On nonsingularity of combinations of two group invertible matrices and two tripotent matrices

Xiaoji Liu [1] Shuxia Wu [1] Julio Benítez [2*]
[1] College of Mathematics and Computer Science,
Guangxi University for Nationalities, Nanning 530006, P.R. China
[2] Departamento de Matemática Aplicada, Instituto de matemática multidisciplinar, Universidad Politécnica de Valencia, Valencia 46022, Spain.

Abstract

Let T_1 and T_2 be two $n \times n$ tripotent matrices and c_1, c_2 two nonzero complex numbers. We mainly study the nonsingularity of combinations $T = c_1T_1 + c_2T_2 - c_3T_1T_2$ of two tripotent matrices T_1 and T_2, and give some formulae for the inverse of $c_1T_1 + c_2T_2 - c_3T_1T_2$ under some conditions. Some of these results are given in terms of group invertible matrices.

Key words: Tripotent matrix, Linear combination, Diagonalization, Nonsingularity, Group invertible matrix.

1 Introduction

Let $\mathbb{C}$ be the field of complex numbers, and let the symbols $\mathbb{C}^*$ and $\mathbb{C}^{n\times n}$ denote the set of nonzero complex numbers and $n \times n$ complex matrices respectively. Moreover, $\mathcal{R}(A)$, $\mathcal{N}(A)$ and A^* stand for the column space, null space and conjugate transpose of $A \in \mathbb{C}^{n\times n}$, respectively. A matrix $A \in \mathbb{C}^{n\times n}$ is said to be idempotent if $A^2 = A$, and tripotent if $A^3 = A$.

Recall that $A \in \mathbb{C}^{n\times n}$ is nonsingular if and only if $\mathcal{N}(A) = \{0\}$. Also notice that if $T \in \mathbb{C}^{n\times n}$ and k is a natural number greater than 1, then T satisfies $T^k = T$ if and only if T is diagonalizable and the spectrum of T is contained in $\sqrt[k-1]{1} \cup 0$, which have been proved in [5].

Recently, the nonsingularity of linear combinations of idempotent matrices, projectors, tripotent matrices and k-potent matrices (see, for example, [1, 2, 4, 6, 7]) have

*Corresponding author.
E-mail: xiaojiliu72@yahoo.com.cn (X. Liu, Tel. +86-0771-3264782), anita623482950@yahoo.com.cn (S. Wu), jbenitez@mat.upv.es (J. Benítez).
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been extensively investigated. In [9], Sarduvan and Özdemir have considered the nonsingularity of linear combinations $T = c_1T_1 + c_2T_2$, where T_1, T_2 are two commuting $n \times n$ tripotent matrices and $c_1, c_2 \in \mathbb{C}^*$. In [10], Zuo studied the nonsingularity of combinations $c_1P + c_2Q - c_3PQ$ of two idempotent matrices P and Q, and the same author generalized the results in [11].

In this paper, we discuss the nonsingularity of combinations $c_1T_1 + c_2T_2 - c_3T_1T_2$ of two tripotent matrices and give some formulae for the inverse of $c_1T_1 + c_2T_2 - c_3T_1T_2$ under some conditions. We point out that the main results of this article are similar to the ones obtained in [10]. Notice that an idempotent matrix is always a tripotent matrix, but a tripotent matrix may not be idempotent. Special types of matrices, such as idempotents, tripotents, etc, are very useful in many contexts and they have been extensively studied in the literature. For example, quadratic forms with idempotent matrices are used extensively in statistical theory. So it is worth to stress and spread these kinds of results.

A matrix $A \in \mathbb{C}^{n\times n}$ is said to be group invertible if there exists $X \in \mathbb{C}^{n\times n}$ such that

$$AXA = A, \quad XAX = X, \quad AX = XA. \tag{1.1}$$

See [3, Chapter 4] for more information on this kind of generalized inverse. It can be proved that the set of matrices X satisfying (1.1) is or empty or a singleton and when is a singleton, it is customary to denote its unique element by $A^{\#}$. Also, it is known [8, Exercise 5.10.12] that a matrix $A \in \mathbb{C}^{n\times n}$ is group invertible if and only if there exist nonsingular $S \in \mathbb{C}^{n\times n}, C \in \mathbb{C}^{r\times r}$ such that $A = S(C \oplus 0)S^{-1}$, being r the rank of A. In this situation, one has $A^{\#} = S(C^{-1} \oplus 0)S^{-1}$.

Evidently, if T is a tripotent matrix, then T is group invertible and $T^{\#} = T$. Many of the results given in this paper will be given in terms of group invertible matrices.

2 Main results

In [10, Corollary 2.6], it had been proved that $P - Q$ is nonsingular if and only if $aP + bQ - cPQ$ and $I_n - PQ$ are nonsingular for any two idempotent matrices $P, Q \in \mathbb{C}^{n\times n}$, $a, b \in \mathbb{C}^*$. In the following theorem, a similar result is established for tripotent matrices.

Theorem 2.1. *Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be two commuting tripotent matrices. Then $T_1 - T_2$ is nonsingular if and only if $I_n - T_1T_2$ and $T_1^2 + (I_n - T_1^2)T_2$ are nonsingular.*

Proof. By a suitable simultaneous diagonalization, there exists $S \in \mathbb{C}^{n\times n}$ such that $T_1 = S\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)S^{-1}$ and $T_2 = S\,\mathrm{diag}(\mu_1, \ldots, \mu_n)S^{-1}$ being $\{\lambda_i\}_{i=1}^n$ and $\{\mu_i\}_{i=1}^n$, the sets of eigenvalues of T_1 and T_2, respectively. Observe that $\lambda_i, \mu_j \in \{-1, 0, 1\}$ for all $1 \le i, j \le n$ since T_1 and T_2 are tripotent. Moreover,

$$T_1 - T_2 = S\,\mathrm{diag}(\lambda_1 - \mu_1, \ldots, \lambda_n - \mu_n)S^{-1},$$

$$I_n - T_1T_2 = S\,\mathrm{diag}(1 - \lambda_1\mu_1, \ldots, 1 - \lambda_n\mu_n)S^{-1},$$

and

$$T_1^2 + (I_n - T_1^2)T_2 = S\,\mathrm{diag}(\lambda_1^2 + (1-\lambda_1^2)\mu_1, \ldots, \lambda_n^2 + (1-\lambda_n^2)\mu_n)S^{-1}. \qquad (2.1)$$

Assume that $T_1 - T_2$ is nonsingular. Then $\lambda_i \neq \mu_i$ for all $i \in \{1, \ldots, n\}$. Hence

$$(\lambda_i, \mu_i) \in \{(1,-1), (1,0), (-1,1), (-1,0), (0,1), (0,-1)\} \quad \text{for all } i = 1, \ldots, n.$$

Easily we have that $1 - \lambda_i\mu_i \neq 0$ and $\lambda_i^2 + (1-\lambda_i^2)\mu_i \neq 0$ for all $1 \leq i \leq n$. Therefore, $I_n - T_1T_2$ and $T_1^2 + (I_n - T_1^2)T_2$ are nonsingular.

Assume that $I_n - T_1T_2$ and $T_1^2 + (I_n - T_1^2)T_2$ are nonsingular. Since $I_n - T_1T_2$ is nonsingular, then $\lambda_i\mu_i \neq 1$ for all $1 \leq i \leq n$. If $T_1 - T_2$ were singular, then there would exist $j \in \{1, \ldots, n\}$ such that $\lambda_j = \mu_j$. Having in mind that $\lambda_j\mu_j \neq 1$, we would get $\lambda_j = \mu_j = 0$. But now, $\lambda_j^2 + (1-\lambda_j^2)\mu_j = 0$, which would yield the singularity of $T_1^2 + (I_n - T_1^2)T_2$. □

Remark: Let $p : \mathbb{C}^2 \to \mathbb{C}$ be the following complex polynomial:

$$p(z,w) = a_{1,0}z + a_{2,0}z^2 + a_{0,1}w + a_{1,1}zw + a_{2,1}z^2w + a_{0,2}w^2 + a_{1,2}zw^2 + a_{2,2}z^2w^2, \quad (2.2)$$

where $a_{i,j}$ are complex numbers. We have

$$p(T_1, T_2) = S\,\mathrm{diag}(p(\lambda_1, \mu_1), \ldots, p(\lambda_n, \mu_n))S^{-1}.$$

Now, if $T_1^2 + (I_n - T_1^2)T_2$ were singular, then by (2.1) there would exist $j \in \{1, \ldots, n\}$ such that

$$\lambda_j^2 + (1-\lambda_j^2)\mu_j = 0. \qquad (2.3)$$

This expression is contradicted by $\lambda_j = \pm 1$. Therefore, $\lambda_j = 0$ and by using again (2.3) we get $\mu_j = 0$. Therefore, $p(T_1, T_2)$ is singular because $p(0,0) = 0$. Hence we can formulate the following corollary:

Corollary 2.1. *Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be two commuting tripotent matrices. If $I_n - T_1T_2$ is nonsingular and there exists a polynomial as in (2.2) such that $p(T_1, T_2)$ is nonsingular, then $T_1 - T_2$ is nonsingular.*

We weaken the hypotheses of the commutativity and the tripotency in the following result.

Theorem 2.2. *Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be two group invertible matrices such that $T_2T_1T_1^{\#} = T_1T_1^{\#}T_2$. If $I_n - T_1^{\#}T_2$ is nonsingular and there exists a polynomial p in two noncommuting variables such that $p(0,0) = 0$ and $p(T_1, T_2)$ is nonsingular, then $T_1 - T_2$ is nonsingular.*

Proof. Let $x \in \mathcal{N}(T_1 - T_2)$, i.e., $T_1x = T_2x$. Premultiplying by $T_1T_1^{\#}$ we get

$$T_1x = T_1T_1^{\#}T_2x. \qquad (2.4)$$

Premultiplying $T_1x = T_2x$ by $T_2T_1^{\#}$ and using $T_2T_1T_1^{\#} = T_1T_1^{\#}T_2$ yield $T_1T_1^{\#}T_2x = T_2T_1^{\#}T_2x$; but having in mind that $T_1T_1^{\#}T_2x = T_1x = T_2x$ we get

$$T_2T_1^{\#}T_2x = T_2x. \tag{2.5}$$

From (2.4) and (2.5) we have

$$T_1(I_n - T_1^{\#}T_2)x = 0, \qquad T_2(I_n - T_1^{\#}T_2)x = 0. \tag{2.6}$$

Since $p(0,0) = 0$, there exist two polynomials in noncommuting variables, say p_1 and p_2 such that $p(T_1,T_2) = p_1(T_1,T_2)T_1 + p_2(T_1,T_2)T_2$. Thus from (2.6)

$$\begin{aligned} p(T_1,T_2)(I_n - T_1^{\#}T_2)x &= (p_1(T_1,T_2)T_1 + p_2(T_1,T_2)T_2)(I_n - T_1^{\#}T_1)x \\ &= p_1(T_1,T_2)T_1(I_n - T_1^{\#}T_2)x + p_2(T_1,T_2)T_2(I_n - T_1^{\#}T_2)x \\ &= 0. \end{aligned}$$

Under the assumption that $I_n - T_1^{\#}T_2$ and $p(T_1,T_2)$ are nonsingular, the above computation yields $x = 0$, which means that $T_1 - T_2$ is nonsingular. $\square$

Remark. Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be group invertible and r the rank of T_1. If $T_1T_2 = T_2T_1$, then by writing $T_1 = S(C \oplus 0)S^{-1}$, where $S \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{r\times r}$ are nonsingular we get that T_2 can be written as $T_2 = S(D \oplus E)S^{-1}$, where $CD = DC$ and $D \in \mathbb{C}^{r\times r}$. Hence, $T_1T_1^{\#}T_2 = S(D \oplus 0)S^{-1} = T_2T_1T_1^{\#}$. However, observe that the condition $T_1T_1^{\#}T_2 = T_2T_1T_1^{\#}$ is more general than $T_1T_2 = T_2T_1$ (it is enough to consider the case when T_1 is nonsingular and $T_1T_2 \neq T_2T_1$).

Assume in this paragraph that $T_1, T_2 \in \mathbb{C}^{n\times n}$ are two commuting group invertible matrices. Now, if we use the condition $T_1^2T_2 = T_2^2T_1$, then we can give some kind of the converse of Theorem 2.2. From $T_1^2T_2 = T_2^2T_1$ and $T_1T_2 = T_2T_1$, we have $(T_1-T_2)T_1T_2 = 0$, hence the invertibility of $T_1 - T_2$ leads to $T_1T_2 = 0$. Thus, $c_1T_1 + c_2T_2 - c_3T_1T_2 = c_1T_1 + c_2T_2$, and we will give the explicit expression of $(c_1T_1 + c_2T_2)^{-1}$ in terms of $(T_1 - T_2)^{-1}$ under mild conditions.

Theorem 2.3. *Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be two group invertible matrices and $c_1, c_2 \in \mathbb{C}^*$. If $T_2T_1 = 0$ and $T_1 - T_2$ is nonsingular. Then $c_1T_1 + c_2T_2$ is nonsingular and*

$$(c_1T_1 + c_2T_2)^{-1} = [(c_1^{-1} + c_2^{-1})T_1T_1^{\#} - c_2^{-1}I_n](T_1 - T_2)^{-1}.$$

Proof. It follows from the following computations:

$$\begin{aligned} &(c_1T_1 + c_2T_2)\left[(c_1^{-1} + c_2^{-1})T_1T_1^{\#} - c_2^{-1}I_n\right] = \\ &\quad = \left(1 + c_1c_2^{-1}\right)T_1 - c_1c_2^{-1}T_1 + \left(c_2c_1^{-1} + 1\right)T_2T_1T_1^{\#} - T_2 = T_1 - T_2. \end{aligned}$$

$\square$

Remark. In Theorem 2.3, the situation $c_1 + c_2 = 0$ is trivial and it does not need the hypothesis $T_2T_1 = 0$.

In [10], it was showed that for two idempotent matrices $P,Q \in \mathbb{C}^{n\times n}$, and $a, b \in \mathbb{C}^*$, one has that matrix $aP + bQ - (a+b)PQ$ is nonsingular if and only if $\mathbb{C}^n = \mathcal{R}(P(I_n - Q)) \oplus \mathcal{R}((I_n - P)Q)$ if and only if $\mathbb{C}^n = \mathcal{N}(P(I_n - Q)) \oplus \mathcal{N}((I_n - P)Q)$. We give similar results in the following theorem.

Theorem 2.4. *Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be two matrices and $c_1, c_2, r_1, r_2 \in \mathbb{C}$. If $c_1T_1 + c_2T_2 + (r_1c_1 + r_2c_2)T_1T_2$ is nonsingular, then we have*

$$\mathcal{N}[T_1(I_n + r_1T_2)] \cap \mathcal{N}[(I_n + r_2T_1)T_2] = 0, \tag{2.7}$$

and

$$\mathcal{R}(T_1(I_n + r_1T_2)) + \mathcal{R}((I_n + r_2T_1)T_2) = \mathbb{C}^n. \tag{2.8}$$

Proof. Denote $\alpha = r_1c_1 + r_2c_2$. Let $x \in \mathcal{N}(T_1(I_n + r_1T_2)) \cap \mathcal{N}((I_n + r_2T_1)T_2)$. Since $T_1(I_n + r_1T_2)x = 0$ and $(I_n + r_2T_1)T_2x = 0$, we have

$$\begin{aligned}[][c_1T_1 + c_2T_2 + \alpha T_1T_2]x &= (c_1T_1 + c_2T_2 + r_1c_1T_1T_2 + r_2c_2T_1T_2)x \\ &= c_1T_1(I_n + r_1T_2)x + c_2(I_n + r_2T_1)T_2x = 0.\end{aligned}$$

The nonsingularity of $c_1T_1 + c_2T_2 + \alpha T_1T_2$ leads to $x = 0$. Hence (2.7) holds.

Since $c_1T_1 + c_2T_2 + \alpha T_1T_2$ is nonsingular, then $\overline{c}_1T_1^* + \overline{c}_2T_2^* + \overline{\alpha}T_2^*T_1^*$ is nonsingular. Applying the first part of the proof we have

$$\mathcal{N}(T_2^*(I_n + \overline{r}_2T_1^*)) \cap \mathcal{N}((I_n + \overline{r}_1T_2^*)T_1^*) = 0. \tag{2.9}$$

Recalling that $[\mathcal{N}(X^*)]^\perp = \mathcal{R}(X)$ holds for any matrix X, by taking perp in (2.9) we get that (2.8) holds. $\square$

In the following result, it is given an expression of the inverse of $c_1T_1 + c_2T_2 - c_3T_1T_2$ under some condition.

Theorem 2.5. *Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be two nonzero tripotent matrices such that $T_1^2T_2 = T_2^2T_1$ and $c_1, c_2 \in \mathbb{C}^*$, $c_3 \in \mathbb{C}$. Assume that T_1 or T_2 are nonsingular. If $(c_1 + c_2)^2 = c_3^2$, then $c_1T_1 + c_2T_2 - c_3T_1T_2$ or $c_1T_1 + c_2T_2 + c_3T_1T_2$ is singular. If $(c_1 + c_2)^2 \neq c_3^2$, then $c_1T_1 + c_2T_2 - c_3T_1T_2$ is nonsingular; and in this case,*

(i) *If T_1 is nonsingular, then*

$$\begin{aligned}& [(c_1 + c_2)^2 - c_3^2](c_1T_1 + c_2T_2 - c_3T_1T_2)^{-1} \\ = \ & (c_1 + c_2)T_1 + c_3T_2^2 + c_1^{-1}c_2c_3(T_2^2 - T_1T_2) + c_1^{-1}c_3^2(T_2 - T_1T_2^2) \\ & + c_1^{-1}(c_2^2 + c_1c_2 - c_3^2)(T_1 - T_1T_2^2).\end{aligned} \tag{2.10}$$

(ii) *If T_2 is nonsingular, then*

$$\begin{aligned}& [(c_1 + c_2)^2 - c_3^2](c_1T_1 + c_2T_2 - c_3T_1T_2)^{-1} \\ = \ & (c_1 + c_2)T_2 - c_3(2T_1^2 - T_2T_1) + c_2^{-1}(c_1^2 + c_1c_2 - c_3^2)(T_2 - T_2T_1^2).\end{aligned} \tag{2.11}$$

Proof. We split the proof depending if T_1 or T_2 is nonsingular.

(i) Let us assume that T_1 is nonsingular. First, let us prove the first part of the Theorem. The nonsingularity of T_1 implies $T_1^2 = I_n$, hence $T_1^2 T_2 = T_2^2 T_1$ reduces to $T_2^2 T_1 = T_2$. Since T_2 is tripotent, there exists a nonsingular $S \in \mathbb{C}^{n\times n}$ such that

$$T_2 = S \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} S^{-1}, \qquad A \in \mathbb{C}^{r\times r},$$

being r the rank of T_2. Since A is nonsingular and $T_2^3 = T_2$, then $A^2 = I_r$. Let us write

$$T_1 = S \begin{pmatrix} B & C \\ D & E \end{pmatrix} S^{-1}, \qquad B \in \mathbb{C}^{r\times r}.$$

From $T_2^2 T_1 = T_2$ we conclude $B = A$ and $C = 0$. Therefore

$$T_1 = S \begin{pmatrix} A & 0 \\ D & E \end{pmatrix} S^{-1} \tag{2.12}$$

and

$$c_1T_1 + c_2T_2 - c_3T_1T_2 = S \begin{pmatrix} (c_1+c_2)A - c_3I_r & 0 \\ c_1D - c_3DA & c_1E \end{pmatrix} S^{-1}. \tag{2.13}$$

Since $T_1^2 = I_n$, we obtain that E is nonsingular and $E^2 = I_{n-r}$. From (2.13) we get that $c_1T_1 + c_2T_2 - c_3T_1T_2$ is nonsingular if and only if $(c_1+c_2)A - c_3I_r$ is nonsingular (recall that the first row in the block matrix appearing in (2.13) must be present, since otherwise, $T_2 = 0$).

Since we have

$$[(c_1+c_2)A - c_3I_r]\,[(c_1+c_2)A + c_3I_r] = \left[(c_1+c_2)^2 - c_3^2\right] I_r, \tag{2.14}$$

we get that if $(c_1+c_2)^2 - c_3^2 = 0$, then $(c_1+c_2)A - c_3I_r$ or $(c_1+c_2)A + c_3I_r$ is singular, which in view of (2.13) we get that $c_1T_1 + c_2T_2 - c_3T_1T_2$ or $c_1T_1 + c_2T_2 + c_3T_1T_2$ is singular.

Now, let us prove the second part of the Theorem, i.e., we shall prove (2.10) for any $c_1, c_2 \in \mathbb{C}^*$ satisfying $(c_1+c_2)^2 \neq c_3^2$. From (2.14) we get that $(c_1+c_2)^2 - c_3^2 \neq 0$ leads to the nonsingularity of $(c_1+c_2)A - c_3I_r$ and

$$[(c_1+c_2)A - c_3I_r]^{-1} = \frac{1}{(c_1+c_2)^2 - c_3^2}\,[(c_1+c_2)A + c_3I_r]\,.$$

Since $T_1^2 = I_n$, we have $T_1^{-1} = T_1$ and from (2.12) we have

$$T_1^{-1} = S \begin{pmatrix} A & 0 \\ -EDA & E \end{pmatrix} S^{-1}, \tag{2.15}$$

from them we can conclude that $D = -EDA$, i.e., $-DA = ED$.

If $c_1, c_2 \in \mathbb{C}^*$ satisfy $(c_1+c_2)^2 \neq c_3^2$, then by using (2.13)

$$\begin{aligned} & (c_1T_1+c_2T_2-c_3T_1T_2)^{-1} \\ = & S\begin{pmatrix} [(c_1+c_2)A-c_3I_r]^{-1} & 0 \\ -c_1^{-1}E(c_1D-c_3DA)[(c_1+c_2)A-c_3I_r]^{-1} & c_1^{-1}E \end{pmatrix}S^{-1} \\ = & S\begin{pmatrix} [(c_1+c_2)^2-c_3^2]^{-1}[(c_1+c_2)A+c_3I_r] & 0 \\ -c_1^{-1}E(c_1D-c_3DA)[(c_1+c_2)^2-c_3^2]^{-1}[(c_1+c_2)A+c_3I_r] & c_1^{-1}E \end{pmatrix}S^{-1}. \end{aligned}$$

Hence

$$\begin{aligned} & [(c_1+c_2)^2-c_3^2](c_1T_1+c_2T_2-c_3T_1T_2)^{-1} \\ = & (c_1+c_2)S\begin{pmatrix} A & 0 \\ -EDA & E \end{pmatrix}S^{-1}+c_3S\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}S^{-1}+c_1^{-1}c_2c_3S\begin{pmatrix} 0 & 0 \\ ED & 0 \end{pmatrix}S^{-1} \\ & +c_1^{-1}c_3^2S\begin{pmatrix} 0 & 0 \\ EDA & 0 \end{pmatrix}S^{-1}+c_1^{-1}(c_2^2+c_1c_2-c_3^2)S\begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix}S^{-1}. \end{aligned} \tag{2.16}$$

On the other hand, we have

$$T_2^2 = S\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}S^{-1}, \tag{2.17}$$

$$T_2^2-T_1T_2 = S\begin{pmatrix} 0 & 0 \\ ED & 0 \end{pmatrix}S^{-1}, \tag{2.18}$$

$$T_2-T_1T_2^2 = S\begin{pmatrix} 0 & 0 \\ EDA & 0 \end{pmatrix}S^{-1}, \tag{2.19}$$

$$T_1-T_1T_2^2 = S\begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix}S^{-1}. \tag{2.20}$$

From (2.16)-(2.20), we get that (2.10) holds.

(ii) Assume that T_2 is nonsingular. First, let us prove the first part of the Theorem. As in (i), but now interchanging the roles of T_1 and T_2, we can write

$$T_1 = S\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}S^{-1}, \qquad T_2 = S\begin{pmatrix} A & 0 \\ D & E \end{pmatrix}S^{-1}, \qquad A \in \mathbb{C}^{r\times r},$$

being r the rank of T_1. Since $T_2^2 = I_n$ we have $A^2 = I_r$ and $E^2 = I_{n-r}$. The difference from (i) is

$$c_1T_1+c_2T_2 \pm c_3T_1T_2 = S\begin{pmatrix} (c_1+c_2)A \pm c_3I_r & 0 \\ c_2D & c_2E \end{pmatrix}S^{-1}. \tag{2.21}$$

As in (i), if $(c_1+c_2)^2 = c_3^2$, by (2.14) and (2.21), one has that $c_1T_1+c_2T_2+c_3T_1T_2$ or $c_1T_1+c_2T_2-c_3T_1T_2$ is singular.

Now, we shall prove (2.11) for any $c_1, c_2 \in \mathbb{C}^*$ satisfying $(c_1+c_2)^2 \neq c_3^2$. By using (2.21) one has

$$\begin{aligned}
& (c_1T_1 + c_2T_2 - c_3T_1T_2)^{-1} \\
= \; & S\begin{pmatrix} [(c_1+c_2)A - c_3I_r]^{-1} & 0 \\ -ED[(c_1+c_2)A - c_3I_r]^{-1} & c_2^{-1}E \end{pmatrix} S^{-1} \\
= \; & S\begin{pmatrix} [(c_1+c_2)^2 - c_3^2]^{-1}[(c_1+c_2)A + c_3I_r] & 0 \\ -ED[(c_1+c_2)^2 - c_3^2]^{-1}[(c_1+c_2)A + c_3I_r] & c_2^{-1}E \end{pmatrix} S^{-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
& [(c_1+c_2)^2 - c_3^2](c_1T_1 + c_2T_2 - c_3T_1T_2)^{-1} \\
= \; & S\begin{pmatrix} (c_1+c_2)A + c_3I_r & 0 \\ -ED[(c_1+c_2)A + c_3I_r] & c_2^{-1}[(c_1+c_2)^2 - c_3^2]E \end{pmatrix} S^{-1} \\
= \; & (c_1+c_2)S\begin{pmatrix} A & 0 \\ -EDA & E \end{pmatrix} S^{-1} + S\begin{pmatrix} c_3I_r & 0 \\ -c_3ED & c_2^{-1}(c_1^2 + c_1c_2 - c_3^2)E \end{pmatrix} S^{-1} \\
= \; & (c_1+c_2)T_2 + c_3S\begin{pmatrix} I_r & 0 \\ -ED & 0 \end{pmatrix} S^{-1} + c_2^{-1}(c_1^2 + c_1c_2 - c_3^2)S\begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix} S^{-1}.
\end{aligned}$$

On the other hand, we easily have

$$2T_1^2 - T_2T_1 = S\begin{pmatrix} I_r & 0 \\ ED & 0 \end{pmatrix} S^{-1} \qquad \text{and} \qquad T_2 - T_2T_1^2 = S\begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix} S^{-1}.$$

The proof is completed. □

If $c_3 = 0$, then we have the following corollary.

Corollary 2.2. [4, Theorem 3.1] *Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be two nonzero tripotent matrices such that $T_1^2T_2 = T_2^2T_1$ and $c_1, c_2 \in \mathbb{C}^*$. If T_1 or T_2 are nonsingular, then $c_1T_1 + c_2T_2$ is nonsingular if and only if $c_1 + c_2 \neq 0$. In this case,*
(i) *If T_1 is nonsingular, then*

$$(c_1+c_2)(c_1T_1 + c_2T_2)^{-1} = T_1 + c_2c_1^{-1}T_1(I_n - T_2^2).$$

(ii) *If T_2 is nonsingular, then*

$$(c_1+c_2)(c_1T_1 + c_2T_2)^{-1} = T_2 + c_2c_1^{-1}T_2(I_n - T_1^2).$$

The following theorem shows that the nonsingularity of $c_1T_1 + c_2T_2 - c_3T_1T_2$ is also related to the nonsingularity of the same combination of $T_1^2T_2$ and $T_2^2T_1$ or $T_2T_1^2$ and $T_1T_2^2$.

Theorem 2.6. *Let $T_1, T_2 \in \mathbb{C}^{n\times n}$ be two tripotent matrices and any $c_1, c_2 \in \mathbb{C}^*$. The following statements are equivalent:*

(i) *$c_1T_2^2T_1 + c_2T_1^2T_2 - c_3T_2^2T_1T_2$ is nonsingular.*

(ii) $c_1T_1T_2^2 + c_2T_2T_1^2 - c_3T_1T_2T_1^2$ *is nonsingular.*

(iii) $c_1T_1 + c_2T_2 - c_3T_1T_2$ *and* $I_n - T_1^2 - T_2^2$ *are nonsingular.*

Proof. The results follow quite easily from the equalities

$$(I_n - T_1^2 - T_2^2)(c_1T_1 + c_2T_2 - c_3T_1T_2) = -(c_1T_2^2T_1 + c_2T_1^2T_2 - c_3T_2^2T_1T_2)$$

and

$$(c_1T_1 + c_2T_2 - c_3T_1T_2)(I_n - T_1^2 - T_2^2) = -(c_1T_1T_2^2 + c_2T_2T_1^2 - c_3T_1T_2T_1^2).$$

□

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