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On the Construction of Analytic-Numerical Approximations for a Class of Coupled Differential Models in Engineering

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Abstract

In this paper, a method to construct an analytic-numerical solution for homogeneous parabolic coupled systems with homogeneous boundary conditions of the type $u_t = Au_{xx}$, $A_1u(0,t)+B_1u_x(0,t)=0$, $A_2u(1,t)+B_2u_x(1,t)=0$, $0<x<1$, $t>0$, $u(x,0)=f(x)$, where A is a positive stable matrix and A_1, A_2, B_1, B_2 are arbitrary matrices for which the block matrix $\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}$ is non-singular, is proposed.

Keywords

Coupled Diffusion Problems, Coupled Boundary Conditions, Vector Boundary-Value Differential Systems, Sturm-Liouville Vector Problems, Analytic-Numerical Solution

1. Introduction

Coupled partial differential systems with coupled boundary-value conditions are frequent in different areas of science and technology, as in scattering problems in Quantum Mechanics [1]-[3], in Chemical Physics [4]-[6], coupled diffusion problems [7]-[9], modelling of coupled thermoelastoplastic response of clays subjected to

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nuclear waste heat [10], etc. The solution of these problems has motivated the study of vector and matrix Sturm-Liouville problems, see [11]-[14] for example.

Recently [15] [16], an exact series solution for the homogeneous initial-value problem

$$u_t(x,t) - Au_{xx}(x,t) = 0, \quad 0 < x < 1, \quad t > 0 \tag{1}$$

$$A_1 u(0,t) + B_1 u_x(0,t) = 0, \quad t > 0 \tag{2}$$

$$A_2 u(1,t) + B_2 u_x(1,t) = 0, \quad t > 0 \tag{3}$$

$$u(x,0) = f(x), \quad 0 \le x \le 1 \tag{4}$$

where $u = (u_1, u_2, \cdots, u_m)^T$ and $f(x) = (f_1(x), f_2(x), \cdots, f_m(x))^T$ are a m-dimensional vectors, was constructed under the following hypotheses and notation:

1. The matrix coefficient A is a matrix which satisfies the following condition

$$\mathrm{Re}(z) > 0, \quad \forall z \in \sigma(A) \tag{5}$$

where $\sigma(C)$ denotes the set of all the eigenvalues of a matrix C in $\mathbb{C}^{m\times m}$. Thus, A is a *positive stable matrix* (where $\mathrm{Re}(z)$ denotes the real part of $z \in \mathbb{C}$).

2. Matrices A_i, B_i, $i = 1, 2$, are $m \times m$ complex matrices, and we assume that the block matrix

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \text{ is regular,} \tag{6}$$

and also that the matrix pencil

$$A_1 + \rho B_1 \text{ is regular.} \tag{7}$$

Condition (7) is well known in the literature of singular systems of differential equations, see [17], and involves the existence of some $\rho_0 \in \mathbb{C}$ so that matrix $A_1 + \rho_0 B_1$ is invertible. In this case, matrix $A_1 + \rho B_1$ is invertible with the possible exception of at most a finite number of complex numbers ρ. In particular, we may assume that $\rho_0 \in \mathbb{R}$.

Using condition (7) we can introduce the following matrices $\tilde{A}_1$ and $\tilde{B}_1$ defined by

$$\tilde{A}_1 = (A_1 + \rho_0 B_1)^{-1} A_1, \quad \tilde{B}_1 = (A_1 + \rho_0 B_1)^{-1} B_1 \tag{8}$$

which satisfy the condition $\tilde{A}_1 + \rho_0 \tilde{B}_1 = I$, where matrix I denotes, as usual, the identity matrix. Under hypothesis (6), is it easy to show that matrix $B_2 - (A_2 + \rho_0 B_2)\tilde{B}_1$ is regular (see [18] for details) and we can introduce matrices $\tilde{A}_2$ and $\tilde{B}_2$ defined by

$$\tilde{A}_2 = \left[B_2 - (A_2 + \rho_0 B_2)\tilde{B}_1\right]^{-1} A_2, \quad \tilde{B}_2 = \left[B_2 - (A_2 + \rho_0 B_2)\tilde{B}_1\right]^{-1} B_2 \tag{9}$$

that satisfy the conditions $\tilde{B}_2 - (\tilde{A}_2 + \rho_0 \tilde{B}_2)\tilde{B}_1 = I$, $\tilde{B}_2\tilde{A}_1 - \tilde{A}_2\tilde{B}_1 = I$.

Under the above assumptions, the homogeneous problem (1)-(4) was solved in [15] [16] in two different cases:

(a) If we consider the following hypotheses:

$$\text{exist } b_1 \in \sigma(\tilde{B}_1) - \{0\},\ b_2 \in \sigma(\tilde{B}_2), \text{ and } v \in \mathbb{C}^m - \{0\}, \text{ such that } (\tilde{B}_1 - b_1 I)v = (\tilde{B}_2 - b_2 I)v = 0 \tag{10}$$

Then, if the vector valued function $f(x)$ satisfies hypotheses

$$\left.\begin{array}{c} f \in C^2([0,1]) \\ (1-\rho_0 b_1) f(0) + b_1 f'(0) = 0 \\ -\left(\dfrac{1 - b_2 + \rho_0 b_1 b_2}{b_1}\right) f(1) + b_2 f'(1) = 0 \end{array}\right\} \tag{11}$$

with the additional condition:

$$f(x) \in \mathrm{Ker}\left(\tilde{B}_1 - b_1 I\right) \bigcap \mathrm{Ker}\left(\tilde{B}_2 - b_2 I\right), \quad 0 \le x \le 1$$
$$\text{and} \tag{12}$$
$$\mathrm{Ker}\left(\tilde{B}_1 - b_1 I\right) \bigcap \mathrm{Ker}\left(\tilde{B}_2 - b_2 I\right) \text{ is an invariant subspace with respect to matrix } A,$$

where a subspace E of $\mathbb{C}^m$ is invariant by the matrix $A \in \mathbb{C}^{m\times m}$ if $A(E) \subset E$, we can construct an exact series solution $u(x,t)$ of homogeneous problem (1)-(4). This construction was made in Ref. [15].

(b) If we consider the following hypotheses:

$$0 \in \sigma\left(\tilde{B}_1\right),\ a_2 \in \sigma\left(\tilde{A}_2\right), \text{ and we have } w \in \mathbb{C}^m - \{0\}, \text{ so that } \tilde{B}_1 w = \left(\widetilde{A_2} - a_2 I\right) w = 0 \tag{13}$$

Then, if the vector valued function $f(x)$ satisfies the hypotheses

$$\left.\begin{array}{r} f \in \mathcal{C}^2\left([0,1]\right) \\ f(0) = 0 \\ a_2 f(1) + f'(1) = 0 \end{array}\right\} \tag{14}$$

under the additional condition:

$$f(x) \in \mathrm{Ker}\left(\tilde{B}_1\right) \cap \mathrm{Ker}\left(\tilde{A}_2 - a_2 I\right), \quad 0 \le x \le 1$$
$$\text{and} \tag{15}$$
$$\mathrm{Ker}\left(\tilde{B}_1\right) \cap \mathrm{Ker}\left(\tilde{A}_2 - a_2 I\right) \text{ is an invariant subspace respect to matrix } A,$$

then we can construct an exact series solution $u(x,t)$ of homogeneous problem (1)-(4). This construction was made in Ref. [16].

Observe that under the different hypotheses (a) and (b), the exact solution of problem (1)-(1) is given by the series

$$u(x,t) = \alpha\left(\left(1 - \rho_0 b_1\right) x - b_1\right) C(0) + \sum_{\lambda_n \in \mathcal{F}} \mathrm{e}^{-\lambda_n^2 A t} X_{\lambda_n}(x) C\left(\lambda_n\right), \quad x \in [0,1], \quad t \ge 0 \tag{16}$$

where, under hypothesis (a), the value of α is given by

$$\alpha = \begin{cases} 1 & \text{if } \dfrac{\left(1 - b_2 + \rho_0 b_1 b_2\right)\left(1 - \rho_0 b_1\right)}{b_1} = 1 \\ 0 & \text{if } \dfrac{\left(1 - b_2 + \rho_0 b_1 b_2\right)\left(1 - \rho_0 b_1\right)}{b_1} \ne 1 \end{cases} \tag{17}$$

and $\mathcal{F}$ is the set of eigenvalues $\lambda_n \in \left(n\pi, (n+1)\pi\right)$, where λ_n is the solution of the equation

$$\lambda \cot(\lambda) = \frac{\left(1 - b_2 + \rho_0 b_1 b_2\right)\left(1 - \rho_0 b_1\right)}{b_1} - b_1 b_2 \lambda^2 \tag{18}$$

with an additional solution $\lambda_0 \in (0, \pi)$ if

$$\frac{\left(1 - b_2 + \rho_0 b_1 b_2\right)\left(1 - \rho_0 b_1\right)}{b_1} < 1 \tag{19}$$

and under hypothesis (b), the value of α is given by

$$\alpha = \begin{cases} 1 & \text{if } -a_2 = 1 \\ 0 & \text{if } -a_2 \ne 1 \end{cases} \tag{20}$$

and $\mathcal{F}$ is the set of eigenvalues $\lambda_n \in (n\pi,(n+1)\pi)$, where λ_n is the solution of the equation

$$\lambda \cot(\lambda) = -a_2 \tag{21}$$

with an additional solution $\lambda_0 \in (0,\pi)$ if

$$-a_2 < 1 \tag{22}$$

Under both hypotheses (a) and (b), the value of $X_{\lambda_n}(x)$, $C(\lambda_n)$ and $C(0)$ are given by

$$X_{\lambda_n}(x) = \left((1-\rho_0 b_1)\sin(\lambda_n x) - b_1\lambda_n \cos(\lambda_n x)\right) \tag{23}$$

$$C(\lambda_n) = \frac{\int_0^1 \left((1-\rho_0 b_1)\sin(\lambda_n x) - b_1\lambda_n \cos(\lambda_n x)\right) f(x)\,\mathrm{d}x}{\int_0^1 \left((1-\rho_0 b_1)\sin(\lambda_n x) - b_1\lambda_n \cos(\lambda_n x)\right)^2 \mathrm{d}x} \tag{24}$$

and

$$C(0) = \frac{\int_0^1 \left((1-\rho_0 b_1)x - b_1\right) f(x)\,\mathrm{d}x}{\int_0^1 \left((1-\rho_0 b_1)x - b_1\right)^2 \mathrm{d}x} \tag{25}$$

taking $b_1 = 0$ in Formulaes (23)-(25) if we consider hypothesis (b).

The series solution of problem (1)-(4) given in (16) presents some computational difficulties:

(a) The infiniteness of the series.

(b) Eigenvalues λ_n are not exactly computable because Equation (18) (or Equation (21) under hypothesis (b) holds) is not solvable in a closed form, although well known and efficient algorithms for approximation, see references [13] [19] [20].

(c) Other problem is the calculation of the matrix exponential, which may present difficulties, see [21] [22] for example.

For this reason we propose in this paper to solve the following problem:

Given an admissible error $\varepsilon > 0$ and a bounded subdomain $D[t_0,t_1] = [0,1]\times[t_0,t_1]$, $t_0 > 0$. How do we construct an approximation that avoids the above-quoted difficulties and whose error with respect to the exact solution (16) is less than ε uniformly in $D[t_0,t_1]$?

This paper deals with the construction of analytic-numerical solutions of problem (1)-(4) in a subdomain $D[t_0,t_1] = [0,1]\times[t_0,t_1]$, $t_0 > 0$, with *a priori* error $\varepsilon > 0$. The work is organized as follows: in Section 2 we construct the approximate solution. In Section 3 we will introduce an algorithm and give an illustrative example.

Throughout this paper we will assume the results and nomenclature given in [15] [16]. If $B = (b_{ij})$ is a matrix in $\mathbb{C}^{m\times m}$, its 2-norm denoted by $\|B\|$ is defined by ([23], p. 56)

$$\|B\| = \sup_{z\neq 0} \frac{\|Bz\|_2}{\|z\|_2}$$

where for a vector y in $\mathbb{C}^m$, $\|z\|_2$ is the usual euclidean norm of y, and the 2-norm satisfies

$$\max_{i,j} |b_{ij}| \le \|B\| \le m \max_{i,j} |b_{ij}|$$

Let us introduce the notation

$$\alpha(C) = \max\{\mathrm{Re}(z); z \in \sigma(C)\} \tag{26}$$

and by ([23], p. 556) it follows that

$$\left\|\mathrm{e}^{tB}\right\| \le \mathrm{e}^{\alpha(B)t} \sum_{k=0}^{m-1} \frac{\left\|\sqrt{m}B\right\|^k t^k}{k!} \tag{27}$$

2. The Proposed Approximation

Let $(x,t)\in D[t_0,t_1]=[0,1]\times[t_0,t_1]$, $t_0>0$, be and we take an admissible error $\varepsilon>0$. Observe first that given (24), using Parseval's identity for scalar Sturm-Liouville problems, see [24] and ([11], p. 223), one gets that

$$\left\|C(\lambda_n)\right\|^2\le\int_0^1\left\|f(x)\right\|^2\mathrm{d}x,\quad \lambda_n\in\mathcal{F}$$

Thus, we can take a positive constant $M>0$, defined by

$$M=\int_0^1\left\|f(x)\right\|^2\mathrm{d}x \tag{28}$$

satisfying

$$\left\|C(\lambda_n)\right\|^2\le M,\quad \lambda_n\in\mathcal{F} \tag{29}$$

Moreover, by (23), we have

$$\left|X_{\lambda_n}(x)\right|^2=\left|\left((1-\rho_0 b_1)\sin(\lambda_n x)-b_1\lambda_n\cos(\lambda_n x)\right)\right|^2\le\left|1-\rho_0 b_1\right|^2+\left|b_1\right|^2\lambda_n^2+2\left|1-\rho_0 b_1\right|\left|b_1\right|\lambda_n.$$

If we define $\beta>0$ by

$$\beta=\max\left\{\left|1-\rho_0 b_1\right|^2,\left|b_1\right|^2,\left|1-\rho_0 b_1\right|\left|b_1\right|\right\} \tag{30}$$

we have that

$$\left|X_{\lambda_n}(x)\right|^2\le\beta(1+\lambda_n)^2,\quad \lambda_n\in\mathcal{F} \tag{31}$$

On the other hand, we know from (27) that

$$\left\|\mathrm{e}^{-A\lambda_n^2 t}\right\|\le\mathrm{e}^{-\alpha(A)\lambda_n^2 t}\sum_{k=0}^{m-1}\frac{\left\|\sqrt{m}A\right\|^k t^k\lambda_n^{2k}}{k!}$$

where, as $\lambda_n\ge1$, $n\ge1$, we have for $t\in[t_0,t_1]$:

$$\left\|\mathrm{e}^{-A\lambda_n^2 t}\right\|^2\le\mathrm{e}^{-2\alpha(A)\lambda_n^2 t_0}\left(\sum_{k=0}^{m-1}\frac{\left\|\sqrt{m}A\right\|^k t_1^k}{k!}\right)^2\lambda_n^{4m-4}=L^2\lambda_n^{4m-4}\mathrm{e}^{-2\alpha(A)\lambda_n^2 t_0}=L^2\lambda_n^{-4}\left(\lambda_n^{4m}\mathrm{e}^{-2\alpha(A)\lambda_n^2 t_0}\right) \tag{32}$$

where

$$L=\sum_{k=0}^{m-1}\frac{\left\|\sqrt{m}A\right\|^k t_1^k}{k!}>0 \tag{33}$$

Observe that for a fixed $m\ge0$ the numerical series $\sum_{\lambda_n\in\mathcal{F}}\lambda_n^{4m}\mathrm{e}^{-2\alpha(A)\lambda_n^2 t_0}$ is convergent, because using Lemma 1 of Ref. [15] if hypothesis (a) holds, or Lemma 2 of Ref. [16] if hypothesis (b) holds, one gets $\lim_{n\to\infty}\lambda_n=\infty$, $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n)=\pi$, and by application of D'Alembert's criterion for series:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\left(\frac{\lambda_{n+1}}{\lambda_n}\right)^{4m}\mathrm{e}^{-2\alpha(A)t_0\left(\lambda_{n+1}^2-\lambda_n^2\right)}\le\lim_{n\to\infty}\mathrm{e}^{-\alpha(A)t_0\left(\lambda_{n+1}^2-\lambda_n^2\right)}\left(\frac{n+2}{n}\right)^{4m}=\mathrm{e}^{\lim_{n\to\infty}-\alpha(A)t_0\pi(\lambda_{n+1}+\lambda_n)}=0$$

then

$$\lim_{n\to\infty}\lambda_n^{4m}\mathrm{e}^{-2\alpha(A)\lambda_n^2 t_0}=0. \tag{34}$$

Taking into account that $(1+\lambda_n)^2>1$ and $M>0$, $\beta>0$, it follows that

$$\frac{1}{M\beta(1+\lambda_n)^2 L^2} < \frac{1}{M\beta L^2} < 1 \tag{35}$$

and by (34) there is a positive integer n_0 so that

$$\lambda_n^{4m} \mathrm{e}^{-2\alpha(A)\lambda_n^2 t_0} < \frac{1}{M\beta(1+\lambda_n)^2 L^2}, \quad \forall n \geq n_0 \tag{36}$$

Using (29), (31), (32) and (36), if $n \geq n_0$, we have

$$\left\| \mathrm{e}^{-A\lambda_n^2 t} X_{\lambda_n}(x) C(\lambda_n) \right\|^2 \leq \left\| \mathrm{e}^{-A\lambda_n^2 t} \right\|^2 \left| X_{\lambda_n}(x) \right|^2 \left\| C(\lambda_n) \right\|^2 = M\beta(1+\lambda_n)^2 L^2 \lambda_n^{-4} \left(\lambda_n^{4m} \mathrm{e}^{-2\alpha(A)\lambda_n^2 t_0} \right) \leq \lambda_n^{-4}$$

As eigenvalues $\lambda_n \in (n\pi, (n+1)\pi)$, then, for $n > 1$ it follows that

$$\frac{1}{\lambda_n^4} < \frac{1}{n^4} \tag{37}$$

Taking into account that $\sum_{n\geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}$, from (37) one gets that

$$\left\| \sum_{\substack{\lambda_n \in \mathcal{F} \\ n \geq n_0}} \mathrm{e}^{-A\lambda_n^2 t} X_{\lambda_n}(x) C(\lambda_n) \right\|^2 \leq \sum_{\substack{\lambda_n \in \mathcal{F} \\ n \geq n_0}} \left\| \mathrm{e}^{-A\lambda_n^2 t} X_{\lambda_n}(x) C(\lambda_n) \right\|^2 \tag{38}$$

$$\leq \sum_{\substack{\lambda_n \in \mathcal{F} \\ n \geq n_0}} \frac{1}{\lambda_n^4}$$

$$\leq \sum_{n \geq n_0} \frac{1}{n^4}$$

$$= \frac{\pi^4}{90} - \sum_{n=1}^{n_0} \frac{1}{n^4} \tag{39}$$

We take the first positive integer n_1 so that

$$\sum_{n=1}^{n_1} \frac{1}{n^4} \geq \frac{\pi^4}{90} - \frac{\varepsilon}{3}, \quad n_1 \geq n_0 \tag{40}$$

We define the vector valued function $u(x,t,n_1)$ as

$$u(x,t,n_1) = \alpha\left((1-\rho_0 b_1) x - b_1 \right) C(0) + \sum_{\substack{\lambda_n \in \mathcal{F} \\ n \leq n_1}} \mathrm{e}^{-\lambda_n^2 A t} X_{\lambda_n}(x) C(\lambda_n), \quad (x,t) \in D[t_0, t_1] \tag{41}$$

Using (38) one gets that

$$\left\| u(x,t) - u(x,t,n_1) \right\|^2 \leq \left\| \sum_{\substack{\lambda_n \in \mathcal{F} \\ n \geq n_1}} \mathrm{e}^{-\lambda_n^2 A t} X_{\lambda_n}(x) C(\lambda_n) \right\|^2$$

$$\leq \frac{\pi^4}{90} - \sum_{n=1}^{n_1} \frac{1}{n^4}$$

$$\leq \frac{\pi^4}{90} - \frac{\pi^4}{90} + \frac{\varepsilon}{3}$$

$$= \frac{\varepsilon}{3},$$

thus

$$\left\| u(x,t) - u(x,t,n_1) \right\|^2 \le \frac{\varepsilon}{3}, \quad (x,t) \in D[t_0,t_1] \tag{42}$$

Remark 1. *Note that to determine the positive integer* n_0 *we need to check condition* (36), *which requires knowledge the exact eigenvalues* λ_n. *From Ref.* [15] [16] *it is well know that* $\lambda_n \in (n\pi,(n+1)\pi)$, *then*

$$\lambda_n^{4m} \mathrm{e}^{-2\alpha(A)\lambda_n^2 t_0} < (n\pi)^{4m} \mathrm{e}^{-2\alpha(A)n^2\pi^2 t_0}$$

and by (35), *we can replace condition* (36) *by take the first positive integer* n_0 *satisfying*

$$(n\pi)^{4m} \mathrm{e}^{-2\alpha(A)n^2\pi^2 t_0} < \frac{1}{M\beta L^2}. \tag{43}$$

Approximation $u(x,t,n_1)$ defined by (41) involves computation of the exact eigenvalues λ_n, $n \le n_1$ which is not easy in practice. Now we study the admissible tolerance when one considers approximate eigenvalues $\widetilde{\lambda_n}$, $n \le n_1$ in expression (41), taking

$$\tilde{u}(x,t,n_1) = \alpha\left((1-\rho_0 b_1)x - b_1\right)C(0) + \sum_{n \le n_1} \mathrm{e}^{-\widetilde{\lambda_n}^2 At} X_{\widetilde{\lambda_n}}(x) C\left(\widetilde{\lambda_n}\right) \tag{44}$$

where

$$X_{\widetilde{\lambda_n}}(x) = \left((1-\rho_0 b_1)\sin\left(\widetilde{\lambda_n}x\right) - b_1\widetilde{\lambda_n}\cos\left(\widetilde{\lambda_n}x\right)\right), \quad x \in [0,1] \tag{45}$$

$$C\left(\widetilde{\lambda_n}\right) = \frac{\int_0^1 \left((1-\rho_0 b_1)\sin\left(\widetilde{\lambda_n}x\right) - b_1\widetilde{\lambda_n}\cos\left(\widetilde{\lambda_n}x\right)\right) f(x)\,\mathrm{d}x}{\int_0^1 \left((1-\rho_0 b_1)\sin\left(\widetilde{\lambda_n}x\right) - b_1\widetilde{\lambda_n}\cos\left(\widetilde{\lambda_n}x\right)\right)^2 \mathrm{d}x} \tag{46}$$

with $C(0)$ defined by (25). Note that

$$\begin{aligned}
&\mathrm{e}^{-\widetilde{\lambda_n}^2 At} X_{\widetilde{\lambda_n}}(x) C\left(\widetilde{\lambda_n}\right) - \mathrm{e}^{-\lambda_n^2 At} X_{\lambda_n}(x) C(\lambda_n) \\
&= \left(\mathrm{e}^{-\widetilde{\lambda_n}^2 At} - \mathrm{e}^{-\lambda_n^2 At}\right)\left\{(1-\rho_0 b_1)\sin\left(\widetilde{\lambda_n}x\right) - b_1\widetilde{\lambda_n}\cos\left(\widetilde{\lambda_n}x\right)\right\} C\left(\widetilde{\lambda_n}\right) \\
&\quad + \mathrm{e}^{-\lambda_n^2 At}\left\{(1-\rho_0 b_1)\sin\left(\widetilde{\lambda_n}x\right) - b_1\widetilde{\lambda_n}\cos\left(\widetilde{\lambda_n}x\right) - (1-\rho_0 b_1)\sin(\lambda_n x) + b_1\lambda_n\cos(\lambda_n x)\right\} C\left(\widetilde{\lambda_n}\right) \\
&\quad + \mathrm{e}^{-\lambda_n^2 At}\left\{(1-\rho_0 b_1)\sin(\lambda_n x) - b_1\lambda_n\cos(\lambda_n x)\right\}\left(C\left(\widetilde{\lambda_n}\right) - C(\lambda_n)\right).
\end{aligned} \tag{47}$$

It is easy to see that

$$\left|(1-\rho_0 b_1)\sin\left(\widetilde{\lambda_n}x\right) - b_1\widetilde{\lambda_n}\cos\left(\widetilde{\lambda_n}x\right)\right| \le |1-\rho_0 b_1| + |b_1|\widetilde{\lambda_n}, \tag{48}$$

$$\left|(1-\rho_0 b_1)\sin(\lambda_n x) - b_1\lambda_n\cos(\lambda_n x)\right| \le |1-\rho_0 b_1| + |b_1|\lambda_n, \tag{49}$$

and

$$\begin{aligned}
&\left|(1-\rho_0 b_1)\sin\left(\widetilde{\lambda_n}x\right) - b_1\widetilde{\lambda_n}\cos\left(\widetilde{\lambda_n}x\right) - (1-\rho_0 b_1)\sin(\lambda_n x) + b_1\lambda_n\cos(\lambda_n x)\right| \\
&\le \left(|1-\rho_0 b_1| + |b_1|(1+\lambda_n)\right)\left|\lambda_n - \widetilde{\lambda_n}\right|.
\end{aligned} \tag{50}$$

Replacing in (47) and taking norms, one gets

$$\begin{aligned}
\left\|\mathrm{e}^{-\widetilde{\lambda_n}^2 At} X_{\widetilde{\lambda_n}}(x) C\left(\widetilde{\lambda_n}\right) - \mathrm{e}^{-\lambda_n^2 At} X_{\lambda_n}(x) C(\lambda_n)\right\| &\le \left\|\mathrm{e}^{-\widetilde{\lambda_n}^2 At} - \mathrm{e}^{-\lambda_n^2 At}\right\|\left(|1-\rho_0 b_1| + |b_1|\widetilde{\lambda_n}\right)\left\|C\left(\widetilde{\lambda_n}\right)\right\| \\
&\quad + \left\|\mathrm{e}^{-\lambda_n^2 At}\right\|\left(\left(|1-\rho_0 b_1| + |b_1|(1+\lambda_n)\right)\left|\lambda_n - \widetilde{\lambda_n}\right|\right)\left\|C\left(\widetilde{\lambda_n}\right)\right\| \\
&\quad + \left\|\mathrm{e}^{-\lambda_n^2 At}\right\|\left(|1-\rho_0 b_1| + |b_1|\lambda_n\right)\left\|C\left(\widetilde{\lambda_n}\right) - C(\lambda_n)\right\|.
\end{aligned} \tag{51}$$

We define $I(\rho)$ for $\rho > 0$ by

$$I(\rho) = \int_0^1 \left((1 - \rho_0 b_1) \sin(\rho x) - b_1 \rho \cos(\rho x) \right)^2 \mathrm{d}x \tag{52}$$

by applying the Cauchy-Schwarz inequality for integrals and (28), one gets:

$$\int_0^1 \| f(x) \| \mathrm{d}x \le \left(\int_0^1 \| f(x) \|^2 \mathrm{d}x \right)^{\frac{1}{2}} = \sqrt{M}$$

We have

$$\begin{aligned}
\left\| C\left(\widetilde{\lambda_n}\right) \right\| &\le \frac{1}{I\left(\widetilde{\lambda_n}\right)} \int_0^1 \left| (1 - \rho_0 b_1) \sin\left(\widetilde{\lambda_n} x\right) - b_1 \widetilde{\lambda_n} \cos\left(\widetilde{\lambda_n} x\right) \right| \| f(x) \| \mathrm{d}x \\
&\le \frac{1}{I\left(\widetilde{\lambda_n}\right)} \int_0^1 \left(|1 - \rho_0 b_1| + |b_1| \widetilde{\lambda_n} \right) \| f(x) \| \mathrm{d}x \\
&\le \frac{|1 - \rho_0 b_1| + |b_1| \widetilde{\lambda_n}}{I\left(\widetilde{\lambda_n}\right)} \int_0^1 \| f(x) \| \mathrm{d}x \\
&\le \frac{|1 - \rho_0 b_1| + |b_1| \widetilde{\lambda_n}}{I\left(\widetilde{\lambda_n}\right)} \sqrt{M}.
\end{aligned}$$

Taking $\gamma > 0$ satisfying

$$\min_{n \le n_1} \left\{ I(\rho), \rho = \lambda_n, \rho = \widetilde{\lambda_n} \right\} \ge 1/\gamma \tag{53}$$

it follows that

$$\left\| C\left(\widetilde{\lambda_n}\right) \right\| \le \gamma \left(|1 - \rho_0 b_1| + |b_1| \widetilde{\lambda_n} \right) \sqrt{M}. \tag{54}$$

Moreover, working component by component:

$$C\left(\widetilde{\lambda_n}\right)_i - C(\lambda_n)_i \tag{55}$$

$$\begin{aligned}
&= \frac{\int_0^1 X_{\widetilde{\lambda_n}}(x) f_i(x) \mathrm{d}x}{I\left(\widetilde{\lambda_n}\right)} - \frac{\int_0^1 X_{\lambda_n}(x) f_i(x) \mathrm{d}x}{I(\lambda_n)} \\
&= \frac{I(\lambda_n) \int_0^1 X_{\widetilde{\lambda_n}}(x) f_i(x) \mathrm{d}x - I\left(\widetilde{\lambda_n}\right) \int_0^1 X_{\lambda_n}(x) f_i(x) \mathrm{d}x}{I\left(\widetilde{\lambda_n}\right) I(\lambda_n)} \\
&= \frac{\left(I(\lambda_n) - I\left(\widetilde{\lambda_n}\right) \right) \int_0^1 X_{\widetilde{\lambda_n}}(x) f_i(x) \mathrm{d}x - I\left(\widetilde{\lambda_n}\right) \int_0^1 \left(X_{\lambda_n}(x) - X_{\widetilde{\lambda_n}}(x) \right) f_i(x) \mathrm{d}x}{I\left(\widetilde{\lambda_n}\right) I(\lambda_n)}
\end{aligned} \tag{56}$$

Applying the Cauchy-Schwarz inequality for integrals again:

$$\int_0^1 \left| X_{\widetilde{\lambda_n}}(x) f_i(x) \right| \mathrm{d}x \le \left(\int_0^1 |f_i(x)|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_0^1 \left| X_{\widetilde{\lambda_n}}(x) \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} = \left(\int_0^1 |f_i(x)|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(I\left(\widetilde{\lambda_n}\right) \right)^{\frac{1}{2}} \tag{57}$$

and

$$\int_0^1 \left| \left(X_{\lambda_n}(x) - X_{\widetilde{\lambda_n}}(x) \right) f_i(x) \right| \mathrm{d}x \tag{58}$$

$$\le \left(|1 - \rho_0 b_1| + |b_1| (1 + \lambda_n) \right) \left| \lambda_n - \widetilde{\lambda_n} \right| \int_0^1 |f_i(x)| \mathrm{d}x$$

$$\leq \left(\left|1-\rho_0 b_1\right|+\left|b_1\right|\left(1+\lambda_n\right)\right)\left|\lambda_n-\widetilde{\lambda_n}\right|\left(\int_0^1\left|f_i(x)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} \tag{59}$$

By (55) and taking into account (57) and (58):

$$\begin{aligned}
&\left|C\left(\widetilde{\lambda_n}\right)_i-C\left(\lambda_n\right)_i\right| \\
&\leq \frac{1}{I\left(\widetilde{\lambda_n}\right)I\left(\lambda_n\right)}\left(\left|I\left(\lambda_n\right)-I\left(\widetilde{\lambda_n}\right)\right|\int_0^1\left|X_{\widetilde{\lambda_n}}(x)f_i(x)\right|\mathrm{d}x+I\left(\widetilde{\lambda_n}\right)\int_0^1\left|\left(X_{\lambda_n}(x)-X_{\widetilde{\lambda_n}}(x)\right)f_i(x)\right|\mathrm{d}x\right) \\
&=\frac{\left(\int_0^1\left|f_i(x)\right|^2\mathrm{d}x\right)^{\frac{1}{2}}}{I\left(\lambda_n\right)}\left(\frac{\left|I\left(\lambda_n\right)-I\left(\widetilde{\lambda_n}\right)\right|}{\left(I\left(\widetilde{\lambda_n}\right)\right)^{\frac{1}{2}}}+\left|I\left(\lambda_n\right)-I\left(\widetilde{\lambda_n}\right)\right|\left(\left|1-\rho_0 b_1\right|+\left|b_1\right|\left(1+\lambda_n\right)\right)\right).
\end{aligned} \tag{60}$$

Note that from the definition of $I(\rho)$, (52), it follows that

$$\left|I\left(\lambda_n\right)-I\left(\widetilde{\lambda_n}\right)\right|\leq\left(\left|1-\rho_0 b_1\right|+\left|b_1\right|\left(1+\widetilde{\lambda_n}\right)\right)\left(2\left|1-\rho_0 b_1\right|+\left|b_1\right|\left(\widetilde{\lambda_n}+\lambda_n\right)\right)\left|\widetilde{\lambda_n}-\lambda_n\right| \tag{61}$$

then, replacing in (60) one gets

$$\begin{aligned}
\left|C\left(\widetilde{\lambda_n}\right)_i-C\left(\lambda_n\right)_i\right|\leq&\frac{\left(\int_0^1\left|f_i(x)\right|^2\mathrm{d}x\right)^{\frac{1}{2}}}{I\left(\lambda_n\right)}\left\{\left(\left|1-\rho_0 b_1\right|+\left|b_1\right|\left(1+\lambda_n\right)\right)\right. \\
&\left.+\left(\left|1-\rho_0 b_1\right|+\left|b_1\right|\left(1+\widetilde{\lambda_n}\right)\right)\left(2\left|1-\rho_0 b_1\right|+\left|b_1\right|\left(\widetilde{\lambda_n}+\lambda_n\right)\right)\left(I\left(\widetilde{\lambda_n}\right)\right)^{-\frac{1}{2}}\right\}\left|\widetilde{\lambda_n}-\lambda_n\right|.
\end{aligned} \tag{62}$$

We take

$$\Lambda\geq\max_{n\leq n_1}\left\{\lambda_n,\widetilde{\lambda_n}\right\} \tag{63}$$

then, if we define

$$\mathcal{A}=\left|1-\rho_0 b_1\right|+\left|b_1\right|\left(1+\Lambda\right),\quad \mathcal{B}=2\left|1-\rho_0 b_1\right|+2\left|b_1\right|\Lambda \tag{64}$$

from (54) we have that

$$\left\|C\left(\widetilde{\lambda_n}\right)\right\|\leq\gamma\mathcal{A}\sqrt{M} \tag{65}$$

and from (62) and (53):

$$\begin{aligned}
\left|C\left(\widetilde{\lambda_n}\right)_i-C\left(\lambda_n\right)_i\right|&\leq\frac{\left(\int_0^1\left|f_i(x)\right|^2\mathrm{d}x\right)^{\frac{1}{2}}}{I\left(\lambda_n\right)}\left\{\mathcal{A}+\mathcal{A}\mathcal{B}\left(I\left(\widetilde{\lambda_n}\right)\right)^{-\frac{1}{2}}\right\}\left|\widetilde{\lambda_n}-\lambda_n\right| \\
&=\left(\int_0^1\left|f_i(x)\right|^2\mathrm{d}x\right)^{\frac{1}{2}}\gamma\left\{\mathcal{A}+\mathcal{A}\mathcal{B}\gamma^{\frac{1}{2}}\right\}\left|\widetilde{\lambda_n}-\lambda_n\right| \\
&\leq\sqrt{M}\gamma\left\{\mathcal{A}+\mathcal{A}\mathcal{B}\gamma^{\frac{1}{2}}\right\}\left|\widetilde{\lambda_n}-\lambda_n\right|.
\end{aligned} \tag{66}$$

Using the 2-norm properties, from (66) we have

$$\left\|C\left(\widetilde{\lambda_n}\right)-C\left(\lambda_n\right)\right\|\leq\left(\int_0^1\left|f_i(x)\right|^2\mathrm{d}x\right)^{\frac{1}{2}}\gamma\left\{\mathcal{A}+\mathcal{A}\mathcal{B}\gamma^{\frac{1}{2}}\right\}\left|\widetilde{\lambda_n}-\lambda_n\right| \tag{67}$$

By other hand, we can write

$$\mathrm{e}^{-\tilde{\lambda}_n^2 At} - \mathrm{e}^{-\lambda_n^2 At} = \mathrm{e}^{-\tilde{\lambda}_n^2 At}\left(\mathrm{e}^{\left(\lambda_n^2 - \tilde{\lambda}_n^2\right)At} - I\right)$$

where taking norm, applying (32) and (33) together the mean value theorem, under the hypothesis $\left|\tilde{\lambda}_n - \lambda_n\right| < 1$, one gets

$$\begin{aligned}
\left\|\mathrm{e}^{-\tilde{\lambda}_n^2 At} - \mathrm{e}^{-\lambda_n^2 At}\right\| &\le \left\|\mathrm{e}^{-\tilde{\lambda}_n^2 At}\right\|\left(\mathrm{e}^{\left(\lambda_n^2 - \tilde{\lambda}_n^2\right)\|A\|t_1} - 1\right) \\
&\le \mathrm{e}^{-t_0\alpha(A)\Lambda^2}\left(\sum_{k=0}^{m-1}\frac{\left(\sqrt{m}\Lambda^2\|A\|t_1\right)^k}{k!}\right)\left(\mathrm{e}^{\left(\lambda_n^2 - \tilde{\lambda}_n^2\right)\|A\|t_1} - 1\right) \\
&\le \mathrm{e}^{-t_0\alpha(A)\Lambda^2} L^\star t_1 \|A\| 4\Lambda \mathrm{e}^{t_1\|A\|2\Lambda}\left|\tilde{\lambda}_n - \lambda_n\right|.
\end{aligned}$$

where

$$L^\star = \sum_{k=0}^{m-1}\frac{\left(\sqrt{m}\Lambda^2\|A\|t_1\right)^k}{k!} > 0 \tag{68}$$

Replacing in (51) we obtain

$$\begin{aligned}
\left\|\mathrm{e}^{-\tilde{\lambda}_n^2 At} X_{\tilde{\lambda}_n}(x) C\left(\tilde{\lambda}_n\right) - \mathrm{e}^{-\lambda_n^2 At} X_{\lambda_n}(x) C\left(\lambda_n\right)\right\| &\le \mathrm{e}^{-t_0\alpha(A)\Lambda^2} L^\star t_1 \|A\| 4\Lambda \mathrm{e}^{t_1\|A\|2\Lambda}\gamma\sqrt{M}\mathcal{A}^2\left|\tilde{\lambda}_n - \lambda_n\right| \\
&\quad + \mathrm{e}^{-t_0\alpha(A)\Lambda^2} L\gamma\sqrt{M}\mathcal{A}^2\left|\tilde{\lambda}_n - \lambda_n\right| \\
&\quad + \mathrm{e}^{-t_0\alpha(A)\Lambda^2} L\mathcal{A}\sqrt{M}\gamma\left\{\mathcal{A} + \mathcal{A}\mathcal{B}\gamma^{\frac{1}{2}}\right\}\left|\tilde{\lambda}_n - \lambda_n\right| \\
&= S\left|\tilde{\lambda}_n - \lambda_n\right|,
\end{aligned} \tag{69}$$

where

$$S = \mathcal{A}^2\gamma\sqrt{M}\mathrm{e}^{-t_0\alpha(A)\Lambda^2}\left(L + L\left(1 + \mathcal{B}\sqrt{\gamma}\right) + L^\star 4t_1\|A\|\Lambda \mathrm{e}^{2t_1\|A\|\Lambda}\right) \tag{70}$$

Given $\varepsilon > 0$ and n_1, consider approximations $\tilde{\lambda}_n$ of λ_n for $n \le n_1$ satisfiying

$$\left|\tilde{\lambda}_n - \lambda_n\right| < \min_{n \le n_1}\left\{1, \frac{\sqrt{\varepsilon}}{\sqrt{3n_1 S}}\right\} \tag{71}$$

then

$$\begin{aligned}
\left\|u\left(x,t,n_1\right) - \tilde{u}\left(x,t,n_1\right)\right\| &= \left\|\sum_{n \le n_1}\left(\mathrm{e}^{-\tilde{\lambda}_n^2 At} X_{\tilde{\lambda}_n}(x) C\left(\tilde{\lambda}_n\right) - \mathrm{e}^{-\lambda_n^2 At} X_{\lambda_n}(x) C\left(\lambda_n\right)\right)\right\| \\
&\le \sum_{n \le n_1}\left\|\mathrm{e}^{-\tilde{\lambda}_n^2 At} X_{\tilde{\lambda}_n}(x) C\left(\tilde{\lambda}_n\right) - \mathrm{e}^{-\lambda_n^2 At} X_{\lambda_n}(x) C\left(\lambda_n\right)\right\| \\
&\le \sum_{n \le n_1} S\left|\tilde{\lambda}_n - \lambda_n\right| \\
&< S n_1 \frac{\sqrt{\varepsilon}}{\sqrt{3n_1 S}} \\
&= \frac{\sqrt{\varepsilon}}{\sqrt{3}},
\end{aligned}$$

and therefore

$$\left\| u(x,t,n_1) - \tilde{u}(x,t,n_1) \right\|^2 \le \frac{\varepsilon}{3}, \quad (x,t) \in D[t_0,t_1]. \tag{72}$$

Remark 2. *From* (61), *and taking into account the definition of* $\mathcal{A}$ *and* $\mathcal{B}$ *given in* (64), *it follows that*

$$\left| I(\lambda_n) - I(\widetilde{\lambda_n}) \right| \le \mathcal{AB} \left| \widetilde{\lambda_n} - \lambda_n \right|$$

so that, if $\left| \widetilde{\lambda_n} - \lambda_n \right|$ *is enough small, it can take* $I(\lambda_n) \approx I(\widetilde{\lambda_n})$ *in the computation of* γ.

Similarly, can be taken in practice

$$\Lambda \ge \max_{1 \le n \le n_1} \left\{ \widetilde{\lambda_n} \right\} \tag{73}$$

instead of the definition (63).

Approximation $\tilde{u}(x,t,n_1)$ need to compute the exact value of the matrix exponential $e^{-\widetilde{\lambda_n}^2 At}$. However, the approximate calculation of the exponential matrix $e^{-\widetilde{\lambda_n}^2 At}$ can be performed by methods such as those based on the Taylor series, [25] [26], based on Hermite matrix polynomials, [27], and other existing methods in the literature, see [22] [23] for example. Suppose we take the matrix $\mathrm{App}\left(e^{-\widetilde{\lambda_n}^2 At}\right)$ as an approximation of matrix $e^{-\widetilde{\lambda_n}^2 At}$, so that

$$\left\| e^{-\widetilde{\lambda_n}^2 At} - \mathrm{App}\left(e^{-\widetilde{\lambda_n}^2 At}\right) \right\| \le \varepsilon_n t_1, \quad t \in [t_0,t_1], \ \ \varepsilon_n > 0, \ \ n \le n_1 \tag{74}$$

We define the approximation $\mathcal{U}(x,t,n_1)$ by:

$$\mathcal{U}(x,t,n_1) = \alpha\left((1-\rho_0 b_1)x - b_1\right)C(0) + \sum_{n \le n_1} \mathrm{App}\left(e^{-\widetilde{\lambda_n}^2 At}\right) X_{\widetilde{\lambda_n}}(x) C\left(\widetilde{\lambda_n}\right) \tag{75}$$

and from (65), (64) and (45) one gets that

$$\begin{aligned} \left\| \tilde{u}(x,t,n_1) - \mathcal{U}(x,t,n_1) \right\| &\le \left\| \sum_{n \le n_1} \left(e^{-\widetilde{\lambda_n}^2 At} - \mathrm{App}\left(e^{-\widetilde{\lambda_n}^2 At}\right) \right) \right\| \left\| X_{\widetilde{\lambda_n}}(x) \right\| \left\| C\left(\widetilde{\lambda_n}\right) \right\| \\ &\le \sum_{n \le n_1} \left\| e^{-\widetilde{\lambda_n}^2 At} - \mathrm{App}\left(e^{-\widetilde{\lambda_n}^2 At}\right) \right\| \gamma \mathcal{A}^2 \sqrt{M} \\ &\le \gamma \mathcal{A}^2 \sqrt{M} t_1 \sum_{n \le n_1} \varepsilon_n . \end{aligned}$$

We take

$$\mathcal{K} = \max_{1 \le n \le n_1} \{\varepsilon_n\} \tag{76}$$

and suppose we make the approximation accurate enough satisfying condition

$$\mathcal{K} < \frac{\sqrt{\varepsilon}}{\sqrt{3} n_1 t_1 \gamma \mathcal{A}^2 \sqrt{M}} \tag{77}$$

Thus, if $\mathcal{K}$ satisfies (77) it follows that

$$\left\| \tilde{u}(x,t,n_1) - \mathcal{U}(x,t,n_1) \right\|^2 \le \frac{\varepsilon}{3}, \tag{78}$$

and from (42), (72) and (78):

$$\begin{aligned}\left\|u(x,t)-\mathcal{U}(x,t,n_1)\right\|^2 &= \left\|u(x,t)-u(x,t,n_1)+u(x,t,n_1)-\tilde{u}(x,t,n_1)+\tilde{u}(x,t,n_1)-\mathcal{U}(x,t,n_1)\right\|^2 \\ &\le \left\|u(x,t)-u(x,t,n_1)\right\|^2+\left\|u(x,t,n_1)-\tilde{u}(x,t,n_1)\right\|^2+\left\|\tilde{u}(x,t,n_1)-\mathcal{U}(x,t,n_1)\right\|^2 \\ &\le \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} \\ &= \varepsilon.\end{aligned}$$

Summarizing, the following results has been established:

Theorem 1. *We consider problem* (1)-(4) *satisfying hypotheses* (5), (6) *and* (7). *Let* $\varepsilon > 0$, $D[t_0,t_1]=[0,1]\times[t_0,t_1]$. *Suppose that the hypothesis* (a) *is verified, this ensures that there is an exact solution* $u(x,t)$ *of problem* (1)-(4), *see Ref.* [15]. *Let* α, $\alpha(A)$, M, β *and* L *be the constant defined by* (17), (26), (28), (30) *and* (68) *respectively. Let* n_0 *and* n_1 *be positive integers satisfying conditions* (43) *and* (40). *Let* $\widetilde{\lambda_n}$ *be the* n_1*-first approximate roots of the equation* (18), *each one in the interval* $(n\pi,(n+1)\pi)$, $n \le n_1$, *and let* $\widetilde{\lambda_0}$ *be the approximation of the additional solution* $\lambda_0 \in (0,\pi)$ *to be consider if condition* (19) *holds. Let* $\gamma > 0$ *be satisfying* (53) *and let* Λ, $\mathcal{A}$, $\mathcal{B}$ *and* $L^\star$ *be the positive constants defined by* (63), (64) *and* (68) *respectively. Suppose that the approximations* $\widetilde{\lambda_n}$ *satisfy* (71), *where* S *is the constant defined by* (70). *Suppose that the approximations* $\operatorname{App}\left(e^{-\widetilde{\lambda_n}^2 At}\right)$ *of matrices* $e^{-\widetilde{\lambda_n}^2 At}$, *for* $n \le n_1$ *satisfy that the approximation error is less than* $\mathcal{K}$, *where* $\mathcal{K}$ *is a positive constant which satisfies* (77). *Consider the functions* $X_{\widetilde{\lambda_n}}(x)$, $n \le n_1$ *defined by* (45) *and vectors* $C\left(\widetilde{\lambda_n}\right)$, $n \le n_1$, *defined by* (46), *joint the vector* $C(0)$ *defined by (24) if* $\alpha \ne 0$. *Then, the vector valued function* $\mathcal{U}(x,t,n_1)$ *defined by* (75) *satisfies*

$$\left\|u(x,t)-\mathcal{U}(x,t,n_1)\right\|^2 \le \varepsilon, \quad (x,t)\in D[t_0,t_1]$$

Theorem 2. *We consider problem* (1)-(4) *satisfying hypotheses* (5), (6) *and* (7). *Let* $\varepsilon > 0$, *and we consider the subdomain* $D[t_0,t_1]=[0,1]\times[t_0,t_1]$. *Suppose that the hypothesis* (b) *is verified, this ensures that there is an exact solution* $u(x,t)$ *of problem* (1)-(4), *see Ref.* [16]. *Let* α, $\alpha(A)$, M *and* L *be the constant defined by* (20), (26), (28) *and* (68) *respectively. Let* n_0 *and* n_1 *be positive integers satisfying conditions* (43) *and* (40). *Take* $\beta = 1$ *and* $b_1 = 0$. *Let* $\widetilde{\lambda_n}$ *be the* n_1*-first approximate roots of the equation* (21), *each one in the interval* $(n\pi,(n+1)\pi)$, $n \le n_1$, *and let* $\widetilde{\lambda_0}$ *be the approximation of the additional solution* $\lambda_0 \in (0,\pi)$ *to be consider if condition* (22) *holds. Let* $\gamma > 0$ *be satisfying* (53) *and let* Λ, $\mathcal{A}$, $\mathcal{B}$ *and* $L^\star$ *be the positive constants defined by* (63), (64) *and* (68) *respectively. Suppose that the approximations* $\widetilde{\lambda_n}$ *satisfy* (71), *where* S *is the constant defined by* (70). *Suppose that the approximations* $\operatorname{App}\left(e^{-\widetilde{\lambda_n}^2 At}\right)$ *of matrices* $e^{-\widetilde{\lambda_n}^2 At}$, *for* $n \le n_1$ *satisfy that the approximation error is less than* $\mathcal{K}$, *where* $\mathcal{K}$ *is a positive constant which satisfies* (77). *Consider the functions* $X_{\widetilde{\lambda_n}}(x)$, $n \le n_1$ *defined by* (45) *and vectors* $C\left(\widetilde{\lambda_n}\right)$, $n \le n_1$, *defined by* (46), *joint the vector* $C(0)$ *defined by (24) if* $\alpha \ne 0$. *Then, the vector valued function* $\mathcal{U}(x,t,n_1)$ *defined by* (75) *satisfies*

$$\left\|u(x,t)-\mathcal{U}(x,t,n_1)\right\|^2 \le \varepsilon, \quad (x,t)\in D[t_0,t_1]$$

3. Algorithm 1, Algorithm 2 and Example

We can give the following algorithms, according to the hypothesis (a) or (b) is satisfied, to construct the approximation $\mathcal{U}(x,t,n_1)$.

Algorithm 1. Construction of the analytic-numerical solution of problem (1)-(4) under hypotheses (a) in the subdomain $D[t_0,t_1]=[0,1]\times[t_0,t_1]$, $t_0>0$, with *a priori* error bound $\varepsilon>0$.

1: Compute the constant ρ_0 satisfying (7).

2: Determine b_1 and b_2 satisfying (10). Compute constant α defined by (17).

3: Compute constants $\|A\|$, $\alpha(A)$, M, β, L defined by (26), (28), (30) and (68) respectively.

4: Determine the first positive integer n_0 which satisfies (43).

5: Determine the first positive integer n_1 which satisfies (40).

6: Determine approaches $\widetilde{\lambda}_n$ of the n_1-first roots of Equation (18) each one in the interval $(k\pi,(k+1)\pi)$, $k\le n_1$, joint the approximation of the additional solution $\lambda_0\in(0,\pi)$ if condition (19) holds.

7: Compute $I(\rho)$ for $\rho=\widetilde{\lambda}_n$, $n\le n_1$ and determine $\gamma>0$ satisfying (53).

8: Compute Λ, $\mathcal{A}$, $\mathcal{B}$ and L^{*} defined by (63), (64) and (68) respectively.

9: Compute S defined by (70)

10: Check that approximations $\widetilde{\lambda}_n$ satisfy (71). Otherwise return to step 6 and calculate approximations $\widetilde{\lambda}_n$ more precisely.

11: Compute $\mathcal{K}$ satisfying (77).

12: Compute approximations $\mathrm{App}\left(\mathrm{e}^{-\widetilde{\lambda}_n^2 At}\right)$ of matrices $\mathrm{e}^{-\widetilde{\lambda}_n^2 At}$, for $n\le n_1$ so that the error in each one approach is less than $\mathcal{K}$.

13: Compute functions $X_{\widetilde{\lambda}_n}(x)$, $n\le n_1$, defined by (45).

14: Compute vectors $C(\widetilde{\lambda}_n)$, $n\le n_1$, defined by (46). If $\alpha\ne 0$, compute $C(0)$ defined by (24).

15: Compute the approximation $\mathcal{U}(x,t,n_1)$ defined by (75).

Algorithm 2. Construction of the analytic-numerical solution of problem (1)-(4) under hypotheses (b) in the subdomain $D[t_0,t_1]=[0,1]\times[t_0,t_1]$, $t_0>0$, with *a priori* error bound $\varepsilon>0$.

1: Compute the constant ρ_0 satisfying (7).

2: Determine a_2 satisfying (13). Compute constant α defined by (20). Take $b_1=0$ and $\beta=1$.

3: Compute constants $\|A\|$, $\alpha(A)$, M, L defined by (26), (28) and (68) respectively.

4: Determine the first positive integer n_0 which satisfies (43).

5: Determine the first positive integer n_1 which satisfies (40).

6: Determine approaches $\widetilde{\lambda}_n$ of the n_1-first roots of Equation (21) each one in the interval $(k\pi,(k+1)\pi)$, $k\le n_1$, joint the approximation of the additional solution $\lambda_0\in(0,\pi)$ if condition (22) holds.

Continue with the step 7 of Algorithm 1

Example 1. We will construct an approximate solution in the subdomain $D[0,1]=[0,1]\times[0.1,1]$, with a priori error bound $\varepsilon=10^{-2}$, of the homogeneous parabolic problem with homogeneous conditions (1)-(4), where the matrix $A\in\mathbb{C}^{4\times4}$ is chosen

$$A=\begin{pmatrix}2&0&0&-1\\1&2&1&-2\\-1&0&2&1\\0&0&0&1\end{pmatrix} \tag{79}$$

and the 4×4 matrices A_i, B_i, $i\in\{1,2\}$, are

$$A_1=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad A_2=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix},$$
$$B_1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad B_2=\begin{pmatrix}1&0&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \tag{80}$$

Also, the vectorial valued function $f(x)$ will be defined as

$$f(x)=\begin{pmatrix}0\\x^2-1\\0\\0\end{pmatrix} \tag{81}$$

This is precisely the example 1 of Ref. [15] whose exact solution is given by:

$$u(x,t)=\left(\sum_{n\geq 0}-\frac{32(-1)^n \mathrm{e}^{-\frac{\pi}{2}(2n+1)^2 t}\cos\left(\frac{\pi}{2}(2n+1)x\right)}{\pi^3(2n+1)^3}\right)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \tag{82}$$

We will follow algorithm 1 step by step:

1. Hypothesis (a) holds with $m=4$. Note that although A_1 is singular, taking $\rho_0=1\in\mathbb{R}$, the matrix pencil

$$A_1+\rho_0 B_1=I_{4\times 4} \tag{83}$$

is regular. Therefore, we take $\rho_0=1$.

2. Performing calculations similar to those made in Ref. [15], one gets that $b_1=1$, $b_2=0$ and $\alpha=0$.

3. It is easy to calculate $\|A\|=3.67571$, $\sigma(A)=\{1,2\}$, thus $\alpha(A)=2$. Similarly $M=8/15$, $\beta=1$ and $L=101.589$.

4. Note that

$$\frac{1}{M\beta L^2}=0.00018168.$$

Then, by (43):

$$n=3\Rightarrow(3\pi)^{16}\mathrm{e}^{-0.4(3^2)\pi^2}=1.43749>\frac{1}{M\beta L^2},$$
$$n=4\Rightarrow(4\pi)^{16}\mathrm{e}^{-0.4(4^2)\pi^2}=1.428708\times 10^{-10}<\frac{1}{M\beta L^2},$$

then we take $n_0=4$.

5. We have

$$n=4\Rightarrow\left(\sum_{k=1}^{4}\frac{1}{k^4}-\frac{\pi^4}{90}+\frac{10^{-2}}{3}\right)=-0.000237971,$$
$$n=5\Rightarrow\left(\sum_{k=1}^{5}\frac{1}{k^4}-\frac{\pi^4}{90}+\frac{10^{-2}}{3}\right)=0.00136203,$$

then we can take $n_1=5>n_0=4$.

6. We need to determinate the n_1-first roots of equation

$$\lambda \cot(\lambda) = 0$$

We can solve exactly this equation, $\lambda_n = \frac{\pi}{2} + n\pi$, $n = 1, \cdots, 5$, with an additional solution $\lambda_0 \in]0, \pi[$, because

$$\frac{(1 - b_2 + \rho_0 b_1 b_2)(1 - \rho_0 b_1)}{b_1} = 0 < 1$$

and then $\lambda_0 = \frac{\pi}{2}$.

In summary, $\lambda_0 = \frac{\pi}{2}$, $\lambda_1 = \frac{3\pi}{2}$, $\lambda_2 = \frac{5\pi}{2}$, $\lambda_3 = \frac{7\pi}{2}$, $\lambda_4 = \frac{9\pi}{2}$, $\lambda_5 = \frac{11\pi}{2}$. We take the approximate values (50 exact decimal)

$$\widetilde{\lambda_0} = 1.5707963267948966192313216916397514420985846996876,$$
$$\widetilde{\lambda_1} = 4.7123889803846898576939650749192543262957540990627,$$
$$\widetilde{\lambda_2} = 7.8539816339744830961566084581987572104929234984378,$$
$$\widetilde{\lambda_3} = 10.995574287564276334619251841478260094690092897813,$$
$$\widetilde{\lambda_4} = 14.137166941154069573081895224757762978887262297188,$$
$$\widetilde{\lambda_5} = 17.278759594743862811544538608037265863084431696563.$$

7. We calculate $I(\rho)$ for $\rho = \widetilde{\lambda_n}$:

$$I\left(\widetilde{\lambda_0}\right) = 1.2337005501361698273543113749845188919142124259051,$$
$$I\left(\widetilde{\lambda_1}\right) = 11.103304951225528446188802374860670027227911833146,$$
$$I\left(\widetilde{\lambda_2}\right) = 30.842513753404245683857784374612972297855310647627,$$
$$I\left(\widetilde{\lambda_3}\right) = 60.451326956672321540361257374241425703796408869350,$$
$$I\left(\widetilde{\lambda_4}\right) = 99.929744561029756015699221373746030245051206498313,$$
$$I\left(\widetilde{\lambda_5}\right) = 149.27776656647654910987167637312678592161970353452.$$

the smallest of them is $I\left(\widetilde{\lambda_0}\right)$, as $1/\widetilde{\lambda_0} \approx 0.810569$, we take $\gamma = 0.82$.

8. We have that $\Lambda = 17.3 > \widetilde{\lambda_5}$, $\mathcal{A} = 18.3$, $\mathcal{B} = 34.6$ and $L^\star = 1.77759 \times 10^9$.

9. We have that $S = 1.56631 \times 10^{43}$.

10. To be applicable the algorithm 1, the approximations $\widetilde{\lambda_n}$ may satisfy:

$$\left|\lambda_n - \widetilde{\lambda_n}\right| < \min\left\{1, \frac{\sqrt{\varepsilon}}{\sqrt{3 n_1 S}}\right\} = 7.37211 \times 10^{-46}$$

As the roots were calculated with 50 decimal accurate, we accept these approximations of the roots.

11. We have to take $\mathcal{K}$ satisfying (77). In our case

$$\mathcal{K} < \frac{\sqrt{\varepsilon}}{\sqrt{3 n_1 t_1 \gamma \mathcal{A}^2 \sqrt{M}}} = 0.0000472137.$$

12. We have to compute approximations $\mathrm{App}\left(e^{-\lambda_n^2 A t}\right)$ of matrices $e^{-\lambda_n^2 A t}$, for $n = 0, 1, 2, 3, 4, 5$ with a maximum error $\mathcal{K}$. In this case, using minimal theorem ([28], p. 571), we can determine the exact value of e^{As} given by:

$$
\mathrm{e}^{As} = \begin{pmatrix} \mathrm{e}^{-2s} & 0 & 0 & \mathrm{e}^{-2s}\left(-1+\mathrm{e}^{s}\right) \\ -\frac{1}{2}\mathrm{e}^{-2s}s(2+s) & \mathrm{e}^{-2s} & -\mathrm{e}^{-2s}s & \frac{1}{2}\mathrm{e}^{-2s}\left(-2+2\mathrm{e}^{s}+2s+s^{2}\right) \\ \mathrm{e}^{-2s}s & 0 & \mathrm{e}^{-2s} & -\mathrm{e}^{-2s}s \\ 0 & 0 & 0 & \mathrm{e}^{-s} \end{pmatrix} \tag{84}
$$

then, we can obtain $\mathrm{App}\left(\mathrm{e}^{-\widetilde{\lambda}_n{}^2 At}\right)$ for $n = 0,1,2,3,4,5$ replacing in (84).

13. Functions $X_{\widetilde{\lambda}_n}(x)$, $n = 0,1,\cdots,4$, defined by (45) are given by:

$$X_{\widetilde{\lambda}_0}(x) = -1.5707963267948966192\cos(1.5707963267948966192x),$$

$$X_{\widetilde{\lambda}_1}(x) = -4.7123889803846898577\cos(4.7123889803846898577x),$$

$$X_{\widetilde{\lambda}_2}(x) = -7.8539816339744830962\cos(7.8539816339744830962x),$$

$$X_{\widetilde{\lambda}_3}(x) = -10.995574287564276335\cos(10.995574287564276335x),$$

$$X_{\widetilde{\lambda}_4}(x) = -14.137166941154069573\cos(14.137166941154069573x),$$

$$X_{\widetilde{\lambda}_5}(x) = -17.278759594743862812\cos(17.278759594743862812x).$$

14. Vectors $C\left(\widetilde{\lambda}_n\right)$, $n = 1,\cdots,5$, defined by (46) are given by:

$$C\left(\widetilde{\lambda}_0\right) = \begin{pmatrix} 0 \\ 0.65702286429979745210577812909559642508 \\ 0 \\ 0 \end{pmatrix},$$

$$C\left(\widetilde{\lambda}_1\right) = \begin{pmatrix} 0 \\ -0.0081113933864172524951330633221678570997 \\ 0 \\ 0 \end{pmatrix},$$

$$C\left(\widetilde{\lambda}_2\right) = \begin{pmatrix} 0 \\ 0.0010512365828796759233692450065529542801 2 \\ 0 \\ 0 \end{pmatrix},$$

$$C\left(\widetilde{\lambda}_3\right) = \begin{pmatrix} 0 \\ -0.00027364550782998644402573016622057327 16 \\ 0 \end{pmatrix},$$

$$C\left(\widetilde{\lambda}_4\right) = \begin{pmatrix} 0 \\ 0.00010014065909157101845843288052059082 84 \\ 0 \\ 0 \end{pmatrix},$$

$$C\left(\widetilde{\lambda}_5\right) = \begin{pmatrix} 0 \\ -0.000044875545679926060522216933890826885 12 \\ 0 \\ 0 \end{pmatrix}.$$

We don't compute $C(0)$ defined by (25) because $\alpha = 0$.

15. Compute $\mathcal{U}(x,t,n_1)$ defined by (75), obtaining:

$$\mathcal{U}(x,t,5) = \begin{pmatrix} 0 \\ \mathcal{W}(x,t) \\ 0 \\ 0 \end{pmatrix}$$

where

$$\begin{aligned} \mathcal{W}(x,t) = & -1.03204910\mathrm{e}^{-4.93480220t}\cos(1.57079633x) + 0.0382240408\mathrm{e}^{-44.4132198t}\cos(4.71238898x) \\ & -0.00825639281\mathrm{e}^{-123.370055t}\cos(7.85398163x) + 0.00300888951\mathrm{e}^{-241.805308t}\cos(10.9955743x) \\ & -0.00141570522\mathrm{e}^{-399.718978t}\cos(14.1371669x) + 0.000775393765\mathrm{e}^{-597.111066t}\cos(17.2787596x). \end{aligned}$$

and our approximation satisfies

$$\left\| u(x,t) - \mathcal{U}(x,t,5) \right\|^2 < 10^{-2}, \quad (x,y) \in D[0.1,1]$$

As an example, consider the point $(x,t) = (0.27, 0.9) \in D[0.1,1]$. We have the approximation

$$\mathcal{U}(0.27,0.9,5) = \begin{pmatrix} 0 \\ -0.0110808 \\ 0 \\ 0 \end{pmatrix}$$

It is easy to check that, from (82), one gets

$$\left\| u(0.27,0.9) - \mathcal{U}(0.27,0.9,5) \right\| < 10^{-18}$$

4. Conclusion

In this paper, a method to construct an analytic-numerical solution for homogeneous parabolic coupled systems with homogeneous boundary conditions of the type (1)-(4) has been presented. An algorithm with an illustrative example is given.

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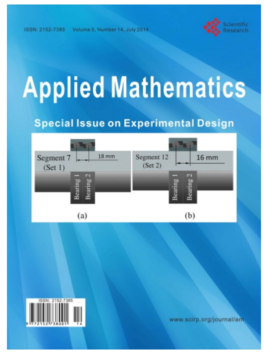
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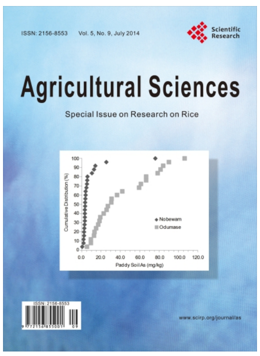
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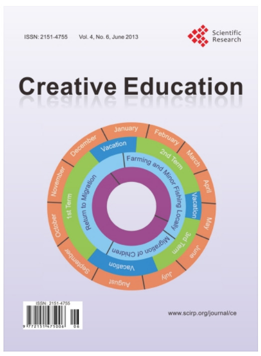
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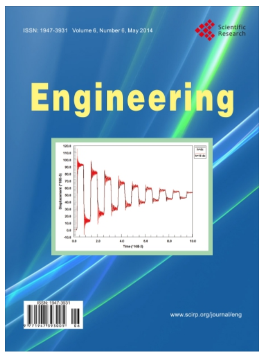
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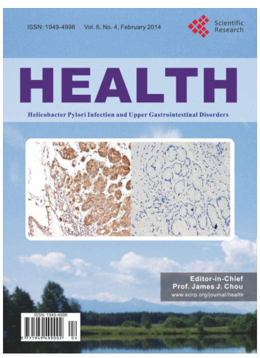
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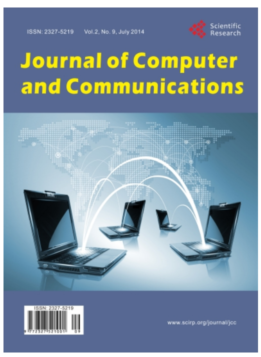
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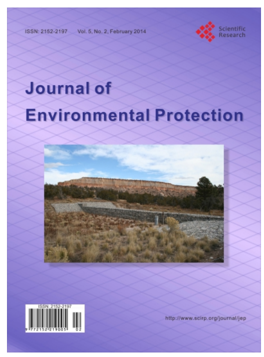
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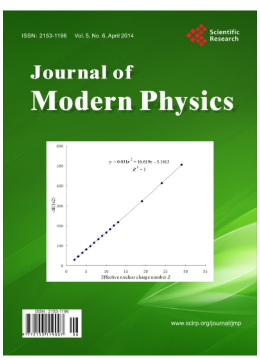
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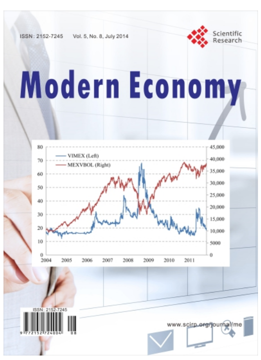
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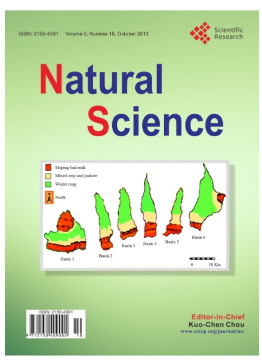
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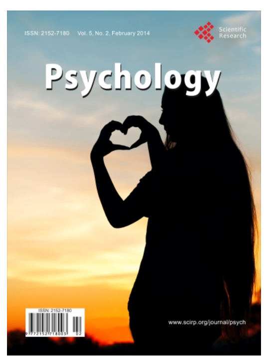
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