

Chaos in King's iterative family [☆]

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Abstract

In this paper, the dynamics of King's family of iterative schemes for solving nonlinear equations is studied. The parameter spaces are presented, showing the complexity of the family. The analysis of the parameter space allows us to find elements of the family that have bad convergence properties, and also other ones with stable behavior.

Keywords: Nonlinear equations; Iterative methods; Dynamical behavior; Quadratic polynomials; Fatou and Julia sets; King's family; Non-convergence regions.

1. Introduction

The application of iterative methods for solving nonlinear equations $f(z) = 0$, with $f : \mathbb{C} \rightarrow \mathbb{C}$, gives rise to rational functions whose dynamics are not well-known.

From the numerical point of view, the dynamical behavior of the rational function associated with an iterative method give us important information about its stability and reliability. In these terms, Varona in [1] and Amat et al. in [2] described the dynamical behavior of several well-known iterative methods. More recently, in [3, 4, 5, 6, 7, 8, 9], the authors study the dynamics of different iterative families. In most of these studies, interesting dynamical planes, including some periodical behavior and other anomalies, have been obtained. However, in contrast to dynamical planes, the parameter planes associated to a family of methods allow us to understand the behavior of the different members of the family of methods, helping us in the election of a particular one.

The family under study is, in this case, King's family of iterative methods (see [10]). It is an uniparametric set of fourth-order iterative schemes to estimate simple roots of nonlinear equations $f(z) = 0$:

$$z_{k+1} = y_k - \frac{f(z_k) + (2 + \beta)f(y_k) \frac{f(y_k)}{f'(z_k)}}{f(z_k) + \beta f(y_k)}, \quad k = 0, 1, \dots, \quad (1)$$

where $y_k = z_k - \frac{f(z_k)}{f'(z_k)}$ and β is a parameter. This family includes the known Ostrowski's method for $\beta = -2$ (see [11]).

In [12], the authors described the conjugacy classes of King's iterative methods and made an excellent initial analysis of their dynamics when they are applied to generic low-degree polynomials. In this work, we deep in the study of the dynamics of this set of methods when they are applied to quadratic polynomials, characterizing completely the stability of all the fixed points.

It is known that the roots of a polynomial can be transformed by an affine map with no qualitative changes on the dynamics of family (1) (see [12]). So, we can use the quadratic polynomial $p(z) = z^2 + c$. For $p(z)$, the operator of the

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family corresponds to the rational function:

$$K_{p,\beta}(z) = \frac{c^3(2 + \beta) + 3cz^4(10 + \beta) + z^6(10 + 3\beta) - c^2z^2(10 + 7\beta)}{8z^3(-c\beta + z^2(4 + \beta))},$$

depending on the parameters β and c .

P. Blanchard, in [13], by considering the conjugacy map

$$h(z) = \frac{z - i\sqrt{c}}{z + i\sqrt{c}}, \quad (2)$$

with the following properties:

$$\text{i) } h(\infty) = 1, \quad \text{ii) } h(i\sqrt{c}) = 0, \quad \text{iii) } h(-i\sqrt{c}) = \infty,$$

proved that, for quadratic polynomials, the Newton's operator is always conjugate to the rational map z^2 . In an analogous way, the operator $K_{p,\beta}$ on quadratic polynomials is conjugated to the operator $O_\beta(z)$,

$$O_\beta(z) = (h \circ K_{p,\beta} \circ h^{-1})(z) = z^4 \frac{5 + 4z + z^2 + \beta(2 + z)}{1 + (4 + \beta)z + (5 + 2\beta)z^2}. \quad (3)$$

We observe that the parameter c has been obviated in $O_\beta(z)$.

We will study the general convergence of methods (1) for quadratic polynomials. To be more precise (see [14, 15]), a given method is generally convergent if the scheme converges to a root for almost every starting point and for almost every polynomial of a given degree.

Now, we are going to recall some dynamical concepts of complex dynamics (see [16]) that we use in this work. Given a rational function $R : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, the *orbit of a point* $z_0 \in \hat{\mathbb{C}}$ is defined as:

$$\{z_0, R(z_0), R^2(z_0), \dots, R^n(z_0), \dots\}.$$

We analyze the phase plane of the map R by classifying the starting points from the asymptotic behavior of their orbits. A $z_0 \in \hat{\mathbb{C}}$ is called a *fixed point* if $R(z_0) = z_0$. A *periodic point* z_0 of period $p > 1$ is a point such that $R^p(z_0) = z_0$ and $R^k(z_0) \neq z_0$, for $k < p$. A *pre-periodic point* is a point z_0 that is not periodic but there exists a $k > 0$ such that $R^k(z_0)$ is periodic. A *critical point* z_0 is a point where the derivative of the rational function vanishes, $R'(z_0) = 0$. Moreover, a fixed point z_0 is called *attractor* if $|R'(z_0)| < 1$, *superattractor* if $|R'(z_0)| = 0$, *repulsor* if $|R'(z_0)| > 1$ and *parabolic* if $|R'(z_0)| = 1$.

The *basin of attraction* of an attractor α is defined as:

$$\mathcal{A}(\alpha) = \{z_0 \in \hat{\mathbb{C}} : R^n(z_0) \rightarrow \alpha, n \rightarrow \infty\}.$$

The Fatou set of the rational function R , $\mathcal{F}(R)$, is the set of points $z \in \hat{\mathbb{C}}$ whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complement in $\hat{\mathbb{C}}$ is the *Julia set*, $\mathcal{J}(R)$. That means that the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.

The rest of the paper is organized as follows: in Section 2 we analyze the fixed and critical points of the operator $O_\beta(z)$ and in Section 3 we study the stability of the fixed points. The dynamical behavior of the family (1) is analyzed in Section 4, by using one of the associated parameter spaces. We finish the work with some remarks and conclusions.

2. Analysis of the fixed and critical points

The fixed points of $O_\beta(z)$ are the roots of the equation $O_\beta(z) = z$, that is, $z = 0$, $z = \infty$ and the *strange fixed points*

- $z = 1$ for $\beta \neq \frac{-10}{3}$,
- $ex_1 = \frac{-1}{4}(5 + \beta) - \frac{1}{4}B_1 - \frac{1}{2}\sqrt{-12 - 3\beta + (1/2)(5 + \beta)^2 - B_2}$,
- $ex_2 = \frac{-1}{4}(5 + \beta) - \frac{1}{4}B_1 + \frac{1}{2}\sqrt{-12 - 3\beta + (1/2)(5 + \beta)^2 - B_2}$,
- $ex_3 = \frac{-1}{4}(5 + \beta) + \frac{1}{4}B_1 - \frac{1}{2}\sqrt{-12 - 3\beta + (1/2)(5 + \beta)^2 + B_2}$,
- $ex_4 = \frac{-1}{4}(5 + \beta) + \frac{1}{4}B_1 + \frac{1}{2}\sqrt{-12 - 3\beta + (1/2)(5 + \beta)^2 + B_2}$,

where $B_1 = \sqrt{\beta^2 - 2\beta - 7}$ and $B_2 = (-1/2)(5 + \beta)B_1$.

Some relations between the strange fixed points are described in the following lemma.

Lemma 1. *The number of simple strange fixed points of operator $O_\beta(z)$ is five, except in cases:*

- i) *If $\beta = -2$, then $ex_1 = ex_2 = -1$. So, there exist 4 strange fixed points, one of them with multiplicity two.*
- ii) *If $\beta = \frac{-10}{3}$, there are four simple strange fixed points.*
- iii) *When $\beta = 1 \pm 2\sqrt{2}$, $ex_1 = ex_3$ and $ex_2 = ex_4$. There are only 3 strange fixed points, two of them double.*
- iv) *If $\beta = -22/5$, then $ex_3 = ex_4 = 1$ and there are two simple strange points and a triple one.*
- v) *Moreover, if $\beta = -5$, then $ex_1 = -ex_4$ and $ex_2 = -ex_3$, and*
- vi) *if $\beta = -8/3$, then $ex_3 = -ex_4$.*

In order to determine the critical points, we calculate the first derivative of $O_\beta(z)$,

$$O'_\beta(z) = 2z^3(1+z)^2 \frac{10 + 4\beta + 20z + 14\beta z + 3\beta^2 z + 10z^2 + 4\beta z^2}{(1 + 4z + \beta z + 5z^2 + 2\beta z^2)^2}.$$

A classical result establishes that there is at least one critical point associated with each invariant Fatou component. It is clear that the roots of the polynomial, $z = 0$ and $z = \infty$, are critical points and give rise to their respective Fatou components, but there exist in the family some *free critical points*, that is, critical points different from the roots, some of them depending on the value of the parameter.

Lemma 2. *Analyzing the equation $O'_\beta(z) = 0$, we obtain*

- a) *If $\beta = -2$, there is no free critical points of operator $O_\beta(z)$.*
- b) *If $\beta = -5/2$ or $\beta = 0$, then $z = -1$ is the only free critical point.*
- c) *In any other case,*
 - (i) $z = -1$
 - (ii) $cr_1 = \frac{-20 - 14\beta - 3\beta^2 - \sqrt{3\beta(\beta+2)(\beta+4)(3\beta+10)}}{4(2\beta+5)}$
 - (iii) $cr_2 = \frac{-20 - 14\beta - 3\beta^2 + \sqrt{3\beta(\beta+2)(\beta+4)(3\beta+10)}}{4(2\beta+5)} = \frac{1}{cr_1}$*are free critical points.*

Some of these properties determine the complexity of the operator, as we can see in the following result.

Theorem 1. *The only member of King's family whose operator is always conjugated to the rational map z^4 is Ostrowski's method.*

Proof. From (3), we denote $p(z) = z^2 + (4 + \beta)z + (5 + 2\beta)$ and $q(z) = (5 + 2\beta)z^2 + (4 + \beta)z + 1$. By factorizing both polynomials, we can observe that the unique value of β verifying $p(z) = q(z)$ is $\beta = -2$. ■

From the previous results, let us remark that

- At most, there are only two different free critical points.
- When $\beta = \frac{-10}{3}$, $cr_1 = cr_2 = 1$ that is not a fixed point, and the associated operator is $O_\beta(z) = z^4 \frac{z+5/3}{z+3/5}$.
- If $\beta = \frac{-5}{2}$, then the associated operator is $O_\beta(z) = z^5 \frac{z+3/2}{3/2z+1}$. So, for quadratic polynomials, the corresponding method has order of convergence five.
- if $\beta = -4$, then $cr_1 = cr_2 = 1$ that is an strange fixed point.

As we will see in the following section, not only the number but also the stability of the fixed points depend on the parameter of the family.

3. Stability of the fixed points

It is clear that the origin and ∞ are always superattractive fixed points, but the stability of the other fixed points changes depending on the values of the parameter β . In the following results we establish the stability of the strange fixed points.

Theorem 2. *The character of the strange fixed point $z = 1$ is:*

- i) *If $|\beta + \frac{226}{55}| < \frac{16}{55}$, then $z = 1$ is an attractor and it is a superattractor if $\beta = -4$.*
- ii) *When $|\beta + \frac{226}{55}| = \frac{16}{55}$, $z = 1$ is a parabolic point.*
- iii) *If $|\beta + \frac{226}{55}| > \frac{16}{55}$, then $z = 1$ is a repulsor.*

Proof. It is easy to proof that

$$O'_\beta(1) = \frac{8(4 + \beta)}{10 + 3\beta}.$$

So,

$$\left| \frac{8(4 + \beta)}{10 + 3\beta} \right| \leq 1 \quad \text{is equivalent to} \quad 8|4 + \beta| \leq |10 + 3\beta|.$$

Let us consider $\beta = a + ib$ an arbitrary complex number. Then,

$$8^2 (16^2 + 8a + a^2 + b^2) \leq 100 + 60a + 9a^2 + 9b^2.$$

By simplifying

$$55a^2 + 55b^2 + 452a + 924 \leq 0,$$

that is,

$$\left(a + \frac{226}{55} \right)^2 + b^2 \leq \left(\frac{16}{55} \right)^2.$$

Therefore,

$$|O'_\beta(1)| \leq 1 \quad \text{if and only if} \quad \left| \beta + \frac{226}{55} \right| \leq \frac{16}{55}.$$

Finally, if β satisfies $|\beta + \frac{226}{55}| > \frac{16}{55}$, then $|O'_\beta(1)| > 1$ and $z = 1$ is a repulsive point. ■

Similar results can be proved for the rest of strange fixed points.

Theorem 3. *The real analysis of the stability of strange points ex_1 and ex_2 shows that:*

- i) *If $\beta \in] -1.83917, 1 - 2\sqrt{2}[\cup] 1 + 2\sqrt{2}, 3.96186[$, then both points are attractors and they are superattractors when $\beta \approx -1.83192$ and $\beta \approx 3.86866$.*
- ii) *If $\beta = 1 \pm 2\sqrt{2}$, $\beta \approx -1.83917$ or $\beta \approx 3.96186$, then ex_1 and ex_2 are parabolic.*
- iii) *In any other case, both are repulsors.*

Theorem 4. *The stability of ex_3 and ex_4 for real values of β verify:*

- i) *If $\beta \in] -4.97983, -22/5[$, then ex_3 and ex_4 are attractors, and superattractors if $\beta \approx -4.7034$.*
- ii) *When $\beta = 1 \pm 2\sqrt{2}$, $\beta = -22/5$ or $\beta \approx -4.97983$, they are parabolic.*
- iii) *In any other case, both are repulsors.*

In Figure 1, we show the behavior of the strange and critical points for real values of β .

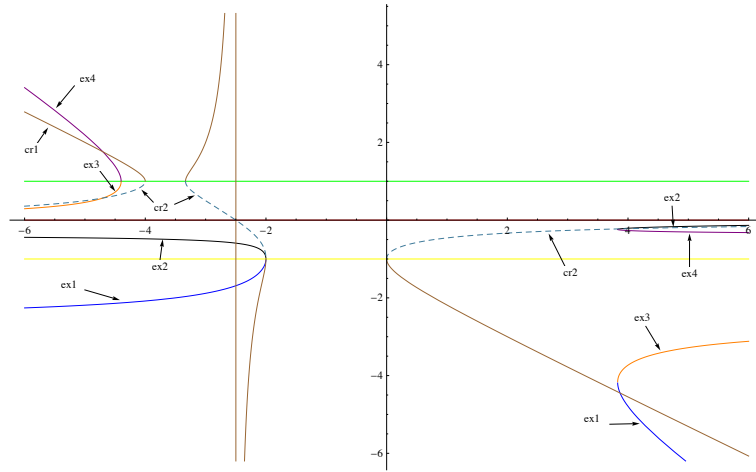


Figure 1: Bifurcation diagram of fixed and critical points

4. The parameter space

The dynamical behavior of operator $O_\beta(z)$ depends on the values of the parameter β . The parameter space associated with a free critical point of family (1) is obtained by associating each point of the parameter plane with a complex value of β , i.e., with an element of family (1). Every value of β belonging to the same connected component of the parameter space give rise to subsets of schemes of family (1) with similar dynamical behavior. So, it is interesting to find regions of the parameter plane as much stable as possible, because these values of β will give us the best members of the family in terms of numerical stability.

As $cr_1 = \frac{1}{cr_2}$, we have at most two free critical points, so we can obtain different parameter planes, with complementary information. When we consider the free critical point $z = -1$ as a starting point of the iterative scheme of the family associated to each complex value of β , we paint this point of the complex plane in red if the method converges to any of the roots (zero and infinity) and they are black in other cases. Then, the parameter plane P_1 is obtained; it is showed in Figure 2.

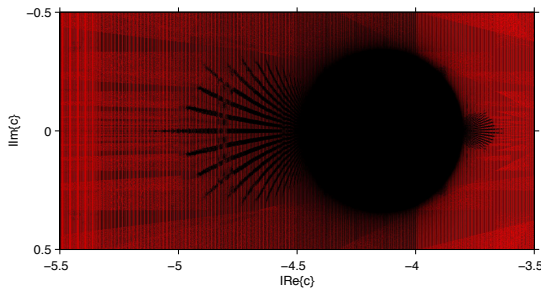


Figure 2: Parameter plane P_1 associated to $z = -1$

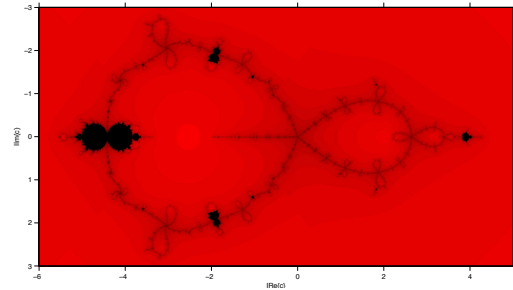


Figure 3: Parameter space P_2 associated to $z = cr_i, i = 1, 2$

A similar procedure can be carried out with the free critical points, $z = cr_i, i = 1, 2$, obtaining the same parameter space P_2 , showed in Figure 3. In the left of this parameter space we observe a black figure with a certain similarity with the known Mandelbrot set (see [17]) that we call *the mask*.

In the following, we will focus our attention on parameter plane P_2 . In the mask, we observe two large disks (the right disk is denoted by D_1 and the left one by D_2): D_1 corresponds to values of β for which $z = 1$ is attractive or superattractive (see Theorem 2) and D_2 is the region where strange fixed points ex_3 and ex_4 are attractive or superattractive (see Theorem 4). In Figure 4, the dynamical plane of the iterative method corresponding to $\beta = -4.5$ is presented, showing the existence of four different basins of attraction, two of them of the superattractors 0 and ∞ and the other two corresponding the attractors ex_3 and ex_4 .

Other regions of the parameter plane P_2 with stability problems are the right central and outer antennas, where Mandelbrot sets appear, corresponding to the stability regions of ex_1 and ex_2 (Theorem 3). A detail of these region is presented in Figure 5.

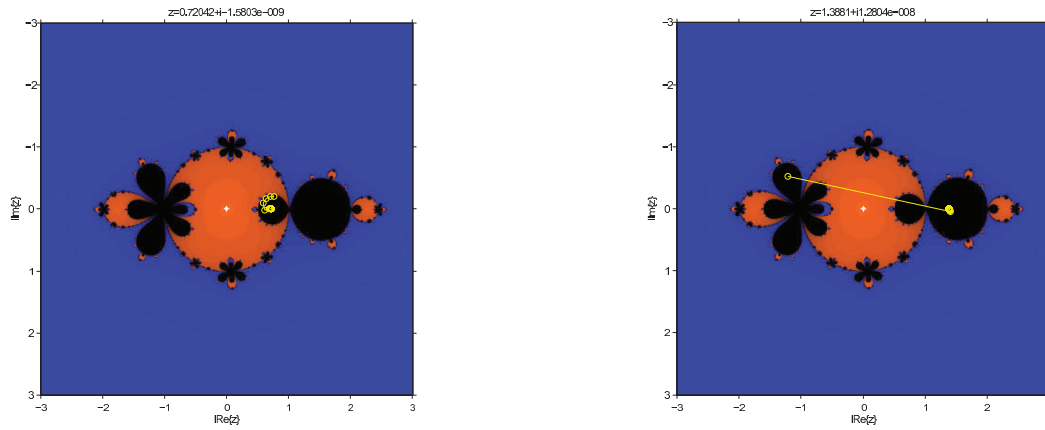


Figure 4: Dynamical plane associated to $\beta = -4.5$

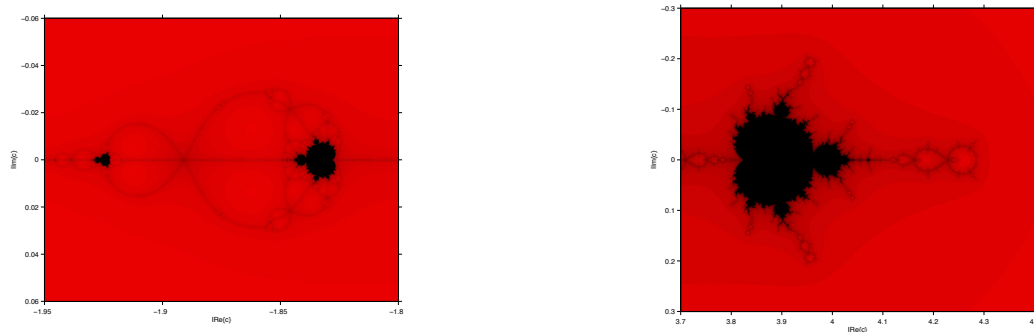


Figure 5: Details of the right central and outer antennas of P_2

A particular value of β in this region is $\beta = 3.9 + 0.1i$. It is placed in a bulb around the Mandelbrot set. The corresponding dynamical plane is showed in Figure 6.

We can observe the existence of three basin of attraction, two of them associated to the roots and the other to an attracting periodic orbit of period 3, $\{-4.5579 + i0.37156, -4.4582 + i0.05353, -4.7681 + i0.27481\}$. Sharkovsky's Theorem ([17]), states that the existence of orbits of period 3, guaranties orbits of any period.

Cases $\beta = -4.5$ and $\beta = 3.9 + 0.1i$ are only two of many situations that highlight the dynamical wealth of this family and also that the analysis of parameter planes are a useful tool for its study.

5. Conclusions

The dynamical behavior of King's family is very rich. It has been proved that there are many regions with no convergence to the roots, and the existence of periodic orbits of arbitrary period has been showed. Nevertheless, there are wide regions in parameter space whose corresponding iterative methods have a good numerical behavior, in terms of stability and efficiency even in terms of order, as $\beta = \frac{-5}{2}$. These regions are associated to non-black regions in both parameter planes simultaneously. Interesting future studies deal with the interaction between free critical points in black regions of P_1 and P_2 .

6. References

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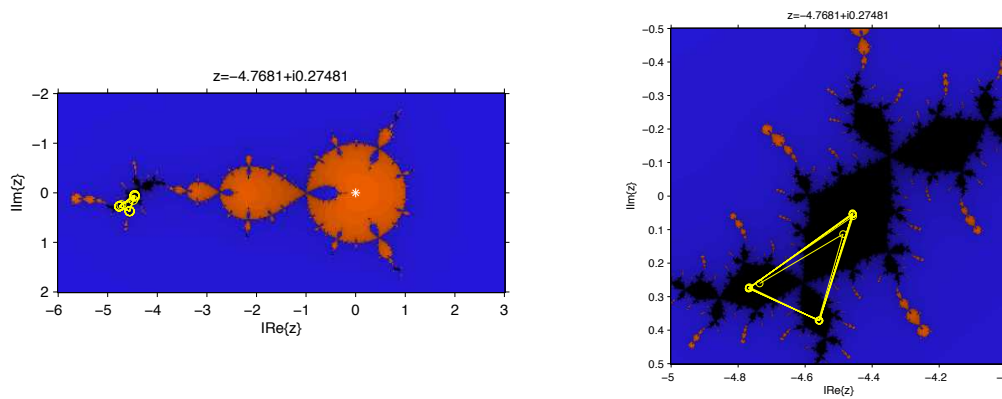


Figure 6: Periodic orbit for $\beta = 3.9 + 0.1i$

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