \leq^* B$, if one of the following equivalent conditions is satisfied:

(SO1) $A^*A = A^*B$ and $AA^* = BA^*$

(SO2) $A^\dagger A = A^\dagger B$ and $AA^\dagger = BA^\dagger$

(SO3) $A^\dagger A = B^\dagger A$ and $AA^\dagger = AB^\dagger$

In this case, it is also said that A is a predecessor of B or B is a successor of A.

Throughout this paper, a and b will stand for the rank of the matrices A and B, respectively.

The index of a matrix $A \in \mathbb{C}^{n\times n}$, denoted by $\mathrm{ind}(A)$, is the smallest nonnegative integer k such that $\mathcal{R}(A^k) = \mathcal{R}(A^{k+1})$. For nonsingular matrices, $\mathrm{ind}(A) = 0$. Let $\mathbb{C}_k^n$ denote the set of all matrices $A \in \mathbb{C}^{n\times n}$ of index k, for a given integer $k \geq 0$.

In [9, 10, 11, 12], the authors worked with the concept of eigenprojection at 0. Specifically, for $A \in \mathbb{C}^{n\times n}$ of index at most 1, it is possible to define its eigenprojection at 0 as $A^\pi = I - AA^\#$, which is the projection onto $\mathcal{N}(A)$ along $\mathcal{R}(A)$. The symbol $A^\#$ denotes the group inverse of A, which exists if and only if A has index at most 1. The group inverse of $A \in \mathbb{C}^{n\times n}$ is the only matrix $A^\#$ that satisfies $AA^\#A = A$, $A^\#AA^\# = A^\#$ and $AA^\# = A^\#A$. For two given matrices $A, B \in \mathbb{C}^{m\times n}$ it is said that A is below B under the sharp partial order, denoted by $A \leq^\# B$, if and only if $A^\#A = A^\#B$ and $AA^\# = BA^\#$. If A is *EP* then $A^\dagger = A^\#$. In the class of *EP* matrices the star partial order coincides with the sharp partial order.

This paper is organized as follows. In Section 2, some properties of EP matrices are given. In Section 3, the star partial order on the class of EP matrices is studied and different characterizations for predecessors and successors of a given EP matrix are obtained. In Section 4, the eigenprojection at 0 is related to the star and sharp partial orders on the class of EP matrices.

2 EP matrices properties

In this section some properties of EP matrices are given. The first one relates EP matrices with commutativity [32]. Notice that if $A, B \in \mathbb{C}^{n\times n}$ are non-zero EP matrices, from (EP4) we can assume that they are given by

$$A = U_A(C_A \oplus O)U_A^* \tag{1}$$

and

$$B = U_B(C_B \oplus O)U_B^*, \tag{2}$$

where $U_A, U_B \in \mathbb{C}^{n\times n}$ are unitary and $C_A \in \mathbb{C}^{a\times a}$, $C_B \in \mathbb{C}^{b\times b}$ are nonsingular.

Theorem 2.1 *Let $A, B \in \mathbb{C}^{n\times n}$ be such that A is EP. The following conditions are equivalent:*

(a) $AB = BA$

(b) *If A is given by (1) then there exist $X \in \mathbb{C}^{a\times a}$ and $T \in \mathbb{C}^{(n-a)\times(n-a)}$ such that $B = U_A(X \oplus T)U_A^*$, with $C_AX = XC_A$.*

Proof. Let us consider the following decomposition of B:

$$B = U_A \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} U_A^*$$

where the partition has been carried out according to the size of the blocks of A. The equality $AB = BA$ is equivalent to

$$(C_A \oplus O)\begin{pmatrix} X & Y \\ Z & T \end{pmatrix} = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}(C_A \oplus O).$$

By making some algebraic computations, this last expression leads to $C_AX = XC_A$, $Y = O$, $Z = O$. ■

When both matrices A and B are EP, we have the following result for commutativity.

Theorem 2.2 *Let $O \neq A, B \in \mathbb{C}^{n\times n}$ be EP matrices in the form given by (1) and (2), respectively. If $U_A^* U_B = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$, with $X \in \mathbb{C}^{a\times b}$, $Y \in \mathbb{C}^{a\times(n-b)}$, $Z \in \mathbb{C}^{(n-a)\times b}$ and $T \in \mathbb{C}^{(n-a)\times(n-b)}$, then the following conditions are equivalent:*

(a) $AB = BA$

(b) $(X^* C_A X)\, C_B = C_B\,(X^* C_A X)$, $X^* C_A Y = O$, $Y^* C_A X = O$

(c) $C_A(X C_B X^*) = (X C_B X^*) C_A$, $Z C_B X^* = O$, $X C_B Z^* = O$

Proof. (a) $\Leftrightarrow$ (b) Substituting decompositions of A and B into the equality $AB = BA$ and replacing $U_A^* U_B$ with $\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$, we get $U_A(C_A X C_B \oplus O)U_B^* = U_B(C_B X^* C_A \oplus O)U_A^*$. Hence, pre-multiplying by U_B^* and post-multiplying by U_B both sides of the equality and replacing again $U_A^* U_B$ with its block decomposition we have

$$\begin{pmatrix} X^* C_A X C_B & O \\ Y^* C_A X C_B & O \end{pmatrix} = \begin{pmatrix} C_B X^* C_A X & C_B X^* C_A Y \\ O & O \end{pmatrix}.$$

This equality is equivalent to $X^* C_A X C_B = C_B X^* C_A X$, $X^* C_A Y = O$ and $Y^* C_A X = O$ because C_A and C_B are nonsingular matrices.

(a) $\Leftrightarrow$ (c) Substituting $U_B = U_A \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$ into the decomposition of B we obtain

$$B = U_A \begin{pmatrix} X C_B X^* & X C_B Z^* \\ Z C_B X^* & Z C_B Z^* \end{pmatrix} U_A^*.$$

Now, some computations show that $AB = BA$ is equivalent to the condition (c). ∎

Theorem 2.3 *Let $O \neq A \in \mathbb{C}^{n\times n}$. The following conditions are equivalent:*

(a) *A is EP (in the form given by (1)).*

(b) *There exist a nonsingular matrix $T \in \mathbb{C}^{(n-a)\times(n-a)}$ and $Z \in \mathbb{C}^{(n-a)\times a}$ such that $A = A^* Q$ with $Q = U_A \begin{pmatrix} (C_A^*)^{-1} C_A & O \\ Z & T \end{pmatrix} U_A^*$.*

(c) *There exist a nonsingular matrix $T \in \mathbb{C}^{(n-a)\times(n-a)}$ and $Z \in \mathbb{C}^{(n-a)\times a}$ such that $A = A^\dagger Q$ with $Q = U_A \begin{pmatrix} C_A^2 & O \\ Z & T \end{pmatrix} U_A^*$.*

Proof. (a) $\Rightarrow$ (b) There is a nonsingular matrix $Q \in \mathbb{C}^{n\times n}$ such that $A = A^*Q$ by (EP2). Let us consider the following decomposition of Q:

$$Q = U_A \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} U_A^*$$

where the partition has been carried out according to the size of the blocks of A. Some calculations yield $Q = U_A \begin{pmatrix} (C_A^*)^{-1}C_A & O \\ Z & T \end{pmatrix} U_A^*$. The nonsingularity of T is derived from the nonsingularity of Q.

(b) $\Rightarrow$ (a) If there is a nonsingular matrix Q such that $A = A^*Q$ then A and A^* are column equivalent, i.e., $\mathcal{R}(A) = \mathcal{R}(A^*)$, hence A is EP.

(a) $\Leftrightarrow$ (c) It is similar to the above equivalence taking into account that $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*)$. ■

The following result is similar to Theorem 2.3.

Theorem 2.4 *Let $O \neq A \in \mathbb{C}^{n\times n}$. The following conditions are equivalent:*

(a) *A is EP (in the form given by (1)).*

(b) *There exist a nonsingular matrix $T \in \mathbb{C}^{(n-a)\times(n-a)}$ and $Y \in \mathbb{C}^{a\times(n-a)}$ such that $A = PA^*$ with $P = U_A \begin{pmatrix} C_A(C_A^*)^{-1} & Y \\ O & T \end{pmatrix} U_A^*$.*

(c) *There exist a nonsingular matrix $T \in \mathbb{C}^{(n-a)\times(n-a)}$ and $Y \in \mathbb{C}^{a\times(n-a)}$ such that $A = PA^\dagger$ with $P = U_A \begin{pmatrix} C_A^2 & Y \\ O & T \end{pmatrix} U_A^*$.*

Proof. It is similar to the proof of Theorem 2.3 considering the fact that A and B are row equivalent if and only if $\mathcal{N}(A) = \mathcal{N}(B)$ and the property $\mathcal{N}(A^*) = \mathcal{N}(A^\dagger)$. ■

3 The star partial order and *EP* matrices

In this section several characterizations for the star partial order on the class of EP matrices are given. Some known results are obtained by means of direct proofs where the canonical form (under unitary similarity) for these matrices is used.

A first observation is the following: Let $A, B \in \mathbb{C}^{n\times n}$ be such that $A \leq^* B$. Then the matrices A^*B, BA^*, $A^\dagger B$ and $BA^\dagger$ are Hermitian.

Remark 3.1 By Theorems 2.3 and 2.4 we derive the following observation when the star partial order is also involved. Let $O \neq A, B \in \mathbb{C}^{n\times n}$ be such that A is EP and $A \leq^* B$. Then there are nonsingular matrices $P, Q \in \mathbb{C}^{n\times n}$ such that the following statements hold: (a) $A^*(A^*Q - B) = O$, (b) $(A^*Q - B)A^* = O$, (c) $A^*(PA^* - B) = O$, (d) $(PA^* - B)A^* = O$. Notice that under the same assumptions as in Corollary 3.1, items (a)-(d) remain valid if we replace $*$ with $\dagger$.

We now characterize the predecessors of an EP matrix under the star partial order.

Theorem 3.1 *Let $A, B \in \mathbb{C}^{n\times n}$ be such that B is a non-zero EP matrix. If B is written as in (2) then the following conditions are equivalent:*

(a) $A \leq^* B$

(b) *There exists $X \in \mathbb{C}^{b\times b}$ such that*

$$A = U_B(X \oplus O)U_B^* \qquad \textit{with} \qquad X \leq^* C_B. \tag{3}$$

Proof. (a) $\Rightarrow$ (b) Let us consider the following decomposition of A

$$A = U_B \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} U_B^*$$

where the partition has been carried out according to the size of the blocks of B. Then

$$A^*A = U_B \begin{pmatrix} X^*X + Z^*Z & X^*Y + Z^*T \\ Y^*X + T^*Z & Y^*Y + T^*T \end{pmatrix} U_B^*, \quad A^*B = U_B \begin{pmatrix} X^*C_B & O \\ Y^*C_B & O \end{pmatrix} U_B^*.$$

The equality $A^*A = A^*B$ yields $X^*X + Z^*Z = X^*C_B$, $X^*Y + Z^*T = O$, $Y^*X + T^*Z = Y^*C_B$, $Y^*Y + T^*T = O$. Then $Y = O$ and $T = O$. A similar computation gives $Z = O$. Hence, $X \leq^* C_B$.

(b) $\Rightarrow$ (a) It is trivial. ■

In general, if B is an EP matrix and $A \leq^* B$, then A is not necessarily EP. For example, we can consider

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

In what follows, some equivalent conditions for A to be an EP matrix are given when B is EP and A is a predecessor of B.

Theorem 3.2 *Let $A, B \in \mathbb{C}^{n\times n}$ be such that $B \neq O$ is EP and $A \leq^* B$. Let us consider the decomposition of B given by (2) and the decomposition of A given by (3). The following conditions are equivalent:*

(a) *A is an EP matrix.*

(b) $XC_B = C_B X$

(c) $X^\dagger C_B = C_B X^\dagger$

(d) $X(X^* - C_B) = C_B(X^* - X)$

(e) $(X^* - C_B)X = (X^* - X)C_B$

(f) *X is an EP matrix.*

Proof. Let B be as in (2). By Theorem 3.1, there exists $X \in \mathbb{C}^{b\times b}$ such that $A = U_B(X \oplus O)U_B^*$ with $X \leq^* C_B$.

(a) $\Leftrightarrow$ (b) Since $A \leq^* B$, the property (SO3) implies that A is EP if and only if A and $B^\dagger$ commute. Thus, replacing in $AB^\dagger = B^\dagger A$ each matrix with its decomposition and by making some computations, it results $XC_B = C_B X$, because $C_B^\dagger = C_B^{-1}$ and U_B is unitary.

In a similar way, the equivalences (a) $\Leftrightarrow$ (c), (b) $\Leftrightarrow$ (d), (b) $\Leftrightarrow$ (e) and (c) $\Leftrightarrow$ (f) can be shown by using (SO1) and (SO2). ■

Remark 3.2 If one of the equivalent conditions (a)-(f) of Theorem 3.2 is satisfied, then the following statements hold: $X = XC_B X^\dagger$ and $X^* = X^* C_B X^\dagger = C_B^* X X^\dagger = C_B^* X^\dagger C_B = C_B^* C_B X^\dagger$.

Corollary 3.1 *Let $A, B \in \mathbb{C}^{n\times n}$ be such that B is EP and $A \leq^* B$. The following conditions are equivalent:*

(a) *A is an EP matrix.*

(b) $AB = BA$

(c) $A(A^* - B) = B(A^* - A)$

(d) $(A^* - B)A = (A^* - A)B$

When the predecessor is an EP matrix, a similar result to that of Theorem 3.1 is given in the following theorem.

Theorem 3.3 *Let $A, B \in \mathbb{C}^{n\times n}$ be such that $A \neq O$ is an EP matrix written as in* (*1*). *The following conditions are equivalent:*

(a) $A \leq^* B$

(b) *There exists $T \in \mathbb{C}^{(n-a)\times(n-a)}$ such that*

$$B = U_A(C_A \oplus T)U_A^*. \tag{4}$$

Proof. The proof is similar to that of Theorem 3.1. ■

As shown before, if A is an EP matrix and $A \leq^* B$, then B is not necessarily EP. For example, we can consider

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

In what follows, some equivalent conditions for B to be an EP matrix are given when A is EP and B is a successor of A.

Theorem 3.4 *Let $A, B \in \mathbb{C}^{n\times n}$ be such that $A \neq O$ is EP and $A \leq^* B$. Let us consider the decompositions of A and B given by* (*1*) *and* (*4*)*, respectively. The following conditions are equivalent:*

(a) *B is an EP matrix.*

(b) *T is an EP matrix.*

(c) $B(A^\dagger - B^\dagger) = (A^\dagger - B^\dagger)B$

(d) $B^\dagger(A - B) = (A - B)B^\dagger$

Proof. Let $A = U_A(C_A \oplus O)U_A^*$ and $B = U_A(C_A \oplus T)U_A^*$.

(a) $\Leftrightarrow$ (b) It follows from the properties (M-P) and (EP5).

(a) $\Rightarrow$ (c) Using (SO2) and (EP5) one has $B(A^\dagger - B^\dagger) = BA^\dagger - BB^\dagger = A^\dagger B - B^\dagger B = (A^\dagger - B^\dagger)B$.

(c) $\Rightarrow$ (a) Since $A \leq^* B$ and A is EP by (SO2) it results that $A^\dagger$ and B commute, then from (c) it is obtained $BB^\dagger = B^\dagger B$, i.e., B is EP.

(a) $\Leftrightarrow$ (d) Since $A \leq^* B$ and A is EP by (SO3) it results that A and $B^\dagger$ commute. Now, it is easy to see that item (d) is equivalent to the condition $BB^\dagger = B^\dagger B$. ■

Remark 3.3 Let $A, B \in \mathbb{C}^{n\times n}$. If $AB = BA$ then the following conditions are trivially equivalent:

(a) $A \leq^* B$.

(b) $A(A^* - B) = B(A^* - A)$, $(A^* - B)A = (A^* - A)B$.

Remark 3.4 Let $A, B \in \mathbb{C}^{n\times n}$ be such that $A \leq^* B$ and A is EP. Then by Theorem 2.1 in [2] we get that A and B commute. So, the following properties hold:

(a) $A(A^* - B) = B(A^* - A)$

(b) $(A^* - B)A = (A^* - A)B$

(c) $BA^\dagger = A^\dagger A$

(d) $A(A^\dagger - B^\dagger) = (A^\dagger - B^\dagger)A$

(e) $A^\dagger(A - B) = (A - B)A^\dagger$

The next result generalizes Theorem 2.1 of Merikoski et al. [21].

Theorem 3.5 *Let $A, B \in \mathbb{C}^{n\times n}$ be EP with $A \neq O$. The following statements are equivalent:*

(a) $A \leq^* B$

(b) *There exist $V \in \mathbb{C}^{n\times n}$, $C \in \mathbb{C}^{a\times a}$ and $T \in \mathbb{C}^{(b-a)\times(b-a)}$ such that $A = V(C \oplus O \oplus O)V^*$ and $B = V(C \oplus T \oplus O)V^*$ where V is unitary, C is nonsingular and T is nonsingular or $T = O$.*

Proof. (a)$\Rightarrow$(b) Let $B = U_B(C_B \oplus O)U_B^*$. By Theorem 3.1, there exists $X \in \mathbb{C}^{b\times b}$ such that $A = U_B(X \oplus O)U_B^*$ with $X \leq^* C_B$. Theorem 3.2 assures that X is an EP matrix, so we can consider $X = U_X(C_X \oplus O)U_X^*$, where $U_X \in \mathbb{C}^{b\times b}$ is unitary and $C_X \in \mathbb{C}^{a\times a}$ is a nonsingular matrix. Since $X \leq^* C_B$, by Theorem 3.3 there exists $T \in \mathbb{C}^{(b-a)\times(b-a)}$ such that $C_B = U_X(C_X \oplus T)U_X^*$. Hence, we get that T is nonsingular when $T \neq O$. By setting $C = C_X$, we obtain $B = V(C \oplus T \oplus O)V^*$, where $V = U_B(U_X \oplus I)$ is a unitary matrix. A similar computation gives $A = V(C \oplus O \oplus O)V^*$.

(b) $\Rightarrow$ (a) It is easy to see that (SO2) holds. $\blacksquare$

Remark 3.5 When Theorem 3.5 is restricted to normal (resp. Hermitian) matrices, the blocks C and T have to be normal (resp. Hermitian) matrices.

Hartwig, Katz and Koliha studied when the product of two EP matrices is EP [16, 20]. What can we say about the relationship between the product of two EP matrices and the partial star order? The next results state that if $A \leq^* B$ then AB is a successor of A (or equivalently a predecessor of B) only when A is an orthogonal projector.

Theorem 3.6 *Let $A, B \in \mathbb{C}^{n\times n}$ be EP such that $A \leq^* B$. Then*

(a) *$ind(AB) = ind(A)$, $rank(AB) = rank(A)$, AB is EP and $AB = A^2 = BA$.*

(b) *AB is a successor of A (or equivalently a predecessor of B) if and only if A is an orthogonal projector (or equivalently $AB = A$).*

Proof. If $A = O$ the conclusion is evident. Let assume $A \neq O$. Then the matrices A and B have the form indicated in item (b) of Theorem 3.5. Thus, it is easy to see that $AB = V(C^2 \oplus O \oplus O)V^*$.

(a) It follows directly from definitions.

(b) The conditions: (i) $A \leq^* AB$, (ii) $AB \leq^* B$, (iii) A is an orthogonal projector, (iv) $A = AB$ are equivalent, which can be shown taking into account that A is an orthogonal projector if and only if there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $A = U(I_a \oplus O \oplus O)U^*$. $\blacksquare$

Remark 3.6 If $A \in \mathbb{C}^{n\times n}$ is EP then A^j is also EP, for all $j \in \mathbb{N}$. Moreover, $A^k \leq^* A^s$ if and only if A^{s-k} is an orthogonal projector for all $k, s \in \mathbb{N}$ with $s \geq k$.

4 On the eigenprojection at 0

The purpose of this section is to study the eigenprojection at 0 and its relation to the considered partial orders. Specifically, we show that A is EP if and only if A^π is EP and we derive a characterization for $(A^\pi)^\pi = A$ to be valid. Predecessors and successors are obtained when the eigenprojection at 0 of two comparable matrices is considered.

The following result is known.

Lemma 4.1 *[8, 9] Let $A \in \mathbb{C}_0^n \cup \mathbb{C}_1^n$.*

(a) *A^π is idempotent.*

(b) *$AA^\pi = A^\pi A = A^\# A^\pi = A^\pi A^\# = O$.*

(c) *$A^\pi = O$ if and only if A is nonsigular.*

(d) *If $A \in \mathbb{C}_1^n$ and $a > 0$ then there exist nonsingular matrices $C \in \mathbb{C}^{a\times a}$ and $P \in \mathbb{C}^{n\times n}$ such that*

$$A = P(C \oplus O)P^{-1}. \tag{5}$$

In this case, $A^\# = P(C^{-1} \oplus O)P^{-1}$ and $A^\pi = P(O \oplus I_{n-a})P^{-1}$.

(e) *If A^π is nonsingular then $A = O$. Conversely, if $A = O$ then $A^\pi = I_n$.*

(f) *A^π has index at most 1.*

(g) *$(A^\pi)^\pi = O$ if and only if $A = O$.*

(h) *$rank(A^\pi) = n - rank(A)$.*

Lemma 4.2 *Let $A, B \in \mathbb{C}_0^n \cup \mathbb{C}_1^n$.*

(a) *If $A \leq^\# B$ then $rank(A) \leq rank(B)$.*

(b) *$A \leq^\# A^\pi$ if and only if $A = O$.*

(c) *$A^\pi \leq^\# A$ if and only if A is nonsingular.*

(d) *If A is a non-zero singular matrix then A and A^π are incomparable under the sharp order.*

(e) *If $B^\pi \leq^\# A^\pi$ then $rank(A) \leq rank(B)$. However, in general, $A \not\leq^\# B$.*

(f) *$A \leq^\# A + A^\pi$.*

Proof.

(a) $\mathcal{R}(A) = \mathcal{R}(AA^\# A) \subseteq \mathcal{R}(AA^\#) = \mathcal{R}(BA^\#) \subseteq \mathcal{R}(B)$, thus $\mathrm{rank}(A) \leq \mathrm{rank}(B)$.

(b) If $A \leq^\# A^\pi$ then $A^\# A = A^\# A^\pi = A^\# - A^\# AA^\# = O$, so $A = AA^\# A = O$. Conversely, if $A = O$, $A^\# = O$ and $A^\pi = I_n$, then $A \leq^\# A^\pi$.

(c) By Lemma 4.1 (d), it is clear that $(A^\pi)^\# = A^\pi$. If $A^\pi \leq^\# A$ then $A^\pi = A^\pi A = O$, therefore A is nonsingular by Lemma 4.1 (c). Conversely, if A is nonsingular then $A^\pi = O$, thus $A^\pi \leq^\# A$.

(d) Follows from (b) and (c).

(e) Follows from (a) and by Lemma 4.1 (h). It is enough to consider $A = \mathrm{diag}(1,0)$ and $B = \mathrm{diag}(2,0)$ to check that, in general, $A \not\leq^{\#} B$.

(f) Follows from definitions. $\blacksquare$

Let $\mathcal{EP}$ be the set of all square complex EP matrices of size $n \times n$. It is well known that $\mathcal{EP} \subseteq \mathbb{C}_0^n \cup \mathbb{C}_1^n$; moreover, if $A \in \mathcal{EP}$ then $A^{\pi} = I - AA^{\dagger}$.

Lemma 4.3 *Let $A \in \mathbb{C}_0^n \cup \mathbb{C}_1^n$.*

(a) *If $A \in \mathcal{EP}$ then A^{π} is Hermitian (therefore, $A^{\pi} \in \mathcal{EP}$).*

(b) *If $A^{\pi} \in \mathcal{EP}$ then $A \in \mathcal{EP}$.*

Proof.

(a) It follows directly from the property $(AA^{\dagger})^* = AA^{\dagger}$.

(b) By Lemma 4.1 (c), if $A^{\pi} = O$ then A is nonsingular. Hence, A is EP. If A^{π} is a non-zero EP matrix, there exist a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a nonsingular matrix $C \in \mathbb{C}^{a\times a}$ such that $A^{\pi} = U(C \oplus O)U^*$. Because A^{π} is idempotent, we get $C^2 = C$, that is $C = I_{n-a}$. Since, $A^{\#}A = AA^{\#} = I_n - A^{\pi}$, we get $AA^{\#} = U(O \oplus I_a)U^*$. Let us consider the decomposition of A

$$A = U \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} U^*$$

where the partition has been carried out according to the size of the blocks of A^{π}. From $A = (AA^{\#})A$, it is obtained $X = O$, $Y = O$. Similarly, from $A = A(A^{\#}A)$ it is obtained $Z = O$, therefore $A = U(O \oplus T)U^*$. Then $AA^{\#} = U(O \oplus TT^{\#})U^*$. Hence, T is nonsingular. $\blacksquare$

Notice that if A^{π} is EP then A could not be Hermitian (it is enough to consider a nonsingular and non-Hermitian matrix A).

Let $f : \mathbb{C}_0^n \cup \mathbb{C}_1^n \longrightarrow \mathbb{C}_0^n \cup \mathbb{C}_1^n$ be the function defined by $f(A) = A^{\pi}$ for each $A \in \mathbb{C}_0^n \cup \mathbb{C}_1^n$. By Lemma 4.1, f is well defined and $f(f(O)) = O$. However, in general, $f(f(A)) \neq A$. For example, let us consider the matrix $A = \mathrm{diag}(2,0)$. Then $A^{\#} = \mathrm{diag}(1/2,0)$, $A^{\pi} = \mathrm{diag}(0,1) = (A^{\pi})^{\#}$ and $f(f(A)) = \mathrm{diag}(1,0)$.

In Lemma 4.1 (g), we have shown that $f(f(A)) = O$ if and only if $A = O$.

Lemma 4.4 *Let f be the function previously defined and $A \in \mathbb{C}_0^n \cup \mathbb{C}_1^n$ a non-zero matrix decomposed as in (5). Then $f(f(A)) = A$ if and only if $C = I_a$.*

Proof. It is easy to check that $AA^{\#} = P(I_a \oplus O)P^{-1}$ and $f(A) = P(O \oplus I_{n-a})P^{-1}$. The equalities $f(A)^{\#} = f(A)$ and $f(f(A)) = P(I_a \oplus O)P^{-1}$ imply $A = P(I_a \oplus O)P^{-1}$. ■

Remark 4.1 If $A \in \mathcal{EP}$ then $f(f(A)) = A$ if and only if A is an orthogonal projector.

By using Lemma 4.1 (c)-(e), it is easy to prove the following assertion by induction on k. Let f^k denote $f^1 = f$ and $f^{k+1} = f \circ f^k$, for every integer $k \geq 1$.

Corollary 4.1 *Let $k \geq 1$ be an integer and $A \in \mathbb{C}_0^n \cup \mathbb{C}_1^n$. The following statements hold:*

(a) *If $A \in \mathbb{C}_0^n$ then* $f^k(A) = \begin{cases} O & \text{if } k \text{ is odd} \\ I_n & \text{if } k \text{ is even} \end{cases}$.

(b) *If $A \in \mathbb{C}_1^n - \{O\}$ then* $f^k(A) = \begin{cases} f(A) & \text{if } k \text{ is odd} \\ I_n - f(A) & \text{if } k \text{ is even} \end{cases}$.

(c) *If $A = O$ then* $f^k(A) = \begin{cases} I_n & \text{if } k \text{ is odd} \\ O & \text{if } k \text{ is even} \end{cases}$.

Let us consider the sets

$$\mathcal{EEP} = \{A \in \mathbb{C}_0^n \cup \mathbb{C}_1^n : f(A) \in \mathcal{EP}\}$$

and

$$\mathcal{EEP}_0 = \{A \in \mathbb{C}_0^n \cup \mathbb{C}_1^n : f(A) \in \mathcal{EP} \text{ and } f(A) \neq O\}.$$

The next result provides a characterization of EP matrices that follows by Lemma 4.1 and Lemma 4.3.

Theorem 4.1 *The following statements hold.*

(a) $\mathcal{EP} = \mathcal{EEP}$.

(b) $\mathcal{EP} \cap \mathbb{C}_1^n = \mathcal{EEP}_0$.

By Lemma 4.3 (a), it is clear that $f(\mathcal{EP}) \subseteq \mathcal{EP}$, but in general the equality is not true as the matrix $A = \text{diag}(2, 0)$ shows. Let us suppose that there exists $M \in \mathbb{C}^{2\times 2} \cap \mathcal{EP}$ such that $A = f(M)$. By Lemma 4.1 (h), $\text{rank}(M) = 1$ and so $M \in \mathcal{EP}$ by Lemma 4.3 (b). If

$M = U\mathrm{diag}(c,0)U^*$, with $U \in \mathbb{C}^{2\times 2}$ a unitary matrix and c a nonzero complex number, it is easy to see that $f(M) = U\mathrm{diag}(0,1)U^*$, which is a contradiction because $2 \in \sigma(A)$ and $2 \notin \sigma(f(M))$, where $\sigma(A)$ denotes the spectrum of A. Therefore, $\mathcal{EP} \nsubseteq f(\mathcal{EP})$.

Remark 4.2 Let $\mathcal{OP}_n$ be the set of all orthogonal projectors of size $n \times n$. By Lemma 4.1, Remark 4.1 and Theorem 4.1 we obtain

(a) $f(\mathbb{C}_1^n - \{O\}) \subseteq \mathbb{C}_1^n - \{O\}$.

(b) $f(\mathcal{EEP}_0 - \{O\}) = \mathcal{OP}_n \cap (\mathbb{C}_1^n - \{O\}) = (\mathcal{OP}_n \cap \mathcal{EEP}_0) - \{O\}$.

(c) $f(\mathcal{EEP}_0) = (\mathcal{OP}_n \cap (\mathbb{C}_1^n - \{O\})) \cup \{I_n\} = ((\mathcal{OP}_n \cap \mathcal{EEP}_0) - \{O\}) \cup \{I_n\}$.

Let $g : \mathcal{EP} \longrightarrow \mathcal{EP}$ be the restriction of the function f to the set $\mathcal{EP}$. By Lemma 4.3 (a), g is well defined. It is clear that g is not surjective. Moreover, g is not injective as the matrices $A = \mathrm{diag}(2,0)$ and $B = \mathrm{diag}(3,0)$ show.

The next lemma characterizes the interval

$$[O, g(A)] = \{B \in \mathbb{C}_0^n \cup \mathbb{C}_1^n : O \leq^* B \leq^* g(A)\}.$$

Lemma 4.5 *Let $A \in \mathbb{C}^{n\times n}$ be an EP matrix given by (1). Then*

$$[O, g(A)] = \{U_A(O \oplus T)U_A^* : T \in \mathcal{OP}_{n-a}\} \subseteq \mathcal{OP}_n.$$

Proof. Let $A = U_A(C_A \oplus O)U_A^*$, with $U_A \in \mathbb{C}^{n\times n}$ unitary and $C_A \in \mathbb{C}^{a\times a}$ nonsingular. Then $f(A) = U_A(O \oplus I_{n-a})U_A^*$. If we assume $B \leq^* g(A)$, then $B = U_A(O \oplus T)U_A^*$, where $T \leq^* I_{n-a}$, that is, $T = T^* = T^2$. ■

Theorem 4.2 *The function g above defined is monotone decreasing.*

Proof. Let $A, B \in \mathbb{C}^{n\times n}$ be EP matrices such that $A \leq^* B$. By using properties (SO2), (SO3), (EP5) and Theorem 2.1 in [2] we get $AA^\dagger BB^\dagger = ABA^\dagger B^\dagger = BAA^\dagger B^\dagger = BA^\dagger AB^\dagger = BB^\dagger AB^\dagger = BB^\dagger AA^\dagger$ and $BB^\dagger AA^\dagger = BA^\dagger AA^\dagger = BA^\dagger = AA^\dagger$. These two last equations are equivalent to $g(B) = g(B)g(A)$ and $g(B) = g(A)g(B)$. Therefore, $g(B) \leq^* g(A)$. ■

However, the EP matrices $A = \mathrm{diag}(1,0)$ and $B = \mathrm{diag}(2,0)$ satisfy $g(A) \leq^* g(B)$ but $B \nleq^* A$. We can state the following theorem for a smaller class of matrices.

Theorem 4.3 *Let $A, B \in \mathcal{OP}_n$. If $g(B) \leq^* g(A)$ then $A \leq^* B$.*

Proof. The proof follows applying Theorem 4.2 and Remark 4.1. ■

Theorem 4.4 *Let $A \in \mathcal{EP}$ and $B \in \mathbb{C}_0^n \cup \mathbb{C}_1^n$ be such that $A \leq^* B$. Then $f(B) \leq^* f(A)$ if and only if $f(B) \in \mathcal{OP}_n$.*

Proof. Let A given by (1). By Theorem 3.3, $B = U(C_A \oplus T)U^*$ and so $f(B) = U(O \oplus f(T))U^*$. If we suppose $f(B) \leq^* f(A)$, by Theorem 3.1, we get $f(B) = U(O \oplus X)U^*$ with $X \leq^* I_{n-a}$. Therefore, $X = f(T)$. Hence, $f(T) \in \mathcal{OP}_{n-a}$.

Conversely, if we suppose that $f(B) \in \mathcal{OP}_n$ then B is EP. Now, Theorem 4.2 yields $f(B) \leq^* f(A)$. ■

For two given matrices $A, B \in \mathbb{C}_0^n \cup \mathbb{C}_1^n$, necessary and sufficient conditions for the equality $f(A) = f(B)$ to be satisfied are given in [9, 11]. Some results about inequalities are given below.

If $A \in \mathbb{C}_0^n \cup \mathbb{C}_1^n$ is the zero matrix then the only matrix $B \in \mathbb{C}^{n\times n}$ such that $A^\pi \leq^\# B^\pi$ is $B = O$. The remaining matrices satisfying this last inequality are found in the next result.

Theorem 4.5 *Let $O \neq A \in \mathbb{C}_0^n \cup \mathbb{C}_1^n$ be as in (5) and $B \in \mathbb{C}_0^n \cup \mathbb{C}_1^n$. Then $f(A) \leq^\# f(B)$ if and only if there exists $Q \in \mathbb{C}_0^a \cup \mathbb{C}_1^a$ such that $B = P(Q \oplus O)P^{-1}$.*

Proof. Suppose that $f(A) \leq^\# f(B)$. Then $f(A)f(B) = f(B)f(A) = f(A)$. Let us consider

$$f(B) = P \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} P^{-1},$$

where the partition has been carried out according to the size of the blocks of A. By making some algebraic manipulations we obtain $Y = O, Z = O$ and $T = I_{n-a}$. Thus, $f(B) = P(X \oplus I_{n-a})P^{-1}$. So, $BB^\# = B^\# B = I_n - f(B) = P((I_a - X) \oplus O)P^{-1}$. If we partition

$$B = P \begin{pmatrix} Q & W \\ R & S \end{pmatrix} P^{-1}$$

according to the blocks of $f(B)$, the equality $B = (BB^\#)B$ yields $R = O$, $S = O$, $XQ = O$, $XW = O$. Analogously, from $B = B(B^\# B)$, we get $W = O$, $QX = O$. That is, $B = P(Q \oplus O)P^{-1}$ with $QX = O$, $XQ = O$. Now, it is easy to see that $X = f(Q)$. It is clear that $Q \in \mathbb{C}^{a\times a}$ has index at most 1 and the conditions $QX = O$, $XQ = O$ are always true. The converse is evident. ■

The next result can be shown by means of a similar technique to that used in the previous theorem.

Theorem 4.6 *Let $O \neq A \in \mathbb{C}_0^n \cup \mathbb{C}_1^n$ be as in (5) and $B \in \mathbb{C}_0^n \cup \mathbb{C}_1^n$. Then $f(B) \leq^{\#} f(A)$ if and only if there exist $S \in \mathbb{C}_0^{n-a} \cup \mathbb{C}_1^a$ and B_i of adequate sizes, $i \in \{1,2,3,4\}$, such that*

$$f(B) = P(O \oplus S)P^{-1}, \qquad B = P \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} P^{-1},$$

where $B_2S = O$, $B_4S = O$, $SB_3 = O$, $SB_4 = O$.

In what follows, we analyze how to locate all the linear combinations between two given EP matrices. The trivial case $A = B$ is discarded and we shall denote $A <^* B$ when $A \leq^* B$ and $A \neq B$.

Theorem 4.7 *Let $A, B \in \mathbb{C}^{n\times n}$ be two EP matrices such that $A <^* B$. Then the linear combinations $C_{\alpha,\beta} = \alpha A + \beta B$, with $\alpha, \beta \in \mathbb{C}$, are EP matrices with rank equals 0 if $\alpha = \beta = 0$; rank equals $b - a$ if $\alpha + \beta = 0 \neq \beta$; rank equals a if $\alpha \neq 0 = \beta$; rank equals b if $\alpha + \beta \neq 0$, $\beta \neq 0$. Moreover,*

(a) *$A \leq^* C_{\alpha,\beta}$ if and only if $A = O$ or $\alpha + \beta = 1$.*

(b) *$C_{\alpha,\beta} \leq^* B$ if and only if*

  (I) *$A \neq O$ and one of the following conditions holds: (i) $\alpha = \beta = 0$, (ii) $\beta = 1$, $\alpha = -1$, (iii) $\beta = 0$, $\alpha = 1$, (iv) $\beta = 1$, $\alpha = 0$.*

  (II) *$A = O$ and one of the following conditions holds: (i) $\beta = 0$, (ii) $\beta = 1$.*

(c) *$C_{\alpha,\beta} \leq^* C_{\gamma,\delta}$ if and only if*

  (I) *$A \neq O$ and one of the following conditions holds: (i) $\alpha = \beta = 0$, (ii) $\alpha = -\beta = -\delta$, (iii) $\alpha = \gamma + \delta$, $\beta = 0$, (iv) $\alpha = \gamma$, $\beta = \delta$.*

  (II) *$A = O$ and one of the following conditions holds: (i) $\beta = 0$, (ii) $\beta = \delta$.*

(d) *$f(C_{\alpha,\beta}) \leq^* f(A)$ if and only if $A = O$, or $A \neq O$ and one of the following conditions holds: (i) $\alpha + \beta \neq 0$, $\beta \neq 0$, (ii) $\alpha \neq 0$, $\beta = 0$.*

(e) $f(B) \leq^* f(C_{\alpha,\beta})$ *for every* $\alpha, \beta \in \mathbb{C}$.

Proof. By Theorem 3.5, there exist $V \in \mathbb{C}^{n\times n}$, $C \in \mathbb{C}^{a\times a}$ and $T \in \mathbb{C}^{(b-a)\times(b-a)}$ such that $A = V(C \oplus O \oplus O)V^*$ and $B = V(C \oplus T \oplus O)V^*$, where V is unitary, C is nonsingular and $T = O$ or nonsingular. Then $C_{\alpha,\beta} = V((\alpha+\beta)C \oplus \beta T \oplus O)V^*$. Notice that $C_{\alpha,\beta}$ is an EP matrix for all $\alpha, \beta \in \mathbb{C}$ and the statement about the rank follows directly. It is easy to see that all the remaining items can be shown using the above decompositions of A, B and $C_{\alpha,\beta}$. ■

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References

[1] J. K. Baksalary, O. M. Baksalary, X. Liu, Further relationships between certain partial orders of matrices and their squares, *Linear Algebra and its Applications*, 375, 171-180, 2003.

[2] J. K. Baksalary, J. Hauke, X. Liu, S. Liu, Relationships between partial orders of matrices and their powers, *Linear Algebra and its Applications*, 379, 277–287, 2004.

[3] J. K. Baksalary, J. Hauke, G. P. H. Styan, On some distributional properties of quadratic forms in normal variables and on some associated matrix partial orderings, *Multivariate Analysis and its Applications*, 24, 111–121, 1994.

[4] J. K. Baksalary, S. Puntanen, Characterizations of the best linear unbiased estimator in the general Gauss-Markov model with the use of matrix partial orderings. *Linear Algebra and its Applications*, 127, 363–370, 1990.

[5] R. B. Bapat, S. K. Jain, L. E. Snyder, Nonnegative idempotent matrices and minus partial order, *Linear Algebra and its Applications*, 261, 143–154, 1997.

[6] B. Blackwood, S. K. Jain, Nonnegative group-monotone matrices and the minus partial order, *Linear Algebra and its Applications*, 430, 121–132, 2009.

[7] A. Ben-Israel, T. Greville, *Generalized Inverses: Theory and Applications.* John Wiley & Sons, Second Edition, 2003.

[8] S.L. Campbell, C.D. Meyer Jr., *Generalized Inverse of Linear Transformations.* Dover, New York, Second Edition, 1991.

[9] N. Castro-González, J.J. Koliha, Y. Wei, Perturbation of the Drazin inverse for matrices with equal eigenprojections at zero, *Linear Algebra and its Applications*, 312, 335–347, 2000.

[10] N. Castro-González, J. J. Koliha, Y. Wei, Error bounds for perturbation of the Drazin inverse of closed operators with equal spectral projections, *Applied Analysis*, 81, 915–928, 2002.

[11] N. Castro-Gonzlez, J- Vélez-Cerrada. Characterizations of matrices which eigenprojections at zero are equal to a fixed perturbation, *Applied Mathematics and Computation*, 159, 613–623, 2004.

[12] D. S. Djordjevic, Y. Wei, Operators with equal projections related to their generalized inverses, *Applied Mathematics and Computations*, 155, 3, 655–664, 2004.

[13] M. P. Drazin, Natural structures on semigroups with involution, *Bull. Amer. Math. Soc.*, 84, 139–141, 1978.

[14] J. Groß, Remarks on the sharp partial order and the ordering of squares of matrices, *Linear Algebra and its Applications*, 417, 87–93, 2006.

[15] R.E. Hartwig, How to partially order regular elements? *Math. Japon.* 25, 1-13, 1980.

[16] R. E. Hartwig, I. J. Katz, On products of EP matrices, *Linear Algebra and its Applications*, 252, 1–3, 339–345, 1997.

[17] R. E. Hartwig, M. P. Drazin, Lattice properties of the $*$-order for complex matrices, *Journal of Mathematical Analysis and Applications*, 86, 2, 359–378, 1982.

[18] R. E. Hartwig, G. P. H. Styan, On some characterizations of the star partial ordering for matrices and rank sustractivity, *Linear Algebra and its Applications*, 82, 145–161, 1986.

[19] J. Hauke, A. Markiewicz, T. Szulc, Inter- and extrapolary properties of matrix partial orderings, *Linear Algebra and its Applications*, 332–334, 437–445, 2001.

[20] J. J. Koliha, A simple proof of the product theorem for EP matrices, *Linear Algebra and its Applications*, 294, 213–215, 1999.

[21] J. K. Merikoski, X. Liu, On the star partial ordering of normal matrices, *Journal of Inequalities in Pure and Applied Mathematics*, 7, 1, Art. 17, 2006.

[22] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra.* SIAM, Philadelphia, 2000.

[23] S.K. Mitra, On group inverses and the sharp order, *Linear Algebra and its Applications*, 92, 17-37, 1987.

[24] S. K. Mitra, P. Bhimasankaram, S. B. Malik, *Matrix partial orders, shorted operators and applications.* World Scientific Publishing Company, 2010.

[25] V.V. Prasolov, *Problems and Theorems in Linear Algebra.* American Mathematical Society, Providence, RI, 1994.

[26] S. Puntanen, G. P.H. Styan, Best linear unbiased estimation in linear models (version 8). *StatProb: The Encyclopedia Sponsored by Statistics and Probability Societies.* Freely available at http://statprob.com/-encyclopedia/BestLinearUnbiasedEstimatinInLinearModels.html

[27] C. R. Rao, S. K. Mitra, *Generalized inverse of matrices and its applications.* John Wiley & Sons, New York, 1971.

[28] C. Stepniak, Ordering of nonnegative definite matrices with application to comparison of linear models, *Linear Algebra and its Applications*, 70, 67–71, 1987.

[29] Y. Tian, H. Wang, Characterizations of EP matrices and weighted-EP matrices, *Linear Algebra and its Applications*, 434, 5, 1295–1318, 2011.

[30] M. Tošić, D. S. Cvetković-Ilić, *Invertibility of a linear combination of two matrices and partial orderings*, *Applied Mathematics and Computation*, 218, 9, 4651–4657, 2012.

[31] Y. Wei, C. Deng, A note on additive results for the Drazin inverse, *Linear and Multilinear Algebra*, 59, 1319–1329, 2011.

[32] H. Zha, Z. Zhang, Computing the optimal commuting matrix pairs, *BIT*, 37, 202–220, 1997.