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FURTHER RESULTS ON THE REVERSE ORDER LAW FOR THE GROUP INVERSE IN RINGS *

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Abstract. In this paper, we use the Drazin inverse to derive some new equivalences of the reverse order law for the group inverse in unitary rings. Moreover, if the ring has an involution, we present more equivalences when both involved elements are EP.

Keywords: Group inverse; Drazin inverse; Moore-Penrose inverse; reverse order law

AMS-Subject Classification: 15A09, 16W10.

1. Introduction. Let \mathcal{R} be a unitary ring whose unity is $\mathbb{1}$. For $a, b \in \mathcal{R}$, the *commutator* of a, b is defined as $[a, b] = ab - ba$. Let $a \in \mathcal{R}$. It can be easily proved that the set of $x \in \mathcal{R}$ satisfying the following conditions

$$axa = a, \quad xax = x, \quad ax = xa \quad (1.1)$$

is either empty or a singleton. When there exists such x , then a is said to be *group invertible* and x is denoted by $a^\#$. A useful fact about group inverses is the following: If a is a group invertible element of a unitary ring \mathcal{R} , then $a^\#$ double commutes with a , that is, if $z \in \mathcal{R}$ satisfies $[a, z] = 0$, then $[a^\#, z] = 0$ (see e.g. [6, Lemma 1.4.5]). We shall denote by $\mathcal{R}^\#$ the subset of \mathcal{R} consisting of group invertible elements and by \mathcal{R}^{-1} the set of standard invertible elements. If $a \in \mathcal{R}^\#$, the *spectral idempotent* of a is defined as $a^\pi = \mathbb{1} - aa^\#$. Also, the following result on group inverses will be used. For the proof, the interested reader can consult [15, Proposition 8.22].

THEOREM 1.1. *Let a be an element of a unitary ring \mathcal{R} . Then a is group invertible if and only if exist $x, y \in \mathcal{R}$ such that $a^2x = a$ and $ya^2 = a$. In this case, one has $a^\# = yax$.*

With each element a of a unitary ring \mathcal{R} we associate two right ideals:

$$a\mathcal{R} = \{ax : x \in \mathcal{R}\}, \quad a^\circ = \{x \in \mathcal{R} : ax = 0\}.$$

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Let a, b be elements of a unitary ring \mathcal{R} . The element b is a *Drazin inverse* of a if

$$ab = ba, \quad b = ab^2, \quad a^k = a^{k+1}b$$

for some nonnegative integer k . It can be proved (see [7, Theorem 1]) that such b is unique and it is customarily denoted a^d . The least nonnegative integer k for which these equalities hold is the *Drazin index* $i(a)$ of a . In [7, Theorem 4] it was proved that an element $a \in \mathcal{R}$ is Drazin invertible if and only if there are nonnegative integers p, q and $u, v \in \mathcal{R}$ such that $a^{p+1}u = a^p$ and $va^{q+1} = a^q$. The smallest value of p for which $\{u \in \mathcal{R} : a^{p+1}u = a^p\} \neq \emptyset$ is called the *left index* of a , denoted by $l(a)$. In a similar way the *right index* of a is defined, and is denoted by $r(a)$. In a remark following [7, Theorem 4] it was shown that in case that a is Drazin invertible, then $i(a) = l(a) = r(a)$.

An involution in a ring \mathcal{R} is a map $a \mapsto a^*$ such that for any $a, b \in \mathcal{R}$,

$$(a^*)^* = a, \quad (a + b)^* = a^* + b^*, \quad (ab)^* = b^* a^*.$$

If the ring \mathcal{R} has an involution, then we can define another class of generalized inverse. An element $a \in \mathcal{R}$ is *Moore-Penrose invertible* if there exists $x \in \mathcal{R}$ such that

$$axa = a, \quad xax = x, \quad (ax)^* = ax, \quad (xa)^* = xa. \quad (1.2)$$

Such x , when exists, is unique and is denoted by a^\dagger .

An element $a \in \mathcal{R}$, where \mathcal{R} is a ring with involution, is said to be *self-adjoint* when $a = a^*$. We say that a is *EP* when a is Moore-Penrose invertible and $aa^\dagger = a^\dagger a$. Observe that when a is EP, then a is group invertible and $a^\dagger = a^\#$, and that any self-adjoint element, Moore-Penrose invertible or group invertible, is EP.

The reverse order law for generalized inverses plays an important role in many areas including singular matrix problem, ill-posed problems, optimization, and statistics (see e.g. [2, 8, 16, 17, 18, 19, 20]). These problems have attracted considerable attention since the middle 1960s and many interesting results have been obtained. T.N.E. Greville [9] proved that $(AB)^\dagger = B^\dagger A^\dagger$ if and only if $R(A^*AB) \subseteq R(B)$ and $R(BB^*A^*) \subseteq R(A^*)$, for complex matrices A and B , where $R(\cdot)$ denotes the column space. This result was extended to linear bounded operators on Hilbert spaces in [10]. Another characterization of the reverse order law for the Moore-Penrose inverse is due to Arghiriade [1]: For complex matrices A, B such that AB exists, $(AB)^\dagger = B^\dagger A^\dagger$ if and only if A^*ABB^* is EP. The interested reader can consult [2, Section 4.4] for the proof of the original results of Greville and Arghiriade. Later, the reverse order law for the Moore-Penrose inverse was considered in rings with involution (see [11]). C.Y. Deng [4] presented some equivalent conditions concerning the reverse order law $(AB)^\# = B^\# A^\#$ for group invertible operators A, B on a Hilbert space. N.Č. Dinčić

and D.S. Djordjević [5] gave new equivalences of the reverse order law for the Moore-Penrose inverse for operators on Hilbert spaces. D. Mosić and D.S. Djordjević [13] investigated some necessary and sufficient conditions for the reverse order law for the group inverse in rings. The hybrid reverse order law $(ab)^\# = b^\dagger a^\dagger$ in rings was studied in [14].

In this paper, we give new equivalent conditions of the reverse order law for the group inverse in unitary rings. Later, we state some new results related to the reverse order law for the group inverse when both involved elements are EP.

The word *idempotent* will be reserved for an element p of a unitary ring \mathcal{R} such that $p^2 = p$. Also, we will write $\bar{p} = \mathbf{1} - p$. If in addition \mathcal{R} has an involution, then we will say that an element p is a *projection* when $p = p^2 = p^*$.

2. Preliminary results. If \mathcal{R} is a unitary ring and $p \in \mathcal{R}$ is an idempotent, then every $x \in \mathcal{R}$ has the following matrix representation

$$x = \begin{bmatrix} pxp & px\bar{p} \\ \bar{p}xp & \bar{p}x\bar{p} \end{bmatrix}_p$$

If, in addition, \mathcal{R} has an involution and p is a projection, then the above matrix representation preserves this involution. More precisely, if $x \in \mathcal{R}$ is represented as $x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_p$, then $x^* = \begin{bmatrix} x_1^* & x_3^* \\ x_2^* & x_4^* \end{bmatrix}_p$.

If a is an element of a ring \mathcal{R} , then

$$a \in \mathcal{R}^\# \iff \text{there is an idempotent } p \in \mathcal{R} \text{ such that } a + p \in \mathcal{R}^{-1} \text{ and } ap = pa = 0. \quad (2.1)$$

Such a p , when it exists, is unique (see [15, Proposition 8.24]). This unique idempotent p is the spectral idempotent of a —recall that the spectral idempotent of a is customarily written by a^π and $a^\pi = \mathbf{1} - aa^\#$ holds. Hence, if $a \in \mathcal{R}^\#$, we can represent

$$a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{aa^\#}, \quad a^\# = \begin{bmatrix} a^\# & 0 \\ 0 & 0 \end{bmatrix}_{aa^\#}, \quad a^\pi = \begin{bmatrix} 0 & 0 \\ 0 & a^\pi \end{bmatrix}_{aa^\#}. \quad (2.2)$$

When (2.1) is applied to a group invertible matrix $A \in \mathbb{C}^{n \times n}$, by writing the idempotent A^π as $U(0 \oplus I_k)U^{-1}$, where $U \in \mathbb{C}^{n \times n}$ is nonsingular and I_k denotes the identity matrix of order k (see e.g. [21, Theorem 5.1]), one easily gets the existence of a nonsingular matrix $B \in \mathbb{C}^{(n-k) \times (n-k)}$ such that $A = U(B \oplus 0)U^{-1}$. Obviously, we have also $A^\# = U(B^{-1} \oplus 0)U^{-1}$.

LEMMA 2.1. *Let \mathcal{R} be a unitary ring and $m \in \mathcal{R}$. If $p \in \mathcal{R}$ is an idempotent,*

$$m = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}_p, \quad (2.3)$$

then

- (i) If m is group invertible and c is Drazin invertible, then a, c are group invertible and $a^\pi bc^\pi = 0$.
- (ii) If a, c are group invertible and $a^\pi bc^\pi = 0$, then m is group invertible and

$$m^\# = \begin{bmatrix} a^\# & (a^\#)^2 bc^\pi + a^\pi b(c^\#)^2 - a^\# bc^\# \\ 0 & c^\# \end{bmatrix}_p.$$

Proof. (i) Assume that m is group invertible and c is Drazin invertible. Let us write

$$m^\# = \begin{bmatrix} u & v \\ w & z \end{bmatrix}_p. \quad (2.4)$$

From $m^2 m^\# = m$ we get

$$c^2 w = 0 \quad (2.5)$$

and

$$c^2 z = c. \quad (2.6)$$

Equality (2.6) implies $l(c) \leq 1$. Since c is Drazin invertible we get $i(c) = l(c) \leq 1$. Thus, c is group invertible. By premultiplying by $c^\#$ the equality (2.5) we get

$$cw = 0. \quad (2.7)$$

By using $m(m^\#)^2 = m^\#$ and (2.7) we get $czw = w$. From (2.6) and $c \in \mathcal{R}^\#$ we get $cz = c^\# c$. Hence (2.7) leads to $w = czw = c^\# cw = 0$. By looking at the ‘‘north-west’’ and ‘‘south-east’’ blocks of $mm^\#m = m$, $m^\#mm^\# = m^\#$, and $mm^\# = m^\#m$ when (2.3), (2.4), and $w = 0$ are used, one gets the group invertibility of a , $a^\# = u$, and $c^\# = z$.

From $m^2 m^\# = m$ we have

$$a^2 v + abc^\# + bcc^\# = b, \quad (2.8)$$

Premultiplication of (2.8) by $(a^\#)^2$ yields $aa^\#v + a^\#bc^\# + (a^\#)^2bcc^\# = (a^\#)^2b$, or equivalently

$$aa^\#v = (a^\#)^2bc^\pi - a^\#bc^\#. \quad (2.9)$$

From $m^\#m^2 = m$ we get $a^\#ab + a^\#bc + vc^2 = b$. Similarly as before, postmultiplication of this last equality by $(c^\#)^2$ gives

$$vcc^\# = a^\pi b(c^\#)^2 - a^\#bc^\#. \quad (2.10)$$

Finally, from $m^\#mm^\# = m^\#$, (2.9), and (2.10) one obtains

$$v = aa^\#v + a^\#bc^\# + vcc^\# = (a^\#)^2bc^\pi + a^\pi b(c^\#)^2 - a^\#bc^\#.$$

Insertion of this expression of v in (2.8) and a further simplification lead to $a^\pi bc^\pi = 0$.

(ii) follows by verifying that the expression for $m^\#$ obeys (1.1), which has a unique solution, namely the group inverse of m . \square

REMARK 2.2. Evidently, Lemma 2.1 is valid for complex matrices $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ for $A \in \mathbb{C}^{n \times n}$ and $C \in \mathbb{C}^{m \times m}$. This case was obtained in [12]. In this setting, the assumption of the Drazin invertibility of C can be removed because any square complex matrix has a unique Drazin inverse (see e.g., [2, Section 4.6]).

REMARK 2.3. A special case of Lemma 2.1 is the following: Let \mathcal{R} be a unitary ring and $m \in \mathcal{R}$. If $p \in \mathcal{R}$ is an idempotent, $m = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}_p$, then $m \in \mathcal{R}^\#$ if and only if $a \in \mathcal{R}^\#$ and $b = aa^\#b$. In this case,

$$m^\# = \begin{bmatrix} a^\# & (a^\#)^2b \\ 0 & 0 \end{bmatrix}_p.$$

REMARK 2.4. The hypothesis “ c is Drazin invertible” in item (i) of Lemma 2.1 can be changed by “ a is Drazin invertible”. The sketch of the proof is as follows: From (2.3), (2.4), and $m^\#m^2 = m$ we get $ua^2 = a$ and $wa^2 = 0$. Hence $r(a) \leq 1$, and the Drazin invertibility of a leads to $i(a) = 1$. Thus, $a \in \mathcal{R}^\#$ and so, $ua = aa^\#$ and $wa = 0$. From $m^\# = (m^\#)^2m$ we get $w = wua$; which in conjunction with $ua = aa^\#$ and $wa = 0$ yields $w = 0$. The proof finishes now as in Lemma 2.1.

REMARK 2.5. There is a version of Lemma 2.1 when the zero element is located at the “north-east” position. Assume that all the hypothesis of Lemma 2.1 hold except that $m = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}_p$. Since $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}_p = \begin{bmatrix} c & b \\ 0 & a \end{bmatrix}_{\bar{p}}$, we get

$$\begin{aligned} m^\# &= \begin{bmatrix} c^\# & (c^\#)^2ba^\pi + c^\pi b(a^\#)^2 - c^\#ba^\# \\ 0 & a^\# \end{bmatrix}_{\bar{p}} \\ &= \begin{bmatrix} a^\# & 0 \\ (c^\#)^2ba^\pi + c^\pi b(a^\#)^2 - c^\#ba^\# & c^\# \end{bmatrix}_p \end{aligned}$$

and $c^\pi ba^\pi = 0$.

LEMMA 2.6. *Let \mathcal{R} be a unitary ring and $m \in \mathcal{R}$. If $p \in \mathcal{R}$ is an idempotent and $m = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}_p$, then $m \in \mathcal{R}^\#$ if and only if $a, c \in \mathcal{R}^\#$. In this case, one has $m^\# = \begin{bmatrix} a^\# & 0 \\ 0 & c^\# \end{bmatrix}_p$.*

Proof. Assume that m is group invertible. Let us write $m^\# = \begin{bmatrix} x & y \\ z & t \end{bmatrix}_p$. From $m^2m^\# = m$ and $m^\#m^2 = m$ we get $a^2x = a = xa^2$ and $c^2t = c = tc^2$. By Theorem 1.1, we obtain that a and c are group invertible. Conversely, if a and c are group invertible, it is enough to apply Lemma 2.1 to get this Lemma. \square

LEMMA 2.7. *Let \mathcal{R} be a unitary ring and $a, b \in \mathcal{R}$ such that a, b are group invertible. If a is represented as in (2.2) and*

$$b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{aa^\#}, \quad (2.11)$$

then the following conditions are equivalent:

- (i) ab is group invertible.
- (ii) ab_1 is group invertible and $ab_2 = ab_1(ab_1)^\#ab_2$.

Under this equivalence, one has

$$(ab)^\# = \begin{bmatrix} (ab_1)^\# & ((ab_1)^\#)^2ab_2 \\ 0 & 0 \end{bmatrix}_{aa^\#}. \quad (2.12)$$

Proof. It follows from Remark 2.3. \square

Let \mathcal{R} be a unitary ring and $a, b \in \mathcal{R}$. If a is group invertible and b is represented as in (2.11), then some equalities that will be used many times in the sequel are

$$b_1aa^\# = aa^\#b_1 = b_1 \quad \text{and} \quad aa^\#b_2 = b_2, \quad (2.13)$$

since $b_1 = pbp$ and $b_2 = pb\bar{p}$, where $p = aa^\#$. In particular, observe that a is invertible in the subring $p\mathcal{R}p$, $a^\#$ is the inverse of a in $p\mathcal{R}p$, and p is the unity in this subring. But, we will avoid the use of the subring $p\mathcal{R}p$ along the paper by utilizing the equalities (2.13).

LEMMA 2.8. *Let \mathcal{R} be a unitary ring and $a, b \in \mathcal{R}$ such that a, b, ab are group invertible. If b is represented as in (2.11), then $(ab_1)^\#ab_2 = 0$ if and only if $b_2 = 0$.*

Proof. Let us assume that $(ab_1)^\#ab_2 = 0$ holds. Lemma 2.7 and the group invertibility of ab yield $ab_2 = ab_1(ab_1)^\#ab_2 = 0$, which implies $b_2 = 0$ in view of the second equality of (2.13). The converse is trivial. \square

LEMMA 2.9. *Let \mathcal{R} be a unitary ring and $a, b \in \mathcal{R}$ such that a, b, ab are group invertible. If a is represented as in (2.2),*

$$b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{aa^\#}, \quad b^\# = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}_{aa^\#}, \quad (2.14)$$

and b_1 or b_4 are Drazin invertible, then the following conditions are equivalent:

- (i) $(ab)^\# = b^\#a^\#$.
- (ii) $(ab_1)^\# = c_1a^\#$, $((ab_1)^\#)^2ab_2 = 0$, and $c_3 = 0$.
- (iii) b_1 is group invertible, $b_2 = b_3 = 0$, and $(ab_1)^\# = b_1^\#a^\#$.

Proof. Let us denote $p = aa^\#$.

(i) \Leftrightarrow (ii): We apply Lemma 2.7. Combining (2.2), (2.12), and (2.14), it is easy to conclude that

$$(ab)^\# = b^\#a^\# \quad \Leftrightarrow \quad (ab_1)^\# = c_1a^\#, \quad ((ab_1)^\#)^2ab_2 = 0, \quad c_3a^\# = 0.$$

From (2.14) we have $c_3 = \bar{p}b^\#p$. Thus, $c_3a^\# = \bar{p}b^\#pa^\# = \bar{p}b^\#aa^\#a^\# = \bar{p}b^\#a^\#$. If $c_3a^\# = 0$, then $c_3 = \bar{p}b^\#p = \bar{p}b^\#a^\#a = 0$. Trivially, if $c_3 = 0$, then $c_3a^\# = 0$. Therefore, the equivalence between items (i) and (ii) of the theorem has been proved.

(ii) \Rightarrow (iii): By premultiplying $((ab_1)^\#)^2ab_2 = 0$ by ab_1 , we get $(ab_1)^\#ab_2 = 0$. Since ab is group invertible, Lemma 2.8 yields $b_2 = 0$. Thus, $b = \begin{bmatrix} b_1 & 0 \\ b_3 & b_4 \end{bmatrix}_p$. Since b is group invertible and b_1 or b_4 are Drazin invertible, by using Lemma 2.1 and the representation of $b^\#$ given in (2.14), we get $c_2 = 0$. Recall that one hypothesis of (ii) is $c_3 = 0$. Therefore $b^\# = \begin{bmatrix} c_1 & 0 \\ 0 & c_4 \end{bmatrix}_p$. By Lemma 2.6 applied for $m = b^\#$, we get $b = (b^\#)^\# = \begin{bmatrix} c_1^\# & 0 \\ 0 & c_4^\# \end{bmatrix}_p$. So, $b_3 = 0$ and $b_1 = c_1^\#$, and this proves (iii).

The implication (iii) \Rightarrow (i) is trivial. \square

3. Reverse order law for the group inverse in rings. Now, we state the main result of this paper. Observe that if a, b are elements in a ring such that a is group invertible and b is represented as in (2.11), then $b_1 = aa^\#baa^\#$ and $b_4 = a^\pi ba^\pi$, which makes the connection between Lemma 2.9 and next result clearer.

THEOREM 3.1. *Let a, b be elements in a unitary ring \mathcal{R} such that a, b are group invertible. If $aa^\#baa^\#$ or $a^\pi ba^\pi$ are Drazin invertible, then the following statements are equivalent:*

- (i) $ab \in \mathcal{R}^\#$ and $(ab)^\# = b^\#a^\#$.
- (ii) $ab \in \mathcal{R}^\#$, $(ab)^\# = b^\#a^\#abb^\#a^\#$, $aa^\#ba^\pi = 0$, and $a^\pi baa^\# = 0$.
- (iii) $ab \in \mathcal{R}^\#$, $(ab)^\# = b^\#a^\#abb^\#a^\#$, and $a^\pi baa^\# = 0$.
- (iv) $a^\#ab, abb^\# \in \mathcal{R}^\#$, $(a^\#ab)^\# = b^\#a^\#a$, and $(abb^\#)^\# = bb^\#a^\#$.
- (v) $ab, a^\#abb^\# \in \mathcal{R}^\#$, $(ab)^\# = b^\#(a^\#abb^\#)^\#a^\#$, $(a^\#abb^\#)^\# = bb^\#a^\#a$.
- (vi) $ab, a^\#ab \in \mathcal{R}^\#$, $(ab)^\# = (a^\#ab)^\#a^\#$, and $(a^\#ab)^\# = b^\#a^\#a$.
- (vii) $ab, abb^\# \in \mathcal{R}^\#$, $(ab)^\# = b^\#(abb^\#)^\#$, $(abb^\#)^\# = bb^\#a^\#$.
- (viii) $ab \in \mathcal{R}^\#$, $ab(ab)^\# = abb^\#a^\# = b^\#a^\#ab$, $a^\pi baa^\# = 0$.
- (ix) $ab \in \mathcal{R}^\#$, $(b^\#a^\#)\mathcal{R} \subseteq (ab)^\#\mathcal{R}$, $(b^\#a^\#)^\circ \subseteq ((ab)^\#)^\circ$, and $abb^\# = bb^\#a$.

Proof. If we write a as in (2.2) and we denote $p = aa^\#$, then we can represent b and $b^\#$ as in (2.14).

(i) \Rightarrow (ii): Observe that $(ab)^\# = (ab)^\# ab(ab)^\# = b^\# a^\# abb^\# a^\#$. Lemma 2.9 yields $b_2 = b_3 = 0$, which is equivalent to say $aa^\# ba^\pi = a^\pi baa^\# = 0$.

(ii) \Rightarrow (iii) is evident.

(iii) \Rightarrow (i): Notice that $b_3 = a^\pi baa^\# = 0$. We can apply Lemma 2.1 to get $c_3 = 0$, $b_1^\# = c_1$, and $b_4^\# = c_4$. By employing $b = \begin{bmatrix} b_1 & b_2 \\ 0 & b_4 \end{bmatrix}_p$, $b^\# = \begin{bmatrix} b_1^\# & c_2 \\ 0 & b_4^\# \end{bmatrix}_p$, (2.13), and the representation of a given in (2.2), after a straightforward computation we get

$$b^\# a^\# abb^\# a^\# = \begin{bmatrix} b_1^\# a^\# & 0 \\ 0 & 0 \end{bmatrix}_p.$$

From $(ab)^\# = b^\# a^\# abb^\# a^\#$, it follows that $(ab_1)^\# = b_1^\# a^\#$ and $((ab_1)^\#)^2 ab_2 = 0$. By the equivalence (i) \Leftrightarrow (ii) of Lemma 2.9 we get $(ab)^\# = b^\# a^\#$.

(i) \Rightarrow (iv): Since $(ab)^\# = b^\# a^\#$, from the equivalence (i) \Rightarrow (iii) of Lemma 2.9, we deduce that $b = \begin{bmatrix} b_1 & 0 \\ 0 & b_4 \end{bmatrix}_p$, $ab_1, b_1 \in \mathcal{R}^\#$, and $(ab_1)^\# = b_1^\# a^\#$. It is evident from (2.13) that

$$a^\# ab = \begin{bmatrix} b_1 & 0 \\ 0 & 0 \end{bmatrix}_p \quad \text{and} \quad b^\# a^\# a = \begin{bmatrix} b_1^\# & 0 \\ 0 & 0 \end{bmatrix}_p.$$

Hence, $a^\# ab$ is group invertible and $(a^\# ab)^\# = b^\# a^\# a$. Now, let us prove that $ab_1 b_1^\#$ is group invertible and $(ab_1 b_1^\#)^\# = b_1 b_1^\# a^\#$. To this end, we shall apply the uniqueness of the group inverse and $pb_1 = b_1 p = b_1$, $pb_1^\# = b_1^\# p = b_1^\#$. Let $x = ab_1 b_1^\#$ and $y = b_1 b_1^\# a^\#$. Now,

$$xy = ab_1 b_1^\# a^\# = ab_1 (ab_1)^\# = (ab_1)^\# ab_1 = b_1^\# a^\# ab_1 = b_1^\# pb_1 = b_1^\# b_1$$

and

$$yx = b_1 b_1^\# pb_1 b_1^\# = b_1 b_1^\# b_1 b_1^\# = b_1^\# b_1$$

prove that $xy = yx$. Also we have

$$xyx = x(yx) = ab_1 b_1^\# b_1 b_1^\# = ab_1 b_1^\# = x$$

and

$$yxy = (yx)y = b_1 b_1^\# b_1 b_1^\# a^\# = b_1 b_1^\# a^\# = y.$$

Since

$$abb^\# = \begin{bmatrix} ab_1b_1^\# & 0 \\ 0 & 0 \end{bmatrix}_p \quad \text{and} \quad bb^\#a^\# = \begin{bmatrix} b_1b_1^\#a^\# & 0 \\ 0 & 0 \end{bmatrix}_p,$$

we get that $abb^\#$ is group invertible and $(abb^\#)^\# = bb^\#a^\#$.

(iv) \Rightarrow (i): We have

$$a^\#ab = \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix}_p. \quad (3.1)$$

By Remark 2.3 and since $a^\#ab \in \mathcal{R}^\#$, we get that $b_1 \in \mathcal{R}^\#$, $b_2 = b_1b_1^\#b_2$, and

$$(a^\#ab)^\# = \begin{bmatrix} b_1^\# & (b_1^\#)^2b_2 \\ 0 & 0 \end{bmatrix}_p. \quad (3.2)$$

By using $(a^\#ab)^\# = b^\#a^\#a$ one gets $b_1^\# = c_1$, $(b_1^\#)^2b_2 = 0$, and $0 = c_3$. Now we have $b_2 = b_1b_1^\#b_2 = b_1^2(b_1^\#)^2b_2 = 0$. Since $b^\#$ is group invertible, c_1 is Drazin invertible (because $c_1 = b_1^\#$), and $c_3 = 0$, Lemma 2.1 allows us to get $b_3 = 0$. Thus, $b = \begin{bmatrix} b_1 & 0 \\ 0 & b_4 \end{bmatrix}_p$, and then

$$(abb^\#)^\# = \begin{bmatrix} (ab_1b_1^\#)^\# & 0 \\ 0 & 0 \end{bmatrix}_p \quad \text{and} \quad bb^\#a^\# = \begin{bmatrix} b_1b_1^\#a^\# & 0 \\ 0 & 0 \end{bmatrix}_p.$$

Since $(abb^\#)^\# = bb^\#a^\#$, we get $(ab_1b_1^\#)^\# = b_1b_1^\#a^\#$. In particular, $ab_1b_1^\#$ and $b_1b_1^\#a^\#$ commute. Hence $ab_1b_1^\#b_1b_1^\#a^\# = b_1b_1^\#a^\#ab_1b_1^\#$, which reduces in view of (2.13) to $ab_1b_1^\#a^\# = b_1b_1^\#$. Now we prove that ab_1 is group invertible and $(ab_1)^\# = b_1^\#a^\#$. Let $x = ab_1$ and $y = b_1^\#a^\#$. From $ab_1b_1^\#a^\# = b_1b_1^\#$ we get $xy = yx$. Now, $xyx = x(yx) = ab_1b_1b_1^\# = ab_1 = x$ and $yx = (yx)y = b_1b_1^\#b_1^\#a^\# = b_1^\#a^\# = y$. The equivalence (i) \Leftrightarrow (iii) of Lemma 2.9 allows us to prove that ab is group invertible and $(ab)^\# = b^\#a^\#$.

(i) \Rightarrow (v): From the hypothesis and Lemma 2.9 we get $b_2 = b_3 = 0$, $ab_1, b_1 \in \mathcal{R}^\#$, and $(ab_1)^\# = b_1^\#a^\#$. Let us recall that any idempotent is group invertible and its group inverse is itself, hence $(b_1b_1^\#)^\# = b_1b_1^\#$. From (2.13) we have

$$a^\#abb^\# = \begin{bmatrix} a^\#ab_1b_1^\# & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} b_1b_1^\# & 0 \\ 0 & 0 \end{bmatrix}_p,$$

hence $a^\#abb^\# \in \mathcal{R}^\#$ and $(a^\#abb^\#)^\# = b_1b_1^\#$; which yields

$$\begin{aligned} b^\#(a^\#abb^\#)^\#a^\# &= \begin{bmatrix} b_1^\#b_1b_1^\#a^\# & 0 \\ 0 & 0 \end{bmatrix}_p \\ &= \begin{bmatrix} b_1^\#a^\# & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} (ab_1)^\# & 0 \\ 0 & 0 \end{bmatrix}_p = (ab)^\#. \end{aligned}$$

Also we have $(a^\#abb^\#)^\# = bb^\#a^\#a$.

(v) \Rightarrow (i): Obviously, $(ab)^\# = b^\#(a^\#abb^\#)^\#a^\# = b^\#bb^\#a^\#aa^\# = b^\#a^\#$.

(i) \Rightarrow (vi): Since we have proved (i) \Rightarrow (iv), we can use also (iv). Now, $(ab)^\# = b^\#a^\# = b^\#a^\#aa^\# = (a^\#ab)^\#a^\#$.

(vi) \Rightarrow (i): It is easy to see $(ab)^\# = (a^\#ab)^\#a^\# = b^\#a^\#aa^\# = b^\#a^\#$.

(i) \Rightarrow (vii): We can use (iv), and therefore, $(ab)^\# = b^\#a^\# = b^\#bb^\#a^\# = b^\#(abb^\#)^\#$.

(vii) \Rightarrow (i): Notice that $(ab)^\# = b^\#(abb^\#)^\# = b^\#bb^\#a^\# = b^\#a^\#$.

(i) \Rightarrow (viii): We will use the equivalence (i) \Rightarrow (iii) of Lemma 2.9. The equality $ab(ab)^\# = abb^\#a^\#$ is obvious from the hypothesis, and $a^\pi baa^\# = 0$ follows from $b_3 = 0$. Now, we have $ab(ab)^\# = (ab)^\#ab = b^\#a^\#ab$.

(viii) \Rightarrow (i): Observe that from $a^\pi baa^\# = 0$, we get $b_3 = 0$. Thus, $b = \begin{bmatrix} b_1 & b_2 \\ 0 & b_4 \end{bmatrix}_p$. From (2.12) and $ab(ab)^\# = abb^\#a^\#$, we get $(ab_1)^\#ab_2 = 0$. Lemma 2.8 leads to $b_2 = 0$. Further, from $abb^\#a^\# = b^\#a^\#ab$, we get $ab_1b_1^\#a^\# = b_1^\#b_1$. This implies that $b_1^\#a^\#$ obeys the equations to be the group inverse of ab_1 . Hence $(ab_1)^\# = b_1^\#a^\#$. Then (i) holds.

(i) \Rightarrow (ix): Since $(ab)^\# = b^\#a^\#$, we have $abb^\#a^\# = b^\#a^\#ab$, which by using Lemma 2.9 reduces to $ab_1b_1^\#a^\# = b_1b_1^\#$. A postmultiplication by a yields $ab_1b_1^\#a = b_1b_1^\#a$, which is equivalent to $abb^\# = bb^\#a$.

(ix) \Rightarrow (i): Observe that the expression of $(ab)^\#$ given in (2.12) can be used. Hence $a^\pi \in (b^\#a^\#)^\circ \subseteq [(ab)^\#]^\circ$ implies $(ab)^\#a^\pi = 0$, which leads to $((ab_1)^\#)^2ab_2 = 0$, or equivalently, $(ab_1)^\#ab_2 = 0$. Lemma 2.8 proves $b_2 = 0$. Hence (2.12) reduces to $(ab)^\# = \begin{bmatrix} (ab_1)^\# & 0 \\ 0 & 0 \end{bmatrix}_p$. Lemma 2.1 implies that $b_1^\# = c_1$ and $b_4^\# = c_4$.

Since $b^\#a^\# \in (b^\#a^\#)\mathcal{R} \subseteq (ab)^\#\mathcal{R}$, there exists $u \in \mathcal{R}$ such that

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}_p \begin{bmatrix} a^\# & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} (ab_1)^\# & 0 \\ 0 & 0 \end{bmatrix}_p \begin{bmatrix} pup & pu\bar{p} \\ \bar{p}up & \bar{p}u\bar{p} \end{bmatrix}_p.$$

Therefore, $0 = c_3a^\#$, which in conjunction of $c_3 = \bar{p}b^\#p$, implies $0 = c_3a^\#a = c_3p = c_3$. Lemma 2.1 yields $b_3 = 0$.

By Lemma 2.9, to prove $(ab)^\# = b^\#a^\#$, it is enough to prove $(ab_1)^\# = b_1^\#a^\#$. From $b = \begin{bmatrix} b_1 & 0 \\ 0 & b_4 \end{bmatrix}_p$ and $abb^\# = bb^\#a$, we get $ab_1b_1^\# = b_1b_1^\#a$, which by (2.13), leads to $ab_1b_1^\#a^\# = b_1b_1^\#aa^\# = b_1b_1^\# = b_1aa^\#b_1^\#$. The equalities

$$(ab_1)(b_1^\#a^\#)(ab_1) = ab_1 \quad \text{and} \quad (b_1^\#a^\#)(ab_1)(b_1^\#a^\#) = b_1^\#a^\#$$

are trivial to prove. Hence, by the uniqueness of the group inverse, $(ab_1)^\# = b_1^\# a^\#$. \square

REMARK 3.2. Let \mathcal{R} be a unitary ring and $a, b \in \mathcal{R}^\#$. The following two equivalences were proved in [13, Theorems 2.4 and 2.5]:

- $a^\# ab \in \mathcal{R}^\#$ and $(a^\# ab)^\# = b^\# aa^\# \Leftrightarrow a^\# ab = ba^\# a$.
- $ab^\# b \in \mathcal{R}^\#$ and $(abb^\#)^\# = bb^\# a^\# \Leftrightarrow abb^\# = b^\# ba$.

The following result was established in [4] when the setting is the Banach algebra of all bounded linear operators on a Hilbert space. However, we will establish this result in rings by using only algebraic techniques. Notice that, in particular, involution and norms will not be used (which were employed in [4]).

THEOREM 3.3. *Let \mathcal{R} be a unitary ring and $a, b \in \mathcal{R}^\#$ such that $ab \in \mathcal{R}^\#$.*

- (i) *If $aa^\# b \in \mathcal{R}^\#$, then $(ab)^\# = (aa^\# b)^\# a^\# \Leftrightarrow aba^\pi = 0$ and $[a, (aa^\# b)^\pi] = 0$.*
- (ii) *If $abb^\# \in \mathcal{R}^\#$, then $(ab)^\# = b^\# (abb^\#)^\# \Leftrightarrow b^\pi ab = 0$ and $[b, (abb^\#)^\pi] = 0$.*

Proof. Assume that $(ab)^\# = (aa^\# b)^\# a^\#$. From (2.12) and (3.2) we get $(ab_1)^\# = b_1^\# a^\#$ and $(ab_1)^\# ab_2 = 0$. By using Lemma 2.8 we get $0 = b_2 = pb\bar{p}$, which yields $0 = aba^\pi$. Also, since any group invertible element commutes with its group inverse, $(ab_1)^\# = b_1^\# a^\#$ implies $ab_1 b_1^\# a^\# = b_1^\# a^\# ab_1$, which reduces in view of (2.13) to $ab_1 b_1^\# a^\# = b_1^\# b_1 a^\#$, which by a postmultiplication by a , simplifies to $ab_1 b_1^\# = b_1^\# b_1 a$. By using this last equality, (3.1), (3.2), and $b_2 = 0$ we get that $(aa^\# b)(aa^\# b)^\#$ commutes with a , which is equivalent to $[a, (aa^\# b)^\pi] = 0$.

Let us prove the converse. Observe that $b_2 = a^\# aba^\pi = 0$. By using the representations (3.1), (3.2), and the hypotheses we get $ab_1 b_1^\# = b_1 b_1^\# a$. By the definition of the group inverse we easily get $(ab_1)^\# = b_1^\# a^\#$. Now, it is trivial to get $(ab)^\# = (aa^\# b)^\# a^\#$.

(ii) Let $q = bb^\#$. The proof is easy if we proceed as in previous item and use the following representations:

$$a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_q, \quad b = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}_q, \quad b^\# = \begin{bmatrix} b^\# & 0 \\ 0 & 0 \end{bmatrix}_q, \quad ab = \begin{bmatrix} a_1 b & 0 \\ a_3 b & 0 \end{bmatrix}_q,$$

$$(ab)^\# = \begin{bmatrix} (a_1 b)^\# & 0 \\ a_3 b ((a_1 b)^\#)^2 & 0 \end{bmatrix}_q, \quad abb^\# = \begin{bmatrix} a_1 & 0 \\ a_3 & 0 \end{bmatrix}_q, \quad (abb^\#)^\# = \begin{bmatrix} a_1^\# & 0 \\ a_3 (a_1^\#)^2 & 0 \end{bmatrix}_q.$$

We omit the details. \square

This last result appeared in [13] (Theorem 2.2 (i),(iii) and Theorem 2.3 (i), (iii)) with another proof.

COROLLARY 3.4. *Let \mathcal{R} be a unitary ring and $a, b \in \mathcal{R}$ be such that $a, b \in \mathcal{R}^\#$*

and $aa^\#baa^\#$ or $a^\pi ba^\pi$ are Drazin invertible. Then the following statements are equivalent:

- (i) $ab \in \mathcal{R}^\#$ and $(ab)^\# = b^\#a^\#$.
- (ii) $[a^\pi, b] = 0$, $[a, b^\pi] = 0$.
- (iii) $ab, aa^\#b \in \mathcal{R}^\#$, $aba^\pi = 0$, $[a, (a^\#ab)^\pi] = 0$, $[a^\pi, b] = 0$.
- (iv) $ab, abb^\# \in \mathcal{R}^\#$, $b^\pi ab = 0$, $[b, (abb^\#)^\pi] = 0$, $[a, b^\pi] = 0$.

Proof. The equivalence of item (iv) Theorem 3.1 and (ii) of this corollary has been noticed in Remark 3.2. The equivalence of item (vi) Theorem 3.1 and (iii) of this corollary has been proved in Theorem 3.3. In the same way, we have that item (vii) of Theorem 3.1 and (iv) of this corollary are equivalent. \square

4. Reverse order law for the group inverse in rings with involution.

In this section we will assume that the unitary ring \mathcal{R} has an involution. Let us recall that if the ring \mathcal{R} has an involution, then the Moore-Penrose invertibility in \mathcal{R} can be defined.

Taking involution in the definition of the group inverse we immediately have $a \in \mathcal{R}^\# \Leftrightarrow a^* \in \mathcal{R}^\#$ and in this case, $(a^*)^\# = (a^\#)^*$.

As we remarked, if an element a of a ring \mathcal{R} with involution is EP, then $a \in \mathcal{R}^\#$ and $a^\dagger = a^\#$. There is a version of (2.1) for this class of elements (see [3, Theorem 2.1]):

$$a \text{ is EP} \Leftrightarrow \text{there is a projection } p \in \mathcal{R} \text{ such that } a + p \in \mathcal{R}^{-1} \text{ and } ap = pa = 0. \quad (4.1)$$

The representation (2.2) is also valid. But, observe that this representation preserves the involution since aa^\dagger is self-adjoint. When (4.1) is applied to an EP matrix $A \in \mathbb{C}^{n \times n}$, by writing the idempotent A^π as $U(0 \oplus I_k)U^*$, where $U \in \mathbb{C}^{n \times n}$ is unitary (see e.g. [21, Theorem 5.4]), there exists a nonsingular matrix $B \in \mathbb{C}^{(n-k) \times (n-k)}$ such that $A = U(B \oplus 0)U^*$. Obviously, $A^\dagger = U(B^{-1} \oplus 0)U^*$.

For a complex matrix $A \in \mathbb{C}^{n \times m}$, the matrix A^*A has been extensively studied (it appears, for example, when one uses a least squares technique). This matrix, evidently, is Hermitian (self-adjoint in the nomenclature used in this paper), hence it is unitarily diagonalizable, i.e. there exist a unitary matrix $U \in \mathbb{C}^{m \times m}$ and a diagonal matrix $D \in \mathbb{C}^{m \times m}$ such that $A^*A = UDU^*$. Hence A^*A is group invertible. In next result, under the additional hypothesis “ A is EP” we see that the reverse order law for the group inverse holds for A^*A in rings.

THEOREM 4.1. *Let \mathcal{R} be a unitary ring with involution. If $a \in \mathcal{R}$ is EP, then a^*a is EP and $(a^*a)^\# = a^\#(a^*)^\#$.*

Proof. Observe that $aa^\# = a^\#a$ is self-adjoint because a is EP. We shall see that

$a^\#(a^*)^\#$ satisfies the equations to be the group inverse of a^*a . Since

$$a^*aa^\#(a^*)^\# = a^*(aa^\#)^*(a^\#)^* = (a^\#aa^\#a)^* = (a^\#a)^* = a^\#a,$$

$$a^\#(a^*)^\#a^*a = a^\#(a^\#)^*a^*a = a^\#(aa^\#)^*a = a^\#aa^\#a = a^\#a,$$

we get $[a^*a][a^\#(a^*)^\#] = [a^\#(a^*)^\#][a^*a]$. Now, we have

$$a^*aa^\#(a^*)^\#a^*a = a^\#aa^*a = (a^\#a)^*a^*a = (aa^\#a)^*a = a^*a.$$

The equality $a^\#(a^*)^\#a^*aa^\#(a^*)^\# = a^\#(a^*)^\#$ is easy to prove. Since a^*a is self-adjoint and group invertible, then a^*a is EP. \square

LEMMA 4.2. *Let \mathcal{R} be a unitary ring with involution. If $a \in \mathcal{R}$ is EP, then $(aa^*)^k[(aa^*)^\#]^k = aa^\#$ for any $k \in \mathbb{N}$.*

Proof. We shall use Theorem 4.1 and $(aa^\#)^* = aa^\#$ because a is EP. The lemma holds for $k = 1$ since

$$(aa^*)(aa^*)^\# = aa^*(a^\#)^*a^\# = a(a^\#a)^*a^\# = aa^\#aa^\# = aa^\#.$$

Now pick any $n \in \mathbb{N}$. Now, by the previous computation and the definition of the group inverse of aa^* we have $(aa^*)^{n+1}[(aa^*)^\#]^{n+1} = aa^*(aa^*)^\# = aa^\#$. \square

THEOREM 4.3. *Let \mathcal{R} be a unitary ring with involution. If a, b are EP, then the following statements are equivalent:*

- (i) $ab \in \mathcal{R}^\#$ and $(ab)^\# = b^\#a^\#$.
- (ii) $(a^\#)^*b \in \mathcal{R}^\#$ and $((a^\#)^*b)^\# = b^\#a^*$.
- (iii) $a(b^\#)^* \in \mathcal{R}^\#$ and $(a(b^\#)^*)^\# = b^*a^\#$.
- (iv) $ab, a^*ab \in \mathcal{R}^\#$ and $(ab)^\# = (a^*ab)^\#a^*$, $(a^*ab)^\# = b^\#(a^*a)^\#$.
- (v) $ab, abb^* \in \mathcal{R}^\#$ and $(ab)^\# = b^*(abb^*)^\#$, $(abb^*)^\# = (bb^*)^\#a^\#$.
- (vi) $ab \in \mathcal{R}^\#$ and for any $m, n \in \mathbb{N} \cup \{0\}$, $(a^*a)^{m+1}(bb^*)^{n+1} \in \mathcal{R}^\#$,

$$[(a^*a)^{m+1}(bb^*)^{n+1}]^\# = [(bb^*)^\#]^{n+1} [(a^*a)^\#]^{m+1}, \quad (4.2)$$

and

$$b^*(bb^*)^n [(a^*a)^{m+1}(bb^*)^{n+1}]^\# (a^*a)^m a^* = (ab)^\#. \quad (4.3)$$

- (vii) $ab \in \mathcal{R}^\#$ and exist $m, n \in \mathbb{N} \cup \{0\}$ such that $(a^*a)^{m+1}(bb^*)^{n+1} \in \mathcal{R}^\#$ and (4.2), (4.3) are satisfied.

Furthermore, under any of the previous conditions, one has that ab is EP.

Proof. Let us represent $a, a^\#$ as in (2.2) and $b, b^\#$ as in (2.14). Let $p = aa^\#$. Notice that since a is EP, the idempotent p is a projection and the representations (2.2) and (2.14) preserve the involution.

(i) \Rightarrow (ii): By Lemma 2.9 we have $b_2 = b_3 = 0$ and $(ab_1)^\# = b_1^\# a^\#$. It is obvious that

$$(a^\#)^* b = \begin{bmatrix} (a^\#)^* b_1 & 0 \\ 0 & 0 \end{bmatrix}_p \quad \text{and} \quad b^\# a^* = \begin{bmatrix} b_1^\# a^* & 0 \\ 0 & 0 \end{bmatrix}_p. \quad (4.4)$$

We will prove that $b_1^\# a^*$ verifies the three conditions to be the group inverse of $(a^\#)^* b_1$. Since $(ab_1)^\# = b_1^\# a^\#$, in particular from (2.13) we have

$$ab_1 b_1^\# a^\# = b_1^\# a^\# ab_1 = b_1^\# b_1. \quad (4.5)$$

Observe that b is EP and $b = \begin{bmatrix} b_1 & 0 \\ 0 & b_4 \end{bmatrix}_p$ imply that b_1 is also EP. Also, let us remark that since a and b_1 are EP, then $aa^\#$ and $b_1 b_1^\#$ are self-adjoint. From (4.5) we have

$$[(a^\#)^* b_1] [b_1^\# a^*] = (a^\#)^* (b_1 b_1^\#)^* a^* = [ab_1 b_1^\# a^\#]^* = b_1 b_1^\#$$

and

$$[b_1^\# a^*] [(a^\#)^* b_1] = b_1^\# (a^* (a^\#)^*) b_1 = b_1^\# (a^\# a)^* b_1 = b_1^\# a^\# ab_1 = b_1^\# b_1.$$

Now it is easy to prove

$$[(a^\#)^* b_1] [b_1^\# a^*] [(a^\#)^* b_1] = (a^\#)^* b_1 \quad \text{and} \quad [b_1^\# a^*] [(a^\#)^* b_1] [b_1^\# a^*] = b_1^\# a^*,$$

which yields that $((a^\#)^* b_1)^\# = b_1^\# a^*$. Hence (4.4) proves that $(a^\#)^* b$ is group invertible and $((a^\#)^* b)^\# = b^\# a^*$.

(ii) \Rightarrow (i): Let $c = (a^\#)^* = (a^*)^\#$. As we have proved (i) \Rightarrow (ii), we can use this implication. The hypotheses are $cb \in \mathcal{R}^\#$ and $(cb)^\# = b^\# c^\#$. Thus, $(c^\#)^* b \in \mathcal{R}^\#$ and $((c^\#)^* b)^\# = b^\# c^*$. Hence, (i) holds.

(i) \Leftrightarrow (iii): Let $x = b^*$ and $y = a^*$. By the previously proved equivalence (i) \Leftrightarrow (ii) we have

$$\begin{aligned} (ab)^\# = b^\# a^\# &\Leftrightarrow [(ab)^\#]^* = [b^\# a^\#]^* &\Leftrightarrow (xy)^\# = y^\# x^\# &\Leftrightarrow [(x^\#)^* y]^\# = y^\# x^* \\ &\Leftrightarrow (y^* x^\#)^\# = x(y^\#)^* &\Leftrightarrow (a(b^\#)^*) = b^* a^\#. \end{aligned}$$

(i) \Rightarrow (iv): By Lemma 2.9, we get $b = \begin{bmatrix} b_1 & 0 \\ 0 & b_4 \end{bmatrix}_p$ and $b_1^\# a^\# = (ab_1)^\#$. By using the forms of a and b , we get

$$a^* ab = \begin{bmatrix} a^* ab_1 & 0 \\ 0 & 0 \end{bmatrix}_p \quad \text{and} \quad b^\# (a^* a)^\# = \begin{bmatrix} b_1^\# (a^* a)^\# & 0 \\ 0 & 0 \end{bmatrix}_p.$$

From the equivalence (i) \Leftrightarrow (ix) of Theorem 3.1 we get $abb^\# = bb^\# a$, or equivalently, $ab_1 b_1^\# = b_1^\# b_1 a$. We shall prove that $a^* ab_1$ is group invertible and $(a^* ab_1)^\# =$

$b_1^\#(a^*a)^\#$. To this end, we use Theorem 4.1 and $(aa^\#)^* = aa^\#$, $(b_1b_1^\#)^* = b_1b_1^\#$. We have from (2.13)

$$\begin{aligned} a^*ab_1b_1^\#(a^*a)^\# &= a^*(ab_1b_1^\#)a^\#(a^*)^\# = a^*b_1^\#b_1aa^\#(a^*)^\# = a^*(b_1^\#b_1)^*(a^\#)^* \\ &= (a^\#b_1^\#b_1a)^* = (a^\#ab_1^\#b_1)^* = (b_1^\#b_1)^* = b_1^\#b_1 \end{aligned}$$

and

$$\begin{aligned} b_1^\#(a^*a)^\#a^*ab_1 &= b_1^\#a^\#(a^*)^\#a^*ab_1 \\ &= b_1^\#a^\#(aa^\#)^*ab_1 = b_1^\#a^\#aa^\#ab_1 = b_1^\#a^\#ab_1 = b_1^\#b_1, \end{aligned} \quad (4.6)$$

which prove $[a^*ab_1][b_1^\#(a^*a)^\#] = [b_1^\#(a^*a)^\#][a^*ab_1]$. Now by using (4.6) we can easily check the two remaining equalities for the group inverse of a^*ab_1 .

Furthermore, we have

$$\begin{aligned} (a^*ab_1)^\#a^* &= b_1^\#(a^*a)^\#a^* \\ &= b_1^\#a^\#(a^\#)^*a^* = b_1^\#a^\#(aa^\#)^* = b_1^\#a^\#aa^\# = b_1^\#a^\# = (ab_1)^\#, \end{aligned}$$

which proves $(a^*ab)^\#a^* = (ab)^\#$.

(iv) \Rightarrow (i) follows from

$$\begin{aligned} (ab)^\# &= (a^*ab)^\#a^* \\ &= b^\#(a^*a)^\#a^* = b^\#a^\#(a^\#)^*a^* = b^\#a^\#(aa^\#)^* = b^\#a^\#aa^\# = b^\#a^\#. \end{aligned}$$

(i) \Leftrightarrow (v): It is enough to apply (i) \Leftrightarrow (iv) by substituting $a \leftrightarrow b^*$ and $b \leftrightarrow a^*$.

(i) \Rightarrow (vi): By Lemma 2.9 we have $b_2 = b_3 = 0$. As we have established in the proof of (i) \Rightarrow (iv), we can use

$$ab_1b_1^\# = b_1^\#b_1a \quad (4.7)$$

Taking adjoint of (4.7) and using that b is EP we get

$$b_1b_1^\#a^* = a^*b_1^\#b_1. \quad (4.8)$$

We have for any $m, n \in \mathbb{N} \cup \{0\}$

$$(a^*a)^{m+1}(bb^*)^{n+1} = \begin{bmatrix} (a^*a)^{m+1}(b_1b_1^\#)^{n+1} & 0 \\ 0 & 0 \end{bmatrix}_p$$

and

$$[(bb^*)^\#]^{n+1}[(a^*a)^\#]^{m+1} = \begin{bmatrix} [(b_1b_1^\#)^\#]^{n+1}[(a^*a)^\#]^{m+1} & 0 \\ 0 & 0 \end{bmatrix}_p.$$

Observe that $(b_1 b_1^*)^{n+1} [(b_1 b_1^*)^\#]^{n+1} = b_1 b_1^* (b_1 b_1^*)^\#$ and by using Lemma 4.2 (for b_1 and $k = 1$), we have $b_1 b_1^* (b_1 b_1^*)^\# = b_1 b_1^\#$. From Lemma 4.2 (applied to a), (2.13), (4.7), and (4.8) we have

$$\begin{aligned} (a^* a)^{m+1} (b_1 b_1^*)^{n+1} [(b_1 b_1^*)^\#]^{n+1} [(a^* a)^\#]^{m+1} &= (a^* a)^{m+1} b_1 b_1^\# [(a^* a)^\#]^{m+1} \\ &= b_1 b_1^\# (a^* a)^{m+1} [(a^* a)^\#]^{m+1} \\ &= b_1 b_1^\# a a^\# \\ &= b_1 b_1^\#. \end{aligned} \quad (4.9)$$

By using (2.13) and Lemma 4.2 we trivially get

$$[(b_1 b_1^*)^\#]^{n+1} [(a^* a)^\#]^{m+1} (a^* a)^{m+1} (b_1 b_1^*)^{n+1} = b_1 b_1^\#. \quad (4.10)$$

If we define $x = (a^* a)^{m+1} (b_1 b_1^*)^{n+1}$ and $y = [(b_1 b_1^*)^\#]^{n+1} [(a^* a)^\#]^{m+1}$, then equalities (4.9) and (4.10) prove $xy = yx$.

Now, the next computation

$$b_1 b_1^\# (b_1 b_1^*)^\# = (b_1 b_1^\#)^* (b_1^\#)^* b_1^\# = (b_1^\# b_1 b_1^\#)^* b_1^\# = (b_1^\#)^* b_1^\# = (b_1 b_1^*)^\#$$

proves

$$b_1 b_1^\# [(b_1 b_1^*)^\#]^{n+1} = [(b_1 b_1^*)^\#]^{n+1}. \quad (4.11)$$

Equalities (4.10) and (4.11) allow to get $xyx = y$.

From $b_1 b_1^* b_1 b_1^\# = b_1 b_1^* (b_1 b_1^\#)^* = b_1 (b_1 b_1^\# b_1)^* = b_1 b_1^*$, we get

$$(b_1 b_1^*)^{q+1} b_1 b_1^\# = (b_1 b_1^*)^{n+1}. \quad (4.12)$$

Equalities (4.10) and (4.12) prove $xyx = x$. Thus, $x, y \in \mathcal{R}^\#$ and $x^\# = y$. Hence we have proved

$$[(a^* a)^{m+1} (bb^*)^{n+1}]^\# = [(bb^*)^\#]^{n+1} [(a^* a)^\#]^{m+1}.$$

Finally, by Theorem 4.1

$$\begin{aligned} b^* (bb^*)^n [(a^* a)^{m+1} (bb^*)^{n+1}]^\# (a^* a)^m a^* &= b^* (bb^*)^n [(bb^*)^\#]^{n+1} [(a^* a)^\#]^{m+1} (a^* a)^m a^* \\ &= b^* (bb^*)^\# (a^* a)^\# a^* \\ &= b^* (b^\#)^* b^\# a^\# (a^\#)^* a^* \\ &= (b^\# b)^* b^\# a^\# (a^\#)^* = b^\# b b^\# a^\# a a^\# \\ &= b^\# a^\# = (ab)^\#. \end{aligned}$$

(vi) \Rightarrow (vii) is evident.

(vii) \Rightarrow (i): By the hypotheses we have $(ab)^\# = b^*(bb^*)^\#(a^*a)^\#a^*$. Furthermore, by Theorem 4.1 and the EP-ness of b we have $b^*(bb^*)^\# = b^*(b^\#)^*b^\# = (b^\#b)^*b^\# = b^\#bb^\# = b^\#$. In the same way we can prove $(a^*a)^\#a^* = a^\#$. Hence $(ab)^\# = b^\#a^\#$.

To prove that ab is EP, observe that by the used representations of a and b we have $ab = \begin{bmatrix} ab_1 & 0 \\ 0 & 0 \end{bmatrix}_p$. Hence ab is EP if and only if ab_1 is EP. From the proof of (i) \Rightarrow (iii) we get $(ab_1)^\#ab_1 = b_1^\#a^\#ab_1 = b_1^\#b_1$ is self-adjoint. Hence $(ab_1)^\# = (ab_1)^\dagger$, and thus, ab_1 is EP. \square

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