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The influence of p-regular class sizes on normal subgroups

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Abstract. Let G be a finite group and N be a normal subgroup of G and suppose that the p-regular elements of N have exactly two G-conjugacy class sizes. It is shown that N is solvable and that if H is a p-complement of N, then either H is abelian or H is the product of an r-group for some prime $r \neq p$ and a central subgroup of G.

1 Introduction

The study of the effect on group structure of conjugacy class sizes is a classical topic in finite group theory. Class sizes do not behave well with respect to normal subgroups and quotients and thus one of the handicaps appearing in this study is the impossibility of using inductive arguments. In fact, regarding normal subgroups, the only available information is basically that each class size of the normal subgroup is a divisor of its respective class size in the whole group. Moreover, there is no relation between the number of class sizes of the normal subgroup and the number of class sizes of the group.

Let G be a finite group and let N be a normal subgroup of G. Since N is the union of classes of G (called G-classes of N), it is natural to wonder if these sizes exert an influence on the structure of N. This new point of view has the advantage of allowing us to work inductively, since for instance the number of G-class sizes of N is trivially bounded by the number of class sizes of G. Recent results have indicated that there is indeed a close connection between G-class sizes and the normal structure of G. In [3], the nilpotency of normal subgroups with two G-class sizes is shown, thereby giving a generalization for normal subgroups of Itô's celebrated theorem on groups having two class sizes ([5]). We remark that

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while the proof of Itô's result is quite elementary, the proof of the above extension requires the classification of the finite simple groups.

In this paper, our attention is focused on those elements of the group whose orders are not divisible by a fixed prime p (p-regular elements) and on their class sizes. The importance of the results on class sizes of p-regular elements is that the obtained information can be used in a more general context of ordinary classes. However, this approach increases the difficulty, especially when no p-solvability hypothesis is assumed. The first results on groups having two p-regular class sizes appeared in [4], where such groups were proved to be solvable. Afterwards, the proof was simplified in [2] and the structure of a group G with two p-regular class sizes was determined: either G has abelian p-complements or, up to central factors, G is a $\{p, r\}$-group for a prime $r \neq p$, so in particular, G is solvable. On the other hand, in [1] it is proved that if G is p-solvable and N is a normal subgroup of G with two p-regular G-class sizes, then the p-complements of N are nilpotent. The significant goal of this note is to improve this result by eliminating the p-solvability condition.

Theorem A. *Let N be a normal subgroup of a finite group G. Let p be a prime number and suppose that the G-conjugacy class of every p-regular element of N has size 1 or m for some fixed integer m. Then N has abelian p-complements or $N = RP \times A$, where R and P are a Sylow r-subgroup for some prime r and a Sylow p-subgroup of N respectively, and A is a central subgroup of G.*

Theorem A is an extension of the main result of [3], when p does not divide the order of N, which is the following

Corollary B. *Let N be a normal subgroup of a finite group G and suppose that the G-conjugacy class of every element of N has size 1 or m for some fixed integer m. Then either N is abelian or $N = R \times A$, where R is a Sylow r-subgroup of N for some prime r and A is central in G.*

At the same time, Theorem A generalizes and provides an alternative proof of the main result of [2], when $N = G$. One of the key parts of the proof of Theorem A consists of analysing the Schur multiplier of some simple groups satisfying certain conditions on the orders of their elements, which can be formulated in the terminology of graph theory as those groups whose prime graph is a forest. The classification of such groups was obtained by M. S. Lucido in [8].

Throughout this paper all groups are finite. If x is any element of a group G, we denote by x^G the conjugacy class of x in G and $|x^G|$ is called the class size of x in G or the G-class size of x.

2 Proofs

First, we state the preliminary results that we are going to use.

Lemma 2.1. *Let $P \times Q$ be the direct product of a p-group P and a p'-group Q and suppose that $P \times Q$ acts on a p-group G. If*

$$\mathbf{C}_G(P) \subseteq \mathbf{C}_G(Q),$$

then Q acts trivially on G.

Proof. This is Thompson's $P \times Q$-Lemma. See for instance [7, (8.2.8)]. □

We also recall the definition of the prime graph associated to a group, which is one of the main tools of the proofs.

Definition 2.2. Let G be a finite group. The prime graph $\Gamma(G)$ of G is defined as follows. The vertices of $\Gamma(G)$ are the primes dividing the order of G and two distinct vertices r and s are joined by an edge if there is an element in G of order rs.

In [8] a complete classification is provided of the finite non-abelian simple groups whose prime graph is a forest. These groups are exactly twelve specific simple groups or belong to five families of Lie type groups under certain arithmetical properties on the parameters. However, we do not need to employ the complete classification, and for the purpose of brevity, we will only state the part we need. The notation for the simple groups is the one appearing in the original article.

Theorem 2.3. *Let G be a finite non-abelian simple group. If $\Gamma(G)$ is a forest, then G is either one of the simple groups*

$$A_5,\ A_6,\ A_7,\ A_8,\ M_{11},\ M_{22},\ \mathrm{PSL}_4(3),\ B_2(3),\ G_2(3),\ U_4(3),\ U_5(2),\ {}^2F_4(2)',$$

or G belongs to one of the families

$$\mathrm{PSL}_2(q),\ \mathrm{PSL}_3(q),\ \mathrm{PSU}_3(q),\ \mathrm{Sz}(q^2)$$

with $q^2 = 2^f$ or $q = 2^{f^2}$ with f an odd prime, or $\mathrm{Ree}(3^f)$*, with f an odd prime.*

Proof. See [8, Proposition 4]. □

We show now how this property of the prime graph arises with the hypothesis under study, that is, when a normal subgroup has two p-regular G-class sizes. The following property is one of the main results in [1].

Theorem 2.4. *Let N be a normal subgroup of a group G such that the G-conjugacy class size of every p-regular element of N is 1 or m, for some integer m. Then either N has abelian p-complements or all p-regular elements of $N/(N \cap \mathbf{Z}(G))$ have prime-power order.*

Proof. See [1, Theorem 1]. □

Corollary 2.5. *Let N be a normal subgroup of a group G such that the G-conjugacy class size of every p-regular element of N is 1 or m, for some integer m. Then either N has abelian p-complements or the prime graph of $\overline{N} = N/(N \cap \mathbf{Z}(G))$ is a forest.*

Proof. By Theorem 2.4, either N has abelian p-complements or all p-regular elements of $\overline{N}$ have prime-power order. This implies that each prime dividing $|\overline{N}|$ distinct from p can only be connected by an edge to p, so $\Gamma(\overline{N})$ is a forest. □

Theorem 2.6. *If N is a solvable normal subgroup of a group G with two G-class sizes of p-regular elements, then N has nilpotent p-complements.*

Proof. We argue by minimal counter-example. Let N be a normal solvable subgroup of G of minimal order which satisfies the hypotheses and does not have a nilpotent p-complement. So by Theorem 2.4, all p-regular elements of the group $\overline{N} = N/(N \cap \mathbf{Z}(G))$ have prime-power order.

Let N/K be a chief factor of G such that $N \cap \mathbf{Z}(G) \subseteq K$. Notice that K is a normal subgroup of G and thus, the hypotheses of the theorem are inherited by K and so, by minimality, K has nilpotent p-complements. Moreover, we know that $\overline{K} = K/(N \cap \mathbf{Z}(G))$ does not have any p-regular element of order divisible by two distinct prime numbers, since $\overline{K} \subseteq \overline{N}$. We conclude that $\pi(\overline{K}) \subseteq \{p, r\}$, where r is a prime number distinct from p.

Notice that N/K is an (elementary abelian) s-group, for some prime number $s \notin \{p, r\}$, otherwise $\overline{N}$ is a $\{r, p\}$-group and consequently, N has nilpotent p-complements, which is a contradiction. Then, there exists an s-element $x \in N$ which is noncentral in G and we take $R \in \mathrm{Syl}_r(\mathbf{C}_G(x))$. Let us consider $NR \leqslant G$ and note that NR is solvable. We can then take a p-complement of NR, say H, such that $H \cap \mathbf{C}_{RN}(x)$ is a p-complement of $\mathbf{C}_{NR}(x)$. We assert that $x \in H$. This is because $\langle x\rangle \mathbf{C}_H(x)$ is a p'-subgroup of $\mathbf{C}_{NR}(x)$ containing $\mathbf{C}_H(x)$. Now we show that $\mathbf{O}_r(H \cap N) \subseteq \mathbf{Z}(G)$. We take $R_0 \in \mathrm{Syl}_r(\mathbf{C}_{RN}(x))$ such that $R_0 \subseteq H$. Since R commutes with x, we have that R is also a Sylow r-subgroup of $\mathbf{C}_{NR}(x)$, and hence $R_0 \in \mathrm{Syl}_r(\mathbf{C}_G(x))$. Then we consider the action of $R_0 \times \langle x\rangle$ on $\mathbf{O}_r(H \cap N)$. We claim that

$$\mathbf{C}_{\mathbf{O}_r(H\cap N)}(R_0) \subseteq \mathbf{C}_{\mathbf{O}_r(H\cap N)}(x).$$

In fact, if $z \in \mathbf{C}_{\mathbf{O}_r(H\cap N)}(R_0) \setminus \mathbf{Z}(G)$, then $\langle R_0, z\rangle \subseteq \mathbf{C}_G(z)$, and this implies that

$$|R_0| \leqslant |\langle R_0, z\rangle| \leqslant |\mathbf{C}_G(z)|_r = |\mathbf{C}_G(x)|_r = |R_0|.$$

Thus, equality holds and hence $z \in R_0$. Therefore

$$z \in \mathbf{C}_G(x) \cap \mathbf{O}_r(H \cap N) = \mathbf{C}_{\mathbf{O}_r(H\cap N)}(x),$$

as claimed. Now, we can apply Lemma 2.1 and we infer that x acts trivially on $\mathbf{O}_r(H\cap N)$, that is, $x \in \mathbf{C}_N(\mathbf{O}_r(H\cap N))$. Now, suppose that $\mathbf{O}_r(H\cap N) \nsubseteq \mathbf{Z}(G)$ and take some $y \in \mathbf{O}_r(H \cap N)$ which is noncentral in G. As x centralizes y, we have that $xy(N \cap \mathbf{Z}(G))$ is a p-regular element of composite order, contradicting the fact that all p-regular elements of $N/(N \cap \mathbf{Z}(G))$ have prime-power orders, observed in the first paragraph. We conclude that $\mathbf{O}_r(H \cap N) \subseteq \mathbf{Z}(G)$.

Finally we get a contradiction. We certainly have that $K \cap H$ is normal in H and observe that

$$K \cap H = R_1 \times (N \cap \mathbf{Z}(G))_{\{p,r\}'},$$

where R_1 is a Sylow r-subgroup of N. Hence, we obtain that

$$R_1 \subseteq \mathbf{O}_r(H \cap N) \subseteq \mathbf{Z}(G),$$

which implies that $K \cap H \subseteq \mathbf{Z}(G)$. This is a contradiction, since N does not have nilpotent p-complements, and the proof is finished. □

Corollary 2.7. *If N is a solvable normal subgroup of a group G with two G-class sizes of p-regular elements, then N has abelian p-complements or $N = RP \times A$, where R and P are a Sylow r-subgroup for a prime $r \neq p$, and a Sylow p-subgroup of N respectively, and A is a central group of G.*

Proof. By Theorem 2.6, we have that N has nilpotent p-complements. Moreover, assuming that N has non-abelian p-complements we know that $N/(N \cap \mathbf{Z}(G))$ does not have any p'-element of order divisible by two different primes. Therefore, if H is a p-complement of N, then all elements in $H/(H \cap \mathbf{Z}(G))$ have prime-power order and, by the nilpotency of H, we conclude that $H = R \times A$, where R is a Sylow r-subgroup of N and A is a central group of G. □

Theorem 2.8. *If N is a normal subgroup of G with two G-class sizes of p-regular elements, then N is solvable.*

Proof. We argue by minimal counter-example and thus, we assume that N is a non-solvable normal subgroup of G of minimal order satisfying the hypothesis of the theorem. First, we notice that, by Theorem 2.4, $\overline{N} = N/(N \cap \mathbf{Z}(G))$ does not have any p'-element whose order is divisible by two primes, since groups with an abelian p-complement are solvable.

Let $x \in N$ be a q-element for some prime $q \neq p$ such that x is noncentral in G. We easily get that any Sylow r-subgroup of $\mathbf{C}_N(x)$ lies in $N \cap \mathbf{Z}(G)$, for every $r \notin \{p, q\}$. Otherwise, there exists an r-element $y \in \mathbf{C}_N(x) \setminus (N \cap \mathbf{Z}(G))$, and so the order of $xy(N \cap \mathbf{Z}(G))$ is not a prime power. Therefore, we obtain that $|\overline{N}|_r$ divides $|x^N|_r$. Set m equal to the size of any noncentral p-regular class of G contained in N. As $|x^N|$ divides $|x^G| = m$, we deduce that $|\overline{N}|_{\{p,q\}'}$ divides $m_{\{p,q\}'}$. Arguing similarly with a noncentral t-element, for any prime divisor t of $|\overline{N}|$ such that $t \notin \{p, q\}$, we conclude that $|\overline{N}|_{p'}$ divides $m_{p'}$. Notice that such a prime and such an element exist, since otherwise the p-complements of N would be nilpotent and consequently, N would be solvable.

Let $q \neq p$ be a prime divisor of $|\overline{N}|$ and let Q be a Sylow q-subgroup of G, so that $Q \cap N$ is a Sylow q-subgroup of N. Suppose that x is an element of $Q \cap N$ which is noncentral in G. Certainly, there exists some $y \in G$ such that $\mathbf{C}_{Q^y}(x)$ is a Sylow q-subgroup of $\mathbf{C}_G(x)$. Hence

$$m_q = \frac{|G|_q}{|\mathbf{C}_G(x)|_q} = \frac{|Q^y|}{|\mathbf{C}_{Q^y}(x)|}.$$

On the other hand, there is some $z \in \mathbf{C}_G(x)$ such that

$$\mathbf{C}_Q(x) \subseteq \mathbf{C}_{Q^y}(x)^z = \mathbf{C}_{Q^{yz}}(x).$$

Then, $m_q = |Q^{yz}|/|\mathbf{C}_{Q^{yz}}(x)|$ divides $|x^Q|$. Now, as $Q \cap N$ is normal in Q, we can use the class equation for this subgroup and this yields

$$|Q \cap N| = |Q \cap N \cap \mathbf{Z}(G)| + km_q,$$

for some positive integer k. Since $|\overline{N}|_q$ divides both $|Q \cap N|$ and m_q, it follows that $|\overline{N}|_q$ divides $|N \cap \mathbf{Z}(G)|$. This is true for every prime $q \neq p$ dividing $|\overline{N}|$, so we conclude that $|\overline{N}|_{p'}$ divides $|N \cap \mathbf{Z}(G)|$.

Now, let N/K be a chief factor of G, such that $N \cap \mathbf{Z}(G) \subseteq K$. By minimality, K is solvable, and since K also has two G-class sizes of p-regular elements, by Theorem 2.6, we obtain that K has nilpotent p-complements. If we denote $\overline{K} = K/(N \cap \mathbf{Z}(G))$, as we know that $\overline{K}$ does not have any p'-element whose order is divisible by two different primes, it follows that a p-complement of $\overline{K}$ is just an r-group for some prime $r \neq p$, that is, $\pi(\overline{K}) \subseteq \{p, r\}$, where $r \neq p$. On the other hand, since N/K is characteristically simple, it follows that

$$N/K \cong S_1 \times \cdots \times S_l,$$

where the subgroups S_i are necessarily non-abelian simple groups and are pairwise isomorphic. Furthermore, by Theorem 2.4, we know that N/K does not have any p'-element whose order is divisible by two primes, whence N/K is isomorphic to just one non-abelian simple group, say S. Moreover, by Corollary 2.5, we know that the prime graph of $\overline{N}$ is a forest, so the prime graph of N/K is a forest too, and thus, N/K is isomorphic to one of the groups appearing in Theorem 2.3.

Regarding the subgroup K, we observe that

$$K \cong \mathbf{O}_{\{r,p\}}(N) \times (N \cap \mathbf{Z}(G))_{\{p,r\}'}.$$

Moreover, notice that N is perfect by minimality, and so the group $N/\mathbf{O}_{\{r,p\}}(N)$ is perfect too. By the fact that

$$\frac{N/\mathbf{O}_{\{r,p\}}(N)}{K/\mathbf{O}_{\{r,p\}}(N)} \cong N/K$$

is simple and

$$(N \cap \mathbf{Z}(G))_{\{r,p\}'} \cong K/\mathbf{O}_{\{r,p\}}(N) \leqslant \mathbf{Z}(N/\mathbf{O}_{\{p,r\}}(N)),$$

we infer that

$$K/\mathbf{O}_{\{r,p\}}(N) = \mathbf{Z}(N/\mathbf{O}_{\{r,p\}}(N)).$$

Hence we can deduce that $N/\mathbf{O}_{\{r,p\}}(N)$ is a quasi-simple group. Now, observe that S is the simple group associated to this quasi-simple group. Consequently, $\mathbf{Z}(N/\mathbf{O}_{\{r,p\}}(N))$ is isomorphic to a subgroup of the Schur multiplier $M(S)$, so in particular, $|(N \cap \mathbf{Z}(G))|_{\{r,p\}'} = |\mathbf{Z}(N/\mathbf{O}_{\{r,p\}}(N))|$ divides $|M(S)|$. Therefore, $|S| = |N/K|$ divides $|M(S)|p^\alpha r^\beta$, for some integers α and β (hence, notice that $|M(S)|$ cannot be trivial). We are going to check (for instance, with the help of [6]) that this property is not possible for the simple groups listed in Theorem 2.3.

Assume first that S is one of the following groups:

$$A_5,\ A_6,\ A_7,\ A_8,\ M_{11},\ M_{22},\ \mathrm{PSL}(4,\ 3),\ B_2(3),$$
$$G_2(3),\ U_4(3),\ U_5(2),\ {}^2F_4(2)',\ \mathrm{Sz}(q^2),\ \mathrm{Ree}(3^f).$$

Then $|M(S)|$ may be 1, 2, 3, 4, 6, 12 or 36, and it is easy to prove that all these possibilities together with the above divisibility property lead to a contradiction.

Suppose that $S \cong \mathrm{PSL}(2,q)$ and then $|M(S)|$ divides $(2, q-1)$, except for the cases $\mathrm{PSL}(2,4) \cong A_5$ and $\mathrm{PSL}(2,9) \cong A_6$. It follows by orders that these two cases are not possible. In the rest of the cases, $|\mathrm{PSL}(2,q)|$ divides $2p^\alpha r^\beta$, which implies a contradiction because $|\mathrm{PSL}(2,q)|_2 > 2$.

Suppose that $S \cong \mathrm{PSL}(3,q)$. Then $|M(S)|$ divides $(3, q-1)$, except for the cases $\mathrm{PSL}(3,2)$ and $\mathrm{PSL}(3,4)$. These are not possible again by order considerations. In the other cases, since $M(S)$ cannot be trivial, we can assume $|M(S)| = 3$

and thus $q \equiv 1(3)$. We get that $|\mathrm{PSL}(3,q)| = q^3(q^2-1)(q^3-1)/3$ divides $3p^\alpha r^\beta$, which is a contradiction again since $|\mathrm{PSL}(3,q)|_3 > 3$ for such q.

Finally, if $S \cong \mathrm{PSU}(3,q)$, then $|M(S)|$ divides $(3, q+1)$, and as above, we can assume $|M(S)| = 3$ and $q \equiv -1(3)$. As $|\mathrm{PSU}(3,q)| = q^3(q^2-1)(q^3+1)/3$ divides $3p^\alpha r^\beta$, we get a contradiction because $|\mathrm{PSU}(3,q)|_3 > 3$ for such q. □

Proofs of Theorem A *and Corollary* B. Theorem A is immediate by Corollary 2.7 and Theorem 2.8. In particular, Corollary B follows by taking any prime p not dividing the order of N. □

Remark. We want to remark that when $N = G$ in Theorem A then m is equal to $p^a q^b$ for some prime q and some nonnegative integers a and b (see the main theorem of [2]). However, this is not true for normal subgroups in general, and we show it with an example. For instance, let $G = \mathrm{SL}(2,3) \times P$, where P is a p-group for a prime $p \neq 2, 3$. Let Q be the Sylow 2-subgroup of $\mathrm{SL}(2,3)$ (quaternion group of order 8) and let $N = Q \times P$. Then, the p-regular G-class sizes of N are 1 and 6.

On the other hand, the fact that each class size of N divides its corresponding G-class size implies that if m is a p'-number, then the normal subgroup N has a Sylow p-subgroup which is a direct factor of N. This also occurs in the above example when $p \neq 2, 3$.

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