A simple formula to find the closest consistent matrix to a reciprocal matrix

J. Benítez[*], J. Izquierdo[†], R. Pérez-García[†], E. Ramos-Martínez[†]

Abstract: Achieving consistency in pair-wise comparisons between decision elements given by experts or stakeholders is of paramount importance in decision-making based on the AHP methodology. Several alternatives to improve consistency have been proposed in the literature. The linearization method (Benítez et al., Achieving matrix consistency in AHP through linearization, Applied Mathematical Modelling 35 (2011) 4449-4457), derives a consistent matrix based on an original matrix of comparisons through a suitable orthogonal projection expressed in terms of a Fourier-like expansion. We propose a formula that provides in a very simple manner the consistent matrix closest to a reciprocal (inconsistent) matrix. In addition, this formula is computationally efficient since it only uses sums to perform the calculations. A corollary of the main result shows that the normalized vector of the vector, whose components are the geometric means of the rows of a comparison matrix, gives the priority vector only for consistent matrices.

1 Introduction

The AHP (Analytic Hierarchy Process) [1, 2] is designed for multi-objective, multi-criteria, and multi-actor decisions, with and without certainty, for any number of alternatives. The AHP approach mainly consists of three stages, construction of the hierarchy of problem ingredients, namely, objective, criteria and alternatives, calculation of the priorities of the elements, and aggregation of results to produce the final decision. Interactions between the elements are considered when building the structure of the problem. The elements are evaluated using pairwise comparisons, by asking experts or stakeholders involved in the decision-making problem about how much importance a criterion has when compared with another criterion with respect to the interests or preferences of respondents. The candidate alternatives are also evaluated by pairwise comparisons with respect to what is the higher degree of satisfaction for each criterion.

Both kinds of related values can be determined by using various scales, in particular the scale of $1-9$ to represent equal importance to extreme importance [1]. Performing such a comparison yields an $n \times n$ matrix $A=(a_{ij})$, whose (positive) entries must adhere to two important properties, namely, $a_{ii}=1$ (homogeneity) and $a_{ji}=1/a_{ij}$ (reciprocity), $i,j=1,\dots,n$. The problem for

[*]Instituto de Matemática Multidisciplinar, Universitat Politècnica de València (Spain)
[†]FluIng-Instituto de Matemática Multidisciplinar, Universitat Politècnica de València (Spain)

matrix A becomes one of producing for the n elements, $E_1, \ldots, E_n$ (criteria or alternatives) under comparison, a set of numerical values $w_1, \ldots, w_n$ that reflect the priorities among the compared elements according to the emitted judgments. If all the judgments are completely consistent, the relations between weights w_i and judgments a_{ij} are simply given by $w_i/w_j = a_{ij}$ (for $i, j = 1, 2, \ldots, n$) and the matrix A is said to be consistent. The following theorem provides equivalent conditions for a matrix A to be consistent.

Firstly, we provide some notation. $M_{n,m}$ will hereinafter denote the set of $n \times m$ real matrices, and $M^+_{n,m}$ will denote the subset of $M_{n,m}$ composed of positive matrices. It will be assumed that the elements of $\mathbb{R}^n$ are column vectors, i.e., $\mathbb{R}^n$ is identified with $M_{n,1}$. For a given $A \in M_{n,m}$, let us write $[A]_{ij}$ the (i, j) entry of the matrix A. The superscript T denotes the matrix transposition. The mapping $J : M^+_{n,m} \to M^+_{n,m}$ defined by $[J(A)]_{ij} = 1/[A]_{ij}$ will play an important role in the sequel.

Theorem 1 [3, Th. 1] *Let $A = (a_{ij}) \in M^+_{n,n}$. The following statements are equivalent.*

(i) *There exists $\mathbf{x} \in M^+_{n,1}$ such that $A = J(\mathbf{x})\mathbf{x}^T$.*

(ii) *There exists $\mathbf{w} = [w_1 \cdots w_n]^T \in M^+_{n,1}$ such that $a_{ij} = w_i/w_j$ for all $i, j \in \{1, \ldots, n\}$.*

(iii) *$a_{ij}a_{ji} = 1$ and $a_{ij}a_{jk} = a_{ik}$ hold for all $i, j, k \in \{1, \ldots, n\}$.*

For a consistent matrix, the leading eigenvalue and the principal (Perron) eigenvector of a comparison matrix provide information to deal with complex decisions, the normalized Perron eigenvector giving the sought priority vector [2]. It is also well known that any consistent matrix has rank one [3], and as a consequence, any of its normalized rows and, in particular, the normalized vector of the geometric means of the rows, also provides the priority vector. Taking into account the (natural lack of) consistency of human thinking, some degree of inconsistency is expected and, as a result, in general A is not consistent. As shown in [4] the eigenvector is necessary for obtaining priorities. The hypothesis that the estimates of these values are small perturbations of the "right" values guarantees a small perturbation of the eigenvalues (see, e.g., [5]). For non-consistent matrices, the problem to solve is the eigenvalue problem $A\mathbf{w} = \lambda_{\max}\mathbf{w}$, where $\lambda_{\max}$ is the unique largest eigenvalue of A that gives the Perron eigenvector as an estimate of the priority vector. As a measurement of inconsistency, Saaty proposed using the consistency index $\mathrm{CI} = (\lambda_{\max} - n)/(n-1)$ and the consistency ratio $\mathrm{CR} = \mathrm{CI}/\mathrm{RI}$, where RI is the so-called average consistency index [2]. If $\mathrm{CR} < 0.1$, the estimate is accepted; otherwise, a new comparison matrix is solicited until $\mathrm{CR} < 0.1$.

Achieving consistency in AHP has become an important issue and different methods have been proposed in the literature [1, 2, 3, 6, 7, 8, 9, 10, 11, 12, 13, 14]. In this paper, we focus on the linearization process [10] not as method to directly obtain the priority vector, but as a method that provides a closed form for achieving complete consistency. Here we use the word closed in contrast with methods relying on optimisation, which is non-linear for this problem, and are iterative by nature. Achieving complete consistency is a feature that may be suitably used for specific purposes.

In section 2 we provide a short review of the linearization method. In section 3 we develop a simple formula to obtain the consistent matrix that is closest to a given comparison matrix and its associated priority vector. This formula involves just sums, a very important computational feature. As a consequence, we show that the row geometric mean method (RGMM) gives the priority vector only for completely consistent matrices. Finally, a section devoted to discussion and conclusions closes the paper.

2 Short review of the linearization method

Let us recall that a reciprocal matrix $A \in M^+_{n,n}$ verifies the condition $A_{ij} = 1/A_{ji}$ for $1 \leq i, j \leq n$, whereas a consistent matrix $A \in M^+_{n,n}$ also satisfies $A_{ij}A_{jk} = A_{ik}$ for $1 \leq i, j, k \leq n$.

As we have mentioned, an important problem in AHP theory is the following: find the closest consistent matrix to a given reciprocal matrix $A \in M^+_{n,n}$. To this end, in [10] the mappings were introduced

$$L : M^+_{n,n} \to M_{n,n}, \qquad [L(A)]_{ij} = \log[A_{ij}]$$

and

$$E : M_{n,n} \to M^+_{n,n}, \qquad [E(A)]_{ij} = \exp[A_{ij}].$$

Each of these mappings is, evidently, one the inverse of the other. Obviously, for a given $A \in M^+_{n,n}$ we have

$$A \text{ is reciprocal} \quad \Longleftrightarrow \quad L(A) \text{ is skew-Hermitian.}$$

Furthermore, in [10, Theorem 2.2] it was proven that

$$\mathscr{L}_n = \{L(A) : A \in M^+_{n,n} \text{ and } A \text{ is consistent } \} \text{ is a linear subspace.} \tag{1}$$

The aforementioned approximation problem was solved by means of a linearization technique [10].

We need some notation to state this solution: We consider all vectors of $\mathbb{R}^n$ as column vectors, by $\mathbf{1}_n$ we denote the vector of $\mathbb{R}^n$ given by $\mathbf{1}_n^T = (1, \ldots, 1)$, the trace operator is denoted by $\text{tr}(\cdot)$, i.e., for a square matrix $A \in M_{n,n}$, $\text{tr}(A) = [A]_{1,1} + \cdots + [A]_{n,n}$, and finally, ϕ_n denotes the linear mapping defined by

$$\phi_n(\mathbf{x}) = \mathbf{x}\mathbf{1}_n^T - \mathbf{1}_n\mathbf{x}^T, \qquad \phi_n : \mathbb{R}^n \to M_{n,n}. \tag{2}$$

The mathematical tool to solve the approximation problem is given by the following result.

Theorem 2 [10, Theorem 2.5] *The orthogonal projection $p_n : M_{n,n} \to \mathscr{L}_n$ is given by*

$$p_n(X) = \frac{1}{2n}\sum_{k=1}^{n-1} \frac{\text{tr}(X^T \phi_n(\mathbf{y}_k))}{\|\mathbf{y}_k\|^2}\phi_n(\mathbf{y}_k), \tag{3}$$

where $\{\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}\}$ *is an orthogonal basis of* $(\text{span}\{\mathbf{1}_n\})^\perp$.

An important property of this projection is that it minimizes certain distance. To be more precise, we first define such distance (see [15]) in $M^+_{n,n}$

$$\mathrm{d}(A,B) = \|L(A) - L(B)\|_{\mathrm{F}}, \tag{4}$$

here $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm (if $X \in M_{n,n}$, then $\|X\|_{\mathrm{F}} = \mathrm{tr}(X^T X)$). Now, for a given matrix $A \in M^+_{n,n}$, the matrix $Y = E(p_n(L(A)))$ is consistent and it satisfies

$$\mathrm{d}(A,Y) \le \mathrm{d}(A,Y') \text{ for any } Y' \in M^+_{n,n} \text{ consistent.}$$

3 A simple formula for equation (3)

As remarked earlier, in AHP, it is of interest to find the closest consistent matrix to a given reciprocal matrix $A \in M^+_{n,n}$. In other words, it is required to apply (3) to $L(A)$, when $A \in M^+_{n,n}$ is reciprocal. The following theorem provides a much simpler approach. As we know, if A is reciprocal, then $L(A)$ is skew-Hermitian.

Theorem 3 *If M is skew-Hermitian and if $p_n : M_{n,n} \to \mathscr{L}_n$ denotes the orthogonal projection, then*

$$p_n(M) = \frac{1}{n}\left[(MU_n) - (MU_n)^T\right], \tag{5}$$

where $U_n = \mathbf{1}_n \mathbf{1}_n^T$.

Proof: Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$ and let us define the following skew-Hermitian matrices:

$$B_{ij} = \mathbf{e}_i \mathbf{e}_j^T - \mathbf{e}_j \mathbf{e}_i^T, \qquad 1 \le i,j \le n. \tag{6}$$

Any skew-Hermitian matrix M can easily be written as

$$M = \sum_{i<j} r_{ij} B_{ij} \tag{7}$$

for some real numbers $\{r_{ij}\}_{i<j}$. Since the orthogonal projection $p_n : M_{n,n} \to \mathscr{L}_n$ is a linear mapping, then

$$p_n(M) = \sum_{i<j} r_{ij} p_n(B_{ij}). \tag{8}$$

Thus, in order to obtain an expression of $p_n(M)$, it is sufficient to simplify $p_n(B_{ij})$. To this end, we will use (3) for B_{ij}. Previously, we shall simplify $\mathrm{tr}(B_{ij}^T \phi_n(\mathbf{y}_k))$. Notice that $\mathbf{e}_i^T \mathbf{1}_n = \mathbf{e}_j^T \mathbf{1}_n = 1$ and $\mathbf{e}_i^T \mathbf{y}_k$, $\mathbf{e}_j^T \mathbf{y}_k$ are scalars, which commute with any matrix.

$$\begin{aligned}
B_{ij}^T \phi_n(\mathbf{y}_k) &= \left(\mathbf{e}_i \mathbf{e}_j^T - \mathbf{e}_j \mathbf{e}_i^T\right)^T \left(\mathbf{y}_k \mathbf{1}_n^T - \mathbf{1}_n \mathbf{y}_k^T\right) \\
&= \left(\mathbf{e}_j \mathbf{e}_i^T - \mathbf{e}_i \mathbf{e}_j^T\right) \left(\mathbf{y}_k \mathbf{1}_n^T - \mathbf{1}_n \mathbf{y}_k^T\right) \\
&= \mathbf{e}_j \mathbf{e}_i^T \mathbf{y}_k \mathbf{1}_n^T - \mathbf{e}_j \mathbf{e}_i^T \mathbf{1}_n \mathbf{y}_k^T - \mathbf{e}_i \mathbf{e}_j^T \mathbf{y}_k \mathbf{1}_n^T + \mathbf{e}_i \mathbf{e}_j^T \mathbf{1}_n \mathbf{y}_k^T \\
&= (\mathbf{e}_i^T \mathbf{y}_k) \mathbf{e}_j \mathbf{1}_n^T - \mathbf{e}_j \mathbf{y}_k^T - (\mathbf{e}_j^T \mathbf{y}_k) \mathbf{e}_i \mathbf{1}_n^T + \mathbf{e}_i \mathbf{y}_k^T
\end{aligned}$$

Bearing in mind that the trace is linear and $\mathrm{tr}(MN) = \mathrm{tr}(NM)$ holds for any pair of matrices M, N such that the products MN and NM are meaningful, we obtain

$$\begin{aligned}
\mathrm{tr}\left(B_{ij}^T \phi_n(\mathbf{y}_k)\right) &= \mathrm{tr}\left((\mathbf{e}_i^T \mathbf{y}_k)\mathbf{e}_j \mathbf{1}_n^T - \mathbf{e}_j \mathbf{y}_k^T - (\mathbf{e}_j^T \mathbf{y}_k)\mathbf{e}_i \mathbf{1}_n^T + \mathbf{e}_i \mathbf{y}_k^T\right) \\
&= (\mathbf{e}_i^T \mathbf{y}_k)\,\mathrm{tr}(\mathbf{e}_j \mathbf{1}_n^T) - \mathrm{tr}(\mathbf{e}_j \mathbf{y}_k^T) - (\mathbf{e}_j^T \mathbf{y}_k)\,\mathrm{tr}(\mathbf{e}_i \mathbf{1}_n^T) + \mathrm{tr}(\mathbf{e}_i \mathbf{y}_k^T) \\
&= (\mathbf{e}_i^T \mathbf{y}_k)\,\mathrm{tr}(\mathbf{1}_n^T \mathbf{e}_j) - \mathrm{tr}(\mathbf{y}_k^T \mathbf{e}_j) - (\mathbf{e}_j^T \mathbf{y}_k)\,\mathrm{tr}(\mathbf{1}_n^T \mathbf{e}_i) + \mathrm{tr}(\mathbf{y}_k^T \mathbf{e}_i) \\
&= 2\mathbf{e}_i^T \mathbf{y}_k - 2\mathbf{e}_j^T \mathbf{y}_k \\
&= 2(\mathbf{e}_i - \mathbf{e}_j)^T \mathbf{y}_k.
\end{aligned}$$

Thus, from (3),

$$\begin{aligned}
p_n(B_{ij}) &= \frac{1}{n} \sum_{k=1}^{n-1} \frac{(\mathbf{e}_i - \mathbf{e}_j)^T \mathbf{y}_k}{\|\mathbf{y}_k\|^2} \phi_n(\mathbf{y}_k) \\
&= \frac{1}{n} \sum_{k=1}^{n-1} \frac{(\mathbf{e}_i - \mathbf{e}_j)^T \mathbf{y}_k}{\|\mathbf{y}_k\|^2} (\mathbf{y}_k \mathbf{1}_n^T - \mathbf{1}_n \mathbf{y}_k^T) \\
&= \frac{1}{n} \left(\sum_{k=1}^{n-1} \frac{(\mathbf{e}_i - \mathbf{e}_j)^T \mathbf{y}_k}{\|\mathbf{y}_k\|^2} \mathbf{y}_k \right) \mathbf{1}_n^T - \frac{1}{n} \sum_{k=1}^{n-1} \frac{(\mathbf{e}_i - \mathbf{e}_j)^T \mathbf{y}_k}{\|\mathbf{y}_k\|^2} \mathbf{1}_n \mathbf{y}_k^T.
\end{aligned}$$

Let us bear in mind that $(\mathbf{e}_i - \mathbf{e}_j)^T \mathbf{y}_k / \|\mathbf{y}_k\|^2$ is a scalar that commutes with any matrix, and therefore in the second summand we can change the order of the matrices and obtain

$$\begin{aligned}
p_n(B_{ij}) &= \frac{1}{n} \left(\sum_{k=1}^{n-1} \frac{(\mathbf{e}_i - \mathbf{e}_j)^T \mathbf{y}_k}{\|\mathbf{y}_k\|^2} \mathbf{y}_k \right) \mathbf{1}_n^T - \frac{1}{n} \mathbf{1}_n \left(\sum_{k=1}^{n-1} \frac{(\mathbf{e}_i - \mathbf{e}_j)^T \mathbf{y}_k}{\|\mathbf{y}_k\|^2} \mathbf{y}_k^T \right) \\
&= \frac{1}{n} \left(\sum_{k=1}^{n-1} \frac{(\mathbf{e}_i - \mathbf{e}_j)^T \mathbf{y}_k}{\|\mathbf{y}_k\|^2} \mathbf{y}_k \right) \mathbf{1}_n^T - \frac{1}{n} \mathbf{1}_n \left(\sum_{k=1}^{n-1} \frac{(\mathbf{e}_i - \mathbf{e}_j)^T \mathbf{y}_k}{\|\mathbf{y}_k\|^2} \mathbf{y}_k \right)^T. \qquad (9)
\end{aligned}$$

Let us remind the classical Fourier expansion in an Euclidean space: If $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is an orthogonal basis of $\mathbb{R}^n$, then any $\mathbf{v} \in \mathbb{R}^n$ can be written as $\mathbf{v} = \frac{\mathbf{v}^T \mathbf{w}_1}{\|\mathbf{w}_1\|} \mathbf{w}_1 + \cdots + \frac{\mathbf{v}^T \mathbf{w}_n}{\|\mathbf{w}_n\|} \mathbf{w}_n$. Therefore, since $\{\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}, \mathbf{1}_n\}$ is an orthogonal basis of $\mathbb{R}^n$ we have

$$\mathbf{v} = \sum_{k=1}^{n-1} \frac{\mathbf{v}^T \mathbf{y}_k}{\|\mathbf{y}_k\|^2} \mathbf{y}_k + \frac{\mathbf{v}^T \mathbf{1}_n}{\|\mathbf{1}_n\|^2} \mathbf{1}_n \qquad \text{for all } \mathbf{v} \in \mathbb{R}^n.$$

Since $\|\mathbf{1}_n\|^2 = n$, we obtain

$$\sum_{k=1}^{n-1} \frac{\mathbf{v}^T \mathbf{y}_k}{\|\mathbf{y}_k\|^2} \mathbf{y}_k = \mathbf{v} - \frac{\mathbf{v}^T \mathbf{1}_n}{n} \mathbf{1}_n \qquad \text{for all } \mathbf{v} \in \mathbb{R}^n. \qquad (10)$$

By employing (9), (10), and $\mathbf{e}_i^T\mathbf{1}_n = \mathbf{e}_i^T\mathbf{1}_n = 1$ we obtain

$$\begin{aligned}
p_n(B_{ij}) &= \frac{1}{n}\left(\mathbf{e}_i - \mathbf{e}_j - \frac{(\mathbf{e}_i-\mathbf{e}_j)^T\mathbf{1}_n}{n}\mathbf{1}_n\right)\mathbf{1}_n^T - \frac{1}{n}\mathbf{1}_n\left(\mathbf{e}_i - \mathbf{e}_j - \frac{(\mathbf{e}_i-\mathbf{e}_j)^T\mathbf{1}_n}{n}\mathbf{1}_n\right)^T \\
&= \frac{1}{n}\left(\mathbf{e}_i-\mathbf{e}_j\right)\mathbf{1}_n^T - \frac{1}{n}\mathbf{1}_n\left(\mathbf{e}_i-\mathbf{e}_j\right)^T \\
&= \frac{1}{n}\phi_n(\mathbf{e}_i-\mathbf{e}_j). \qquad (11)
\end{aligned}$$

By (8) and (11) it follows that

$$p_n(M) = \frac{1}{n}\sum_{i<j} r_{ij}\phi_n(\mathbf{e}_i-\mathbf{e}_j).$$

Moreover, by using (6) and (7) we obtain

$$\begin{aligned}
M\mathbf{1}_n &= \sum_{i<j} r_{ij}B_{ij}\mathbf{1}_n \\
&= \sum_{i<j} r_{ij}(\mathbf{e}_i\mathbf{e}_j^T - \mathbf{e}_j\mathbf{e}_i^T)\mathbf{1}_n = \sum_{i<j} r_{ij}\left(\mathbf{e}_i(\mathbf{e}_j^T\mathbf{1}_n) - \mathbf{e}_j(\mathbf{e}_i^T\mathbf{1}_n)\right) = \sum_{i<j} r_{ij}(\mathbf{e}_i-\mathbf{e}_j).
\end{aligned}$$

Since ϕ_n is a linear mapping, by using the last two identities we obtain

$$p_n(M) = \frac{1}{n}\phi_n\left(\sum_{i<j} r_{ij}(\mathbf{e}_i-\mathbf{e}_j)\right) = \frac{1}{n}\phi_n(M\mathbf{1}_n). \qquad (12)$$

Finally, we will use the definition of ϕ_n given in (2):

$$\phi_n(M\mathbf{1}_n) = (M\mathbf{1}_n)\mathbf{1}_n^T - \mathbf{1}_n(M\mathbf{1}_n)^T = M(\mathbf{1}_n\mathbf{1}_n^T) - (\mathbf{1}_n\mathbf{1}_n^T)M^T.$$

By recalling that U_n is defined as $U_n = \mathbf{1}\mathbf{1}_n^T$ and U_n is symmetric,

$$\phi_n(M\mathbf{1}_n) = MU_n - U_nM^T = MU_n - U_n^TM^T = MU_n - (MU_n)^T. \qquad (13)$$

The expressions (12) and (13) conclude the proof of this theorem. ∎

Let us remark that the use of (5) when M is skew-Hermitian does not require any multiplication, since only sums are required to compute MU_n.

It was proven in [10] that the linear space $\mathcal{L}_n$ defined in (1) and the linear mapping ϕ_n defined in (2) satisfy $\operatorname{Im}\phi_n = \mathcal{L}_n$. Now, if $M \in M_{n,n}$, then there exists $\mathbf{v} \in \mathbb{R}^n$ such that $\phi_n(\mathbf{v}) = p_n(M)$. We shall use Theorem 3 for finding such vectors $\mathbf{v}$ when M is skew-Hermitian.

Theorem 4 *Let $M \in M_{n,n}$ be skew-Hermitian and $\mathbf{v} \in \mathbb{R}^n$. Then*

$$\phi_n(\mathbf{v}) = p_n(M) \quad \Leftrightarrow \quad \mathbf{v} = \frac{1}{n}M\mathbf{1}_n + \alpha\mathbf{1}_n, \;\; \alpha \in \mathbb{R}.$$

PROOF: As in Theorem 3, we define $U_n = \mathbf{1}_n \mathbf{1}_n^T$. In [10, Theorem 2.2] it was proven that the null space of $\phi_n : \mathbb{R}^n \to M_{n,n}$ is span$\{\mathbf{1}_n\}$. By Theorem 3 we obtain

$$\begin{aligned}
\phi_n\left(\frac{1}{n}M\mathbf{1}_n\right) &= \frac{1}{n}\phi_n(M\mathbf{1}_n) = \frac{1}{n}\left[(M\mathbf{1}_n)\mathbf{1}_n^T - \mathbf{1}_n(M\mathbf{1}_n)^T\right] \\
&= \frac{1}{n}\left[M(\mathbf{1}_n\mathbf{1}_n^T) - \mathbf{1}_n\mathbf{1}_n^T M^T\right] = \frac{1}{n}\left[MU_n - U_n M^T\right] \\
&= \frac{1}{n}\left[MU_n - U_n^T M^T\right] = \frac{1}{n}\left[MU_n - (MU_n)^T\right] = p_n(M). \qquad (14)
\end{aligned}$$

$\Leftarrow$: It follows from (14), the linearity of ϕ_n, and $\mathbf{1}_n \in \ker\phi_n$.

$\Rightarrow$: Let $\mathbf{v} \in \mathbb{R}^n$ such that $\phi_n(\mathbf{v}) = p_n(M)$. By (14), one has $\phi_n(\mathbf{v} - \frac{1}{n}M\mathbf{1}_n) = \mathbf{0}$, hence $\mathbf{v} - \frac{1}{n}M\mathbf{1}_n \in \ker\phi_n = \text{span}\{\mathbf{1}_n\}$. ■

As we have mentioned, for a given reciprocal matrix $A \in M_{n,n}$, the matrix $Y = E(p_n(L(A)))$ is the unique consistent matrix that satisfies $\mathrm{d}(A, Y) \leq \mathrm{d}(A, Y')$ for any consistent $Y' \in M_{n,n}$, where the distance $\mathrm{d}(\cdot,\cdot)$ is the distance defined in (4). But observe that $L(A)$ is skew-Hermitian (since A is reciprocal), hence $p_n(L(A))$ can be briefly described by means of Theorem 3. Furthermore, since $p_n(L(A)) \in \mathscr{L}_n$, then $Y = E(p_n(L(A)))$ is a consistent matrix. By Theorem 1, there exists $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x}J(\mathbf{x})^T = Y$. We shall find such priority vector $\mathbf{x}$ in terms of the matrix A.

Theorem 5 *Let $A = (a_{ij}) \in M_{n,n}$ be a reciprocal matrix and $\mathbf{x} \in M_{n,1}^+$. Then $\mathbf{x}J(\mathbf{x})^T = E(p_n(L(A)))$ if and only if there exists $C > 0$ such that $\mathbf{x} = C(x_1, \ldots, x_n)^T$, where $x_i = \sqrt[n]{a_{i1}\cdots a_{in}}$.*

PROOF: Let $\mathbf{v} = (v_1, \ldots, v_n)^T = L(\mathbf{x})$, or in other words, $\mathrm{e}^{v_i} = x_i$ for $i = 1, \ldots, n$. Since

$$\phi_n(\mathbf{v}) = \mathbf{v}\mathbf{1}_n^T - \mathbf{1}_n\mathbf{v}^T = \begin{bmatrix} v_1 - v_1 & v_1 - v_2 & \cdots & v_1 - v_n \\ v_2 - v_1 & v_2 - v_2 & \cdots & v_2 - v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n - v_1 & v_n - v_2 & \cdots & v_n - v_n \end{bmatrix},$$

then

$$E(\mathbf{v}\mathbf{1}_n^T - \mathbf{1}_n\mathbf{v}^T) = \begin{bmatrix} x_1/x_1 & x_1/x_2 & \cdots & x_1/x_n \\ x_2/x_1 & x_2/x_2 & \cdots & x_2/x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n/x_1 & x_n/x_2 & \cdots & x_n/x_n \end{bmatrix} = \mathbf{x}^T J(\mathbf{x}).$$

Thus, $\mathbf{x}J(\mathbf{x})^T = E(p_n(L(A))) \iff \phi_n(\mathbf{v}) = p_n(L(A))$. Since A is reciprocal, then $L(A)$ is skew-Hermitian and Theorem 4 enables us to ensure $\phi_n(\mathbf{v}) = p_n(L(A)) \iff \exists\alpha \in \mathbb{R}$ such that $\mathbf{v} = \frac{1}{n}L(A)\mathbf{1}_n + \alpha\mathbf{1}_n$. To prove the theorem it is enough to simplify $\mathbf{x}$. Recall that $\mathbf{x} = E(\mathbf{v}) =$

$E\left(\frac{1}{n}L(A)\mathbf{1}_n + \alpha\mathbf{1}_n\right)$. Observe that $a_{ij} > 0$ because A is reciprocal. Since

$$\begin{aligned}\frac{1}{n}L(A)\mathbf{1}_n + \alpha\mathbf{1}_n &= \frac{1}{n}\begin{bmatrix} \log(a_{11}) & \cdots & \log(a_{1n}) \\ \vdots & \ddots & \vdots \\ \log(a_{n1}) & \cdots & \log(a_{nn}) \end{bmatrix}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + \alpha\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \alpha + \frac{1}{n}\left[\log(a_{11}) + \cdots + \log(a_{1n})\right] \\ \vdots \\ \alpha + \frac{1}{n}\left[\log(a_{n1}) + \cdots + \log(a_{nn})\right] \end{bmatrix},\end{aligned}$$

we obtain

$$\mathbf{x} = E(\mathbf{v}) = \mathrm{e}^{\alpha}\begin{bmatrix} \sqrt[n]{a_{11}\cdots a_{1n}} \\ \vdots \\ \sqrt[n]{a_{n1}\cdots a_{nn}} \end{bmatrix}.$$

The proof is finished. ■

This theorem clearly shows that the RGMM gives the priority vector for the matrix $E(p_n(L(A)))$ when A is reciprocal, which, according to Theorem 2, is the closest consistent matrix to the original comparison matrix A. Nevertheless, the decision maker may not recognize $E(p_n(L(A)))$ as representative of his or her judgment. This matrix (with synthetic, forced, and artificial consistency) has been obtained in an attempt to improve consistency, but may not be to the decision maker. Consequently, producing the priority vector directly through the RGMM is generally incorrect, since it may be far from representing the real thoughts of the decision maker. The final priority must be obtained after a trade-off process following some feedback with the decision maker until reaching a consensus. As a consequence, Theorem 2 must be considered an auxiliary process within the decision process, a tool that pushes toward consistency. In contrast, Theorem 5, as well as the RGMM, should be never applied, since the final priority vector will be the Perron eigenvector of the consensus matrix –as long as it has an acceptable consistency ratio.

4 Discussion and Conclusions

In order to compare the numeric efficiency of the proposed expression appearing in Theorem 3 with the formula (3), we shall count the number of required operations to find $p_n(M)$ when M is a $n \times n$ matrix.

If we apply (3), we must perform $2(n-1)$ multiplications of $n \times n$ matrices, compute $n-1$ traces of matrices $n \times n$, and finally make n divisions.

If we apply Theorem 3, then only only one matrix multiplication of two $n \times n$ matrices is required (even, if we adequately program the expression MU_n, no matrix multiplication is needed because U_n is a particularly simple matrix), subtract two $n \times n$ matrices and finally only perform one division. Also let us remark that in the computation of $(MU_n) - (MU_n)^T$ only the elements above the main diagonal must be computed because $(MU_n) - (MU_n)^T$ is skew-Hermitian.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
C_1	1	1/3	1/5	1	1/4	2	3
C_2	3	1	1/2	2	1/3	2	3
C_3	5	2	1	4	5	6	5
C_4	1	1/2	1/4	1	1/4	1	2
C_5	4	3	1/5	4	1	3	1
C_6	1/2	1/3	1/6	1	1/3	1	1/3
C_7	1/3	1/3	1/5	1/2	1	3	1

Table 1: Matrix of criteria A

In [10], in the context of a comparison between active leakage control (ALC) and passive leakage control (PLC) in water supply, the matrix of comparison of criteria in Table 1 is used.

As noted in [10], this matrix is inconsistent. The Perron eigenvalue is $\lambda_{\max} \simeq 7.9$. According to [2], the consistency index is $\mathrm{CI} = (\lambda_{\max} - 7)/6 \simeq 0.148$, and the consistency ratio, obtained by comparing CI with Saaty's random consistency index value is $\mathrm{CR} \simeq 10.95\%$, which shows that even when almost acceptable, the matrix consistency is unacceptable. Thus, A lacks a minimum of consistency.

By using the proposed formula, applied to $M = L(A)$, which is skew-hermitian, $p_n(M)$ is obtained, and calculating $E(p_n(M))$ the matrix given in Table 2 is obtained as the consistent matrix closest to A.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
C_1	1	0.526	0.208	1.069	0.445	1.644	1.17
C_2	1.902	1	0.395	2.034	0.847	3.126	2.225
C_3	4.815	2.532	1	5.149	2.144	7.914	5.633
C_4	0.935	0.492	0.194	1	0.416	1.537	1.094
C_5	2.246	1.181	0.466	2.402	1	3.691	2.627
C_6	0.608	0.32	0.126	0.651	0.271	1	0.712
C_7	0.855	0.449	0.178	0.914	0.381	1.405	1

Table 2: Consistent matrix closest to A

For this consistent matrix the normalized Perron eigenvector, the priority vector, is

$$Z = (0.081, 0.154, 0.390, 0.076, 0.182, 0.049, 0.069)^T.$$

This calculation corrects the errata in [10], where wrong values were mistakenly copied in the consistent matrix closest to A, now corrected in Table 2, and its associated (herein correct) priority vector Z. Even though the changes are of no real significance and have no material effect on the paper's message, which we do not repeat here, the authors wish to apologise for any inconvenience that may have been caused by the wrong figures in [10]. Also observe that the aggregation in [10] must be corrected from $W = (0.70, 0.30)^T$ to $W = (0.69, 0.31)^T$.

To close the paper we now present some conclusions. Applications of the analytic hierarchy process (AHP) involve pairwise comparisons. The judgments are used to develop pairwise comparison matrices, which are then used to estimate weights of decision elements (i.e., criteria or alternatives). There are several methods in the AHP literature for deriving weights. In this paper, we have focussed on the Linearization Method [10], and have proposed a simple formula for obtaining the closest consistent matrix to a non-consistent comparison matrix. We note that these methods provide synthetic and artificial consistency. In particular, we have shown that the RGMM is one of those methods that does not necessarily provide an acceptable priority vector. As a result, these methods can somehow distort the original expert opinions by moving figures away from the expert judgments. In this sense, these methods must be integrated within a process of feedback with the expert until consensus is obtained [17]. Given the great simplicity of the proposed formula, involving only sums, computational efficiency is guaranteed and integration in any AHP-based DSS is straightforward and of great interest. Finally, various applications of the Linearization Method (see, for example, [15, 16]) can also benefit in a straightforward way from the obtained simplicity.

Acknowledgments

This work has been performed with the support of the project IDAWAS, DPI2009-11591 of the Dirección General de Investigación del Ministerio de Ciencia e Innovación (Spain), with the supplementary support of ACOMP/2010/146 of the Conselleria d'Educació of the Generalitat Valenciana, and the support given to the first author by the Spanish project MTM2010-18539. The use of English in this paper was revised by John Rawlins; and the revision was funded by the Universitat Politècnica de València, Spain.

References

[1] T.L. Saaty, Fundamentals of Decision Making with the Analytic Hierarchy Process, paperback, RWS Publications, 4922 Ellsworth Avenue, Pittsburgh, PA 15213-2807, original edition 1994, revised 2000.

[2] T.L. Saaty, Relative measurement and its generalization in decision making. Why pairwise comparisons are central in mathematics for the measurement of intangible factors. The analytic hierarchy/network process, Revista de la Real Academia de Ciencias Serie A: Matemáticas 102 (2) (2008) 251-318.

[3] J. Benítez, X. Delgado-Galván, J. Izquierdo, R. Pérez-García, Improving consistency in AHP decision-making processes, Applied Mathematics and Computation 219 (2012) 2432-2441.

[4] T.L. Saaty, Decision-making with the AHP: why is the principal eigenvector necessary, European Journal of Operational Research 145 (2003) 85-91.

[5] G.W. Stewart, Matrix Algorithms, vol. II, SIAM, 2001.

[6] G. Crawford, C. Williams, A note on the analysis of subjective judgement matrices, Journal of Mathematical Psychology 29 (4) (1985) 387-405.

[7] J.S. Finan, W.J. Hurley, The analytic hierarchy process: does adjusting a pairwise compari-

son matrix to improve the consistency ratio help? Computers and Operations Research 24 (1997) 749-755.

[8] R. Ramanathan, U. Ramanahan, A qualitative perspective to deriving weights from pairwise comparison matrices, Omega 38 (2010) 228-232.

[9] J. Barzilai, Deriving weights from pairwise comparison matrices, The Journal of the Operational Research Society 48 (12) (1997) 1226-1232.

[10] J. Benítez, X. Delgado-Galván, J. Izquierdo, R. Pérez-García, Achieving matrix consistency in AHP through linearization, Applied Mathematical Modelling 35 (2011) 4449-4457.

[11] D. Cao, L.C. Leung, J.S. Law, Modifying inconsistent comparison matrix in analytic hierarchy process: a heuristical approach , Decision Support Systems 44 (2008) 944-953.

[12] X. Delgado-Galván, R. Pérez-García, J. Izquierdo, J.Mora-Rodríguez, Analytic hierarchy process for assessing externalities in water leakage management, Mathematical and Computer Modelling 52 (2010) 1194-1202.

[13] C.-C. Lin, An enhanced goal programming method for generating priority vectors, The Journal of the Operational Research Society 57 (12) (2006) 1491-1496.

[14] T.L. Saaty, L.G. Vargas, Comparison of eigenvalue, logarithmic least squares and least squares methods in estimating ratios, Mathematical Modelling 5 (5) (1984) 309-324.

[15] J. Benítez, X. Delgado-Galván, J. Izquierdo, R. Pérez-García, An approach to AHP decision in a dynamic context, Decision Support Systems 53 (2012) 499-506.

[16] J. Benítez, X. Delgado-Galván, J. Izquierdo, R. Pérez-García, Consistent completion of incomplete judgments in decision making using AHP, Decision Support Systems, submitted (2012).

[17] J. Benítez, X. Delgado-Galván, J. Izquierdo, R. Pérez-García, Balancing consistency and expert judgment in AHP, Mathematical and Computer Modelling 54 (2011) 1785-1790.