Optimal Linear Combination of Poisson Variables for Multivariate Statistical Process Control

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ABSTRACT

When the number of defects in a production process has to be monitored there are cases where the Poisson distribution is suitable for modeling the frequency of these defects and for developing a control chart. In this paper we analyze the monitoring of *p* Poisson quality characteristics simultaneously, developing a new multivariate control chart based on the linear combination of the Poisson variables, the LCP control chart. The coefficients of this linear combination are optimized to keep the desired in-control ARL and to minimize the out-of-control ARL. In order to facilitate the use of this new control chart the optimization is carried out employing user-friendly Windows© software, which also makes a comparison of performance between this chart and other schemes based on monitoring a set of Poisson variables; namely: a control chart on the sum of the variables (MP chart), a control chart on their maximum (MX chart) and an optimized set of univariate Poisson charts (Multiple scheme). The LCP control chart shows very good performance. First, the desired in-control ARL (ARL_0) is achieved, which is an advantage over the rest of charts, which cannot in general achieve the required ARL_0 value, because their control limits can only take integer values. Secondly, in the vast majority of cases this scheme detects the process shifts faster than the rest of the charts.

KEYWORDS: Control Chart, Poisson, Genetic Algorithm, Multivariate

1. Introduction

The monitoring of a single Poisson variable employing a quality control chart is an easy task: 3-sigma limits are given in closed form as a function of the in-control mean of the variable; alternatively, probability limits can be determined by inverting the cumulative distribution so as to achieve an acceptable false-alarm probability (although exactly matching a specified false-alarm probability is not in general possible, due to the discreteness of the Poisson variable, which makes its cumulative distribution discontinuous). For details, see, for instance, Montgomery (2012). When several Poisson variables need to be monitored simultaneously, the practitioner has two options: (i) a control scheme based on one chart for each variable (multiple scheme); and (ii) a control scheme based on one single control chart (multivariate scheme). There is a large bibliography about multivariate and multiple statistical process control for continuous variables; see, for example, Bersimis et al. (2006). However, very little research has been done when the variables to be monitored are discrete, and in the specific case of this paper, when they follow the Poisson distribution.

Holgate (1964) investigated the bivariate distribution of correlated Poisson variables. His model assumes that there is a common factor for all the variables, plus a unique factor for each of the observed variables. For example, the common factor in a cutting process can be the rotational speed of a saw when cutting wood panels. The speed affects the frequencies of two types of possible defects in the surface. The influence of this common factor in the number of defects of the observed variables X_i, i = 1, 2, is represented by an unobservable variable Y_0, and the influence of each individual factor on the respective observed variable X_i is represented by an unobservable variable Y_i so that $X_i = Y_0 + Y_i$. The common part Y_0 responds for the correlation between the observed variavles X_i. This model can be easily extended to more than two variables, simply by making i = 1, 2, ... , p, with $p > 2$. This extended Holgate's model is assumed throughout this paper. Let λ_i denote the mean of each Poisson variable Y_i. It is straightforward to obtain:

$$E(X_i)=\lambda_0+\lambda_i \qquad Cov(X_i, X_j)= \lambda_0 \qquad \rho(X_i, X_j) = \frac{\lambda_0}{\sqrt{(\lambda_0+\lambda_i)(\lambda_0+\lambda_j)}} \tag{1}$$

Another reference in multiple process control by attributes is Patel (1973), who developed a multivariate control chart based on the multivariate normal approximation to the binomial distribution. More recently, Skinner et al. (2003) proposed to employ the Deviance Residual of the general linear model as the statistic to monitor several independent Poisson variables. Chiu and Kuo (2008) proposed the multivariate monitoring of several Poisson variables by the sum of all of them, in what they named the MP control chart. They found the distribution of this sum and analyzed the performance under Holgate's model for correlation. Another multivariate proposal was presented by Ho and Costa (2009), for the case of a bivariate Poisson. They proposed monitoring the variables by their difference (DX chart) and also by the maximum of them (MX chart). An exhaustive comparison of performance is presented in their paper.

With respect to the Multiple scheme, designing a set of univariate Poisson control charts is not an easy task. Aparisi et al. (2013) developed a procedure to design this set of charts, considering that the set has to achieve a required in-control ARL. They provided user-friendly software to perform the ARL calculations and optimize the charts parameters. The difficulty is that, since the in-control means of the variables are fixed values, then the only parameter in each chart that one has freedom to change (as a decision variable) is the upper control limit. Since, however, the Poisson variable can take only integer values, its cumulative distribution is discontinuous, which prevents (except by a lucky chance) adjusting the limits of this set of charts for a specified false-alarm probability so as to match the required in-control ARL. Sometimes a close value can be obtained, but the majority of times there is a large distance between the desired value and the closest ARL available. For that reason, in many cases, we have to choose from a scheme where the number of false alarms is high, or from a scheme that is not powerful to detect the process shifts, because the in-control ARL is too large (which is tantamount to saying that the control limit is too high for providing good power for the chart). A similar problem occurs with all previously cited charts based on the Poisson distribution.

Laungrungong, Borror and Montgomery (2011) developed a multivariate EWMA control chart for Poisson variables (the MPEWMA control chart) assuming Holgate’s model as well. The MPEWMA chart was compared with the traditional MEWMA control chart (Lowry et al., 1992) applied to the Poisson variables. They show that the use of the MEWMA chart only produces reasonable results when the mean of the Poisson variables have a value of 5 of more. When this condition is not satisfied, the MEWMA chart tends to produce more false alarms.

One of the applications where often there is the need of monitoring several discrete (and often Poisson) variables is health surveillance. There is a vast bibliography about this field, where the new control chart developed in this paper may be applied. A good review of the use of control charts in health-care and public-health surveillance is Woodall (2006). Other interesting papers, where the interested reader can find more information and references are Joner et al. (2008) and Jiang et al. (2011).

In this paper we propose a new multivariate control chart, the Linear Combination of Poisson counts (LCP) chart, that can be optimized to obtain the required in-control ARL, solving one of the problems of multivariate control charts for Poisson variables. In addition, as it will be shown, the optimized LCP chart is practically always the most powerful one to detect process shifts. In order to promote the use of the LCP chart, user-friendly software (available by request) has been developed.

This software finds the best linear combination of the Poisson variables and the best control limits in order to minimize the out-of-control ARL. Moreover, this computer program also optimizes the multivariate MP and MX control charts and the Multiple scheme, and makes a complete comparison of performance among all the charts. This way, the end-user can determine which is the most efficient control scheme for his/her particular needs.

The remainder of this paper is organized as follows: Section 2 presents the theoretical basis of the (optimized) multiple scheme (several univariate control charts) and of the multivariate schemes focused here (sum, maximum and linear combination of the variables). Section 3 gives a formal definition of the optimization problem. Section 4 describes a computer program that has been developed with the aim of helping the final user with selecting the best control scheme for his/her process. This software allows obtaining the control limits for the charts mentioned in Section 2, when p = 2, 3 or 4 Poisson variables are monitored, in order to obtain the desired in-control ARL, or the closest possible. In addition, the software carries out a complete performance comparison between the different control charts. A sensitivity analysis appears in Section 5. A general comparison of performance is shown in Section 6. Finally, Section 7 summarizes the conclusions of this paper.

2. Controlling several Poisson variables: multivariate and multiple approaches.

In this section all the Poisson control charts that will be considered in the comparison of performance are introduced. The performance measure that will be used is the Average Run Length (ARL), which is the most used one to make this type of comparisons. It is the average number of points on the control chart until the chart signals, i.e., until a point is plots outside the control limits or just on the control limits. A signal will be a false alarm if the process is in control, that is, with no shifts in its parameters. Therefore a large in-control ARL is usually required. On the other hand, shifts in the process parameters (out-of-control state) must be detected quickly, therefore small out-of-control ARLs are desired. In the type of charts analyzed in this paper, where the statistics (points) to be plotted are independent, the number of samples until a signal follows a geometric distribution, and the ARL is the reciprocal of the probability of a signal, formally, ARL = 1 / P(signal).

2.1 The Multiple Scheme: multiple univariate Poisson control charts.

The multiple scheme consists of monitoring each Poisson variable by one control chart. Aparisi et al. (2013) offer a computer program that optimizes this set of univariate control charts to minimize the ARL for a given shift. The authors also analyze the performance of this scheme for a large number of cases. In that paper, as in the present one, it is assumed that each of these Poisson's charts has only the upper control limit, as normally the practitioner is only interested in detecting shifts that increase the number of defects. Therefore, if p Poisson variables are to be monitored, p upper control limits should be determined. The search for their optimal values is conducted taking into account that a specified value, ARL_0, is required for the joint in-control ARL of the set of p control charts. However, it is normally not feasible to achieve the required value of ARL_0 given that the Poisson variables can take only integer values. Aparisi et al. (2013) search the control limits following the objective of obtaining the closest possible value of in-control ARL to the required ARL_0, but always larger than ARL_0 if the exact value cannot be matched. This criterion will be employed in this article as well.

The statistic to be plotted in each chart is the observed value of the variable, X_i. One chart shows a signal when the observed value is greater or equal than the control limit, i.e., $X_i \geq UCL_i$. Hence, this set of p control charts shows an out-of-control signal when one or more control charts signal.

2.2 Multivariate Schemes

As it was previously mentioned, the multivariate approach consists in employing a unique statistic for all the p variables to be monitored. For example, in the case of Poisson variables, Chiu and Kuo (2008) formulated the statistic $D = \sum_{i=1}^{p} X_i$, i. e., the sum of the observed values of the Poisson distributions, known as the MP chart. Ho and Costa (2009) proposed charting the difference, DF = X_1 - X_2, and the maximum of two (p = 2) Poisson variables, MX = max(X_1, X_2).

In this paper we propose a new chart, based on the optimized Linear Combination of Poisson variables (LCP chart). Note that the sum of the variables in the MP chart and the difference of the variables in the DF chart (which can be used only in the two variables case) are particular cases of a linear combination. Therefore, it is expected that the LCP chart will outperform these charts. An important aspect of the LCP chart is that it will need a lower control limit when some of coefficients of the optimum linear combination are negative, because negative values can be produced when the process is in control. To account for this possibility, the optimization problem formulation will consider the chart as two-sided, letting the search algorithm find the best values for the coefficients and control limits.

Let $LCP = a_1 X_1 + a_2 X_2 + \cdots + a_p X_p$ denote the linear combination of p Poisson Correlated Random Variables, where $a_i \in [-1, 1]$ and $X_i = Y_0 + Y_i$. This linear combination can be expressed as a function of the unobservable Poisson variables Y_0 and $\{Y_i\}$, i = 1, 2, ... , p, as follows:

$$LCP = \sum_{i=1}^{p} a_i \cdot X_i = \sum_{i=1}^{p} a_i \cdot (Y_0 + Y_i) = \sum_{i=1}^{p} a_i \cdot Y_0 + \sum_{i=1}^{p} a_i \cdot Y_i = b \cdot Y_0 + \sum_{i=1}^{p} a_i \cdot Y_i \qquad (2)$$

Hence, the probability function $P(LCP = d)$ is given by

$$P(LCP = d) = P(b * Y_0 + \sum_{i=1}^{p} a_i * Y_i = d) = \sum_{i_0=0}^{trunc(d/b)} P(Y_0 = i_0, \sum_{i=1}^{p} a_i * Y_i = d - b * i_0) \qquad (3)$$

$$P(LCP = d) = \sum_{i_0=0}^{trunc(d/b)} \sum_{i_1=0}^{trunc((d-b*i_0)/a_1)} P(Y_0 = i_0, Y_1 = i_1, \sum_{i=2}^{p} a_i * Y_i = d - b * i_0 - a_1 * i_1) \qquad (4)$$

Where $b = \sum_{i=1}^{p} a_i$.

It is noted that the maximum values of each sum with index i_j of the equations (3), (4) and (5) are obtained assuming that the Poisson independent variables Y_j with j = k+1, k+2 ,…., p take the value of 0.

And now, developing equation 4, we obtain the probability function of the Linear Combination in (6):

$$P(LCP = d) = \sum_{i_0=0}^{trunc(d/b)} \sum_{i_1=0}^{trunc((d-b*i_0)/a_1)} \sum_{i_2=0}^{trunc((d-b*i_0-a_1*i_1)/a_2)} \cdots \sum_{i_{p-1}=0}^{trunc((d-b*i_0-\sum_{j=1}^{p-2} a_j*i_j)/a_{p-1})} P(Y_0 = i_0, ..., Y_{p-1} = i_{p-1}, Y_p = (d - b * i_0 - \sum_{j=1}^{p-1} a_j * i_j)/a_p)$$

(5)

$$P(LCP = d) = \sum_{i_0=0}^{trunc(d/b)} \sum_{i_1=0}^{trunc((d-b*i_0)/a_1)} \sum_{i_2=0}^{trunc((d-b*i_0-a_1*i_1)/a_2)} \cdots \sum_{i_{p-1}=0}^{trunc((d-b*i_0-\sum_{j=1}^{p-2} a_j*i_j)/a_{p-1})} \exp\left\{-\sum_{i=0}^{p} \lambda_i\right\} * \frac{\lambda_0^{i_0} * \lambda_1^{i_1} * \cdots * \lambda_{p-1}^{i_{p-1}} * \lambda_p^{(d-b*i_0-\sum_{j=1}^{p-1} a_j*i_j)/a_p}}{i_0! * i_1! * \cdots * i_{p-1}! * ((d - b * i_0 - \sum_{j=1}^{p-1} a_j * i_j)/a_p)!}$$

(6)

Note that, given a set of coefficients $\{a_i\}$, LCP can only take discrete values, namely, the ones that satisfy the following equation:

$$d = \sum_{i=1}^{p} a_i * Y_0 + \sum_{i=1}^{p} a_i * Y_i$$

$$y_i = 0, 1, 2, \ldots, \infty \,; i = 0, 1, 2, \ldots, p$$

The expression for the ARL is

$$ARL = \frac{1}{1 - P(LCL < LCP < UCL)} \qquad (7)$$

where the probability that the statistic LCP falls between the control limits is:

$$P(LCL < LCP < UCL) =$$

$$= \sum_{d=LCL}^{UCL} \sum_{i_0=0}^{trunc(\frac{d}{b})} \sum_{i_1=0}^{trunc(\frac{d-b*i0}{a1})} \sum_{i_2=0}^{trunc(\frac{d-b*i0-a1*i1}{a2})} \ldots . \sum_{i_{p-1}=0}^{trunc((d-b*i0-\sum_{i=1}^{p-2} a_i*(i_i))/a_{p-1})} exp\left\{-\left[\sum_{i=0}^{p}(\lambda_i)\right]\right\} \frac{\lambda_0^{i_0} * \lambda_1^{i_1} * \ldots .* \lambda_{p-1}^{i_{p-1}} * \lambda_p^{d-b*i0-\sum_{i=1}^{p-1} a_i*(i_i))/a_p}}{i_0! * i_1! * \ldots \ldots \ldots .* i_{p-1}! * (d - b * i0 - \sum_{i=1}^{p-1} a_i * (i_i))\,/a_p)!}$$

(8)

Note that the function $(d - b * i_0 - \sum_{i=1}^{p-1} a_i * (i_i))\,/a_p$ only can return integer values. Therefore, whatever combination of the values of i_0, …, i_{p-1} that returns a non-integer value must be discarded in the ARL calculations.

3. Optimization of the LCP control chart

It is a common practice in designing a control chart to set a value, ARL_0, for the in-control ARL, as a constraint, and to minimize the out-of-control ARL for a shift in the process parameters that is considered relevant. In the case of the LCP control chart the optimization problem consists in finding the values of the control limits and of the coefficients a_i of the linear combination: $a_i \in [-1, 1]$, $i = 1, 2, \ldots, p$ and $\text{LCP} = \sum_{i=1}^{p} a_i * X_i$.

The formal definition of this optimization problem is:

Given

In-control ARL: ARL_0

In control means: $\lambda_{0,i}$ i = 0, 1, 2, …, p

Shift for which to minimize the ARL: $d^* = (d_0, d_1, d_2, \ldots, d_p)$

$$d_i = (\lambda_{1,i} - \lambda_{0,i})/ (\lambda_{0,i})^{1/2} \quad i = 0, 1, 2, \ldots, p$$

where: $\lambda_{0,i}$: in-control means

$\lambda_{1,i}$: out-of-control means.

d_i: shift in sigma units for mean $\lambda_{0,i}$

Find

Upper and lower control limits for LCP Chart: LCL and UCL.

The parameters a_i, $a_i \in [-1, 1]$, $i = 1, 2, \ldots, p$ for the linear combination LCP = $\sum_{i=1}^{p} a_i * X_i$.

Minimize: ARL ($d = d^*$)

Such that ARL ($d = 0$) = ARL_0

This is a rather complex optimization problem that has been solved using Genetic Algorithms (GA). The GA employed in this paper has been calibrated to maximize its performance employing the technique of design of experiments. GA have proved to be an efficient tool to optimize quality control charts. For example, some references are: Chen (2007), Kaya (2009) and Aparisi et al. (2009).

4. Software and example of application

To easy the use of the LCP control chart, we have developed user-friendly Windows© software for optimizing its parameters and making a complete comparison of performance with the previously mentioned other control charts based on the Poisson distribution. An example of use of this computer program follows.

According to Sánchez et al. (2002), there are two types of defects that can occur in the production of ceramic vases: blisters and discolorations. These defects are sometimes produced by raw material contaminants, like manganese oxides, iron oxides, and titanium oxides. It is required to monitor the production to control these two types of defects. In order to estimate their means, 100 samples of 10 ceramic vases each were taken when the process was assumed to be in control. The results are shown in Table 1, where X_1 is the number of blisters and X_2 is the number of discolorations.

X_1	X_2	Frecuency
0	0	4
0	1	5
0	2	10
0	3	4
0	4	4
0	5	2
0	6	1
1	0	5
1	1	12
1	2	6
1	3	7
1	4	6
1	6	1
2	0	6
2	1	3
2	3	8
2	4	2
2	5	1
3	0	1
3	2	1
3	3	1
3	4	4
3	5	2
3	6	1
4	1	1
4	3	1
5	4	1
$E(X_1)=1.2$	$E(X_2)=2.29$	$\Sigma = 100$

Table 1. Sampling from the process. X_1 is the number of blisters, and X_2 is the number of discolorations

The observed correlation between these variables is r = 0.15. It is desired to obtain the control limits for all the charts shown on Section 2 in order to obtain an in-control ARL_0 = 370 or the closest possible.

It is possible to estimate the means of the non-observed Y_i Poisson variables, knowing that $E(Y_i) = E(X_i) - Cov(X_1, X_2)$, see equation (1). It is required to find the chart's best parameters to detect rapidly a shift of 1 sigma unit in the mean λ_1. The following relationships are used to determine the out-of-control Y_i mean values:

$\lambda_{1,i} = \lambda_{0,i} + d_i * (\lambda_{0,i})^{1/2}$;. $d_i = (\lambda_{1,i} - \lambda_{0,i}) / (\lambda_{0,i})^{1/2}$ $i = 0, 1, 2$

where: $\lambda_{0,i}$: in-control means

$\lambda_{1,i}$: out-of-control means.

d_i: shift in sigma units for mean $\lambda_{0,i}$

Table 2 shows the results:

In control			Out of control		
$\lambda_{0,0}$	$\lambda_{0,1}$	$\lambda_{0,2}$	$\lambda_{1,0}$	$\lambda_{1,1}$	$\lambda_{1,2}$
0.27	0.93	2.01	0.27	1.89	2.01

Table 2. Means values for the example of application

Table 3 shows the in-control and out-of-control ARLs for all the charts studied. Figure 1 shows the software output.

	Type of Chart					
	MP Chart	MX Chart	LCP Chart		Multi Poisson Chart	
Control Limits	UCL = 11	UCL = 8	LCL =-0.97	UCL = 3.12	$UCL_1 = 7$	$UCL_2 = 8$
ARL (d = 0)	440.58	401.31	369.72		370.24	
ARL (d = d*)	105.49	236.65	36.74		108.08	

Table 3. Control limits for each control scheme

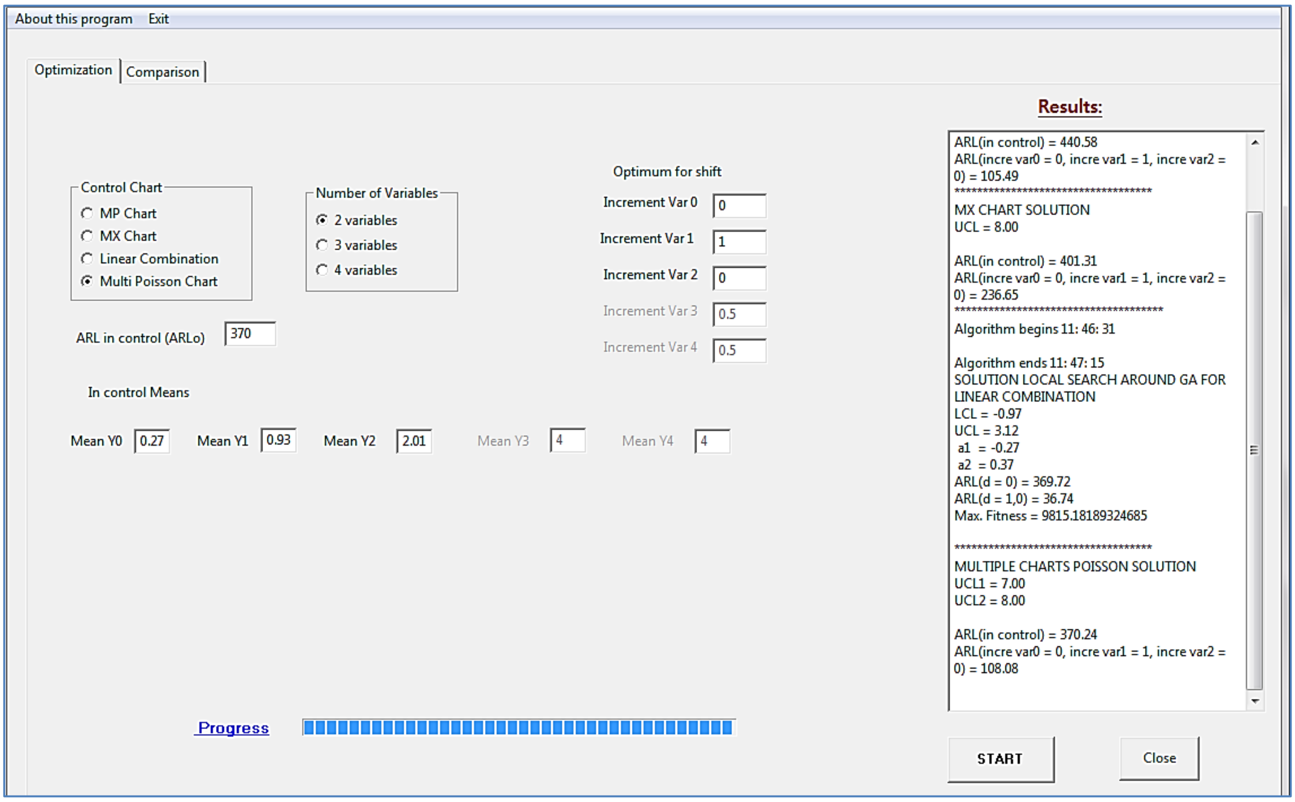

Figure 1. The computer program solving the example of application.

As Table 3 shows, the fact that the LCP values are not integers, but real numbers, allows matching the actual in-control ARL, ARL(d = 0) = 369.72, to the required value ARL_0 = 370. This is one of the advantages of the LCP control chart. The rest of the charts analyzed, in contrast, cannot in general fulfill this requirement. In this example, the in-control ARLs of the MP, MX and Multi-Poisson charts are of 440.58, 401.31 and 370.24, respectively. The last value, of 370.24, happens to be quite close to the required value of 370, but this happens by chance, only occasionally, and cannot be taken as granted in general.

The comparison now focuses on the out-of-control ARLs. Although this comparison is not strictly fair, because the in-control ARLs are not the same for all the charts, it seems clear that the LCP control chart shows the best performance. Employing the parameters a_1 = –0.27, a_2 = 0.37, LCL = –0.97 and UCL = 3.12, the out-of-control ARL is 36.74, while the out-of-control ARLs of the MP, MX and multi-Poisson charts are 105.49, 236.65 and 108.08, respectively. All this information is part of the program output (see Figure 1). The results are summarized in Table 3.

Now follows an example of use of the LCP chart. Table 4 shows several samples of 10 ceramic vases and the number of defects observed in each sample. As before, X_1 is the number of blisters, and X_2 is the number of discolorations. The statistics to be plotted is LCP $= -0.27X_1 + 0.37X_2$.

Figure 2 shows the LCP chart with the values from Table 4 plotted. The last sample (LCP = 3.16) plots above the upper control limit (UCL = 3.12), indicating that we should consider that the process is out of control.

Sample	X_1	X_2	LCP
1	1	4	1.21
2	1	0	-0.27
3	0	4	1.48
4	1	1	0.1
5	0	2	0.74
6	0	1	0.37
7	4	3	0.03
8	1	1	0.1
9	0	2	0.74
10	**2**	**10**	**3.16**

Table 4. Values for the LCP Chart. X_1 is the number of blisters, and X_2 is the number of discolorations

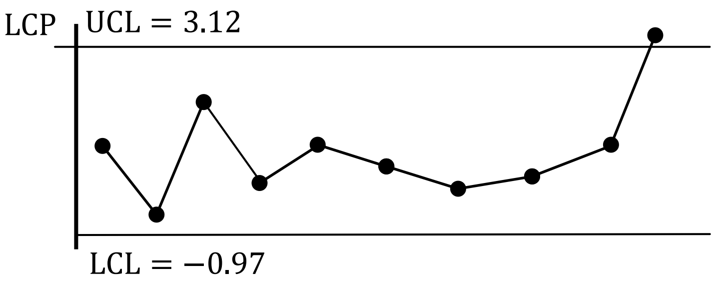

Figure 2. LCP Chart of the example

One of the important features of the developed software is that it can carry out a complete comparison of performance among the MP, MX, LCP charts and the Multiple scheme. This comparison is done after optimizing the charts in the "Optimization" tab. Then the "Comparison" tab is selected by the user to begin the analysis of performance (see Figure 3). The leftmost plot in this tab, "Best Control Chart: Two free shifts", shows which control chart is the best for a given couple of shifts in two of the variables Y_i, i = 0, ..., *p*. The user has to select two of the variables (Y_i, Y_j) for the horizontal and vertical axis. In the case shown in Figure 3 the shifts in Y_1 are on the horizontal axis and the shifts in Y_2 are on the vertical axis.

This plot shows that when, for example, the shifts in the Y_1 and the Y_2 means are of 1.0 and 0.2 sigma units respectively, and the Y_0 mean does not change (the user had set this shift to 0), the best scheme is the LCP chart. The figure only indicates which chart is the best (lowest ARL) for that shift. The exact ARL values for this case can be easily obtained in the “Optimization” tab for the MX, MP and Multiple Poisson charts. The ARL for the LCP chart for a given shift is shown in the right area of the software interface window, “ARL calculations for LCP chart”. In the example, the LCP chart shows an ARL of 45.57, which is 47%, 70% and 38% smaller than the ARLs of the Multiple Scheme, MX chart and MP Chart, respectively. As another example, when the Y_0 mean undergoes no shift, and the Y_1 and Y_2 means have a shift of 0.4 sigma units each, then the plot shows that the best option is the Multiple Scheme. In this case the Multiple scheme has an out-of-control ARL of 98.53; 9,7%, 8% and 42% less than the ARLs of the MX, MP and LCP charts respectively.

The second plot in the interface window (Figure 3), “Best Control Chart: One free shift”, displays the ARL curves (ARLs in the vertical axis, shifts in the horizontal axis) for shifts in Y_1 in, when the means of Y_0 and Y_2 undergo no changes (the user had set these shifts to 0). The chart clearly shows that the best control chart is the LCP (lowermost curve). In general, the user specifies fixed values for the shifts in all variables but one and the program plots the ARL curves as a function of the shifts in the latter. See the entries above the plot, in Figure 3. Let’s repeat that, although the user in this example entered 0 for the shifts in the means of Y_0 and Y_2, he or she could have chosen any other fixed value for them.

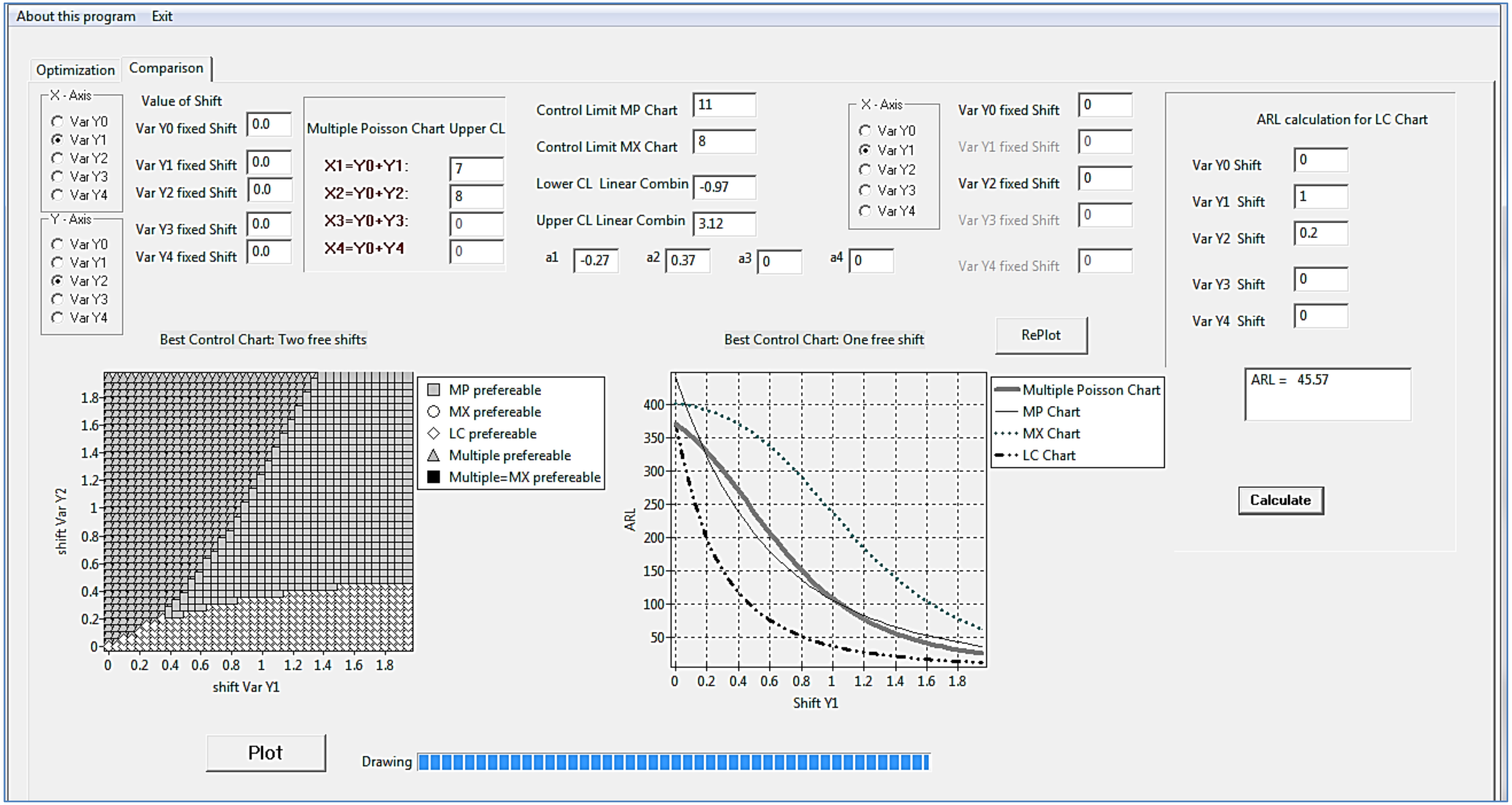

Figure 3. Comparison of the charts for the example of application.

5. Sensitivity Analysis

A common technique for optimally designing a control chart is to specify a shift in the process parameters for which the ARL should be minimized, subject to a constraint in the in-control ARL. This is the procedure employed in this paper and implemented in the software developed. A proper choice of shift for optimization is often the smallest shift that is already relevant to detect because it is enough large to have an impact on the product quality. The ARLs for larger shifts will be, of course, smaller than the ARL for the selected shift, since the chart will be more sensitive to larger shifts. This does not mean, however, that the chart is optimal for these larger shifts. Other charts (and other designs of the same type of chart) may perform better for the larger shifts. No chart is uniformly better for all shifts. The question that arises is, then, what is the performance of the chart for these larger shifts.

Therefore, it results interesting to study the robustness of the optimization with respect to the selected out-of-control shift. The objective is to know whether the optimized LCP control chart shows good performance as well for other shifts than the selected one. This study is not trivial. For example, when $p > 2$ it is difficult to see the variation of the performance for different shifts, due to the combinatorial explosion of the number of possible cases, since the shifts could occur in any subset of the Y_i variables, including the case of all of them. For instance, just with $p = 3$, we should study the sensitivity for different values of Y_0, Y_1, Y_2 and Y_3. Therefore, for better understanding, a sensitivity analysis is here presented for the case $p = 2$, although similar conclusions are found for larger values of p.

First of all, a result that we have found is that two charts optimized for shift vectors such that (in each shift vector) the shifts in the means of two variables are equal have equal performance. no matter the magnitude of these shifts. An example will illustrate this. Suppose an in-control ARL of 500 is required, and the in-control means are $\lambda_{0,0} = 0.5$, $\lambda_{0,1} = 1$, $\lambda_{0,2} = 3$. The next step is to select a set of shift increments in the means for the optimization. Let us optimize for two different shift vectors: A = (0.5, 1, 1) and B = (0.5, 2, 2), where the i-th element is the shift increment in the mean of Y_i. After optimizing for both shift vectors, the optimum LCP control charts will have the same ARLs for whatever shift vector. The conclusion is the same for whatever couple (Y_i, Y_j), $Y_i = Y_j$ that is selected for the optimization. The ARLs for different increments in variables Y_1 and Y_2 are shown on Table 5, and we remind that these ARLs are the same for LCP charts optimized for cases

A and B.

		Shifts for Y_2					
		0.5	**1**	**1.5**	**2**	**2.5**	**3**
	0.5	41.31	21.39	12.27	7.68	5.18	3.73
	1	28.68	15.55	9.33	6.09	4.26	3.17
	1.5	20.73	11.72	7.31	4.94	3.58	2.74
Shifts for Y_1	**2**	15.52	9.10	5.88	4.11	3.06	2.41
	2.5	11.95	7.26	4.85	3.49	2.67	2.15
	3	9.43	5.93	4.08	3.02	2.37	1.95

Table 5. Out-of-control ARLs for LCP charts optimized for increments A = (0.5, 1, 1) and B = (0.5, 2. 2). In-control ARL = 500.

The same ARLs are obtained because both optimizations return equivalent schemes. The LCP chart optimized for shift vector A has these parameters: a_1 = 0.68, a_2 = 0.71, UCL = 9.72 and LCL = -0.01 and the LCP chart optimized for shift vector B has: a_1 = 0.64, a_2 = 0.69, UCL = 9.28 and LCL = -0.01. Although the parameters are not the same, they are equivalent, proportional, yielding the same ARLs for whatever shift. Indeed, if the shifts in the variables keep a constant proportion, it is reasonable that the coefficients of the linear combination also keep a proportion. So both linear combinations are the same, except for a matter of scale. And the constraint on the in-control ARL forces the UCLs of the two charts to be equivalent (i.e., the same but for a matter of scale). Therefore, for any given shift, both schemes will have the same performance. As a practical conclusion, if all shift vectors the final user is concerned with keep a constant proportion between their elements, then the magnitude of the shifts to be entered for the optimization (the norm of the shift vector) does not matter: the result will be the same. This may sometimes simplify the user's decision. In addition, it is worth commenting that the ARLs for large shifts are quite small, what it is a very desirable result.

Another issue that arises from inspection of Table 5 is why the ARLs for shifts that are symmetric are not equal. For example ARL(0.5, 1) = 21.39, different from ARL(1, 0.5) = 28.68. Intuitively, they could be expected to be equal, since the shifts in Y_1 and Y_2 are the same, just interchanged. However, their ARLs are different (Table 5 is not symmetrical). This occurs because the coefficients of the optimal linear combination, a_1 and a_2, are not equal. The reason they are not equal is that there is no way to obtain an ARL(d = 0) = 500 with $a_1 = a_2$. This is easy to check, because the software optimizes as well the sum of variables, the MP chart; the sum is the linear combination, with $a_1 = a_2 = 1$. Therefore, if this chart based on sums if optimized for sets A or B, the closest ARL that is obtained is 1212.4, far away from 500. For that reason, the optimization of the LCP chart returns values of a_1 and a_2 that are not equal, producing the asymmetry found in Table 5.

To continue with the analysis of sensitivity, let's examine another pair of shift vectors for optimization. In this case, in-control ARL = 400, point A = (0.5, 1, 1) and B = (0.5, 1, 1.5). The objective is to see what happens when the shift vectors selected for the optimization are not very different. Table 6 shows the ARLs of both optimal designs.

		Shifts for Y_2					
		0.5	**1**	**1.5**	**2**	**2.5**	**3**
Shifts for Y_1	**0.5**	34.27 (41.53)	19.88 (18.39)	12.3 (9.66)	8.1 (5.8)	5.65 (3.86)	4.15 (2.80)
	1	21.74 (34.62)	13.42 (15.73)	8.78 (8.47)	6.06 (5.19)	4.41 (3.52)	3.36 (2.60)
	1.5	14.59 (29.20)	9.49 (13.62)	6.52 (7.49)	4.7 (4.69)	3.55 (3.24)	2.8 (2.43)
	2	10.29 (24.82)	7.00 (11.89)	5.02 (6.69)	3.76 (4.27)	2.94 (3.00)	2.39 (2.28)
	2.5	7.59 (21.2)	5.36 (10.45)	3.99 (6.02)	3.10 (3.91)	2.50 (2.79)	2.08 (2.15)
	3	5.81 (18.14)	4.25 (9.22)	3.27 (5.44)	2.62 (3.60)	2.17 (2.61)	1.85 (2.01)

Table 6. Comparison of ARLs for LCP control charts optimized for A = (0.5, 1, 1) and B = (0.5, 1, 1.5). ARL values for B in parenthesis. In-control ARL = 400.

The analysis of Table 6 shows that both charts perform well and with very similar ARLs when both variables Y_1 and Y_2 have large shifts. This result is important because, even though the optimizations have been carried out for moderate shifts, the charts have good performance for large shifts, which are always very costly for the company and must be detected quickly. For example, $ARL_A(2.5, 2.5) = 2.50$ and $ARL_B(2.5, 2.5) = 2.79$. Another case is $ARL_A(3, 2.5) = 2.17$ and $ARL_B(3, 2.5) = 2.61$ However, there are differences in the performance for other cases. When the shift in Y_1 is small, $Y_1 = 0.5$, and the shift in Y_2 is moderate or large, the best chart is always the one optimized for case B. For example, $ARL_A(0.5, 2) = 8.1$ and $ARL_B(0.5, 2) = 5.8$. Nevertheless, as the value of increment of Y_1 increases, the differences become smaller. For example, $ARL_A(1, 2) = 6.06$ and $ARL_B(1, 2) = 5.19$. On the other hand, the ARLs for small shifts in Y_2 are always smaller for the chart optimized for the shift vector A. For example, $ARL_A(0.5, 0.5) = 34.27$ and $ARL_B(0.5, 0.5) = 41.53$, or $ARL_A(2, 0.5) = 10.29$ and $ARL_B(2, 0.5) = 24.82$. Therefore, as an expected result, the chart optimized for a larger increment in the mean of Y_2 performs better when the shift in the mean of Y_1 is small for and the shift in the mean of Y_2 is large and the control chart optimized for a smaller increment in Y_2 performs better for small shifts in this variable.

		Shifts for Y_2					
		0.5	**1**	**1.5**	**2**	**2.5**	**3**
Shifts for Y_1	**0.5**	10.03 (9.98)	6.97 (8.92)	5.11 (8.01)	3.91 (7.22)	3.11 (6.55)	2.56 (5.96)
	1	9.04 (6.96)	6.35 (6.31)	4.70 (5.74)	3.63 (5.23)	2.92 (4.80)	2.42 (4.41)
	1.5	8.21 (5.10)	5.82 (4.68)	4.34 (4.31)	3.39 (3.97)	2.74 (3.68)	2.29 (3.412)
	2	7.50 (3.91)	5.36 (3.62)	4.04 (3.37)	3.17 (3.14)	2.59 (2.94)	2.18 (2.76)
	2.5	6.88 (3.11)	4.96 (2.91)	3.76 (2.73)	2.98 (2.57)	2.45 (2.43)	2.08 (2.30)
	3	6.33 (2.56)	4.61 (2.42)	3.53 (2.29)	2.82 (2.17)	2.33 (2.07)	1.99 (1.97)

Table 7. Comparison of ARLs for LCP control charts optimized for A = (1.5, 0.5, 2) and B = (1.5, 2, 0.5). ARL values for B in parenthesis. In-control ARL = 400.

Another interesting combination of shift vectors is when the shifts are symmetric, and very different, for example, in-control ARL = 400, A = (0.5, 0.5, 3) and B = (0.5, 3, 0.5). Table 7 shows the comparison of ARLs. It is easily seen that as the charts are optimized for two points that are symmetric with respect to the line $Y_1 = Y_2$, for shifts in that line (see the main diagonal of Table 7), the ARLs are practically equal. For example, $ARL_A(1, 1) = 10.03$ and $ARL_B(1, 1) = 9.98$ or $ARL_A(2.5, 2.5) = 2.45$ and $ARL_B(2.5, 2.5) = 2.43$. Therefore, for shifts with very similar value for Y_1 and Y_2 the performance of both charts is equivalent. As it can be expected, the performance of both charts is quite different in the area around the point selected for the optimization. The LCP chart optimized for point A shows a better performance in the area around that point, and the same behavior occurs for the chart optimized for point B. For example, $ARL_A(0.5, 2) = 3.91$ and $ARL_B(0.5, 2) = 7.22$, where $ARL_A(2.5, 1) = 4.96$ and $ARL_B(2.5, 1) = 2.95$. Of course, these large differences are due to the fact we are comparing a chart optimized for one region and a chart optimized for a completely different region. Nevertheless, both charts have the same performance on the line $Y_1 = Y_2$. Also, and importantly, both optimized control charts perform well for large shifts.

As a summary for this Section, as expected, the choice of the shift for which to optimize the chart is important. However, if we are expecting a shift with similar values for all the means, it does not matter which point is selected for the optimization. It is also important to note that when comparing two very different vector shifts employed for the optimization, if they are symmetric with respect to $Y_i = Y_j$, the ARL values will be the same in the line $Y_i = Y_j$.

6. Comparison of performance

As commented before, there is a problem when optimizing the MP, MX and Multiple Poisson charts, due to the discrete nature of the Poisson variables. For that reason, in the large majority of the times, the in-control ARL differs noticeably of the desired ARL_0. This problem does not occur with the LCP control chart. This issue complicates a lot our aim of making a fair comparison of performance among the charts, because when there is a large difference in the ARL_0's of two charts, the one with the larger one is in disadvantage regarding its out-of-control performance. Since, for the optimization, a minimum value for the in-control ARL is specified, and the LCP chart is able to match it (this is one of its advantages, as already pointed out) while the other charts in generally are not, this might favor the LCP chart in the comparison. We have tried to avoid this problem showing in this Section only cases where the differences in the in-control ARLs of the optimized charts are not large.

The comparisons are made for p = 2 and 3 variables for mean shifts measured in sigma units $d_i = (\lambda_{1,i}-\lambda_{0,i})/(\lambda_{0,i})^{½}$, where $\lambda_{1,i}$ is the mean $E(Y_i)$ when the process is out of control and $\lambda_{0,i}$ is the mean $E(Y_i)$ when in control. We have also results for p = 4 variables, which are not included in this paper due to space limitations, taking into account that the tables are very large for p = 4 and that the conclusions are very similar. All results shown on the following tables have been obtained employing the software presented in Section 4.

Four different scenarios have been considered for p = 2 and 3: A, B, C, and D. These scenarios are described in Tables 8 and 9. They correspond to different correlations among the non-observed variables, Y_i. For example, Table 8 shows the value of the means when the process is in control for the four scenarios, when p = 2, with correlations 0.15, 0.33, 0.50 and 0.75. The scenarios shown on Tables 8 and 9 are the same considered in Aparisi et al. (2013).

	$\lambda_{0.0}$	$\lambda_{1.0}$	$\lambda_{2.0}$	ρ
Scenario A	0.25	1	2	0.15
Scenario B	0.5	1	1	0.33
Scenario C	1.45	1.45	1.45	0.50
Scenario D	3.94	1.32	1.32	0.75

Table 8. Analyzed cases for *p* = 2

	$\lambda_{0.0}$	$\lambda_{1.0}$	$\lambda_{2.0}$	$\lambda_{3.0}$	ρ_{12}	ρ_{13}	ρ_{23}
Scenario A	0.5	1	1	1	0.33	0.33	0.33
Scenario B	0.7	1.4	0.5	1	0.44	0.37	0.49
Scenario C	0.7	0.7	0.7	0.7	0.50	0.50	0.50
Scenario D	2	0.5	1	0.7	0.73	0.77	0.70

Table 9. Analyzed cases for *p* = 3

We have considered mean shifts, d_i, from 0 to 2. If d_i = 0, for i = 0, 1 and 2, then the process is in control. Table 10 shows the ARL values and the new correlations as a function of the mean shifts for Y_0, Y_1 and Y_2 variables (p = 2). The ARL values in bold represents the best (lowest) ARLs for that shift and case. As found by Aparisi et al. (2013), the MX chart and the Multiple scheme have the same ARLs for scenarios B, C, and D, as Table 10 reflects. The first impression after considering these results is that the LCP control chart, in the large majority of cases, is the chart with best performance. There are 104 out-of-control comparisons on Table 10 and in only 15 of them the LCP chart is not the best option. Even in such cases, the differences between the LCP chart and the best chart are very small. For example: Scenario A, shifts d_0 = 1, d_1 = 1, and d_2 = 1: the best chart is the MP chart, with an ARL = 8.83, whilst the LCP chart has an ARL = 8.98, practically the same value. Another example: the only case where the LCP chart is not the best option for Scenario B: d_0 = 0, d_1 = 0.25, and d_2 = 0.25, where the MX chart and the Multiple scheme have an ARL = 231.40, whist the LCP chart has ARL = 249.24, being this difference the largest one found in the Table.

The largest differences among the ARLs of the LCP chart and the rest of competitors normally are found when there is a shift in only one of the variables and for Scenario A. One example is d_0 = 0, d_1 = 0.75, and d_2 = 0, the LCP chart has an ARL = 45.24 and the best competitor is the MP chart with ARL = 140.98, more than three times larger. Another example: d_0 = 0.25, d_1 = 0, and d_2 = 0: the LCP chart has an ARL of 180.19 and the best competitor is again the MP chart, with ARL = 233.31. Of course, when that shifts are large, and they are present in all the variables, the differences of performance among the charts are smaller, for example, in Scenario C, d_0 = 1.5, d_1 = 1.5, and d_2 = 1.5, where the LCP chart has an ARL = 4.60 and the MX chart and the Multiple scheme have an ARL = 4.97. In general terms, the best performance of the LCP chart is found for small shifts, where the real need of showing a good performance is more important.

Table 11 shows the comparison of performance for three variables (p = 3) and for the four scenarios analyzed. In this case, the MX control chart and the Multiple scheme have the same ARL values for Scenarios A, C and D. There are 140 ARL comparisons on Table 11 and in only 25 of them the LCP chart is not the best option, which again demonstrates the very good performance of the LCP chart. The conclusions are very similar to the case p = 2. These same conclusions are reached from some examples from the Table 11. As before, when the LCP control chart is not the best option, the difference between this chart and the best competitor is normally small. For example, the largest difference is found in Scenario D, with d_0 = 0.25, d_1 = 0.25, d_2 = 0.25, and d_3 = 0.25, where the LCP control chart has an ARL = 177.83 and the Multiple Scheme has an ARL of 168.29. However, in the majority of cases the performance is quite similar, for example, in Scenario C: d_0 = 1, d_1 = 1, d_2 = 1 and d_3 = 1. The ARLs are of 11.66 and 11.28 for the LCP and MX charts, respectively. Again the LCP chart has good performance for detecting small shifts, especially if few variables have shifted. In some cases the reduction is remarkable. For example, if we examine the shift d_0 = 0, d_1 = 0, d_2 =

0.5, and d_3 = 0.5 for the four scenarios we obtain that: in Scenario A, the ARL of the LCP chart is of 56.21: its best competitor has an ARL of 103.25. In Scenario B, the values are of 79.21 against 141.54. Scenario C: 75.52 versus 174.86. Scenario D: 83.02 versus 221.33.

Hence, the conclusion of this Section is that the LCP control chart outperforms all its competitors in the large majority of cases. When the shifts are small the gains in performance are the most significant, where in some cases the ARL is about one third of its best competitor.

Shift in means			Scenario A (ARL(d = d*))					Scenario B (ARL(d = d*))					Scenario C (ARL(d = d*))					Scenario D (ARL(d = d*))				
d_0	d_1	d_2	ρ	MP Chart	MX Chart	LCP Chart	Multiple	ρ	MP Chart	MX Chart	LCP Chart	Multiple	ρ	MP Chart	MX Chart	LCP Chart	Multiple	ρ	MP Chart	MX Chart	LCP Chart	Multiple
0	0	0	0.15	446.89	431.99	370	387.44	0.33	572.23	549.81	545	549.81	0.5	663.66	607.13	600.00	607.13	0.75	682.14	511.96	500.00	511.96
0	0.25	0	0.14	296.3	410.92	**161.02**	314.45	0.31	388.42	325.02	**213.9**	325.02	0.48	485.64	395.46	**268.19**	395.46	0.73	556.74	385.38	**214.68**	385.38
0	0	0.25	0.14	251.94	184.4	**156.10**	175.89	0.31	388.42	325.02	**213.89**	325.02	0.48	485.62	395.44	**247.97**	395.44	0.73	556.73	385.37	**214.73**	385.37
0.25	0	0	0.21	233.31	309.93	**180.19**	271.99	0.4	248.26	297.78	**246.25**	297.78	0.55	284.89	301.52	**264.15**	301.52	0.77	287.44	234.65	**217.69**	234.65
0	0.5	0	0.13	201.83	366.46	**79.71**	224.99	0.29	269.28	185.6	**102.28**	185.6	0.45	359.59	246.21	**119.68**	246.21	0.71	456.42	280.54	**109.11**	280.54
0	0.25	0.25	0.13	173.43	180.5	**143.5**	159.26	0.29	269.28	**231.4**	249.24	**231.4**	0.45	359.62	294.54	**290.78**	294.54	0.71	456.45	311.2	**256.47**	311.2
0	0	0.5	0.13	149.68	90.5	**76.78**	88.44	0.29	269.28	185.6	**113.42**	185.6	0.45	359.59	246.21	**138.05**	246.21	0.71	456.42	280.54	**112.39**	280.54
0.25	0.5	0	0.18	117.82	259.94	**70.99**	159.21	0.35	131.81	116.53	**94.44**	116.53	0.5	167.69	138.16	**114.93**	138.16	0.73	201.8	139.39	**87.17**	139.39
0	0.75	0	0.12	140.98	298.35	**45.24**	147.91	0.27	190.48	108.69	**56.09**	108.69	0.44	269.37	153.13	**65.34**	153.13	0.69	375.87	201.29	**62.30**	201.29
0.25	0	0.75	0.17	60.16	40.95	**38.9**	40.27	0.34	98.37	72.96	**52.36**	72.96	0.48	130.57	91.5	**74.53**	91.5	0.72	170.1	104.73	**70.49**	104.73
0.25	0.5	0.75	0.15	35.58	40.01	**31.36**	36.65	0.29	57.29	53.62	**53.39**	53.62	0.44	81.41	**68.69**	72.29	**68.69**	0.69	122.27	**82.65**	87.26	**82.65**
0.5	0	0	0.26	137.35	227.17	**116.47**	195.42	0.46	127.97	174.95	**127.32**	174.95	0.59	141.02	163.9	**132.7**	163.9	0.79	137.2	119.15	**107.14**	119.15
0.5	0.5	0	0.22	75.35	188.85	**46.21**	115.82	0.41	74.18	77	**65.79**	77	0.54	88.67	82.9	**65.88**	82.9	0.75	100.33	75.66	**67.61**	75.66
0.5	0	0.5	0.23	59.92	58.24	**49.09**	56.07	0.41	74.18	77	**65.83**	77	0.54	88.67	82.9	**65.88**	82.9	0.75	100.33	75.66	**68.89**	75.66
0.5	0.5	1	0.18	19.27	21.24	**16.86**	19.98	0.33	28.99	28.26	**26.29**	28.26	0.47	38.71	33.7	**30.23**	33.7	0.69	55.97	**39.04**	42.11	**39.04**
0	1	0	0.11	100.8	221.55	**28.16**	94.55	0.26	137.35	66.72	**34.02**	66.72	0.42	204.04	97.4	**49.11**	97.4	0.68	310.91	144.35	**38.79**	144.35
0	0	1	0.12	60.62	29.72	**25.72**	29.51	0.26	137.35	66.72	**34.01**	66.72	0.42	204.04	97.4	**49.08**	97.4	0.68	310.88	144.35	**58.31**	144.35
1	0	0	0.34	59.94	129.21	**54.32**	107.54	0.55	46.92	72.74	**46.8**	72.74	0.65	46.71	59.92	**44.83**	59.92	0.82	41.7	39.27	**34.15**	39.27
1	1	0	0.27	23.96	70.03	**13.92**	34.67	0.45	20.91	20.12	**17.92**	20.12	0.57	23.14	20.23	**20.07**	20.23	0.76	25.73	**18.92**	19.32	**18.92**
1	0	1	0.28	17.3	16.18	**13.97**	15.87	0.45	20.91	20.12	**17.92**	20.12	0.57	23.14	20.23	**20.07**	20.23	0.76	25.73	18.92	**17.12**	18.92
1	1	1	0.22	**8.83**	14.85	8.98	12.49	0.38	10.73	12.29	**10.64**	12.29	0.5	12.67	12.92	**11.99**	12.92	0.71	16.62	**13.36**	14.03	**13.36**
1	1.5	0.5	0.22	10.03	23.97	**8.38**	14.43	0.38	10.73	10.43	**9.5**	10.43	0.5	12.67	**11.07**	11.17	**11.07**	0.71	16.61	11.99	**10.06**	**11.99**
1.5	1.5	1.5	0.24	**3.75**	6.09	3.86	5.19	0.38	4.4	5.03	**4.37**	5.03	0.5	4.79	4.97	**4.6**	4.97	0.69	5.98	**5.08**	5.3	**5.08**
0	2	0	0.09	32.3	48.95	**7.62**	20.09	0.22	44.14	15.16	**8.62**	15.16	0.37	74.64	22.25	**12.54**	22.25	0.62	151.49	43.07	**16.97**	43.07
2	0	0	0.46	19.32	50.49	**17.98**	40.46	0.66	12.82	20.7	**12.81**	20.7	0.73	10.81	14.44	**10.59**	14.44	0.86	8.44	8.54	**7.41**	8.54
2	0	2	0.338	4.14	3.74	**3.42**	3.71	0.51	4.81	4.29	**4.14**	4.29	0.6	4.65	**3.88**	4.10	**3.88**	0.77	4.74	**3.58**	3.85	**3.58**
2	2	2	0.25	**2.2**	3.3	2.26	2.88	0.39	**2.49**	2.8	5.2	2.8	0.5	2.56	2.66	**2.48**	2.66	0.69	3	**2.68**	2.76	**2.68**

Table 10. Comparison of performance, two variables, p = 2.

Shift in means				Scenario A (ARL(d = d*))							Scenario B (ARL(d = d*))							Scenario C (ARL(d = d*))							Scenario D (ARL(d = d*))						
d_0	d_1	d_2	d_3	Mp Chart	Mx Chart	Multiple Charts	LCP Chart	ρ_{12}	ρ_{13}	ρ_{23}	Mp Chart	Mx Chart	Multiple Charts	LCP Chart	ρ_{12}	ρ_{13}	ρ_{23}	Mp Chart	Mx Chart	Multiple Charts	LCP Chart	ρ_{12}	ρ_{13}	ρ_{23}	Mp Chart	Mx Chart	Multiple Charts	LCP Chart	ρ_{12}	ρ_{13}	ρ_{23}
0	0	0	0	407.5	372.2	372.21	370.00	0.33	0.33	0.3	561.7	534.7	530.42	530.00	0.44	0.4	0.49	596.32	578.17	578.17	570.00	0.5	0.5	0.5	582.8	631.16	624.38	620.00	0.7	0.77	0.7
0	0.25	0	0	323	254.5	254.5	**154.62**	0.31	0.31	0.3	436.8	272	499.23	**244.28**	0.41	0.4	0.49	496.97	407.66	407.66	**212.04**	0.5	0.47	0.5	526.09	576.36	446.89	**249.71**	0.7	0.74	0.7
0	0	0.25	0	323	254.5	254.5	**155.73**	0.31	0.33	0.3	482.9	520.6	300.81	**210.03**	0.41	0.4	0.46	496.97	407.66	407.66	**215.06**	0.5	0.5	0.5	504.42	414.74	548.95	**243**	0.7	0.77	0.7
0	0	0	0.25	323	254.5	254.5	**154.04**	0.33	0.31	0.3	454	416.8	416.85	**242.03**	0.44	0.4	0.46	496.97	407.66	407.66	**215.03**	0.5	0.47	0.5	516.47	522.37	521.03	**287.51**	0.7	0.74	0.7
0.25	0	0	0	169.2	204.1	204.06	**155.81**	0.4	0.4	0.4	223.9	291.1	248.51	**210.7**	0.5	0.4	0.55	229.84	273.42	273.42	**222.03**	0.6	0.56	0.6	234.65	281.33	268.35	**247.43**	0.8	0.8	0.7
0	0.25	0.25	0	257.5	193.8	193.81	**130.21**	0.29	0.31	0.3	377	268.6	290.77	**169.12**	0.39	0.4	0.46	415.37	315.83	315.83	**170.97**	0.4	0.47	0.5	455.72	391.63	408.31	**186.48**	0.7	0.74	0.7
0.25	0.25	0	0	139.3	147.3	147.28	**108.88**	0.38	0.38	0.4	181	162.9	236.98	**154.42**	0.47	0.4	0.55	197.2	204.16	204.16	**165.56**	0.5	0.53	0.6	214.71	258.41	201.84	**194.63**	0.7	0.77	0.7
0.25	0	0.25	0.25	115.2	115.6	115.59	**93.01**	0.38	0.38	0.4	165	225.5	137	**130.49**	0.48	0.4	0.49	169.56	163.56	163.56	**141.14**	0.5	0.53	0.5	186.55	176.7	210.75	**168.1**	0.7	0.77	0.7
0.25	0.25	0.25	0.25	95.7	95.37	95.37	**88.47**	0.35	0.35	0.4	134.4	140.7	133.67	**127.54**	0.45	0.4	0.49	146.17	**136.91**	**136.91**	139.70	0.5	0.5	0.5	170.96	**168.29**	**168.29**	177.83	0.71	0.75	0.69
0	0.5	0	0	257.5	160.8	160.75	**76.57**	0.29	0.29	0.3	341.8	145.7	439.2	**130.66**	0.39	0.3	0.49	415.39	266.58	266.58	**111.29**	0.4	0.44	0.5	475.29	504.64	317.14	**116.83**	0.7	0.72	0.7
0	0	0.5	0.5	166.7	103.3	103.25	**56.21**	0.29	0.29	0.3	276.7	280	141.54	**79.71**	0.39	0.3	0.38	292.97	174.86	174.86	**75.52**	0.4	0.44	0.4	345.32	221.33	329.4	**83.02**	0.7	0.72	0.6
0.5	0	0	0	86.05	121.6	121.57	**81.63**	0.46	0.46	0.5	109	170.6	132.16	**102.11**	0.55	0.5	0.6	109.52	146.08	146.08	**106.5**	0.6	0.62	0.6	**110.9**	141.21	131.68	116.24	0.8	0.82	0.8
0.5	0.5	0	0	61.77	65.32	65.32	**44.63**	0.41	0.41	0.5	75.72	63.66	117.84	**57.76**	0.5	0.4	0.6	84.31	84.67	84.67	**65.15**	0.6	0.55	0.6	94.95	117.37	80.38	**77.01**	0.8	0.78	0.8
0.5	0	0.5	0.5	45.08	45.23	45.23	**34.46**	0.41	0.41	0.4	65.53	98.79	**50.44**	55.96	0.5	0.4	0.49	65.4	60.49	60.49	**48.48**	0.6	0.55	0.5	74.45	64.63	83.45	**61.34**	0.7	0.77	0.7
0.5	0.5	0.5	0.5	33.43	34.89	34.89	**31.07**	0.36	0.36	0.4	46.72	50.86	48.44	**44.31**	0.45	0.4	0.49	51.19	**47.53**	**47.53**	49.27	0.5	0.5	0.5	64.15	**60.14**	60.99	66.78	0.7	0.73	0.7
0.75	0	0	0	49.95	77.43	77.43	**47.64**	0.51	0.51	0.5	60.79	106.1	77.18	**58.42**	0.59	0.5	0.64	60.16	85.65	85.65	**58.79**	0.7	0.65	0.7	**59.09**	77.98	71.69	61.5	0.8	0.84	0.8
0	0.75	0	0	206.6	99.89	99.89	**49.25**	0.27	0.27	0.3	269.4	83.59	353.34	**77.8**	0.37	0.3	0.49	348.29	170.42	170.42	**63.89**	0.4	0.42	0.5	429.72	423.73	225.96	**73.27**	0.7	0.7	0.7
0	0	0.75	0.75	110.4	58.45	58.45	**38.66**	0.27	0.27	0.2	198.1	180.9	82.75	**39.3**	0.37	0.3	0.34	209.36	101.54	101.54	**38.12**	0.4	0.42	0.4	268.2	139.27	224.47	**51.5**	0.7	0.69	0.6
0.75	0.75	0	0	32.48	32.88	32.88	**22.78**	0.43	0.43	0.5	37.83	30.04	63.43	**27.34**	0.52	0.5	0.64	42.8	40.77	40.77	**32.02**	0.6	0.57	0.7	48.18	58.69	37.82	**35.18**	0.8	0.78	0.8
0.75	0	0	0.75	32.48	32.88	32.88	**22.05**	0.51	0.43	0.4	40.6	51.23	44.85	**26.61**	0.59	0.5	0.56	42.8	40.77	40.77	**31.87**	0.7	0.57	0.6	46.43	48.69	47.07	**34.91**	0.8	0.77	0.7
0.75	0.75	0.75	0.75	15.1	16.18	16.18	**14.29**	0.37	0.37	0.4	20.35	22.94	22.16	**19.43**	0.45	0.40	0.49	22.77	**21.2**	**21.2**	21.98	0.5	0.5	0.5	29.01	**26.58**	27.14	28.54	0.69	0.72	0.67
0	1	0	0	166.7	63.41	63.41	**31.38**	0.26	0.26	0.3	213.5	51.08	260.02	**31.76**	0.35	0.3	0.49	292.97	110.24	110.24	**39.19**	0.4	0.4	0.5	388.71	343.04	162.95	**40.22**	0.6	0.68	0.7
0	0	1	0	166.7	63.41	63.41	**30.36**	0.26	0.33	0.3	311.3	384.9	72.67	**40.02**	0.35	0.37	0.39	292.97	110.24	110.24	**39.60**	0.40	0.50	0.40	329.9	118.14	255.74	**47.64**	0.63	0.77	0.61
0	0	1	1	74.84	35.34	35.34	**17.89**	0.26	0.26	0.2	143.7	115.3	51.58	**28.91**	0.35	0.3	0.31	151.57	62.3	62.3	**25.64**	0.4	0.4	0.3	209.59	91.19	152.07	**35.72**	0.6	0.67	0.5
1	0	0	0	31.84	52.08	52.08	**30.62**	0.55	0.55	0.6	37.28	69.5	48.63	**35.71**	0.63	0.6	0.68	36.6	54.04	54.04	**35.93**	0.7	0.69	0.7	**34.69**	46.59	42.51	36.63	0.8	0.85	0.8
1	1	0	0	19.26	18.53	18.53	**12.93**	0.45	0.45	0.6	21.45	16.36	36.6	**16.45**	0.53	0.5	0.68	24.59	22.27	22.27	**17.33**	0.6	0.59	0.7	27.27	32.12	**20.34**	27.74	0.8	0.78	0.8
1	0	0	1	19.26	18.53	18.53	**12.97**	0.55	0.45	0.5	23.28	27.73	24.73	**14.64**	0.63	0.5	0.57	24.59	22.27	22.27	**17.25**	0.7	0.59	0.6	26.11	26.04	25.29	**21.86**	0.8	0.78	0.7
1	1	1	0	12.25	11.79	11.79	**10.17**	0.38	0.45	0.5	15.84	15.46	14.78	**13.14**	0.46	0.5	0.58	16.95	14.7	14.7	**13.06**	0.5	0.59	0.6	19.66	**16.57**	16.94	18.36	0.7	0.78	0.7
1	1	1	1	8.18	8.88	8.88	**7.81**	0.38	0.38	0.4	10.68	12.17	11.96	**10.65**	0.46	0.4	0.49	12.02	**11.28**	**11.28**	11.66	0.5	0.5	0.5	15.19	**13.8**	14.14	15.3	0.7	0.71	0.7
1	0	2	0	12.25	7.16	7.16	**6.7**	0.4	0.55	0.4	19.39	28.55	**7.29**	12.31	0.48	0.6	0.52	16.95	9.37	9.37	**8.06**	0.5	0.69	0.5	17.93	**8.51**	14.12	9.57	0.7	0.85	0.7
1	2	1	0.5	6.8	5.82	5.82	**5.22**	0.33	0.36	0.4	8.34	5.72	10.1	**5.71**	0.4	0.4	0.53	10.23	**7.64**	**7.64**	8.91	0.4	0.48	0.5	13.94	12.11	**9.61**	10.02	0.6	0.69	0.7
1.5	1.5	1.5	1.5	3.47	3.8	3.8	**3.37**	0.38	0.38	0.4	4.26	4.86	4.89	**4.14**	0.46	0.4	0.49	4.79	**4.59**	**4.59**	5.03	0.5	0.5	0.5	5.78	**5.28**	5.42	5.9	0.7	0.7	0.7
2	0	0	0	9.23	15.72	15.72	**9.05**	0.66	0.66	0.7	9.72	18.97	12.73	**9.47**	0.72	0.67	0.76	9.39	14.09	14.09	**9.29**	0.77	0.77	0.77	**7.8**	10.39	9.45	8.27	0.87	0.89	0.85
2	2	0	0	4.91	4.16	4.16	**3.81**	0.51	0.51	0.7	4.9	3.57	7.16	**3.50**	0.57	0.52	0.76	5.71	**4.64**	**4.64**	4.70	0.62	0.62	0.77	5.78	5.91	**4.11**	4.95	0.77	0.79	0.85
2	0	2	2	2.99	2.77	2.77	**2.59**	0.51	0.51	0.4	3.8	4.58	**2.95**	2.97	0.59	0.53	0.49	3.74	3.17	3.17	**2.97**	0.62	0.62	0.50	3.78	**3.03**	3.66	3.44	0.75	0.78	0.64
2	2	2	2	2.05	2.22	2.22	**2.07**	0.39	0.39	0.4	2.38	2.66	2.71	**2.32**	0.46	0.41	0.49	2.64	**2.57**	**2.57**	2.60	0.50	0.50	0.50	3.01	**2.8**	2.87	3.01	0.66	0.69	0.64

Table 11. Comparison of performance, three variables, p = 3.

7. Conclusions

In this paper a new control chart designed for monitoring correlated Poisson variables has been introduced: the Linear Combination of Poisson variables (LCP) control chart. The statistics to be plotted is a linear combination of the Poisson scores. The values of the coefficients of this linear combination and of the chart's control limits that result in the required in-control ARL (ARL_0) and minimize the out-of-control ARL are obtained by employing user-friendly Windows© software developed by the authors. A main advantage of this chart is that it makes possible to match whatever value of ARL_0 specified by the user because the possible values of the control limits are real numbers, not integers, contrarily to what happens with the rest of Poisson control charts.

The software developed, freely available from the authors, also optimizes the parameters of the MP, MX and multiple univariate Poisson charts. In addition, it carries out a complete comparison of performance among them. Therefore, it is possible to carefully study which is the best control chart under different scenarios. We think that this software makes much easier to employ this new chart, or the previous ones, in real applications.

In the vast majority of cases analyzed in this paper the LCP chart outperforms the other control schemes considered. In some cases, the LCP chart gives ARL values that are about one third of the ones of its best competitor. In the few cases where the LCP chart does not have the best performance, the differences are rather small.

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