Nonnegative singular control systems using the Drazin projector

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Abstract

In this paper we study conditions to guarantee the nonnegativity of a discrete-time singular control system. A first approach can be found in the literature for general systems by using the entire coefficient matrices. Also, the particular case of matrices of index 1 has been treated by using a block decomposition and the group-projector of the matrix that gives the singularity to the system. In order to complete this study, an analysis of the nonnegativity of a singular control system for matrices having arbitrary index is done by means of the core-nilpotent decomposition. This technique allows us to reduce the size of the original matrices improving the results where the entire coefficients are involved.

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1. Introduction

It is well-known that a singular control system is nonnegative when states and outputs take only nonnegative values for nonnegative initial states and nonnegative controls. Applications of this kind of systems appear in a wide range of areas such as engineering, management science, economics, social sciences, biology and medicine, etc. For instance, nonnegative systems are useful in modeling chemical reactors, storage systems, compartmental systems, mechanical systems, water and atmospheric pollution systems, the Leontief economical system, etc. [2, 16, 17].

In [9, 14] a necessary and sufficient condition for a standard linear control system to be nonnegative is analyzed. This characterization states that all the matrices of the system are nonnegative. Some other subjects related to nonnegative systems are robust stability, delays, pointwise completeness, etc. [3, 15]. However, for singular control systems the theory of nonnegativity is more complicated and less developed in the literature. Recently, some results on nonnegativity of singular linear control systems have been obtained in [5, 10, 11] under certain conditions on the matrices of the system. In [5], conditions on the whole matrices are required, while in [10, 11] a block decomposition is used and some conditions on these blocks are assumed. Moreover, positive N-periodic singular systems have been studied in [6].

We focus our attention on nonnegativity of singular discrete-time control systems where the matrix that gives the singularity has index $k \geq 1$. In this sense, we generalize the previous results in [10, 11] where the case $k = 1$ was studied. Specifically, we are going to present some applications to control theory of the theoretical results given in [12]. While in [12] we concentrated our efforts on matrix aspects, this paper put the stress on the study of nonnegativity in a singular control systems setting.

This work is organized as follows. In Section 2 we present some previous results needed in the development of the paper. Later, in Section 3, necessary and sufficient conditions for a singular control system to be nonnegative are obtained in terms of some blocks of the matrices of the system.

2. Background and Preliminaries

The solution of a singular control system requires the Drazin inverse of a real square matrix E. We recall that the smallest nonnegative integer k such that $\operatorname{rank}(E^{k+1}) = \operatorname{rank}(E^k)$ is called the index of E and denoted by $\operatorname{ind}(E)$. The Drazin inverse of E is defined as the only matrix E^D satisfying the properties $E^D E E^D = E^D$, $E^D E = E E^D$ and $E^{k+1} E^D = E^k$, where $k = \operatorname{ind}(E)$ [1, 4]. The special case $k = 1$ is usually called group inverse and denoted by $E^{\#}$.

It is well known that a square matrix $E \in \mathbb{R}^{n\times n}$ of index $k > 0$ can be written as

$$E = S \begin{bmatrix} C & O \\ O & N_1 \end{bmatrix} S^{-1} = B_E + N_E, \tag{1}$$

where

$$B_E = S \begin{bmatrix} C & O \\ O & O \end{bmatrix} S^{-1}, \qquad N_E = S \begin{bmatrix} O & O \\ O & N_1 \end{bmatrix} S^{-1},$$

S and C nonsingular matrices and N_1 a nilpotent matrix of index k. Consequently, B_E has index 1 and N_E is nilpotent of index k. This expression is called the core-nilpotent decomposition of the matrix E and it can be found, for instance, in [4]. This decomposition is a useful tool to study for example perturbations of the Drazin inverse [7], perturbations in linear systems [8], etc. It will be used in the analysis presented in this paper.

A matrix E is said to be nonnegative if all its entries are nonnegative and will be denoted by $E \geq O$.

Moreover, for a matrix $E \in \mathbb{R}^{n\times n}$, its Drazin projector is defined as the product EE^D. Below we present a characterization of the nonnegativity of Drazin projectors that was given in [12].

Theorem 2.1. *Let $E \in \mathbb{R}^{n\times n}$ and $B_E \in \mathbb{R}^{n\times n}$ be the matrix of the core-nilpotent decomposition (1) of E with* $\operatorname{rank}(B_E) = r > 0$. *Then $EE^D \geq O$ if* *and only if there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that the matrix B_E is given by*

$$B_E = P \begin{bmatrix} XTY & XTYM & O \\ O & O & O \\ NXTY & NXTYM & O \end{bmatrix} P^T \tag{2}$$

where diagonal blocks are square, M, N are nonnegative matrices of appropriate size, $T \in \mathbb{R}^{r\times r}$ is nonsingular and $X = \operatorname{diag}(x_1, \ldots, x_r)$, $Y = \operatorname{diag}(y_1^T, \ldots, y_r^T)$ with x_i and y_j positive column vectors, $i, j \in \{1, \ldots, r\}$, such that $YX = I$. In this case, the group inverse of B_E is given by

$$B_E^{\#} = P \begin{bmatrix} XT^{-1}Y & XT^{-1}YM & O \\ O & O & O \\ NXT^{-1}Y & NXT^{-1}YM & O \end{bmatrix} P^T. \tag{3}$$

In addition, under this notation, in [12] it was proved that $T \geq O$ and $T^{-1} \geq O$ when $E \geq O$ and $E^D \geq O$.

This theorem was based on a result by Jain and Tynan [13] which provides a constructive proof. Following the proof of both theorems, a method to compute such a matrix B_E can be obtained.

On the other hand, we consider the discrete-time singular control system

$$\left\{ \begin{array}{rcl} Ex(k+1) & = & Ax(k) + Bu(k) \\ y(k) & = & Cx(k) \end{array} \right. \tag{4}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $x(k) \in \mathbb{R}^{n\times 1}$, $u(k) \in \mathbb{R}^{m\times 1}$, $y(k) \in \mathbb{R}^{p\times 1}$. Throughout, we suppose that (E, A) is a regular matrix pair, that is, there exists a scalar α such that $\det(\alpha E + A) \neq 0$. Sometimes it is also said that the pencil $\{E, A\}$ is regular. This fact assures the existence and uniqueness of solution of the system [4]. This system is denoted by (E, A, B, C).

The output vector of the system (E, A, B, C) is given by $y(k) = \widehat{C}x(k)$ where the state vector is

$$\begin{aligned} x(k) \quad = \quad & (\widehat{E}^D\widehat{A})^k\widehat{E}^D\widehat{E}x(0) + \sum_{i=0}^{k-1}\widehat{E}^D(\widehat{E}^D\widehat{A})^{k-i-1}\widehat{B}u(i) - \\ & - \sum_{i=0}^{q-1}(I - \widehat{E}^D\widehat{E})(\widehat{E}\widehat{A}^D)^i\widehat{A}^D\widehat{B}u(k+i), \end{aligned}$$

with $q = \operatorname{ind}(\widehat{E}) > 0$, $x(0)$ an initial admissible condition and the matrices

$$\widehat{E} = (\alpha E + A)^{-1}E, \quad \widehat{A} = (\alpha E + A)^{-1}A, \quad \widehat{B} = (\alpha E + A)^{-1}B, \quad \widehat{C} = C \tag{5}$$

satisfy $\widehat{A} = I - \alpha\widehat{E}$ and so $\widehat{E}\widehat{A} = \widehat{A}\widehat{E}$. Note that the original system (E, A, B, C) is equivalent to the system $(\widehat{E}, \widehat{A}, \widehat{B}, \widehat{C})$, that is both systems have the same solutions [4, 14].

We close this section giving a characterization of the nonnegativity of a discrete-time singular control system. This characterization was presented in [10] by means of the entire coefficient matrices of the system.

Theorem 2.2. *Let (E, A, B, C) be a discrete-time singular system satisfying $E^D E \geq O$ and $EA = AE$. Then the system (E, A, B, C) is nonnegative if and only if the following conditions hold:*

(a) $E^D A \geq O$,

(b) $E^D B \geq O$,

(c) $CE^D E \geq O$,

(d) $-(I - EE^D)(EA^D)^i A^D B \geq O$,

(e) $-C(I - EE^D)(EA^D)^i A^D B \geq O$,

for each $i = 0, 1, \ldots, \mathrm{ind}(E) - 1$.

3. Main results

In this section we analyze, from a different framework (see Theorem 2.2), the nonnegativity of a discrete-time singular control system (E, A, B, C) whose matrix E is singular, has index greater than or equal to 1 and has nonnegative Drazin projector. We focus our attention on the study of the nonnegativity of such a system by means of Theorem 2.1 where some blocks are used instead of the entire matrices as in Theorem 2.2.

For the singular control system (E, A, B, C) given in (4), we construct the equivalent system

$$\left\{ \begin{array}{rcl} \widehat{E}x(k+1) & = & \widehat{A}x(k) + \widehat{B}u(k) \\ y(k) & = & \widehat{C}x(k) \end{array} \right. \tag{6}$$

where the matrices $\widehat{E}$, $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ have been given in (5). Moreover, since $\mathrm{ind}(\widehat{E}) = q$, from the core-nilpotent decomposition (1) we get

$$\widehat{E} = B_{\widehat{E}} + N_{\widehat{E}} \tag{7}$$

and then $\widehat{A} = I - \alpha\widehat{E} = I - \alpha(B_{\widehat{E}} + N_{\widehat{E}})$ and $\widehat{E}^D = B^{\#}_{\widehat{E}}$. Taking into account this decomposition and the result given in Theorem 2.2 we can analyze the nonnegativity of the system $(\widehat{E}, \widehat{A}, \widehat{B}, \widehat{C})$.

From now on, we will assume that $\widehat{E}\widehat{E}^D \geq O$ with $\mathrm{rank}(B_{\widehat{E}}) = r > 0$. Then we get that $B_{\widehat{E}}$ has the form given in Theorem 2.1. Recall that if we add $\widehat{E} \geq O$ and $\widehat{E}^D \geq O$ in Theorem 2.1 then $T \geq O$ and $T^{-1} \geq O$ hold.

In order to study the nonnegativity of the system $(\widehat{E}, \widehat{A}, \widehat{B}, \widehat{C})$ by means of Theorem 2.2, we have to analyze firstly condition $\widehat{E}^D\widehat{A} \geq O$. Since $\widehat{E}^D = B^{\#}_{\widehat{E}}$ has the form given in (3) and $\widehat{A} = I - \alpha(B_{\widehat{E}} + N_{\widehat{E}})$, we have that

$$\widehat{E}^D\widehat{A} = B^{\#}_{\widehat{E}} - \alpha B_{\widehat{E}} B^{\#}_{\widehat{E}} = P \begin{bmatrix} X(T^{-1} - \alpha I)Y & X(T^{-1} - \alpha I)YM & O \\ O & O & O \\ NX(T^{-1} - \alpha I)Y & NX(T^{-1} - \alpha I)YM & O \end{bmatrix} P^T$$

provided that $B^{\#}_{\widehat{E}}N_{\widehat{E}} = O$. Then, we can conclude that $\widehat{E}^D\widehat{A} \geq O$ if and only if $X(T^{-1} - \alpha I)Y \geq O$ due to M and N are nonnegative matrices. Using that $YX = I$ and $X, Y \geq O$ we get that the last inequality becomes

$$T^{-1} - \alpha I \geq O \tag{8}$$

which is equivalent to

$$\alpha T \leq I \tag{8'}$$

if T and T^{-1} are nonnegative matrices.

Next, we have to analyze the nonnegativity of product $\widehat{E}^D\widehat{B}$. Again, the form of $\widehat{E}^D$ yields

$$\widehat{E}^D\widehat{B} = B^{\#}_{\widehat{E}}\widehat{B} = P \begin{bmatrix} XT^{-1}Y & XT^{-1}YM & O \\ O & O & O \\ NXT^{-1}Y & NXT^{-1}YM & O \end{bmatrix} P^T\widehat{B}.$$

Partitioning $\widehat{B} = P \begin{bmatrix} B_1^T & B_2^T & B_3^T \end{bmatrix}^T$, we have that $\widehat{E}^D\widehat{B} \geq O$ if and only if

$$T^{-1}Y(B_1 + MB_2) \geq O, \tag{9}$$

which is equivalent to

$$Y(B_1 + MB_2) \geq O \tag{9'}$$

if T and T^{-1} are nonnegative matrices.

Now, we study condition $\widehat{C}\widehat{E}^D\widehat{E} \geq O$. The equality $\widehat{E}^D\widehat{E} = B^{\#}_{\widehat{E}}B_{\widehat{E}}$ gives

$$\widehat{C}\widehat{E}^D\widehat{E} = \widehat{C}P \begin{bmatrix} XY & XYM & O \\ O & O & O \\ NXY & NXYM & O \end{bmatrix} P^T$$

and partitioning $\widehat{C} = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} P^T$, we get that $\widehat{C}\widehat{E}^D\widehat{E} \geq O$ if and only if

$$(C_1 + C_3N)X \geq O. \tag{10}$$

In order to study condition $-(I - \widehat{E}\widehat{E}^D)(\widehat{E}\widehat{A}^D)^i\widehat{A}^D\widehat{B} \geq O$, first we have to obtain the Drazin inverse of matrix $\widehat{A}$. In fact, it is well-known that the Drazin inverse of any square matrix is always expressible as a polynomial in that matrix [4]. In particular, there is a polynomial p in $\widehat{A}$ such that

$$\widehat{A}^D = p(\widehat{A}) = \sum_{s=0}^{n-1} \hat{a}_s\widehat{A}^s,$$

where the coefficients $\hat{a}_s$ are obtained in [4, Theorem 7.5.1, pp. 130].

Since $\widehat{A} = I - \alpha\widehat{E}$, the Drazin inverse of $\widehat{A}$ can be written as

$$\widehat{A}^D = \sum_{s=0}^{n-1} \hat{a}_s\widehat{A}^s = \sum_{s=0}^{n-1} \hat{a}_s(I - \alpha\widehat{E})^s = \sum_{s=0}^{n-1} \hat{a}_s \sum_{j=0}^{s} \binom{s}{j}(-\alpha\widehat{E})^j$$

where the Newton binomial formula is used. Thus,

$$\widehat{A}^D = \sum_{j=0}^{n-1} a_j \widehat{E}^j$$

where

$$a_j = \sum_{s=j}^{n-1} \hat{a}_s \binom{s}{j} (-\alpha)^j, \tag{11}$$

for $j = 0, 1, \ldots, n-1$. That is, from the coefficients $\hat{a}_s$ of the polynomial p obtained in [4], we can get the coefficients a_j by means of the relation (11).

Then, using the core-nilpotent decomposition of $\widehat{E}$ and the equality $B_{\widehat{E}} N_{\widehat{E}} = O$, the Drazin inverse of $\widehat{A}$ has the form

$$\widehat{A}^D = \sum_{j=0}^{n-1} a_j (B_{\widehat{E}} + N_{\widehat{E}})^j = a_0 I + \sum_{j=1}^{n-1} a_j (B_{\widehat{E}}^j + N_{\widehat{E}}^j).$$

Substituting this expression in the condition that we are studying we obtain

$$-(I - B_{\widehat{E}} B_{\widehat{E}}^{\#})(B_{\widehat{E}} + N_{\widehat{E}})^i \left[a_0 I + \sum_{j=1}^{n-1} a_j (B_{\widehat{E}}^j + N_{\widehat{E}}^j) \right]^{i+1} \widehat{B} \geq O$$

for all $i = 0, 1, \ldots, \mathrm{ind}(\widehat{E}) - 1$. Two situations should be distinguished:

- If $i = 0$ then

  $$\left[-a_0 (I - B_{\widehat{E}} B_{\widehat{E}}^{\#}) - \sum_{j=1}^{n-1} a_j N_{\widehat{E}}^j \right] \widehat{B} \geq O$$

  and so

  $$-a_0 \begin{bmatrix} B_1 - XY(B_1 + MB_2) \\ B_2 \\ -NXY(B_1 + MB_2) \end{bmatrix} - \sum_{j=1}^{n-1} a_j \bar{N}_{\widehat{E}}^j \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \geq O \tag{12}$$

  where $\bar{N}_{\widehat{E}} = P^T N_{\widehat{E}} P$.

- If $i > 0$ then

  $$-N_{\widehat{E}}^i \left[-a_0 I + \sum_{j=1}^{n-1} a_j (B_{\widehat{E}}^j + N_{\widehat{E}}^j) \right]^{i+1} \widehat{B} \geq O$$

  and so

  $$-\left[a_0 I + \sum_{j=1}^{n-1} a_j \bar{N}_{\widehat{E}}^j \right]^{i+1} \bar{N}_{\widehat{E}}^i \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \geq O. \tag{13}$$

The last condition to be studied is $-\widehat{C}(I-\widehat{E}\widehat{E}^D)(\widehat{E}\widehat{A}^D)^i\widehat{A}^D\widehat{B}\geq O$, which also involves the Drazin inverse of matrix $\widehat{A}$. As before, a similar analysis leads to:

- If $i=0$ then

$$-a_0\left(C_1B_1-(C_1+C_3N)XY(B_1+MB_2)+C_2B_2\right)\\ -\sum_{j=1}^{n-1}a_j\begin{bmatrix} C_1 & C_2 & C_3\end{bmatrix}\bar{N}_{\widehat{E}}^j\begin{bmatrix} B_1\\ B_2\\ B_3\end{bmatrix}\geq O. \tag{14}$$

- If $i>0$ then

$$-\begin{bmatrix} C_1 & C_2 & C_3\end{bmatrix}\left(a_0I+\sum_{j=1}^{n-1}a_j\bar{N}_{\widehat{E}}^j\right)^{i+1}\bar{N}_{\widehat{E}}^i\begin{bmatrix} B_1\\ B_2\\ B_3\end{bmatrix}\geq O. \tag{15}$$

The following result summarizes all this analysis.

Theorem 3.1. *Let (E,A,B,C) be a discrete-time singular system with $\widehat{E}$ defined as in (5) having index $q\geq 1$ and satisfying $\widehat{E}\widehat{E}^D\geq O$. Then the system (E,A,B,C) is nonnegative if and only if conditions (8)-(10) and (12)-(15) hold.*

Proof. Systems (E,A,B,C) and $(\widehat{E},\widehat{A},\widehat{B},\widehat{C})$ have the same solutions. So, we apply Theorem 2.2 to the last one to study their nonnegativity.

From expressions (3) and $\widehat{A}=I-\alpha\widehat{E}$, we have derived that condition (a) in Theorem 2.2 is equivalent to $T^{-1}-\alpha I\geq O$. Next, partitioning $\widehat{B}$ and $\widehat{C}$ adequately, we obtain the equivalence between conditions (b) and (c) in Theorem 2.2 and $T^{-1}Y(B_1+MB_2)\geq O$ and $(C_1+C_3N)X\geq O$, respectively. Expressions (12) and (13) are necessary and sufficient conditions to condition (d) in Theorem 2.2 for the cases $i=0$ and $i>0$, respectively. Analogously, condition (e) is obtained to be equivalent to expressions (14) and (15). ∎

This theorem gives a characterization of the nonnegativity of discrete-time singular control systems with coefficient matrix $\widehat{E}$ having index greater than or equal to 1. In this sense, this result complements the one obtained in [11] for matrices of index 1. In this case, if $\widehat{E}$ has index 1, then $N_{\widehat{E}}=O$, the conditions (13) and (15) are absent and (12) and (14) are simplified as follows:

$$-a_0\begin{bmatrix} B_1-XY(B_1+MB_2)\\ B_2\\ -NXY(B_1+MB_2)\end{bmatrix}\geq O \tag{16}$$

and

$$-a_0\left(C_1B_1-(C_1+C_3N)XY(B_1+MB_2)+C_2B_2\right)\geq O. \tag{17}$$

So, we have the following corollary.

Corollary 3.2. *Let (E, A, B, C) be a discrete-time singular system with $\widehat{E}$ defined as in (5) having index 1 and satisfying $\widehat{E}\widehat{E}^{\#} \geq O$. Then system (E, A, B, C) is nonnegative if and only if conditions (8)-(10), (16) and (17) hold.*

Remark 1. *Assuming that both matrices $\widehat{E}$ and $\widehat{E}^D$ are nonnegative in Theorem 3.1, we can conclude that the system (E, A, B, C) is nonnegative if and only if conditions (8′), (9′), (10) and (12)-(15) hold.*

Notice that, in general, these results allow us to work with matrices of smaller sizes than the original ones in the analysis of the nonnegativity of a discrete-time singular system.

As we have seen, the computation of the matrix $\widehat{A}^D$ consists on finding the coefficients of certain polynomial. However, we can simplify this process by choosing adequately the value of α to be used in (5). In what follows, we discuss how to find the best (more convenient) α.

Clearly, we can find infinite α's such that the matrix $I - \alpha B_{\widehat{E}}$ is nonsingular. Moreover, there are also infinite values of α for which the matrix $\alpha E + A$ is nonsingular. Hence, we can choose a value of α satisfying both conditions simultaneously. For that α, the matrix $\widehat{A} = I - \alpha\widehat{E}$ becomes $\widehat{A} = (I - \alpha B_{\widehat{E}}) - \alpha N_{\widehat{E}}$, which corresponds to the core-nilpotent decomposition of $\widehat{A}$ and then $\widehat{A}^D = (I - \alpha B_{\widehat{E}})^{-1}$. This fact simplifies the study of the two last conditions of Theorem 2.2 for the nonnegativity of the system (E, A, B, C). Those conditions can be rewritten as follows:

(d′) $-(I - B_{\widehat{E}}B_{\widehat{E}}^{\#})((B_{\widehat{E}} + N_{\widehat{E}})(I - \alpha B_{\widehat{E}})^{-1})^i(I - \alpha B_{\widehat{E}})^{-1}\widehat{B} \geq O$,

(e′) $-\widehat{C}(I - B_{\widehat{E}}B_{\widehat{E}}^{\#})((B_{\widehat{E}} + N_{\widehat{E}})(I - \alpha B_{\widehat{E}})^{-1})^i(I - \alpha B_{\widehat{E}})^{-1}\widehat{B} \geq O$,

for all $i = 0, 1, \ldots, \mathrm{ind}(\widehat{E}) - 1$.

Corollary 3.3. *Let (E, A, B, C) be a discrete-time singular system with $\widehat{E} = B_{\widehat{E}} + N_{\widehat{E}}$ defined as in (5) having index $q \geq 1$, satisfying $\widehat{E}\widehat{E}^D \geq O$ and $I - \alpha B_{\widehat{E}}$ is nonsingular. Then the system (E, A, B, C) is nonnegative if and only if conditions (a)-(c) in Theorem 2.2 and (d′)-(e′) hold.*

Moreover, these conditions (d′) and (e′) can be considerably simplified when the spectral radius of $B_{\widehat{E}}$ is less than 1 because, in this case, the von Neumann expression states that

$$(I - \alpha B_{\widehat{E}})^{-1} = \sum_{j=0}^{\infty} \alpha^j B_{\widehat{E}}^j.$$

Substituting this expression in the inequalities (d′) and (e′) they can be expressed in terms of matrices of smaller sizes as:

(18) $XY(B_1 + MB_2) - B_1 \geq O$

(19) $-B_2 \geq O$

(20) $NXY(B_1 + MB_2) - B_3 \geq O$

(21) $-N^i_{\widehat{E}}\widehat{B} \geq O$

(22) $C_1(XY(B_1 + MB_2) - B_1) - C_2B_2 - C_3(NXY(B_1 + MB_2) - B_3) \geq O$

(23) $-\widehat{C}N^i_{\widehat{E}}\widehat{B} \geq O.$

for all $i = 0, 1, \ldots, \mathrm{ind}(\widehat{E}) - 1$.

Summarizing we have the following result.

Theorem 3.4. *Let (E, A, B, C) be a discrete-time singular system. Let α be a complex such that $\widehat{E} = B_{\widehat{E}} + N_{\widehat{E}}$ defined as in (5) has index $q \geq 1$, satisfies $\widehat{E}\widehat{E}^D \geq O$, $I - \alpha B_{\widehat{E}}$ is nonsingular and the spectral radius of $B_{\widehat{E}}$ is less than 1. Then the system (E, A, B, C) is nonnegative if and only if conditions (8)-(10) and (18)-(23) hold.*

Proof. Conditions (8)-(10) have been shown in Theorem 3.1. Using the facts that $B_{\widehat{E}}N_{\widehat{E}} = O$ and $B_{\widehat{E}}B^{\#}_{\widehat{E}}B_{\widehat{E}} = B_{\widehat{E}}$, if we substitute the von Neumann expression given before in the inequalities (d′) and (e′), a simple computation allows us to derive conditions (18)-(23). ∎

We illustrate the results with an example.

Example 3.1. *Let $\widehat{E} = B_{\widehat{E}} + N_{\widehat{E}}$ where*

$$B_{\widehat{E}} = \begin{bmatrix} 1 & 0 & 1 & 1 & 13 & 16 & 0 & 0 & 0 \\ 2 & 0 & 2 & 2 & 26 & 32 & 0 & 0 & 0 \\ 3 & 0 & 2 & 2 & 27 & 34 & 0 & 0 & 0 \\ 3 & 0 & 2 & 2 & 27 & 34 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 26 & 0 & 19 & 19 & 254 & 318 & 0 & 0 & 0 \\ 35 & 0 & 26 & 26 & 347 & 434 & 0 & 0 & 0 \\ 43 & 0 & 37 & 37 & 487 & 604 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} XTY & XTYM & O \\ O & O & O \\ NXTY & NXTYM & O \end{bmatrix}$$

with

$$N = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 9 & 8 & 2 & 4 \end{bmatrix} \qquad M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$X = \left[\begin{array}{c|c} 1 & 0 \\ 2 & 0 \\ \hline 0 & 1 \\ 0 & 1 \end{array}\right] \qquad Y = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1/2 & 1/2 \end{array}\right]$$

and

$$N_{\widehat{E}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\widehat{B} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ *and let* $\widehat{A} = I - \alpha\widehat{E}$ *with* $\alpha \in \mathbb{R}$. *Then, system* $(\widehat{E}, \widehat{A}, \widehat{B}, I)$ *satisfies that* $\operatorname{ind}(\widehat{E}) = 4$ *and* $\widehat{E}\widehat{E}^D \geq O$*, so we can apply Theorems 3.1 or 3.4 to study its nonnegativity.*

Firstly, we have to analyze conditions (8)-(10) (see Theorems 3.1 and 3.4). If we choose $\alpha \leq -2$ *then* $T^{-1} - \alpha I \geq O$ *holds. Notice that, in this case, we have worked with a* 2×2 *matrix instead of the* 9×9 *original one. It can be easily checked that conditions (9) and (10) are satisfied. Moreover, choosing* $\alpha \leq -2$ *and* $\alpha \neq \frac{-5-\sqrt{33}}{4}$*, the matrix* $\widehat{A}$ *is nonsingular. Then, Theorem 3.4. assures the nonnegativity of the system since simple computations show that conditions (18)-(23) are satisfied.*

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