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Additive results for the group inverse in an algebra with applications to block operators *

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Abstract

We derive a very short expression for the group inverse of $a_1+\cdots+a_n$ when $a_1,\ldots,a_n$ are elements in an algebra having group inverse and satisfying $a_ia_j=0$ for $i<j$. We apply this formula in order to find the group inverse of 2×2 block operators under some conditions.

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1 Introduction

Throughout this paper, $\mathcal{A}$ will denote an algebra with unity 1 and we will denote by $\mathcal{A}^{-1}$ the subset of invertible elements of $\mathcal{A}$. An idempotent $p\in\mathcal{A}$ satisfies $p=p^2$. An element $a\in\mathcal{A}$ is said to have a group inverse if there exists $x\in\mathcal{A}$ such that

$$axa=a, \qquad xax=x, \qquad ax=xa. \tag{1.1}$$

It can easily be proved that if $a\in\mathcal{A}$ has a group inverse, then the element x satisfying (1.1) is unique, and under this situation, we shall write $x=a^\#$. The subset of $\mathcal{A}$ consisting of elements of $\mathcal{A}$ that have a group inverse inverse will be denoted by $\mathcal{A}^\#$. For an arbitrary algebra $\mathcal{A}$, it is not true that $\mathcal{A}=\mathcal{A}^\#$. Even when $\mathcal{A}$ is the set of complex matrices of order $n\times n$ (denoted by $\mathbb{C}^{n\times n}$), the equality $\mathcal{A}=\mathcal{A}^\#$ does not hold (an equivalent condition for $A\in\mathbb{C}^{n\times n}$ having a group inverse is that $\operatorname{rank}(A)=\operatorname{rank}(A^2)$, see for example [1]).

The group inverse in a Banach algebra is a particular case of the Drazin inverse (see [12]). Since the group inverse of $a\in\mathcal{A}^\#$ must commute with a, the study of such kind on invertibility resembles to the study of the elements of a C^*-algebra that commute with their Moore–Penrose inverse (see [2, 12]). Following the proof of Theorem 2.1 in [2], we can establish the following result:

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Theorem 1.1. *Let $\mathcal{A}$ be an algebra with unity 1 and $a \in \mathcal{A}$. Then the following conditions are equivalent:*

(i) *There exists a unique idempotent p such that $a + p \in \mathcal{A}^{-1}$ and $ap = pa = 0$.*

(ii) $a \in \mathcal{A}^{\#}$.

Following [13], we denote by a^π the unique idempotent satisfying condition (i) of Theorem 1.1 for a given $a \in \mathcal{A}^{\#}$. Let us remark that $aa^\pi = a^\pi a = 0$ and from the proof of Theorem 1.1 we have

$$a^\pi = 1 - aa^{\#} = 1 - a^{\#}a \qquad \text{and} \qquad a^{\#} = (a + a^\pi)^{-1} - a^\pi. \tag{1.2}$$

The idempotent a^π will be named the spectral idempotent of a corresponding to 0.

2 Some additive results for the group inverse in a algebra

Recently, there has been interest in giving formulae for the Drazin inverse of a sum of two matrices (or two operators in a Hilbert space) under some conditions, see [11, 10, 7, 8] and references therein. In [3] it was proved the following result (among others). If $T_1, T_2 \in \mathbb{C}^{n\times n}$ are two k-potent matrices for some natural $k > 1$ (a matrix X is said to be k-potent when $X^k = K$) such that $T_1T_2 = 0$ and c_1, c_2 are two nonzero complex numbers, then $c_1T_1 + c_2T_2$ is group invertible and

$$(c_1T_1 + c_2T_2)^{\#} = c_1^{-1}(I_n - T_2^{k-1})T_1^{k-2} + c_2^{-1}T_2^{k-2}(I_n - T_1^{k-1}). \tag{2.3}$$

In order to prove (2.3) the authors used block matrices and spectral theory for diagonalizable matrices (which means that the setting of the proof was finite-dimensional).

Next result generalizes the expression (2.3) in a algebra setting without considering the dimension of the space under consideration. First of all, let us see how (2.3) can be generalized. Let us remark that if T is a k-potent matrix for some natural $k > 1$, then T is group invertible and $T^{\#} = T^{k-2}$ as is easily seen from the definition of the group inverse (see equations (1.1)). Observe that under this situation one has from (1.2) that $T^{k-1} = TT^{k-2} = TT^{\#} = I_n - T^\pi$, where I_n denotes the identity matrix of order n. Moreover, it is very simple to check that if A is any group invertible matrix and c is a nonzero complex number, then $(cA)^{\#} = c^{-1}A^{\#}$. Hence, the formula (2.3) can be written

$$(c_1T_1 + c_2T_2)^{\#} = T_2^\pi(c_1T_1)^{\#} + (c_2T_2)^{\#}T_1^\pi.$$

A related result in the setting of operators in a Banach space and Drazin inverses was given in [6].

Theorem 2.1. *Let $\mathcal{A}$ be an algebra with unity. If $a, b \in \mathcal{A}^{\#}$ satisfy $ab = 0$, then $a + b \in \mathcal{A}^{\#}$ and*

$$(a + b)^{\#} = b^\pi a^{\#} + b^{\#}a^\pi.$$

Proof. Let us denote $x = b^\pi a^{\#} + b^{\#}a^\pi$. In order to prove the formula of the theorem, we will check $(a + b)x = x(a + b)$, $(a + b)x(a + b) = a + b$, and $x(a + b)x = x$. Before doing this, let us observe that $a^{\#}b = 0$ (since $a^{\#}b = a^{\#}aa^{\#}b = (a^{\#})^2ab = 0$), $a^\pi b = b$ (since

$a^\pi b = (1 - aa^\#)b = b)$, $ab^\# = 0$ (since $ab^\# = ab^\# bb^\# = ab(b^\#)^2 = 0$), and $ab^\pi = a$ (since $ab^\pi = a(1 - bb^\#) = a$). Next we simplify $(a+b)x$ and $x(a+b)$:

$$(a+b)x = (a+b)(b^\pi a^\# + b^\# a^\pi) = ab^\pi a^\# + ab^\# a^\pi + bb^\pi a^\# + bb^\# a^\pi = aa^\# + bb^\# a^\pi,$$

and

$$\begin{aligned} x(a+b) &= (b^\pi a^\# + b^\# a^\pi)(a+b) = b^\pi a^\# a + b^\pi a^\# b + b^\# a^\pi a + b^\# a^\pi b \\ &= b^\pi a^\# a + b^\# b = (1 - bb^\#)a^\# a + b^\# b = a^\# a + bb^\#(1 - a^\# a) = a^\# a + bb^\# a^\pi. \end{aligned}$$

This proves $(a+b)x = x(a+b)$. Now,

$$(a+b)x(a+b) = (a^\# a + bb^\# a^\pi)(a+b) = a^\# a^2 + a^\# ab + bb^\# a^\pi a + bb^\# a^\pi b = a + b$$

and

$$\begin{aligned} x(a+b)x &= (b^\pi a^\# + b^\# a^\pi)(a^\# a + bb^\# a^\pi) \\ &= b^\pi (a^\#)^2 a + b^\pi a^\# bb^\# a^\pi + b^\# a^\pi a^\# a + b^\# a^\pi bb^\# a^\pi = b^\pi a^\# + b^\# a^\pi. \end{aligned}$$

The theorem is proved. □

Corollary 2.1. *Let $\mathcal{A}$ be an algebra with unity. If $a, b \in \mathcal{A}^\#$ satisfy $ab = 0 = ba$, then $a + b \in \mathcal{A}^\#$ and*

$$(a+b)^\# = a^\# + b^\#.$$

Proof. By Theorem 2.1 we have $a+b \in \mathcal{A}^\#$ and $(a+b)^\# = b^\pi a^\# + b^\# a^\pi$. Let us simplify each summand under the condition $ba = 0$. Let us remark that $ba^\# = ba^\# aa^\# = ba(a^\#)^2 = 0$ and similarly, one has $b^\# a = 0$. Now, $b^\pi a^\# = (1 - b^\# b)a^\# = a^\#$ and $b^\# a^\pi = b^\#(1 - aa^\#) = b^\#$. Hence, the corollary is proved. □

Theorem 2.2. *Let $a_1, \ldots, a_n$ be elements in an algebra having group inverse. If $a_i a_j = 0$, for $i < j$ and $i, j \in \{1, 2, \ldots, n\}$, then $a_1 + \cdots + a_n \in \mathcal{A}^\#$ and*

$$(a_1 + \cdots + a_n)^\# = a_n^\pi \cdots a_2^\pi a_1^\# + a_n^\pi \cdots a_3^\pi a_2^\# a_1^\pi + \cdots + a_n^\pi a_{n-1}^\# a_{n-2}^\pi \cdots a_1^\pi + a_n^\# a_{n-1}^\pi \cdots a_1^\pi.$$

Proof. This formula will be proved by induction on n. For $n = 2$, the formula is simply Theorem 2.1. Assume that the theorem is true when the number of summands is less than n. By the induction hypothesis, we have that $a_2 + \cdots + a_n \in \mathcal{A}^\#$ and

$$(a_2 + \cdots + a_n)^\# = a_n^\pi \cdots a_3^\pi a_2^\# + a_n^\pi \cdots a_4^\pi a_3^\# a_2^\pi + \cdots + a_n^\pi a_{n-1}^\# a_{n-2}^\pi \cdots a_2^\pi + a_n^\# a_{n-1}^\pi \cdots a_2^\pi. \quad (2.4)$$

Since a_1 and $a_2 + \cdots + a_n$ have group inverse and $a_1(a_2 + \cdots + a_n) = 0$, by Theorem 2.1, we get $a_1 + a_2 + \cdots + a_n \in \mathcal{A}^\#$ and

$$(a_1 + a_2 + \cdots + a_n)^\# = (a_2 + \cdots + a_n)^\pi a_1^\# + (a_2 + \cdots + a_n)^\# a_1^\pi.$$

Using (2.4) we get

$$(a_1 + a_2 + \cdots + a_n)^\# = (a_2 + \cdots + a_n)^\pi a_1^\# + a_n^\pi \cdots a_3^\pi a_2^\# a_1^\pi + \cdots + a_n^\# a_{n-1}^\pi \cdots a_1^\pi.$$

Therefore, in order to finish the proof, it is enough to demonstrate

$$(a_2 + \cdots + a_n)^\pi = a_n^\pi \cdots a_2^\pi. \quad (2.5)$$

We are going to prove (2.5) by induction on n. If $n = 2$, there is nothing to prove. Assume that

$$(a_2 + \cdots + a_{n-1})^\pi = a_{n-1}^\pi \cdots a_2^\pi \tag{2.6}$$

holds. By the hypothesis of the Theorem, we easily get when $i < j$ and $i, j \in \{1, \cdots n\}$ that $a_i a_j^\# = 0$ and $a_i a_j^\pi = a_i$. Let us denote

$$x_2 = a_n^\pi \cdots a_3^\pi a_2^\#, \quad \cdots, \quad x_{n-1} = a_n^\pi a_{n-1}^\# a_{n-2}^\pi \cdots a_2^\pi, \quad x_n = a_n^\# a_{n-1}^\pi \cdots a_2^\pi$$

and

$$y_2 = a_{n-1}^\pi \cdots a_3^\pi a_2^\#, \quad \cdots, \quad y_{n-1} = a_{n-1}^\# a_{n-2}^\pi \cdots a_2^\pi.$$

Observe that $a_n^\pi y_i = x_i$ holds for $i = 2, \ldots, n-1$. Now, from (2.4)

$$\begin{aligned}
\left(\sum_{i=2}^n a_i\right)^\pi &= 1 - \left(\sum_{i=2}^n a_i\right)\left(\sum_{i=2}^n a_i\right)^\# \\
&= 1 - \left(\sum_{i=2}^n a_i\right)\left(\sum_{i=2}^n x_i\right) \\
&= 1 - \left(a_n + \sum_{i=2}^{n-1} a_i\right)\left(x_n + \sum_{i=2}^{n-1} x_i\right) \\
&= 1 - a_n x_n - a_n \sum_{i=2}^{n-1} x_i - \sum_{i=2}^{n-1} a_i x_n - \left(\sum_{i=2}^{n-1} a_i\right)\left(\sum_{i=2}^{n-1} x_i\right) \\
&= 1 - a_n x_n - a_n \sum_{i=2}^{n-1} a_n^\pi y_i - \sum_{i=2}^{n-1} a_i x_n - \left(\sum_{i=2}^{n-1} a_i\right)\left(\sum_{i=2}^{n-1} a_n^\pi y_i\right) \\
&= 1 - \sum_{i=2}^{n} a_i x_n - \left(\sum_{i=2}^{n-1} a_i\right)\left(\sum_{i=2}^{n-1} y_i\right).
\end{aligned} \tag{2.7}$$

On the other hand, since (2.6) was assumed, we have

$$1 - \left(\sum_{i=2}^{n-1} a_i\right)\left(\sum_{i=2}^{n-1} a_i\right)^\# = a_{n-1}^\pi \cdots a_2^\pi.$$

Since we assume that the formula stated in the theorem holds when the number of summands is less than n, we have

$$\left(\sum_{i=2}^{n-1} a_i\right)^\# = \sum_{i=2}^{n-1} y_i,$$

hence

$$1 - \left(\sum_{i=2}^{n-1} a_i\right)\left(\sum_{i=2}^{n-1} y_i\right) = a_{n-1}^\pi \cdots a_2^\pi. \tag{2.8}$$

Therefore, from (2.7) and (2.8) and recalling $a_2 a_n^\# = \cdots = a_{n-1} a_n^\# = 0$, we obtain

$$\begin{aligned}
\left(\sum_{i=2}^n a_i\right)^\pi &= a_{n-1}^\pi \cdots a_2^\pi - \left(\sum_{i=2}^n a_i\right) x_n \\
&= a_{n-1}^\pi \cdots a_2^\pi - \left(\sum_{i=2}^n a_i\right) a_n^\# a_{n-1}^\pi \cdots a_2^\pi \\
&= a_{n-1}^\pi \cdots a_2^\pi - a_n a_n^\# a_{n-1}^\pi \cdots a_2^\pi \\
&= (1 - a_n a_n^\#) a_{n-1}^\pi \cdots a_2^\pi \\
&= a_n^\pi a_{n-1}^\pi \cdots a_2^\pi .
\end{aligned}$$

And the theorem is proved. □

3 Applications

We will use Theorem 2.1 to find the group inverse of a block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{3.9}$$

under some conditions on blocks A, B, C, and D. The main idea is to decompose $M = M_1 + M_2$, where matrices M_1 and M_2 contain some blocks of the matrix M, and these matrices M_1 and M_2 satisfy conditions of Theorem 2.1, i.e., M_1, M_2 are group invertible and $M_1 M_2 = 0$. A prior work (for the Drazin inverse) can be founded in [9].

But, instead of establishing the main results in the setting of matrix theory (recall that the algebra composed of complex $n \times n$ matrices has finite dimension), we will work in an arbitrary algebra. We believe that avoiding the use of the spatial arguments in the standard n dimensional Euclidean space in favor of simpler algebraic techniques gives a greater insight into this problem. As simple consequences we obtain results on matrices and bounded operators on Banach spaces.

We generalize the use of block matrices to the setting of unital algebras in the following lines. For any idempotent $p \in \mathcal{A}$, we shall denote $\overline{p} = 1 - p$. Evidently we have that $\overline{p}$ is also an idempotent and $p\overline{p} = \overline{p}p = 0$. Now, any $a \in \mathcal{A}$ has the following matrix representation (see [5, Lemma 2.1]):

$$a \equiv \begin{bmatrix} pap & pa\overline{p} \\ \overline{p}ap & \overline{p}a\overline{p} \end{bmatrix}. \tag{3.10}$$

Also, recall that since p is an idempotent, $p\mathcal{A}p$ and $\overline{p}\mathcal{A}\overline{p}$ are algebras with units p and $\overline{p}$, respectively. Such representation is useful to deal with block matrices or operators in direct sums of Banach spaces.

1. Let $\mathbb{C}^{n\times m}$ denote the set composed of $n \times m$ complex matrices and I_n the identity matrix of order n. If $P \in \mathbb{C}^{n\times n}$ is the idempotent given by $P = S(I_r \oplus 0)S^{-1}$, where $S \in \mathbb{C}^{n\times n}$ is nonsingular, then any $M \in \mathbb{C}^{n\times n}$ can be represented using (3.10). Let us write M as

$$M = S \begin{bmatrix} A & B \\ C & D \end{bmatrix} S^{-1}, \quad A \in \mathbb{C}^{r\times r},\ D \in \mathbb{C}^{(n-r)\times(n-r)}. \tag{3.11}$$

If we compute, for example, PMP we have

$$PMP = S \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} S^{-1} = S \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} S^{-1}.$$

Hence the block A in (3.11) can be identified with PMP. Similarly, we can identify the remaining blocks in (3.11) with the different products $PM\overline{P}$, $\overline{P}MP$, and $\overline{P}M\overline{P}$.

2. If $\mathcal{X}$ and $\mathcal{Y}$ are two Banach spaces, let us denote $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ the algebra composed of bounded operators from $\mathcal{X}$ to $\mathcal{Y}$, and $\mathcal{B}(\mathcal{X}) = \mathcal{B}(\mathcal{X}, \mathcal{X})$. If $P : \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$ denotes the projection given by $P(x, y) = (x, 0)$, then any $M \in \mathcal{B}(\mathcal{X} \times \mathcal{Y})$, written as $M(x, y) = (M_1(x, y), M_2(x, y))$, can be represented using (3.10). For example, if we compute PMP we have

$$PMP(x, y) = PM(x, 0) = P(M_1(x, 0), M_2(x, 0)) = (M_1(x, 0), 0).$$

Thus, PMP can be identified with the operator in $\mathcal{B}(\mathcal{X})$ given by $x \mapsto M_1(x, 0)$. Similarly, $PM\overline{P}$ can be identified with the operator in $\mathcal{B}(\mathcal{Y}, \mathcal{X})$ given by $y \mapsto M_1(0, y)$, and so on.

In next result, we describe the elements having an inverse or a group inverse in the subalgebra $p\mathcal{A}p$, where p is a nontrivial idempotent of $\mathcal{A}$. It is interesting to observe that pap is not invertible in $\mathcal{A}$ for any $a \in \mathcal{A}$, in fact, if there exists $x \in \mathcal{A}$ such that $papx = 1$, then premultiplying by $\overline{p}$ we get $0 = \overline{p}$, which contradicts the non-triviality of p. Contrarily, as we will see, the group inverse behaves differently. We denote $\mathrm{inv}(a, \mathcal{B})$ and $\#(a, \mathcal{B})$ the inverse and the group inverse, respectively in the subalgebra $\mathcal{B} \subset \mathcal{A}$ of $b \in \mathcal{B}$, if exists such inverse.

Theorem 3.1. *Let $\mathcal{A}$ be an algebra with unity, $p \in \mathcal{A}$ an idempotent, and $a \in \mathcal{A}$. Then*

(i) *$pap \in (p\mathcal{A}p)^{-1}$ if and only if $pap+\overline{p} \in \mathcal{A}^{-1}$. Under this situation one has $\mathrm{inv}(pap, p\mathcal{A}p)+\overline{p} = (pap + \overline{p})^{-1}$.*

(ii) *If $pap \in \mathcal{A}^{\#}$, then $(pap)^{\#} \in p\mathcal{A}p$.*

(iii) *$pap \in (p\mathcal{A}p)^{\#}$ if and only if $pap \in \mathcal{A}^{\#}$. Under this situation one has $\#(pap, p\mathcal{A}p) = (pap)^{\#}$.*

Proof. (i) Assume that $pap \in (p\mathcal{A}p)^{-1}$. Having in mind that p is the unity in the subalgebra $p\mathcal{A}p$, there exists $b \in \mathcal{A}$ such that $(pap)(pbp) = p$ and $(pbp)(pap) = p$, i.e., $papbp = p$ and $pbpap = p$. Now we have

$$(pap + \overline{p})(pbp + \overline{p}) = papbp + pap\overline{p} + \overline{p}pbp + \overline{p}^2 = p + \overline{p} = 1,$$

and similarly $(pbp + \overline{p})(pap + \overline{p}) = 1$ holds. This proves $pap + \overline{p} \in \mathcal{A}^{-1}$ and $(pap + \overline{p})^{-1} = pbp + \overline{p} = \mathrm{inv}(pap, p\mathcal{A}p) + \overline{p}$.

Assume now that $pap + \overline{p} \in \mathcal{A}^{-1}$, or in other words, there exists $x \in \mathcal{A}$ such that $(pap + \overline{p})x = 1$ and $x(pap + \overline{p}) = 1$, i.e., $papx + \overline{p}x = 1$ and $xpap + x\overline{p} = 1$. We have

$$(pap)(pxp) = (papx)p = (1 - \overline{p}x)p = p - \overline{p}xp.$$

Premultiplying by p we get $(pap)(pxp) = p$. Similarly we have $(pxp)(pap) = p$.

(ii) Since $pap \in \mathcal{A}^{\#}$, there exists $x \in \mathcal{A}$ such that $x = (pap)^{\#}$, i.e.,

$$papx = xpap, \qquad papxpap = pap, \qquad xpapx = x.$$

We have to prove $x \in p\mathcal{A}p$. Since $\overline{p}x = \overline{p}xpapx = \overline{p}papx^2 = 0$, we obtain $x = px$. In a similar way, since $x\overline{p} = xpapx\overline{p} = x^2pap\overline{p} = 0$, we get $x = xp$. Therefore, $x = pxp$ holds, and this item is proved.

(iii) It is evident from the definition of the group inverse that $pap \in (p\mathcal{A}p)^{\#}$ implies $pap \in \mathcal{A}^{\#}$ and $\#(pap, p\mathcal{A}p) = (pap)^{\#}$.

Assume that $pap \in \mathcal{A}^{\#}$. There exists $b \in \mathcal{A}$ such that

$$(pap)b = b(pap), \qquad (pap)b(pap) = pap, \qquad b(pap)b = b.$$

Pre and postmultiplying the above equalities by p and having in mind $p^2 = p$ we have

$$(pap)(pbp) = (pbp)(pap), \qquad (pap)b(pap) = pap, \qquad pb(pap)bp = pbp.$$

These equalities mean $pap \in (p\mathcal{A}p)^{\#}$. The proof is finished. $\square$

As we said before, one of our purposes is to find the group inverse of the 2×2 block matrix M represented in (3.9) under some conditions on the sub-blocks of M. One idea is to decompose

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix}$$

and apply Theorem 2.1. Of course, we must know when is it possible to find the group inverse of each summand in the right-hand of the above expression. In 2001, Cao [4] gave the following result:

Theorem 3.2. *If $A \in \mathbb{C}^{r\times r}$, $C \in \mathbb{C}^{s\times s}$, and $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$, then there exists $M^{\#}$ if and only if there exist $A^{\#}$, $C^{\#}$, and $(I_r - AA^{\#})B(I_s - CC^{\#}) = 0$. Furthermore, when $M^{\#}$ exists, it is given by*

$$M^{\#} = \begin{bmatrix} A^{\#} & (A^{\#})^2B(I_s - CC^{\#}) + (I_r - AA^{\#})B(C^{\#})^2 - A^{\#}BC^{\#} \\ 0 & C^{\#} \end{bmatrix}.$$

From this theorem we have the following obvious consequence: Let $A \in \mathbb{C}^{r\times r}$, $B \in \mathbb{C}^{r\times s}$, and

$$M = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}.$$

Then M has group inverse if and only if A is group invertible and

$$(I_r - AA^{\#})B = 0. \tag{3.12}$$

Under this equivalence, one has

$$M^{\#} = \begin{bmatrix} A^{\#} & (A^{\#})^2B \\ 0 & 0 \end{bmatrix}. \tag{3.13}$$

The following lemma, that extends the above consideration, will be needed in the sequel.

Lemma 3.1. *Let $\mathcal{A}$ be an algebra with unity, $q \in \mathcal{A}$ an idempotent, and $a \in \mathcal{A}$ such that $qa = a$. Then the following are equivalent*

(i) $qaq \in \mathcal{A}^{\#}$ *and* $\left[1 - a(qaq)^{\#}\right] a\overline{q} = 0$.

(ii) $a \in \mathcal{A}^{\#}$.

Under this equivalence, one has

$$a^{\#} = (qaq)^{\#} + ((qaq)^{\#})^2 a\overline{q}. \tag{3.14}$$

Note (previous to the proof). Since $qa = a$ we have $\overline{q}aq = \overline{q}a\overline{q} = 0$, which means that in the representation (3.10) the two lower entries are zero; this generalizes when a 2×2 block matrix has null its lower blocks. The equality $\left[1 - a(qaq)^{\#}\right] a\overline{q} = 0$ generalizes the matrix relation (3.12). Moreover, the equality (3.14) clearly extends formula (3.13).

Proof. (i) $\Rightarrow$ (ii): Let $b = (qaq)^{\#}$. By definition of group inverse one has

$$bqaqb = b, \qquad qaqbqaq = qaq, \qquad qaqb = bqaq. \tag{3.15}$$

By Theorem 3.1, there exists $x \in \mathcal{A}$ such that $b = qxq$, and having in mind $q^2 = q$, we get $bq = b = qb$, and thus, taking into account $qa = a$, relations (3.15) reduce to

$$bab = b, \qquad abaq = aq, \qquad ab = baq. \tag{3.16}$$

Let $c = b + b^2 a\overline{q}$. We will prove $a^{\#} = c$ by definition of the group inverse. First, we will simplify ac:

$$\begin{aligned} ac &= a(b + b^2 a\overline{q}) = ab + ab^2 a\overline{q} \\ &= ab + (ab)(ba) - (ab)(baq) = ab + (baq)(ba) - (ab)(ab) \\ &= ab + ba(qb)a - a(bab) = ab + baba - ab = ba. \end{aligned}$$

Now, we compute ca. Taking into account that $qa = a$, and thus $\overline{q}a = 0$, we have

$$ca = (b + b^2 a\overline{q})a = ba.$$

These last two equalities show $ac = ca$. Next, we prove $cac = c$:

$$cac = (ca)c = (ba)(b + b^2 a\overline{q}) = (bab)(1 + ba\overline{q}) = b(1 + ba\overline{q}) = c.$$

Finally, we prove $aca = a$: Until now, we have not used $(1 - ab)a\overline{q} = 0$. From this, and the middle relation of (3.16) we have $aba = a$, and thus

$$aca = a(ca) = a(ba) = aba = a.$$

Since we have proved $aca = a$, $cac = b$, and $ac = ca$, we have gotten $a^{\#} = c$.

(ii) $\Rightarrow$ (i): Firstly, let us recall $qa^{\#} = qa^{\#}aa^{\#} = qa(a^{\#})^2 = a(a^{\#})^2 = a^{\#}$. Now, let us prove that $qa^{\#}q = (qaq)^{\#}$ by checking the three conditions of the group inverse. It is easy to see $(qaq)(qa^{\#}q) = aa^{\#}q = a^{\#}aq = (qa^{\#}q)(qaq)$. Moreover, one has

$$(qaq)(qa^{\#}q)(qaq) = aa^{\#}aq = aq = qaq$$

and similarly $(qa^{\#}q)(qaq)(qa^{\#}q) = qa^{\#}q$ holds. Finally, let us prove $\left[1 - a(qaq)^{\#}\right] a\overline{q} = 0$:

$$\left[1 - a(qaq)^{\#}\right] a\overline{q} = (1 - aqa^{\#}q)a\overline{q} = a\overline{q} - aqa^{\#}qa\overline{q} = a\overline{q} - a\overline{q} = 0.$$

The lemma is proved. □

Theorem 3.3. *Let $\mathcal{A}$ be an algebra with unity, $a \in \mathcal{A}$, and $p \in \mathcal{A}$ an idempotent. Assume that pap and $\overline{p}a\overline{p}$ have group inverse and*

$$pa\overline{p}a = 0, \quad (1 - pa(pap)^{\#})pa\overline{p} = 0, \quad (1 - \overline{p}a(\overline{p}a\overline{p})^{\#})\overline{p}ap = 0,$$

then

$$a^{\#} = (pap)^{\#} + (\overline{p}a\overline{p})^{\#} + (1 - (\overline{p}a\overline{p})^{\#}a)[(pap)^{\#}]^2 a\overline{p} + [(\overline{p}a\overline{p})^{\#}]^2 ap(1 - (pap)^{\#}a) - (\overline{p}a\overline{p})^{\#}a(pap)^{\#}.$$

Proof. Let $a_1 = pa$ and $a_2 = \overline{p}a$. By hypothesis we have $a_1 a_2 = pa\overline{p}a = 0$. We will use now Lemma 3.1 to prove that a_1 has group inverse. Since $pa_1 = p(pa) = pa = a_1$, $pa_1p = pap \in \mathcal{A}^{\#}$, and $(1 - a_1(pa_1p)^{\#})a_1\overline{p} = (1 - pa(pap)^{\#})pa\overline{p} = 0$ we have that $a_1 \in \mathcal{A}^{\#}$ and

$$a_1^{\#} = (pa_1p)^{\#} + ((pa_1p)^{\#})^2 a_1\overline{p} = (pap)^{\#} + ((pap)^{\#})^2 pa\overline{p}.$$

In a similar way, we use Lemma 3.1 to prove that $a_2 \in \mathcal{A}^{\#}$; but, we will use the idempotent $\overline{p}$ instead of p as before. In fact, since $\overline{p}a_2 = a_2$, $\overline{p}a_2\overline{p} = \overline{p}a\overline{p} \in \mathcal{A}^{\#}$, and $(1 - a_2(\overline{p}a_2\overline{p})^{\#})a_2p = (1 - \overline{p}a(\overline{p}a\overline{p})^{\#})\overline{p}ap = 0$, we get

$$a_2^{\#} = (\overline{p}a\overline{p})^{\#} + ((\overline{p}a\overline{p})^{\#})^2 \overline{p}ap.$$

By Theorem 2.1 we have

$$a^{\#} = (a_1 + a_2)^{\#} = a_2^{\pi} a_1^{\#} + a_2^{\#} a_1^{\pi}.$$

Next thing we do is simplifying a_1^{π} and a_2^{π}. To this end, let us denote $b = (pap)^{\#}$, and since $b = (pap)^{\#} \in (p\mathcal{A}p)^{\#}$, by item (ii) of Theorem 3.1, we get $bp = b = pb$. Having in mind that $b = (pap)^{\#}$ we get $papb = bpap$, $papbpap = pap$, and $bpapb = b$ or equivalently,

$$pab = bap, \qquad pabap = pap, \qquad bab = b.$$

Now we have

$$a_1^{\pi} = 1 - a_1 a_1^{\#} = 1 - pa(b + b^2 pa\overline{p}) = 1 - pab - pab^2 pa\overline{p} = 1 - bap - ba\overline{p} = 1 - ba.$$

Similarly, if we denote $c = (\overline{p}a\overline{p})^{\#}$, we get $a_2^{\pi} = 1 - ca$. Therefore, recalling that $pb = b = bp$, $\overline{p}c = c = c\overline{p}$, and $cb = bc = 0$, we obtain

$$\begin{aligned}
a^{\#} &= a_2^{\pi} a_1^{\#} + a_2^{\#} a_1^{\pi} \\
&= (1 - ca)(b + b^2 pa\overline{p}) + (c + c^2 \overline{p}ap)(1 - ba) \\
&= b + b^2 pa\overline{p} - cab - cab^2 pa\overline{p} + c + c^2 \overline{p}ap - cba - c^2 \overline{p}apba \\
&= b + b^2 a\overline{p} - cab - cab^2 a\overline{p} + c + c^2 ap - c^2 aba \\
&= b + c + (1 - ca)b^2 a\overline{p} + c^2 ap(1 - ba) - cab.
\end{aligned}$$

This concludes the proof. □

In what follows we shall apply Theorem 3.3 to the operator $M \in \mathcal{B}(X \times Y)$ represented

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{3.17}$$

where X and Y are Banach spaces, $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y, X)$, $C \in \mathcal{B}(X, Y)$, and $D \in \mathcal{B}(Y)$.

Theorem 3.4. *Let M be an operator of the form (3.17). If A and D are group invertible, $BC = 0$, $BD = 0$, $A^\pi B = 0$, $D^\pi C = 0$, then M is group invertible and*

$$M^\# = \begin{bmatrix} A^\# & (A^\#)^2 B \\ -D^\# C A^\# + (D^\#)^2 C A^\pi & D^\# - D^\# C (A^\#)^2 B - (D^\#)^2 C A^\# B \end{bmatrix}.$$

Proof. We will apply Theorem 3.3 identifying $M \leftrightarrow a$ and the idempotent

$$P = \begin{bmatrix} I_X & 0 \\ 0 & 0 \end{bmatrix} \leftrightarrow p,$$

where I_Z denotes the the identity operator in the Banach space Z. Since A is group invertible in $\mathcal{B}(X)$ and $PMP = A \oplus 0$ we obtain that PMP is group invertible in $\mathcal{B}(X \times Y)$ and $(PMP)^\# = A^\# \oplus 0$. In a similar way we get the group invertibility of $\overline{P}M\overline{P}$ and $(\overline{P}M\overline{P})^\# = 0 \oplus D^\#$. From $BC = 0$ and $BD = 0$ we easily get $PM\overline{P}M = 0$. An evident computation shows

$$\left[I_{X\times Y} - PM(PMP)^\# \right] PM\overline{P} = \begin{bmatrix} 0 & A^\pi B \\ 0 & 0 \end{bmatrix}.$$

Hence from the hypothesis of this theorem we get $\left[I_{X\times Y} - PM(PMP)^\# \right] PM\overline{P} = 0$. Analogously, from $D^\pi C = 0$ we obtain $\left[I_{X\times Y} - \overline{P}M(\overline{P}M\overline{P})^\# \right] \overline{P}MP = 0$. We can apply Theorem 3.3 obtaining the formula of this theorem. □

Theorem 3.5. *Let M be an operator of the form (3.17). If A and D are group invertible, $CA = 0$, $CB = 0$, $A^\pi B = 0$, $D^\pi C = 0$, then M is group invertible and*

$$M^\# = \begin{bmatrix} A^\# - (A^\#)^2 B D^\# C - A^\# B (D^\#)^2 C & (A^\#)^2 B D^\pi - A^\# B D^\# \\ (D^\#)^2 C & D^\# \end{bmatrix}.$$

Proof. This theorem has the same proof as the former Theorem 3.4, but now identifying $M \leftrightarrow a$ and the idempotent $P = 0 \oplus I_Y \leftrightarrow p$, □

For the following two results, we need to recall that given an operator $T \in \mathcal{B}(X, Y)$, the adjoint of T (which generalizes the concept of the conjugate transpose of a matrix) is an operator $T^* \in \mathcal{B}(Y^*, X^*)$, where Z^* denotes the dual space of the Banach space Z. It is easy to see that if $T \in \mathcal{X}$ is group invertible, then T^* is also group invertible and $(T^\#)^* = (T^*)^\#$. Also, as is easily seen from the definition, one has $(T^*)^\pi = (T^\pi)^*$.

Theorem 3.6. *Let M be an operator of the form (3.17). If A and D are group invertible, $BC = 0$, $DC = 0$, $CA^\pi = 0$, $BD^\pi = 0$, then M is group invertible and*

$$M^\# = \begin{bmatrix} A^\# & A^\# B D^\# + A^\pi B (D^\#)^2 \\ C (A^\#)^2 & C (A^\#)^2 B D^\# - C A^\# B (D^\#)^2 + D^\# \end{bmatrix}.$$

Proof. It is enough to apply Theorem 3.4 to operator M^*. □

Theorem 3.7. *Let M be an operator of the form (3.17). If A and D are group invertible, $AB = 0$, $CB = 0$, $CA^\pi = 0$, $BD^\pi = 0$, then M is group invertible and*

$$M^\# = \begin{bmatrix} A^\# - B D^\# C (A^\#)^2 - B (D^\#)^2 C A^\# & B (D^\#)^2 \\ D^\pi C (A^\#)^2 - D^\# C A^\# & D^\# \end{bmatrix}.$$

Proof. As before, it is enough to apply Theorem 3.5 to operator M^*. □

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