Köthe dual of Banach lattices generated by vector measures

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Abstract

We study the Köthe dual spaces of Banach function lattices generated by abstract methods having roots in the theory of interpolation spaces. We apply these results to Banach spaces of integrable functions with respect to Banach space valued countably additive vector measures. As an application we derive a description of the Banach dual of a large class of these spaces, including Orlicz spaces of integrable functions with respect to vector measures. *Keywords: Banach lattice; vector measure; integration; Köthe dual space. AMSClassification: 46E30; 47B38; 46B42*

1 Introduction

The integration of scalar valued measurable functions with respect to Banach space valued countably additive vector measures has a very long history. The story begins with the work of R. G. Bartle, N. Dunford and J. Schwartz [2]. After several years, D. R. Lewis [15, 16] fixed the definitions and main results regarding this theory. The spaces of integrable functions with respect to a vector measure find interesting applications in many areas of analysis, including the theory Banach lattices as well as the theory of extension of operators from Banach lattices (see, e.g., [21]).

The interest on this area has grown in the last few years. This is due in part to the fact that an abstract Banach lattices with order continuous norm and having a weak unit is order isomorphic to a space of integrable functions with respect to some vector measure (see [4]; see also Ch.3 in [21]). This result was extended in [8], where it was shown that an abstract Banach p-convex Banach lattice with order continuous norm and a weak unit is order isomorphic to a space $L^p(\nu)$ of p-integrable functions with respect to some vector measure ν. More results in this direction have been obtained in recent years; for instance, in [5] a new representation theorem for a large class of abstract p-convex Banach lattices is proved.

Since the introduction of the spaces $L^p(\nu)$ in [24], some effort has been made for finding a good representation of their Banach duals when $1 < p < \infty$. The first attempt made use of a topological tensor product formalism for giving a description of these dual spaces ([9, 10, 25]) following the original decomposition technique given in [20] for representing the elements of the dual of the space $L^1(\nu)$ as combinations of the elements of the range of ν and some particular integrable functions. However, a representation in terms of Köthe duals of Banach function spaces, i.e. as spaces of integrable functions, was unknown until the recent relevant Galaz-Fontes's paper [11]. In the present paper we provide a general approach which allows to describe the Köthe dual of function spaces that are generated by procedures involving abstract unions and intersections of families of Banach lattices. Applying our general results, we show how to construct classes of Banach function lattices which are subspaces of the space $L^1(\nu)$ of integrable functions with respect to a Banach space valued countably additive vector measure ν. We also describe the Banach duals for a large class of these type of spaces. Actually, we believe that this unified approach will find further applications in the study of topological properties of new classes of Banach lattices of vector measure integrable functions.

The paper is organized as follows. In Section 2 we fix the notation used throughout it and we show some basic results. In Section 3, we prove Köthe duality results between lattices generated by families of Banach function lattices on a given measure space by the so called Δ and Σ-methods. In Section 4, we define new general classes of Banach function spaces of integrable functions with respect to Banach space valued countably additive vector measures that includes well known ones. We present applications of our general results of Section 3 for a new description of the Köthe dual as well as the Banach dual for a large class of these spaces. In particular we derive such description for Orlicz and Nakano spaces of integrable functions with respect to vector-valued measures. In the case of the power functions, we recover the results due to Galaz-Fontes [11] for the spaces of scalarly p-integrable and p-integrable functions with respect to Banach space valued countably additive vector measures.

2 Preliminaries

We start by providing some notions and terminology required later. We use the standard notation from Banach space theory. If E is a Banach space, then by B_E and S_E we denote the unit ball and the unit sphere of E, respectively. As usual E^* denotes the dual Banach space of E. Let E and F be now normed spaces. If E and F are isomorphic, then we write $E \simeq F$. If E is a subspace of F and the inclusion $I\colon E \to F$ is continuous with $\|I\| \le c$, then we write $E \overset{c}{\hookrightarrow} F$. In the particular case $c = 1$ we write $E \hookrightarrow F$. If $E \hookrightarrow F$ and $F \hookrightarrow E$, we write $E \cong F$. As usual by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ we denote the set of real, non-negative real and natural numbers, respectively.

Let $(\Omega, \mathscr{S}, \mu)$ be a σ-finite measure space, and let $L^0 := L^0(\mu)$ be the space of (equivalence classes of μ-a.e. equal) real valued measurable functions on Ω equipped with the topology of convergence in measure on sets of finite measure. We say that $(X, \|\cdot\|_X)$ is a *Banach function lattice* (for short a *Banach lattice*) on $(\Omega, \mathscr{S}, \mu)$ if X is an ideal in L^0 and whenever $f, g \in X$ and $|f| \le |g|$ μ-a.e., then $\|f\|_X \le \|g\|_X$.

We recall that for any Banach lattice X on a measure space $(\Omega, \mathscr{S}, \mu)$ it is possible to construct (see [13, Corollary 1, p. 95] or [26, pp. 454-456]) a set $\Omega_X \in \mathscr{S}$ called the support of X such that every element of X vanishes μ-a.e. out of Ω_X and every measurable set $E \subset \Omega_X$ with $\mu(E) > 0$ has a measurable subset B of finite positive measure with $\chi_B \in X$. Furthermore Ω_X is the union of an increasing sequence of measurable sets (E_n) such that $\chi_{E_n} \in X$ and $\mu(E_n) < \infty$ for each n. Notice that Ω_X is defined up to a null set. As usual the support Ω_X of a Banach function lattice X is also denoted by $\mathrm{supp}(X)$.

Let X be a Banach lattice on $(\Omega, \mathscr{S}, \mu)$. An element $f \in X$ is called order continuous if for every $0 \le f_n \le |f|$ such that $f_n \downarrow 0$ a.e. it holds $\|f_n\|_X \to 0$. We denote by X_a the set of all order continuous elements of X. We will use the easily verified fact that $f \in X_a$ if and only if

$$\lim_{n\to\infty} \|f \chi_{E_n}\|_X = 0$$

for any sequence (E_n) of sets in $\mathscr{S}$ such that $E_n \searrow \emptyset$ (i.e., (E_n) is a decreasing such that $\mu(E_n \cap E) \to 0$ for every set E of finite measure).

The Banach lattice X is said to be *order continuous* if $X = X_a$. It is said that X is *maximal* if the unit ball B_X is closed in $L^0(\mu)$. It is well know that this is equivalent to the so called Fatou property of X, i.e., if for any sequence (f_n) in X such that $0 \le f_n \uparrow f$ and $\sup \|f_n\|_X < \infty$ we have that $f \in X$ and $\|f_n\|_X \to \|f\|_X$. X is said to have the *weak Fatou property* (or equivalently that the norm of X is semi-continuous) whenever if f_n, $f \in X$, $f_n \uparrow f$ a.e., then $\|f_n\|_X \to \|x\|_X$. It is clear that order continuous Banach function lattices have the weak Fatou property.

The Köthe dual space X' of X is the subset of all elements $f \in L^0(\mu)$ such that

$$\|f\|_{X'} := \sup\left\{ \int_\Omega |fg|\, d\mu;\ \|g\|_X \le 1 \right\} < \infty.$$

It is well known that X' equipped with the norm $\|\cdot\|_{X'}$ is a Banach lattice with $\mathrm{supp}(X') = \mathrm{supp}(X)$. Furthermore $X'' = X$ with equality of norms if and only if X has the Fatou property (see [13, Theorem 6, p. 190] or [26, Theorems 3 and 4, p. 472]). We will use the well known fact that $f \in (X'')_a$ if and only if $f \in X_a$ and $\|f\|_{X''} = \|f\|_X$, i.e., $(X'')_a \cong X_a$. We also recall that if a Banach lattice X on $(\Omega, \mathscr{S}, \mu)$ is order continuous, then for every $x^* \in X^*$ there exists a unique $g = g_{x^*} \in X'$ such that

$$x^*(f) = \int_\Omega fg\, d\mu, \quad f \in X,$$

and $\|x^*\|_{X^*} = \|g\|_{X'}$. In particular the map $X^* \ni x^* \mapsto g_{x^*} \in X'$ is an order isometrical isomorphism of X^* onto X'. For more details and more information on Banach lattices we refer to [13, 26].

We will also use the following definition. If v is an $\mathscr{S}$-measurable function $v\colon \Omega \to [0,\infty]$ and F is a Banach lattice on $(\Omega, \mathscr{S}, \mu)$, we denote by $F(v)$ a Banach lattice on $(\Omega, \mathscr{S}, \mu)$ with $\mathrm{supp}(F(v)) = \mathrm{supp}(F) \setminus \{t \in \Omega;\, v(t) = 0 \text{ or } v(t) = \infty\}$ and equipped with the norm $\|f\|_{F(v)} := \|vf\|_F$.

Let us give now some notions on spaces of vector measure integrable functions. Let $(\Omega, \mathscr{S})$ be a measurable space, E a Banach space, and $\nu\colon \mathscr{S} \to E$ a (countably additive) vector measure. The *variation* of the measure ν is denoted by $|\nu|$ and corresponds to the set function $|\nu|\colon \mathscr{S} \to [0,\infty]$ given by $|\nu|(A) := \sup_\pi \sum_{B\in\pi} \|\nu(B)\|_E$ for all $A \in \mathscr{S}$, where the supremum is taken over all the finite (disjoint) partitions π of A. We remark that the variation is the smallest $[0,\infty]$-valued measure dominating ν.

That is $\|\nu(A)\|_E \leq |\nu|(A)$ for every $A \in \mathscr{S}$. As usual, for every $x^* \in X^*$ we denote by $\langle m, x^* \rangle \colon \mathscr{S} \to \mathbb{R}$ the scalar measure given by

$$\langle \nu, x^* \rangle(A) := \langle \nu(A), x^* \rangle, \quad A \in \Sigma.$$

The *semivariation* of ν is the set function $\|\nu\| \colon \mathscr{S} \to \mathbb{R}_+$ defined by

$$\|\nu\|(A) = \sup_{x^* \in B_{E^*}} |\langle \nu, x^* \rangle|(A), \quad A \in \mathscr{S}.$$

Notice that $\|\nu\|(\Omega) < \infty$. Throughout the paper we will denote by λ a Rybakov (control) measure $\lambda \colon \mathscr{S} \to \mathbb{R}_+$ for ν. This means that $\lambda = |\langle \nu, x_0^* \rangle|$ for some $x^* \in B_{E^*}$ and $\lambda(A) = 0$ if and only if $\|\nu\|(A) = 0$. Such a measure always exists (see [7, Chapter IX, Theorem 2.2]). To avoid trivial cases we will suppose that $\|\nu\|(\Omega) > 0$.

Following Lewis [15], a function $f \in L^0(\nu)$ is scalarly integrable if $f \in L^1(|\langle \nu, x^* \rangle|)$ for all $x^* \in E^*$. The space consisting of all scalarly integrable functions with respect to ν is denoted by $L^1_w(\nu)$, which is a Banach function lattice over λ with the Fatou property and a weak unit when equipped with the norm

$$\|f\| = \sup_{x^* \in B_{E^*}} \int_\Omega |f| \, d|\langle \nu, x^* \rangle|.$$

A function $f \in L^1_w(\nu)$ is said to be integrable with respect to ν provided for every $A \in \mathscr{S}$ there exists an element of X denoted by $\int_A f \, d\nu$, such that

$$\left\langle \int_A f \, d\nu, x^* \right\rangle = \int_A f \, d\langle \nu, x^* \rangle, \quad x^* \in E^*.$$

The space of all integrable functions is denoted by $L^1(\nu)$. It is an order continuous Banach function sublattice of $L^1_w(\nu)$.

A function $f \in L^1(\lambda)$ is called scalarly p-integrable ($1 < p < \infty$) with respect to ν if $|f|^p \in L^1_w(\nu)$, and p-integrable with respect to ν whenever $|f|^p \in L^1(\nu)$. We denote by $L^p_w(\nu)$ and $L^p(\nu)$ the corresponding spaces of p-integrable and scalarly p-integrable functions with respect to ν. These spaces are equipped with the norm

$$\|f\| = \sup_{x^* \in B_{E^*}} \left(\int_\Omega |f|^p \, d|\langle \nu, x^* \rangle| \right)^{1/p},$$

for which both spaces are p-convex (for the definition of p-convexity of Banach lattices see [17]). As usual $L^\infty(\nu)$ denotes the Banach lattice of the (classes of) real valued measurable functions that are λ-essentially bounded.

Notice that these spaces are particular instances of Orlicz spaces of integrable functions with respect to vector-valued measures. In the final part of the paper we give some applications of our results to the description of the Köthe dual of Musielak-Orlicz spaces of integrable functions with respect to vector-valued measures.

3 Köthe duality formulas for Δ and Σ-methods

We will start by briefly recalling some basic definitions having their roots in interpolation theory (cf. [1], see also [3, 14]). Let $\{X_\alpha\}_{\alpha\in\mathscr{A}}$ be a family of Banach lattices of measurable functions over the σ-finite measure space $(\Omega,\mathscr{S},\mu)$. In what follows we define the Δ and Σ procedures associated to this family of spaces. Although these constructions can be done for general classes of Banach spaces, we center our attention in the class of Banach lattices on a given measure space. For the sake of simplicity, we also assume that the supports of all the Banach lattices in the family $\{X_\alpha\}_{\alpha\in\mathscr{A}}$ coincide with Ω.

A family $\{X_\alpha\}_{\alpha\in\mathscr{A}}$ of Banach lattices is said to be *compatible* if there is a measure space $(\Omega,\mathscr{S},\mu)$ such that X_α is a Banach lattice over it. A family $\{X_\alpha\}_{\alpha\in\mathscr{A}}$ over a given measure space $(\Omega,\mathscr{S},\mu)$ is called *strongly compatible* if there exists a Banach lattice Y over $(\Omega,\mathscr{S},\mu)$ such that $X_\alpha \overset{c_\alpha}{\hookrightarrow} Y$ with $\sup_{\alpha\in\mathscr{A}} c_\alpha<\infty$.

Let $\{X_\alpha\}_\alpha$ be a compatible family of Banach function lattices over $(\Omega,\mathscr{S},\mu)$, and let

$$\Delta(X_\alpha)=\Big\{x\in\bigcap_{\alpha\in\mathscr{A}}X_\alpha;\ \|x\|_{\Delta(X_\alpha)}:=\sup_{\alpha\in\mathscr{A}}\|x\|_{X_\alpha}<\infty\Big\}.$$

Then $(\Delta(X_\alpha),\|\cdot\|_{\Delta(X_\alpha})$ is a Banach lattice on $(\Omega,\mathscr{S},\mu)$ with the following properties:

(i) $\Delta(X_\alpha)\hookrightarrow X_\alpha$ for every $\alpha\in\mathscr{A}$.

(ii) If F is a Banach space such that $F\overset{c}{\hookrightarrow}X_\alpha$, for every $\alpha\in\mathscr{A}$, then $F\overset{c}{\hookrightarrow}\Delta(X_\alpha)_{\alpha\in\mathscr{A}}$.

In the case where $\{X_\alpha\}_{\alpha\in\mathscr{A}}$ is a strongly compatible family, we let $\Sigma(X_\alpha)$ to be the space of all $x\in L^0(\mu)$ such that

$$|x|\le\sum_{\alpha\in\mathscr{A}}|x_\alpha|\quad(x_\alpha\in X_\alpha),\quad\text{where}\ \sum_\alpha\|x_\alpha\|_{X_\alpha}<\infty. \tag{$*$}$$

It follows from the last condition that there are only countably many summands in $\Sigma_{\alpha\in\mathscr{A}}x_\alpha$ different from zero. We equipped $\Sigma(X_\alpha)$ with the norm

$$\|x\|_{\Sigma(X_\alpha)}=\inf\Big\{\sum_{\alpha\in\mathscr{A}}\|x_\alpha\|_{X_\alpha};\ |x|\le\sum_{\alpha\in\mathscr{A}}|x_\alpha|\Big\}.$$

Note that from $(*)$ it follows that the series $\Sigma_{\alpha\in\mathscr{A}}x_\alpha$ converges absolutely in Y, and so x can be represented in the form

$$x=\sum_{\alpha\in\mathscr{A}}\frac{|x_\alpha|}{\Sigma_{\alpha\in\mathscr{A}}x_\alpha}\cdot x.$$

In particular, this implies that $\Sigma(X_\alpha)$ is the space of all $x\in L^0(\mu)$ such that

$$x=\sum_{\alpha\in\mathscr{A}}x_\alpha\quad(x_\alpha\in X_\alpha),\quad\text{where}\ \sum_\alpha\|x_\alpha\|_{X_\alpha}<\infty$$

equipped with the norm

$$\|x\|_{\Sigma(X_\alpha)}=\inf\Big\{\sum_{\alpha\in\mathscr{A}}\|x_\alpha\|_{X_\alpha};\ x=\sum_{\alpha\in\mathscr{A}}x_\alpha\Big\},$$

where the infimum is taken over all possible representations of x as above.

Let us note that $(\Sigma(X_\alpha), \|\cdot\|_{\Sigma(X_\alpha)})$ is the smallest Banach lattice with the property $X_\beta \hookrightarrow \Sigma(X_\alpha)$ for every $\beta \in \mathscr{A}$.

In what follows we describe Köthe duals of the Banach lattices $\Sigma(X_\alpha)$ and $\Delta(X_\alpha)$ generated by strongly compatible families $\{X_\alpha\}$ of Banach lattices on a given measure space. Recall that we require that the support of all the spaces coincide with Ω.

We start with the following lemma.

Lemma 3.1. *Let $\{X_\alpha\}_{\alpha\in\mathscr{A}}$ be a strongly compatible family of Banach lattices on a measure space $(\Omega, \mathscr{S}, \mu)$. Then the following Köthe duality formula holds*:

$$\Sigma(X_\alpha)' \cong \Delta(X_\alpha').$$

Proof. Since $X_\alpha \hookrightarrow \Sigma(X_\alpha)$, $\Sigma(X_\alpha)' \hookrightarrow X_\alpha'$ for all $\alpha \in \mathscr{A}$. This implies

$$\Sigma(X_\alpha)' \hookrightarrow \Delta(X_\alpha').$$

To prove that the converse continuous inclusion holds we fix $\varepsilon > 0$ and $f \in \Delta(X_\alpha')$. Let $g \in \Sigma(X_\alpha)$ with norm $\|g\| \le 1$. Then $|g|$ is representable in the form $|g| = \Sigma_{\alpha\in\mathscr{A}} |g_\alpha|$ (convergence in $L^0(\mu)$) with

$$\sum_{\alpha\in\mathscr{A}} \|g_\alpha\|_{X_\alpha} \le 1 + \varepsilon.$$

Thus combining with Hölder's inequality yields

$$\begin{aligned}\int_\Omega |fg|\,d\mu &\le \sum_{\alpha\in\mathscr{A}} \int_\Omega |fg_\alpha|\,d\mu \le \sum_{\alpha\in\mathscr{A}} \|f\|_{X_\alpha'} \|g_\alpha\|_{X_\alpha} \\ &\le \|f\|_{\Delta(X_\alpha')} \sum_{\alpha\in\mathscr{A}} \|g_\alpha\|_{X_\alpha} \le (1+\varepsilon)\|f\|_{\Delta(X_\alpha')}.\end{aligned}$$

This shows that $f \in \Sigma(X_\alpha)'$ and

$$\|f\|_{\Sigma(X_\alpha)'} \le (1+\varepsilon)\|f\|_{\Delta(X_\alpha')}.$$

Since $\varepsilon > 0$ was arbitrary, the proof is complete. □

The main corollary of this result is the following.

Lemma 3.2. *Let $\{X_\alpha\}_{\alpha\in\mathscr{A}}$ be a strongly compatible family of Banach lattices on a measure space $(\Omega, \mathscr{S}, \mu)$. Then the following statements are true*:

(i) $(\Sigma(X_\alpha'))'' \hookrightarrow \Delta(X_\alpha)'$.

(ii) *If X_α has the weak Fatou property for all $\alpha \in \mathscr{A}$ and* $\operatorname{supp}(\Delta(X_\alpha)) = \operatorname{supp}(\Sigma(X_\alpha))$ (*in particular if* $\operatorname{supp}(\Delta(X_\alpha)) = \Omega$), *then*

$$\Delta(X_\alpha)' \cong (\Sigma(X_\alpha'))''.$$

(iii) *If X_α is maximal for every $\alpha \in \mathscr{A}$ and $\Sigma(X'_\alpha)$ is maximal, then $\Delta(X_\alpha)$ is maximal, and*

$$\Delta(X_\alpha)' \cong \Sigma(X'_\alpha).$$

Proof. (i). From Lemma 3.1 we have

$$\Delta(X''_\alpha) \cong \Delta((X'_\alpha)') \cong (\Sigma(X'_\alpha))'. \tag{$*$}$$

Since $\Delta(X_\alpha) \hookrightarrow \Delta(X''_\alpha)$,

$$\Delta(X_\alpha) \hookrightarrow (\Sigma(X'_\alpha))'$$

and this yields the required continuous inclusion.

(ii). The weak Fatou property implies that $\Delta(X_\alpha)$ is a closed ideal in $\Delta(X''_\alpha)$ and so in $(\Sigma(X'_\alpha))'$; then our hypothesis gives (by $\operatorname{supp}(\Sigma(X'_\alpha)) = \operatorname{supp}((\Sigma(X'_\alpha))')$) order density of $\Delta(X_\alpha)$ in $(\Sigma(X'_\alpha))'$ and so

$$\Delta(X_\alpha)' \cong \Sigma(X'_\alpha)''.$$

(iii). Since $X_\alpha \cong X''_\alpha$, for every $\alpha \in \mathscr{A}$, $(*)$ gives

$$\Delta(X_\alpha) \cong (\Sigma(X'_\alpha))'.$$

In particular this implies that $\Delta(X_\alpha)$ is maximal; further our hypothesis gives the duality formula. □

A natural question appears: for which classes of strongly compatible families of Banach lattices $\{X_\alpha\}$ has the Banach lattice $\Sigma(X_\alpha)$ the Fatou property or the weak Fatou property? We shall present general cases where the answer is positive. However, we can not expect this to be true in general even in the case of countable families, as the following theorem shows.

Theorem 3.3. *For every $1 \le p < \infty$ there exists a strongly compatible family $\{X_n\}_{n\in\mathbb{N}}$ of maximal Banach lattices over $\mathbb{N}$ with the counting measure such that $X_n \simeq \ell^p$ for each $n \in \mathbb{N}$ and*

$$\Sigma(X_n) \cong c_0\,.$$

In particular, this implies that $\Sigma(X_n)$ is not a maximal Banach lattice even when all X_n are maximal Banach lattices.

Proof. We construct space X_n for each $n \in \mathbb{N}$ by renorming ℓ_p $(1 \le p < \infty)$ as follows:

$$\|x\|_{X_n} := \max_{1\le i\le n} |x_i| + \Big(\sum_{i=n+1}^{\infty} |x_i|^p\Big)^{1/p}, \quad x = (x_i) \in \ell_p.$$

Then $X_n \simeq \ell_p$, by

$$n^{-1/p}\|x\|_{\ell_p} \le 2^{1-1/p}\|x\|_{X_n} \le \|x\|_{\ell_p}.$$

It is straightforward to see that X_n is a maximal Banach lattice and $X_n \hookrightarrow c_0$ for each n.

We claim that $\Sigma(X_n) \cong c_0$. To see this, let (x_i) be a norm one element in c_0 and $\varepsilon > 0$. Consider an increasing sequence of indexes i_j such that $|x_k| \leq \varepsilon/2^j$ for all $k \geq i_j$. Take a decomposition of (x_i) as follows,

$$(x_i) = \sum_{j=1}^{\infty} y_j,$$

where $y_j = \sum_{i=i_{j-1}+1}^{i_j} x_i e_i$ for each $j \in \mathbb{N}$ with $i_0 = 0$, and (e_i) is the canonical basis of ℓ^p. Then we have

$$\sum_{j=1}^{\infty} \|y_j\|_{X_{i_j}} \leq 1 + \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = 1 + \varepsilon.$$

Since ε is arbitrary, $c_0 \hookrightarrow \Sigma(X_n)$. Clearly $\Sigma(X_n) \hookrightarrow c_0$ by $X_n \hookrightarrow c_0$ for each $n \in \mathbb{N}$ and so the claim is proved. To conclude we note that c_0 is not maximal. □

The following lemma shows that order continuity is preserved by the Σ-method.

Lemma 3.4. *If $\{X_\alpha\}_{\alpha \in \mathscr{A}}$ is a strongly compatible family of order continuous Banach lattices on $(\Omega, \mathscr{S}, \mu)$, then $\Sigma(X_\alpha)$ is also order continuous.*

Proof. Fix $f \in \Sigma(X_\alpha)$. We claim that f has order continuous norm. To show this fix a sequence (E_n) in $\mathscr{S}$ such that $E_n \searrow \emptyset$.

Given $\varepsilon > 0$ there exist sequences (α_k) in $\mathscr{A}$ and $(f_k) \in \Pi_{k=1}^{\infty} X_{\alpha_k}$ such that $|f| = \sum_{k=1}^{\infty} |f_k|$ (convergence in $L^0(\mu)$) and

$$\sum_{k=1}^{\infty} \|f_k\|_{X_{\alpha_k}} < \infty.$$

This implies that there exists $m \in \mathbb{N}$ such that

$$\sum_{k=m+1}^{\infty} \|f_k\|_{X_{\alpha_k}} \leq \varepsilon/2.$$

Since $|f\chi_{E_n}| = \sum_{k=1}^{m} |f_k \chi_{E_n}| + \sum_{k=m+1}^{\infty} |f_k \chi_{E_n}|$ for each $n \in \mathbb{N}$, the above inequality yields

$$\begin{aligned} \|f\chi_{E_n}\|_{\Sigma(X_\alpha)} &\leq \sum_{k=1}^{m} \|f_k \chi_{E_n}\|_{X_{\alpha_k}} + \sum_{k=m+1}^{\infty} \|f_k \chi_{E_n}\|_{X_{\alpha_k}} \\ &\leq \sum_{k=1}^{m} \|f_k \chi_{E_n}\|_{X_{\alpha_k}} + \varepsilon/2. \end{aligned}$$

Combining our hypothesis with the estimate above now gives that there exists n_0 such that

$$\|f\chi_{E_n}\|_{\Sigma(X_\alpha)} \leq \varepsilon, \quad n > n_0.$$

In consequence $\lim_{n\to\infty} \|f\chi_{E_n}\|_{\Sigma(X_\alpha)} = 0$ and so the claim is proved. □

Corollary 3.5. *Let $\{X_\alpha\}_{\mathscr{A}}$ be a strongly compatible family of maximal Banach lattices such that X'_α is order continuous for every $\alpha \in \mathscr{A}$. Then the following conditions are equivalent*:

(i) $\Sigma(X'_\alpha)$ *is a maximal Banach lattice.*

(ii) $\Delta(X_\alpha)' \cong \Sigma(X'_\alpha)$.

(iii) $\Delta(X_\alpha)'$ *is order continuous.*

Proof. (i)⇒(ii). From Lemma 3.2(ii) and taking into account that maximality implies the weak Fatou property we have

$$\Delta(X_\alpha)' \cong \Sigma(X'_\alpha)'' \tag{$*$}$$

and so we get the desired formula provided (i) holds. Clearly, (ii)⇒ (iii) follows by Lemma 3.4.

(iii) ⇒ (i). From the formula $(*)$, it follows that

$$(\Delta(X_\alpha)')_a \cong (\Sigma(X'_\alpha)'')_a.$$

Combining our assumption with the general fact that for any Banach function lattice X we have $X_a = (X'')_a$, we obtain

$$\Delta(X_\alpha)' \cong \Sigma(X'_\alpha).$$

Since by Lemma 3.2(iii) $\Delta(X_\alpha)$ is a maximal Banach lattice, $\Sigma(X'_\alpha)$ is also maximal. □

We now provide some sufficient geometrical conditions, which ensure that one of the three equivalent conditions of Corollary 3.5 is satisfied. We first recall that a Banach lattice X is said to satisfy an upper p-estimate (where $1 < p < \infty$), respectively a lower p-estimate, if there exists a constant C such that for any disjoint $f_1, \dots, f_n \in X$ we have

$$\Big\| \sum_{i=1}^n f_i \Big\|_X \le C \Big(\sum_{i=1}^n \|f_i\|_X^p \Big)^{1/p},$$

respectively,

$$\Big(\sum_{i=1}^n \|f_i\|_X^p \Big)^{1/p} \le C \Big\| \sum_{i=1}^n f_i \Big\|_X.$$

The smallest constant C satisfying the above inequalities is called the *upper*, respectively, *lower p-estimate constant* of X and is denoted by $u_p(X)$, respectively, $\ell_p(X)$.

It is well known (see [17, Proposition 1.f.5]) that a Banach lattice satisfies an upper, respectively, lower estimate if and only if its dual X^* satisfies a lower, respectively, upper q-estimate, where $1/p + 1/q = 1$. Notice also that a Banach lattice which satisfies a lower q-estimate is automatically both maximal and order continuous since it cannot contain an isomorphic copy of c_0.

Corollary 3.6. *Let $\{X_\alpha\}_{\mathscr{A}}$ be a strongly compatible family of maximal Banach lattices. Assume that all X_α satisfy an upper p-estimate with $\sup_{\alpha\in\mathscr{A}} u_p(X_\alpha) < \infty$. Then $\Delta(X_\alpha)$ satisfies an upper p-estimate and as a consequence,*

$$\Delta(X_\alpha)' \cong \Sigma(X'_\alpha).$$

Proof. Put $C = \sup_{\alpha\in\mathscr{A}} u_p(X_\alpha)$. Our hypothesis implies that for any disjoint $f_1,\dots,f_n \in \Delta(X_\alpha)$ and all $\alpha \in \mathscr{A}$ we have

$$\Big\|\sum_{i=1}^{n} f_i\Big\|_{X_\alpha} \le u_p(X_\alpha)\Big(\sum_{i=1}^{n} \|f_i\|_{X_\alpha}^p\Big)^{1/p} \le C\Big(\sum_{i=1}^{n} \|f_i\|_{X_\alpha}^p\Big)^{1/p}.$$

This shows that $\Delta(X_\alpha)$ satisfies an upper p-estimate with $u_p(\Delta(X_\alpha)) \le C$. Combining the above mentioned results, we conclude that the dual $\Delta(X_\alpha)^*$ and so also $\Delta(X_\alpha)'$ is order continuous and thus Corollary 3.5 applies. □

Remark 3.7. *Fix $1 \le p < \infty$ and q such that $1 = 1/p + 1/q$, let us define the family $\{Y_n\}_{n\in\mathbb{N}}$ of Banach lattices over $\mathbb{N}$ equipped with the counting measure, where Y_n is ℓ_q equipped with the norm*

$$\|x\|_{Y_n} := \max\Big\{\sum_{i=1}^{n} |x_i|, \Big(\sum_{i=n+1}^{\infty} |x_i|^q\Big)^{1/q}\Big\}, \quad x = (x_i) \in \ell_q.$$

Now observe that $Y'_n \cong X_n$ for each $n \in \mathbb{N}$, where the family $\{X_n\}_{n\in\mathbb{N}}$ was constructed in the proof of Theorem 3.3. A simple calculation shows that $\sup_{n\in\mathbb{N}} u_p(X_n) = \infty$. From Theorem 3.3, $\Sigma(X_n) \cong c_0$. Thus we conclude that the assumption $\sup_{\alpha\in\mathscr{A}} u_p(X_\alpha) < \infty$ is essential in general in Corollary 3.6, by the fact that $\Sigma(X'_\alpha)$ is a maximal Banach lattice under this requirement.

4 The duals of function spaces generated by vector measures

As it was mentioned, Banach spaces of integrable functions with respect to vector valued measures has been intensively studied in the past decades. A particular topic of interest has been the description of the Banach dual spaces. The purpose of the present section is to obtain several new results of this flavor.

Let $(\Omega,\mathscr{S})$ be a measure space and let $\nu\colon \mathscr{S} \to E$ be a countably additive vector valued measure on a Banach space E. Let λ be a Rybakov measure for ν. Throughout the rest of the paper we let $R\colon E^* \to L^1(\lambda)$ to be defined using the Radon-Nikodým derivative of $|\langle \nu, x^*\rangle|$ with respect to λ for every $x^* \in E^*$ as follows.

$$R(x^*) := \frac{d|\langle \nu, x^*\rangle|}{d\lambda} + \chi_\Omega, \quad x^* \in E^*.$$

We note here that we add the function χ_Ω in order to the supports of all the spaces appearing equal Ω, which was a requirement for the results of the previous section.

We will present examples of strongly compatible families of Banach lattices $\{X_{x^*}\}_{x^*\in S_{E^*}}$ on $(\Omega,\mathscr{S},\lambda)$ contained in the Banach lattice $L^1_w(\nu)$ for which the following Köthe duality formula holds,

$$\Delta(X_{x^*})' \cong \Sigma(X'_{x^*}).$$

Given be a Banach lattice F on $(\Omega,\mathscr{S},\lambda)$ with $\mathrm{supp}(F)=\Omega$ and a function $u\colon S_{E^*}\to L^1(\lambda)$, let $F(\varphi,u)$ be the space of all $f\in L^0(\lambda)$ such that $f\in F(u(x^*))$ for all $x^*\in S_{E^*}$ equipped with the seminorm

$$\|f\|_{F(u,\nu)} := \sup_{x^*\in S_{E^*}} \|u(x^*)f\|_F.$$

Notice that in the case $F=L^p$ for $1<p<\infty$, and $u(x^*)=|R(x^*)|^{1/p}$ for all $x^*\in S_{E^*}$ the space $F(u,\nu)=L^p(|R(\cdot)|^{1/p})$ coincides with the space $L^p_w(\nu)$ of scalarly integrable functions with respect to a vector measure ν (see [21, Ch. 3]) with equivalence of norms: if $f\in L^p_w(\nu)$,

$$\begin{aligned}\|f\| &= \sup_{x^*\in B_{E^*}} \Big(\int_\Omega |f|^p\, d|\langle \nu,x^*\rangle|\Big)^{1/p}\\ &\le \sup_{x^*\in B_{E^*}} \Big(\int_\Omega |f|^p\,(\frac{d|\langle \nu,x^*\rangle|}{d\lambda}+\chi_\Omega)d\lambda\Big)^{1/p} \le 2^{1/p}\|f\|.\end{aligned}$$

Notice also that the norm given by the new expression is also a lattice norm.

We will use the following theorem proved by Lozanovskii [18], which states that if X is a Banach function lattice on $(\Omega,\mathscr{S},\mu)$ with $\mathrm{supp}(X)=\Omega$, then every $f\in L^1(\mu)$ has, for every $\varepsilon>0$, a decomposition $f=f_0f_1$ with $f_0\in X$, $f_1\in X'$ and

$$\|f_0\|_X\,\|f_1\|_{X'} \le (1+\varepsilon)\|f\|_{L^1(\mu)}.$$

Moreover in the case when X is maximal there is a decomposition $f=f_0f_1$ such that

$$\|f_0\|_X\,\|f_1\|_{X'} = \|f\|_{L^1(\mu)}.$$

Observe that the set $\{R(x^*);\,x^*\in S_{E^*}\}$ is norm bounded in $L^1(\lambda)$ and

$$\sup_{\|x^*\|_E\le 1}\|R(x^*)\|_{L^1(\lambda)} \le \|\nu\|(\Omega)+\lambda(\Omega).$$

Thus, based on Lozanovskii's result, for a given $\varepsilon>0$ (resp., $\varepsilon=0$) we can find a decomposition such that for all $x^*\in S_{E^*}$ μ-a.e.

$$R(x^*) = R_0(x^*)\,R_1(x^*)$$

with

$$\|R_0(x^*)\|_F = 1 \quad\text{and}\quad \|R_1(x^*)\|_{F'} \le (1+\varepsilon)(\|\nu\|(\Omega)+\lambda(\Omega))$$

(resp., with

$$\|R_0(x^*)\|_F = 1 \quad \text{and} \quad \|R_1(x^*)\|_{F'} \leq \|\nu\|(\Omega) + \lambda(\Omega)$$

provided F is a maximal Banach lattice).

Notice that if F is a strictly convex maximal Banach lattice we have an unique decomposition. We include a proof of this fact for the sake of completeness.

Proposition 4.1. *If X is a strictly convex maximal Banach lattice on $(\Omega, \mathscr{S}, \mu)$ with $\operatorname{supp}(X) = \Omega$, then for every $f \in L^1(\mu)$ with $\|f\|_{L^1(\mu)} = 1$ there exist unique $0 \leq f_0 \in S_X$ and $0 \leq f_1 \in B_{X'}$ such that $|f| = f_0 f_1$.*

Proof. Let $0 \leq f_0, g_0 \in S_X$ and $0 \leq f_1, g_1 \in S_{X'}$ be such that $|f| = f_0\, f_1$ and $|f| = g_0\, g_1$. Thus if we set $g = \sqrt{f_0\, g_0}$ and $h = \sqrt{f_1\, g_1}$, then $0 \leq g \leq (f_0 + g_0)/2$, $0 \leq h \leq (f_1 + g_1)/2$ and whence $\|g\|_X \leq 1$, $\|h\|_{X'} \leq 1$. Since $|f| = gh$,

$$\|f\|_{L^1(\mu)} \leq \|g\|_X\, \|h\|_{X'}.$$

This yields $1 = \|g\|_X \leq \|(f_0 + g_0)/2\|_X$. Thus we conclude $\|f_0 + g_0\|_X = 2$ and so the strict convexity of X gives $f_0 = g_0$. This completes the proof. □

Lemma 4.2. *Let $\nu \colon \mathscr{S} \to E$ be a countably additive vector valued measure in a Banach space E and let λ be a Rybakov measure for ν. Assume that F is a Banach lattice on $(\Omega, \mathscr{S}, \lambda)$ with $\operatorname{supp}(F) = \Omega$. For a given $\varepsilon > 0$, let $R_0 \colon S_{E^*} \to F$ be the first factor of a Lozanovskii's decomposition $R \colon S_{E^*} \to L^1(\lambda)$ as above. Then we have*

$$L^\infty(\lambda) \hookrightarrow F(R_0, \nu) \overset{c}{\hookrightarrow} L^1_w(\nu),$$

where $c = (\|\nu\|(\Omega) + \lambda(\Omega))(1 + \varepsilon)$. As a consequence $\{F(R_0(x^*))\}_{x^* \in S_{E^*}}$ is a strongly compatible family of Banach lattices on $(\Omega, \mathscr{S}, \lambda)$ such that

$$\Delta(F(R_0(x^*))) \cong F(R_0, \nu).$$

Proof. Since $\|R_0(x^*)\|_F = 1$ for every $x^* \in S_{E^*}$, we have the continuous inclusion

$$L_\infty(\lambda) \hookrightarrow F(R_0, \nu).$$

Now observe that for every $f \in F(R_0, \nu)$,

$$\begin{aligned} \int_\Omega |f|\, d|\langle \nu, x^* \rangle| &\leq \int_\Omega |f| R_0(x^*) R_1(x^*)\, d\lambda \leq \|f R_0(x^*)\|_F \|R_1(x^*)\|_{F'} \\ &\leq \|f\|_{F(R_0,\nu)} (\|\nu\|(\Omega) + \lambda(\Omega))(1 + \varepsilon)\,. \end{aligned}$$

and so the second required continuous inclusion holds. As a consequence we get the second statement. □

Combining the Köthe duality results from Section 3 with Lemma 4.2 yields immediately the following result.

Theorem 4.3. *Assume that the assumptions of Lemma* 4.2 *are satisfied for a maximal Banach lattice F having a lower p-estimate for some $1 < p < \infty$. Then for the strongly compatible family $\{F(R_0(x^*))\}_{x^* \in S_{E^*}}$ of Banach lattices on $(\Omega, \mathscr{S}, \lambda)$, we have that*

$$F(R_0, \nu)' \simeq \Sigma(F'(1/R_0(x^*))).$$

As an application, we obtain the description of the dual space of L_p of a vector measure given by Galaz-Fontes in [11]. We note that in this case, $R(x^*)$ is given by $\frac{d|\langle \nu, x^* \rangle|}{d\lambda} + \chi_\Omega$ and $R_0(x^*)$ by $R(x^*)^{1/p}$ for all $x^* \in S_{E^*}$.

Corollary 4.4. *If $1 < p < \infty$ and $1/p + 1/q = 1$, then*

$$L^p_w(\nu)' \simeq \Sigma(L^q(1/R(x^*)^{1/p})).$$

In order to show how far the procedure that we have presented can be used for obtaining representations of dual spaces of vector measure integrable functions, we give in what follows the application of our results to the description of the Köthe dual of Musielak-Orlicz spaces of integrable functions with respect to vector-valued measures.

We need some definitions. A function $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ is said to be an Orlicz function if φ is a convex continuous function at zero such that $\varphi(u) = 0$ if any only if $u = 0$. Let $(\Omega, \mathscr{S}, \mu)$ be a measure space and let $\Phi \colon \mathbb{R}_+ \times \Omega \to [0, \infty]$ be an Orlicz function with parameter or a Musilak-Orlicz function, i.e., $u \mapsto \Phi(u,t)$ is an Orlicz function for μ-a.e. $t \in \Omega$ and $t \mapsto \Phi(t,u) \in L^0(\mu)$ for any $u \in \mathbb{R}_+$.

Given a Musielak-Orlicz function the Musielak-Orlicz space L^Φ is defined to be the space of all $f \in L^0(\mu)$ satisfying

$$I_\Phi(\lambda f) = \int_\Omega \Phi(\lambda |f(t)|, t)\, d\mu < \infty$$

for some $\lambda > 0$. It is well known that $L_\Phi = L_\Phi(\mu)$ is a Banach lattice equipped with the norm

$$\|f\| = \inf\big\{\varepsilon > 0; I_\Phi(f/\varepsilon) \le 1\big\}.$$

If φ is an Orlicz function, then L^φ is called an Orlicz space (we refer to [19] for more details on Musielak-Orlicz spaces). If $p \in L^0(\mu)$ and $p \ge 1$, the space generated by Musielak-Orlicz function $\Phi(u,t) := u^{p(t)}$ for all $(u,t) \in \mathbb{R}_+ \times \Omega$ is called the Nakano space.

The Orlicz spaces of vector measure integrable functions were studied in [6, 23]. In a natural way we introduce the class of the Musielak-Orlicz space of a vector measure. Let E be a real Banach space, $\nu \colon \mathscr{S} \to E$ a vector measure and λ a Rybakov measure for ν.

Let $\Phi \colon \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ be a Musielak-Orlicz function. The *Musielak-Orlicz space* $L^\Phi_w(\nu)$ *of scalarly* Φ*- integrable functions with respect to* ν is defined to be the space of all $f \in L^0(\lambda)$ such that

$$\|f\|_{L^\Phi(\nu)} := \sup_{x^* \in S_{E^*}} \|f\|_{L^\Phi(|\langle \nu, x^* \rangle|)} < \infty.$$

The closure of the set of simple functions in $L^\Phi_w(\nu)$ is said to be the Musielak-Orlicz space of Φ-*integrable* functions with respect to ν and is denoted by $L^\Phi(\nu)$.

Let us use this setting for computing the duals of Orlicz spaces of ν-integrable functions. Given an Orlicz function φ, we define a family of Musielak-Orlicz spaces $\{L^{\varphi_{x^*}}\}_{x^* \in S_{E^*}}$ on $(\Omega, \mathscr{S}, \lambda)$, where φ_{x^*} is given by $\varphi_{x^*}(u,t) := \varphi(u) R(x^*)(t)$ for all $(u,t) \in \mathbb{R}_+ \times \Omega$, where R is the same function used above in this section. It is straightforward to see that $L^\varphi(\nu) \cong \Delta(L^{\varphi_{x^*}})$ by

$$\|f\|_{L^\varphi(\nu)} \leq \|f\|_{\Delta(L^{\varphi_{x^*}})} \leq 2\|f\|_{L^\varphi(\nu)}, \quad f \in L^\varphi(\nu).$$

Theorem 4.5. *Let φ be an Orlicz function such that the Young conjugate function φ^* satisfies condition Δ_2 at ∞ (i.e., there exist $C > 0$ and $u_0 \geq 0$ such that $\varphi(2u) \leq C\varphi(u)$ for all $u \leq u_0$). Then the following duality formulas hold*:

$$L^\varphi_w(\nu)' \simeq \Sigma((L^{\varphi_{x^*}})'), \quad L^\varphi(\nu)^* \simeq \Sigma((L^{\varphi_{x^*}})'),$$

where $\{\varphi_{x^*}\}_{x^* \in S_{E^*}}$ is the family of functions defined above.

Proof. Notice first that if an Orlicz function ψ satisfies the Δ_2-condition, then there exist $1 < q < \infty$, $C = C(\psi) > 0$ and $u_0 \geq 0$ such that $\psi(\lambda u) \geq C\lambda^q \psi(u)$ for all $\lambda \geq 1$ and $u \geq u_0$. It is well known that this inequality implies that an Orlicz space L^ψ on any finite measure space satisfies a lower q-estimate (see [17]).

We now consider the family $\{L^{\varphi_{x^*}}\}_{x^* \in S_{E^*}}$ of Musielak-Orlicz spaces on the finite measure space $(\Omega, \mathscr{S}, \lambda)$. Applying the above fact, we conclude by duality that there exists $1 < p < \infty$ such that all $L^{\varphi_{x^*}}$ satisfy an upper p-estimate with constant uniformly bounded. Since Musielak-Orlicz spaces are maximal, the first result follows from Corollary 3.5.

To conclude, observe that $L^\varphi(\nu)$ is order continuous for any Orlicz function (see [6, Proposition 4.1]) and that $L^\infty(\nu) \hookrightarrow L^\varphi(\nu)$. Thus the order density implies

$$L^\varphi_w(\nu)' \cong L^\varphi(\nu)' \simeq L^\varphi(\nu)^*.$$

This completes the proof. □

Let us recall how to give an explicit description of the dual spaces $(L^{\varphi_{x^*}})'$ that appear in the representation given by Theorem 4.5 of the dual of the Orlicz spaces $L^\Phi(\nu)$. Let Φ be a Musielak-Orlicz function $\Phi\colon \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ (for instance the functions $\varphi_{x^*}(u,t) = \varphi(u)R(x^*)(t)$ appearing in the theorem). Then $(L^\Phi)' \cong L^{\Phi^*}_0$, where $L^{\Phi^*}_0$ is the Musielak-Orlicz space L^{Φ^*} equipped with the Orlicz norm ([19, 22])

$$\|f\| = \inf_{k>0} \frac{1 + I_{\Phi^*}(kf)}{k}, \quad f \in L^{\Phi^*}.$$

Here as usual Φ^* is the Young conjugate of Φ defined by the formula:

$$\Phi^*(u,t) = \sup\{us - \Phi(s,t);\, s \geq 0\}, \quad (u,t) \in \mathbb{R}_+ \times \Omega.$$

The general analysis of the case of the Musielak-Orlicz spaces of a vector measure with lower p-estimates as well as with upper p-estimates (see [12]) yields variants of duality results for Musielak-Orlicz spaces $L_w^{\phi}(\nu)$ and $L^{\Phi}(\nu)$ and is left to the reader. However, notice that the same technique that has been shown above for Orlicz spaces just changing adequately the Musielak-Orlicz functions appearing there gives the result for this general case. We only state here the result for Nakano spaces for finishing the paper. To analyze the corresponding duality formulas for this case we need to observe that if $1 \leq p \leq p(t)$ μ-a.e. (resp., $p(t) \leq q < \infty$ μ-a.e.), then the Nakano space $L^{p(t)}$ satisfies an upper p-estimate (resp., a lower q-estimate) (see [12]). Below the corresponding Nakano spaces generated by a vector measure are denoted by $L_w^{p(t)}(\nu)$ and $L^{p(t)}(\nu)$, respectively.

Theorem 4.6. *Let $\nu\colon \mathscr{S} \to E$ be a countably additive vector valued measure in a Banach space E and let λ be a Rybakov measure for ν. Assume that $p(\cdot) \in L^0(\lambda)$ satisfies $1 < \operatorname{essinf}_{t\in\Omega} p(t)$ and $\operatorname{essup}_{t\in\Omega} p(t) < \infty$. Then the following duality formulas hold*:

$$L_w^{p(t)}(\nu)' \cong \Sigma((L^{\Phi_{x^*}})'), \quad L^{p(t)}(\nu)^* \simeq \Sigma((L^{\Phi_{x^*}})'),$$

where $\{\Phi_{x^*}\}$ is the family of Musielak-Orlicz functions defined by $\Phi_{x^*}(u,t) = u^{p(t)}\, R(x^*)(t)$ for all $(u,t) \in \mathbb{R}_+ \times \Omega$ and $x^* \in S_{E^*}$.

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