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Accelerated iterative methods for finding solutions of nonlinear equations and their dynamical behavior

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Abstract In this paper, we present a family of optimal, in the sense of Kung-Traub's conjecture, iterative methods for solving nonlinear equations with eighth-order convergence. Our methods are based on Chun's fourth-order method. We use the Ostrowski's efficiency index and several numerical tests in order to compare the new methods with other known eighth-order ones. We also extend this comparison to the dynamical study of the different methods.

Keywords Convergence order · Efficiency index · Basin of attraction · Periodic orbit · Dynamical plane · Nonlinear equations · Iterative methods

Mathematics Subject Classification (2000) 65B99 · 65H05 · 65P99 · 37F10

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1 Introduction

One of the most important problems in numerical analysis is solving nonlinear equations. In recent years, much attention have been given to develop a number of iterative methods for solving the nonlinear equations, paying attention to the effectiveness of the schemes, usually analyzed by means of the efficiency index introduced by Ostrowski in [10]. This index is defined as $I = p^{1/d}$, where p is the order of convergence and d is the total number of functional evaluations per step. In this sense, Kung and Traub conjectured in [8] that a multistep method without memory performing $n+1$ functional evaluations per iteration can have at most convergence order 2^n, in which case it is said to be optimal.

Recently, different optimal iterative methods of order of convergence eight have been published. For instance, optimal eighth order methods can be found in [1, 11, 13, 9, 4, 6, 12], all of them with efficiency index 1.682. Some of them will be used in the numerical and dynamical sections, in order to compare them with the new schemes introduced in this paper. In particular, Liu and Wang in [9] present a three-step iterative scheme whose expression is:

$$\begin{cases} y_m = x_m - \frac{f(x_m)}{f'(x_m)}, \\ z_m = y_m - \frac{4f(x_m)-f(y_m)}{4f(x_m)-9f(y_m)} \frac{f(y_m)}{f'(x_m)}, \\ x_{m+1} = z_m - \frac{f(z_m)}{f'(x_m)} \left[\frac{8f(y_m)}{4f(x_m)-11f(y_m)} + \left(1 + \frac{f(z_m)}{3f(y_m)+\beta_1 f(z_m)}\right)^3 + \frac{4f(z_m)}{f(x_m)+\beta_2 f(z_m)} \right], \end{cases} \quad (1)$$

that will be denoted by LW. Moreover, Cordero et. al. designed in [6] the optimal eighth-order method

$$\begin{cases} y_m = x_m - \frac{f(x_m)}{f'(x_m)}, \\ z_m = x_m - \frac{f(x_m)-f(y_m)}{f(x_m)-2f(y_m)} \frac{f(x_m)}{f'(x_m)}, \\ x_{m+1} = z_m - \frac{f(z_m)}{f'(x_m)} \frac{f(y_m)}{f(y_m)-3f(z_m)} \left[\frac{f(x_m)-f(y_m)}{f(x_m)-2f(y_m)} - \frac{f(z_m)}{f(y_m)} \right]^2, \end{cases} \quad (2)$$

based on Ostrowski's fourth-order scheme, that will be denoted by CTV. Finally, Soleymani et. al. in [12] design the following scheme by using weight functions:

$$\begin{cases} y_m = x_m - \frac{f(x_m)}{f'(x_m)}, \\ z_m = y_m - \frac{2f(x_m)-f(y_m)}{2f(x_m)-5f(y_m)} \frac{f(y_m)}{f'(x_m)}, \\ x_{m+1} = z_m - \frac{f(z_m)}{2f[z_m,x_m)]-f'(x_m)} \left[1 + \left(\frac{f(z_m)}{f(x_m)}\right)^2 + \frac{f(z_m)}{f(y_m)} + \left(\frac{f(z_m)}{f(y_m)}\right)^2 \right. \\ \qquad \left. - \frac{3}{2}\left(\frac{f(y_m)}{f(x_m)}\right)^3 - 2\left(\frac{f(y_m)}{f'(x_m)}\right)^2 - \frac{f(z_m)}{f'(x_m)} \right], \end{cases} \quad (3)$$

denoted by SK.

Now, let us recall some basic concepts on complex dynamics (see [2] and [7], for example). Given a rational function $R : \hat{C} \to \hat{C}$, where $\hat{C}$ is the Riemann sphere, the *orbit of a point* $z_0 \in \hat{C}$ is defined as:

$$z_0,\ R(z_0),\ R^2(z_0),...,R^n(z_0),...$$

We are interested in the study of the asymptotic behavior of the orbits depending on the initial condition z_0, that is, we are going to analyze the phase plane of the map R defined by the different iterative methods.

To obtain these phase spaces, the first of all is to classify the starting points from the asymptotic behavior of their orbits.

A $z_0 \in \hat{C}$ is called a *fixed point* if it satisfies $R(z_0) = z_0$. A *periodic point* z_0 of period $p > 1$ is a point such that $R^p(z_0) = z_0$ and $R^k(z_0) \neq z_0,\ k < p$. A *pre-periodic point* is a point z_0 that is not periodic but there exists a $k > 0$ such that $R^k(z_0)$ is periodic. A *critical point* z_0 is a point where the derivative of rational function vanishes, $R'(z_0) = 0$.

On the other hand, a fixed point z_0 is called *attractor* if $|R'(z_0)| < 1$, *superattractor* if $|R'(z_0)| = 0$, *repulsor* if $|R'(z_0)| > 1$ and *parabolic* if $|R'(z_0)| = 1$.

The basin of attraction of an attractor α is defined as the set of pre-images of any order:

$$\mathcal{A}(\alpha) = \{z_0 \in \hat{C}\ :\ R^n(z_0) \to \alpha,\ n \to \infty\}.$$

The set of points $z \in \hat{C}$ such that their families $\{R^n(z)\}_{n \in N}$ are normal in some neighborhood $U(z)$, is the *Fatou set,* $\mathcal{F}(R)$, that is, the Fatou set is composed by the set of points whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complement in $\hat{C}$ is the *Julia set,* $\mathcal{J}(R)$; therefore, the Julia set includes all repelling fixed points, periodic orbits and their pre-images. That means that the basin of attraction of any fixed point belongs to the Fatou set. On the contrary, the boundaries of the basins of attraction belong to the Julia set.

In this paper we present an optimal family of iterative methods which are free of second derivatives and are of eighth-order of convergence. The rest of this paper is organized as follows. The proposed methods are described in Section 2 and the convergence analysis is carried out to establish the order of convergence. In Section 3, some numerical examples confirm the theoretical results and allow us to compare the proposed methods with other known methods mentioned in the Introduction. Section 4 is devoted to the complex dynamical analysis of the designed methods on quadratic and cubic polynomials. In Section 5, we end this paper with some conclusions.

2 Development of the eighth-order family and convergence analysis

In this section, we derive a family of eighth-order methods using an approximation for the last derivative. Let us consider the family of fourth-order methods proposed by Chun in [3]

$$\begin{cases} y_m = x_m - \frac{f(x_m)}{f'(x_m)}, \\ z_m = y_m - \frac{f^3(x_m)}{f^3(x_m)-2f^2(x_m)f(y_m)+2\alpha f(x_m)f^2(y_m)-2\alpha^2 f^3(y_m)} \frac{f(y_m)}{f'(x_m)}, \end{cases}$$

where $\alpha \in R$.

If we compose this scheme with Newton's method, it is known that the resulting algorithm is of eighth-order of convergence, but it is not optimal, since it uses two additional functional evaluations. In order to improve the efficiency, we are going to approximate $f'(z_m)$ trying to hold the order of convergence. By using the Taylor expansion, $f(z_m)$ and $f^{'}(z_m)$ can be approximated by

$$f(z_m) \approx f(y_m) + f'(y_m)(z_m - y_m) + \frac{1}{2} f''(y_m)(z_m - y_m)^2, \tag{4}$$

$$f'(z_m) \approx f'(y_m) + f''(y_m)(z_m - y_m). \tag{5}$$

In order to avoid the computation of the second derivative, we can express $f''(y_m)$ as follows

$$f''(y_m) \approx 2f[z_m, x_m, x_m] = \frac{2(f[z_m, x_m] - f'(x_m))}{z_m - x_m}, \tag{6}$$

where $f[,]$ denotes the divided difference of first order.

From (4), (5) and (6), we have

$$f'(z_m) \approx f[z_m, y_m] + f[z_m, x_m, x_m](z_m - y_m).$$

It can be proved that by using this approximation, the composed scheme only reaches seventh-order of convergence, for any value of α. So, we propose to use a weight function to attain the optimal order. We consider the following three-step iteration scheme

$$\begin{cases} y_m = x_m - \frac{f(x_m)}{f'(x_m)}, \\ z_m = y_m - \frac{f^3(x_m)}{f^3(x_m)-2f^2(x_m)f(y_m)+2\alpha f(x_m)f^2(y_m)-2\alpha^2 f^3(y_m)} \frac{f(y_m)}{f'(x_m)}, \\ x_{m+1} = z_m - \frac{Af(x_m)+Bf(z_m)}{Cf(x_m)+Df(z_m)} \frac{f(z_m)}{f[z_m,y_m]+f[z_m,x_m,x_m](z_m-y_m)}, \end{cases} \tag{7}$$

where A, B, C, D and α are parameters to be determined such that the iterative method defined by (7) has the order of convergence eight.

Theorem 1 *Assume that function $f : D \subseteq \Re \to \Re$ is sufficiently differentiable and f has a simple zero $x^* \in D$. If the initial point x_0 is sufficiently close to $x^* \in D$, then the methods defined by (7) converge to x^* with eighth-order under the conditions $A = C \neq 0$, $2C - B + D = 0$ and $\alpha = -\frac{1}{2}$, and with error equation*

$$e_{m+1} = c_2^2 c_3 \left(3c_2^3 + 2c_2 c_3 - c_4\right) e_m^8 + O(e_m^9).$$

Proof: Let $e_m = x_m - x^*$ be the error at the mth iteration and $c_k = \frac{1}{k!}\frac{f^{(k)}(x^*)}{f'(x^*)}$, $k = 2, 3, \ldots$. By using Taylor expansions, we have:

$$\begin{aligned} f(x_m) = f'(x^*)[&e_m + c_2 e_m^2 + c_3 e_m^3 + c_4 e_m^4 + c_5 e_m^5 + c_6 e_m^6 \\ &+ c_7 e_m^7 + c_8 e_m^8] + O(e_m^9), \end{aligned} \tag{8}$$

$$\begin{aligned} f'(x_m) = f'(x^*)[&1 + 2c_2 e_m + 3c_3 e_m^2 + 4c_4 e_m^3 + 5c_5 e_m^4 + 6c_6 e_m^5 \\ &+ 7c_7 e_m^6 + 8c_8 e_m^7 + 9c_9 e_m^8] + O(e_m^9). \end{aligned} \tag{9}$$

Now, from (8) and (9), we have

$$\begin{aligned} y_m - x^* = {}& c_2 e_m^2 + (2c_3 - 2c_2^2)e_m^3 + (3c_4 - 3c_2c_3 - 2(2c_3 - 2c_2^2)c_2)e_m^4 \\ &+ (4c_5 - 10c_2c_4 - 6c_3^2 + 20c_3c_2^2 - 8c_2^4)e_m^5 \\ &+ (-17c_4c_3 + 28c_4c_2^2 - 13c_2c_5 + 33c_2c_3^2 + 5c_6 - 52c_3c_2^3 + 16c_2^5)e_m^6] + O(e_m^7), \end{aligned} \tag{10}$$

From (10), we get

$$\begin{aligned} f(y_m) = f'(x^*)[&c_2 e^2 + (2c_3 - 2c_2^2)e_m^3 + (3c_4 - 7c_2c_3 + 5c_2^3)e_m^4 \\ &+ (-6c_3^2 + 24c_3c_2^2 - 10c_2c_4 + 4c_5 - 12c_2^4)e_m^5 \\ &+ (-17c_4c_3 + 34c_4c_2^2 - 13c_2c_5 + 5c_6 + 37c_2c_3^2 - 73c_3c_2^3 + 28c_2^5)e_m^6] + O(e_m^7). \end{aligned} \tag{11}$$

Combining (8), (9), (10) and (11), we have

$$\begin{aligned} z_m - x^* = {}& \left((1 + 2\alpha)c_2^3 - c_2c_3\right) e_m^4 \\ &- 2\left(\left(2 + 6\alpha + \alpha^2\right) c_2^4 - 2(2 + 3\alpha)c_2^2c_3 + c_3^2 + c_2c_4\right) e_m^5 \\ &+ \left(2\left(5 + 22\alpha + 7\alpha^2\right) c_2^5 - 2\left(15 + 42\alpha + 8\alpha^2\right) c_2^3c_3 \right. \\ &\left. + 6(2 + 3\alpha)c_2^2c_4 - 7c_3c_4 + 3c_2\left((6 + 8\alpha)c_3^2 - c_5\right)\right) e_m^6 + O(e_m^7). \end{aligned} \tag{12}$$

From (12), we get

$$\begin{aligned} f(z_m) = f'(x^*)[&\left((1 + 2\alpha)c_2^3 - c_2c_3\right) e_m^4 \\ &- 2(\left(\left(2 + 6\alpha + \alpha^2\right) c_2^4 - 2(2 + 3\alpha)c_2^2c_3 + c_3^2 + c_2c_4\right) e_m^5 \\ &+ \left(2\left(5 + 22\alpha + 7\alpha^2\right) c_2^5 - 2\left(15 + 42\alpha + 8\alpha^2\right) c_2^3c_3 \right. \\ &\left. + 6(2 + 3\alpha)c_2^2c_4 - 7c_3c_4 + 3c_2\left((6 + 8\alpha)c_3^2 - c_5\right)\right) e_m^6] + O(e_m^7). \end{aligned} \tag{13}$$

By using the Taylor expansions (8) and (13), we get

$$\begin{aligned} \frac{Af(x_m) + Bf(z_m)}{Cf(x_m) + Df(z_m)} = {}& \frac{A}{C} + \frac{(BC - AD)c_2\left((1 + 2\alpha)c_2^2 - c_3\right) e_m^3}{C^2} \\ &- \frac{(BC - AD)\left(\left(5 + 14\alpha + 2\alpha^2\right) c_2^4 - 3(3 + 4\alpha)c_2^2c_3 + 2c_3^2 + 2c_2c_4\right) e_m^4}{C^2} + O(e_m^5). \end{aligned}$$

Moreover, from (8), (9), (10), (11), (12) and (13) we obtain

$$\begin{aligned} \frac{f(z_m)}{f[z_m, y_m] + f[z_m, x_m, x_m](z_m - y_m)} = {}& \left((1 + 2\alpha)c_2^3 - c_2c_3\right) e_m^4 \\ &- 2\left(\left(2 + 6a + a^2\right) C_2^4 - 2(2 + 3a)C_2^2C_3 + C_3^2 + C_2C_4\right) e_m^5 \\ &+ \left(2\left(5 + 22\alpha + 7\alpha^2\right) c_2^5 - 2\left(15 + 42\alpha + 8\alpha^2\right) c_2^3c_3 \right. \\ &\left. + 6(2 + 3\alpha)c_2^2c_4 - 7c_3c_4 + 3c_2\left((6 + 8\alpha)c_3^2 - c_5\right)\right) e_m^6 + O(e_m^7), \end{aligned}$$

and, then the error equation is

$$\begin{aligned} e_{m+1} = & -\frac{(A-C)c_2\left((1+2\alpha)c_2^2-c_3\right)e_m^4}{C} \\ & +\frac{2(A-C)\left(\left(2+6\alpha+\alpha^2\right)c_2^4-2(2+3\alpha)c_2^2c_3+c_3^2+c_2c_4\right)e_m^5}{C} \\ & -\frac{(A-C)}{C}\left[2\left(5+22\alpha+7\alpha^2\right)c_2^5-2\left(15+42\alpha+8\alpha^2\right)c_2^3c_3\right. \\ & \left.+6(2+3\alpha)c_2^2c_4-7c_3c_4+3c_2\left((6+8\alpha)c_3^2-c_5\right)\right]e_m^6+O(e_m^7), \end{aligned}$$

which shows that the convergence order of any method of the family (7) is at lest seven if $A=C\neq 0$, and the error equation is

$$\begin{aligned} e_{m+1} = & -\frac{c_2^2\left((1+2\alpha)c_2^2-c_3\right)\left((1+2\alpha)(B-D)c_2^2+(-B+2C+D)c_3\right)}{C}e_m^7 \\ & +\frac{c_2}{C}\left((1+2\alpha)\left(\left(9+26\alpha+4\alpha^2\right)B+C+2\alpha C-\left(9+26\alpha+4\alpha^2\right)D\right)C_2^6\right. \\ & +\left(-2\left(13+42\alpha+26\alpha^2\right)B+\left(11+30\alpha+4\alpha^2\right)C+2\left(13+42\alpha+26\alpha^2\right)D\right)c_2^4c_3 \\ & +((21+32\alpha)B-8(3+4\alpha)C-(21+32\alpha)D)c_2^2c_3^2 \\ & +4(-B+2C+D)c_3^3+(1+2\alpha)(4B-3C-4D)c_2^3c_4 \\ & \left.+(-4B+7C+4D)c_2c_3c_4\right)e_m^8+O(e_m^9). \end{aligned}$$

Finally, if $\alpha=-\frac{1}{2}$ and $-B+2C+D=0$, then the error equation is

$$e_{m+1}=c_2^2c_3\left(3c_2^3+2c_2c_3-c_4\right)e_m^8+O(e_m^9),$$

and the proof is completed. □

Therefore, the methods of the new family are optimal in the sense of Kung-Traub's conjecture, so they have efficiency indices $8^{\frac{1}{4}}=1.682$, as well as other eighth-order schemes described in [1, 11, 13, 9, 4, 6, 12].

In what follows, we give some concrete iterative methods of (7), that we are going to use in the following section.

FA1. If $A=C=1$, $D=1$, $B=3$,

$$\begin{cases} y_m = x_m - \frac{f(x_m)}{f'(x_m)}, \\ z_m = y_m - \frac{f^3(x_m)}{f^3(x_m)-2f^2(x_m)f(y_m)-f(x_m)f^2(y_m)-\frac{1}{2}f^3(y_m)}\frac{f(y_m)}{f'(x_m)}, \\ x_{m+1} = z_m - \frac{f(x_m)+3f(z_m)}{f(x_m)+f(z_m)}\frac{f(z_m)}{f[z_m,y_m]+f[z_m,x_m,x_m](z_m-y_m)}. \end{cases}$$

FA2. If $A=C=1$, $D=-1$, $B=1$,

$$\begin{cases} y_m = x_m - \frac{f(x_m)}{f'(x_m)}, \\ z_m = y_m - \frac{f^3(x_m)}{f^3(x_m)-2f^2(x_m)f(y_m)-f(x_m)f^2(y_m)-\frac{1}{2}f^3(y_m)}\frac{f(y_m)}{f'(x_m)}, \\ x_{m+1} = z_m - \frac{f(x_m)+f(z_m)}{f(x_m)-f(z_m)}\frac{f(z_m)}{f[z_m,y_m]+f[z_m,x_m,x_m](z_m-y_m)}. \end{cases}$$

FA3. If $A = C = 1$, $D = -3$, $B = -1$,

$$\begin{cases} y_m = x_m - \frac{f(x_m)}{f'(x_m)}, \\ z_m = y_m - \frac{f^3(x_m)}{f^3(x_m)-2f^2(x_m)f(y_m)-f(x_m)f^2(y_m)-\frac{1}{2}f^3(y_m)} \frac{f(y_m)}{f'(x_m)}, \\ x_{m+1} = z_m - \frac{f(x_m)-f(z_m)}{f(x_m)-3f(z_m)} \frac{f(z_m)}{f[z_m,y_m]+f[z_m,x_m,x_m](z_m-y_m)}. \end{cases}$$

3 Numerical results

We present some examples to illustrate the efficiency of the proposed methods comparing them with the eighth-order methods described in the introduction, specifically, we compare our methods with LW scheme (expression (1)) with $\beta_1 = \beta_2 = 1$, CTV method defined in (2) and SK method showed in (3). The test functions used are:

- $f_1(x) = \sin^2(x) - x^2 + 1; x^* \approx 1.4044916482153$;
- $f_2(x) = \sin(x) - \frac{x}{2}; x^* \approx 1.8954942670339$;
- $f_3(x) = \cos(x) - x; x^* \approx 0.73908513321516$;
- $f_4(x) = 10xe^{-x^2} - 1; x^* \approx 1.6796306104285$;
- $f_5(x) = x^2 - e^x - 3x + 2; x^* \approx 0.2575302854398608$;

f_1, $x_0 = 1$	LW	CTV	SK	$FA1$	$FA2$	$FA3$
$\lvert f(x_n)\rvert$	2.0e-299	1.5e-189	1.2e-299	1.5e-189	2.3e-185	9.5e-181
$\lvert x_{n+1} - x_n\rvert$	2.8e-180	2.6e-24	4.2e-107	3.01e-24	1.0e-23	3.8e-23
$iter$	4	3	4	3	3	3
ρ	8.000	8.338	7.977	8.567	8.610	8.661
f_2, $x_0 = 2$	LW	CTV	SK	$FA1$	$FA2$	$FA3$
$\lvert f(x_n)\rvert$	3.0e-300	3.0e-300	3.0e-300	3.0e-300	3.0e-300	3.0e-300
$\lvert x_{n+1} - x_n\rvert$	4.8e-73	4.5e-75	8.6e-91	7.3e-82	7.4e-82	7.4e-82
$iter$	3	3	3	3	3	3
ρ	7.976	7.973	7.913	7.984	7.984	7.984
f_3, $x_0 = 1$	LW	CTV	SK	$FA1$	$FA2$	$FA3$
$\lvert f(x_n)\rvert$	1.0e-300	1.0e-300	1.0e-300	1.0e-300	1.0e-300	1.0e-300
$\lvert x_{n+1} - x_n\rvert$	9.3e-67	2.4e-67	3.5e-66	2.7e-83	1.5e-83	8.0e-84
$iter$	3	3	3	3	3	3
ρ	7.956	7.967	7.957	7.894	7.892	7.888
f_4, $x_0 = 1$	LW	CTV	SK	$FA1$	$FA2$	$FA3$
$\lvert f(x_n)\rvert$	5.0e-300	5.0e-300	2.1e-209	3.6e-233	2.9e-231	1.3e-229
$\lvert x_{n+1} - x_n\rvert$	5.6e-174	5.1e-68	8.2e-27	9.3e-30	1.6e-29	2.6e-29
$iter$	4	4	3	3	3	3
ρ	8.000	8.017	7.530	7.547	7.552	7.557
f_5, $x_0 = 2$	LW	CTV	SK	$FA1$	$FA2$	$FA3$
$\lvert f(x_n)\rvert$	0	0	1.0e-299	1.0e-299	0	0
$\lvert x_{n+1} - x_n\rvert$	3.5e-94	2.4e-126	1.2e-67	3.7e-86	3.6e-125	9.5e-158
$iter$	4	4	4	4	4	4
ρ	7.967	8.002	7.881	8.010	8.002	7.999

Table 1: Comparison of various iterative methods.

For each method and each test function we show in Table 1 the value of function in the last iteration, the distance between the two last iterations, computed with 300 significant digits, the number of iterations such that $|x_{n+1} - x_n| \leq 10^{-180}$ or $f(x_{n+1}) \leq 10^{-180}$ and the computational order of convergence ρ, approximated using the formula (see [5])

$$\rho \approx \frac{\ln(|x_{m+1} - x_m|/|x_m - x_{m-1}|)}{\ln(|x_m - x_{m-1}|/|x_{m-1} - x_{m-2}|)}.$$

All computations were done by using MAPLE.

4 Dynamical analysis

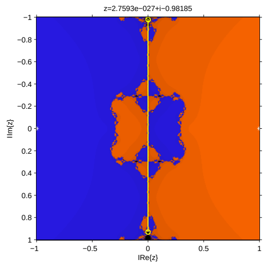
In this section, we study the dynamics of fixed point operators when they are applied on quadratic and cubic polynomials. As we will observe in the following, the dynamics of the proposed methods is very rich, several periodic orbits appear with infinite pre-images, Julia set is connected and the connected components of Fatou set are also infinite.

In our calculations, we usually consider the region $[-2, 2] \times [-2, 2]$ of the complex plane, with 400×400 points and we apply the corresponding iterative method starting in every x_0 in this area. If the sequence generated by iterative method reaches a zero x^* of the polynomial with a tolerance $|x_k - x^*| < 10^{-2}$ and a maximum of 40 iterations, we decide that x_0 is in the basin of attraction of these zero and we paint this point in a color previously selected for this root. In the same basin of attraction, the number of iterations needed to achieve the solution is showed in darker or brighter colors (the less iterations, the brighter color). Black color denotes lack of convergence to any of the roots (with the maximum of iterations established) or convergence to the infinity.

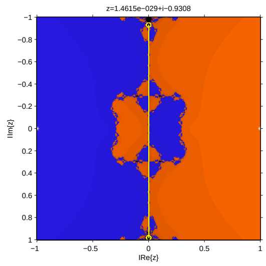
When we apply the fixed point operator of the new methods (see Figures 1a to 1e for FA1, FA2 and FA3, respectively) on the second-degree polynomial $p(z) = z^2 - 1$, a rational function is obtained, with polynomials of degree 30 and 29 in the numerator and denominator, respectively. Analyzing the fixed points, we found that the roots of $p(z)$ are, obviously, superattractive and there exist other 28 repulsive strange fixed points. That is, all strange fixed points are on the Julia set.

Observing Figure 1, we note that the method FA2 never fails, meanwhile two periodic orbits of period 2 appear in the dynamical plane of FA1 (the orbits are $\{0.98185i, -0.9308i\}$ and $\{-0.98185i, 0.9308i\}$) and also of FA3 (the first orbit is $\{0.4496i, -0.5037i\}$ and the second $\{-0.4496i, 0.5037i\}$). In case of double roots, all the dynamical planes are the same as in Newton's case, as can be seen in Figure 1f.

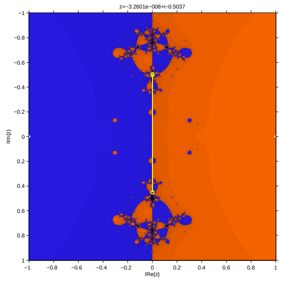
The dynamical planes of the new methods for cubic polynomials are showed in Figure 2. We observe that the dynamical behavior of the three schemes is similar. We found again that the roots of the polynomial are the only superattractive fixed points. If we are near the origin (see Figure 2a), the connected components of the basin of attraction containing the roots seem to expand



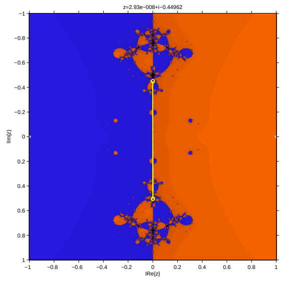
(a) Orbit 1 in FA1 on $z^2 - 1$



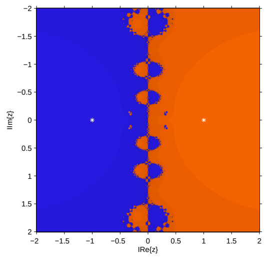
(b) Orbit 2 in FA1 on $z^2 - 1$



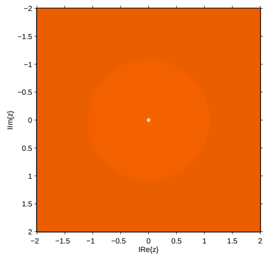
(c) Orbit 1 in FA3 on $z^2 - 1$



(d) Orbit 2 in FA3 on $z^2 - 1$



(e) FA2 on $z^2 - 1$



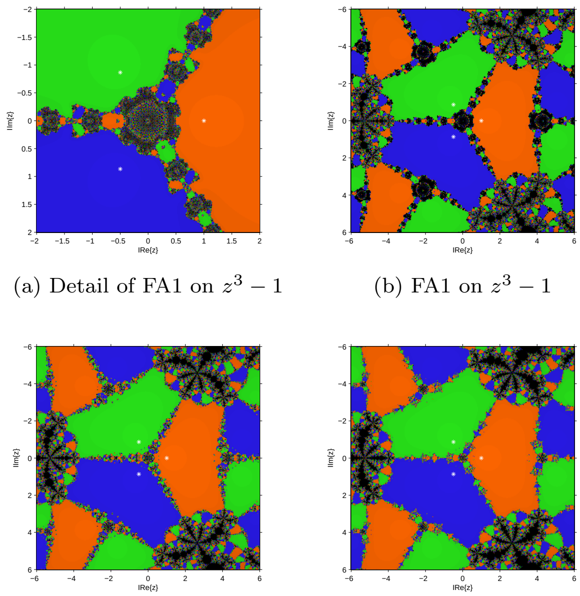
(f) New methods on z^2

Fig. 1: Dynamical planes of the new methods on quadratic polynomials

filling all the plane. However, we can see in Figures 2b to 2d that the structure around the origin is infinitely replicated. Indeed, the black regions between the copies contain three periodic orbits of period 4 in each new method. Two of them are on the diagonals of the plane (conjugated between them) and the third one is on the real axis. The orbits of FA1 are:

$$\{-6.1829 - 10.7092i, 4.5609 + 7.8997i, -5.1414 - 8.9052i, 3.4898 + 6.0445i\}$$
$$\{-6.1829 + 10.7092i, 4.5609 - 7.8997i, -5.1414 + 8.9052i, 3.4898 - 6.0445i\}$$
$$\{-9.1218, 10.2828, -6.9795, 12.3659\}$$

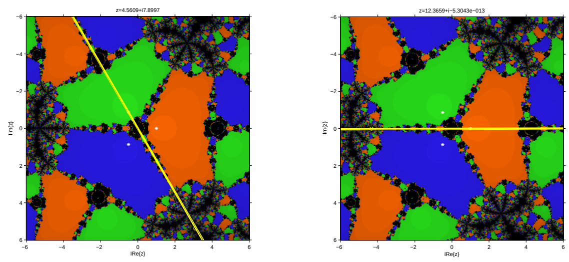
In Figure 3 we can see two of these periodic orbits.



(a) Detail of FA1 on $z^3 - 1$ (b) FA1 on $z^3 - 1$

(c) FA2 on $z^3 - 1$ (d) FA3 on $z^3 - 1$

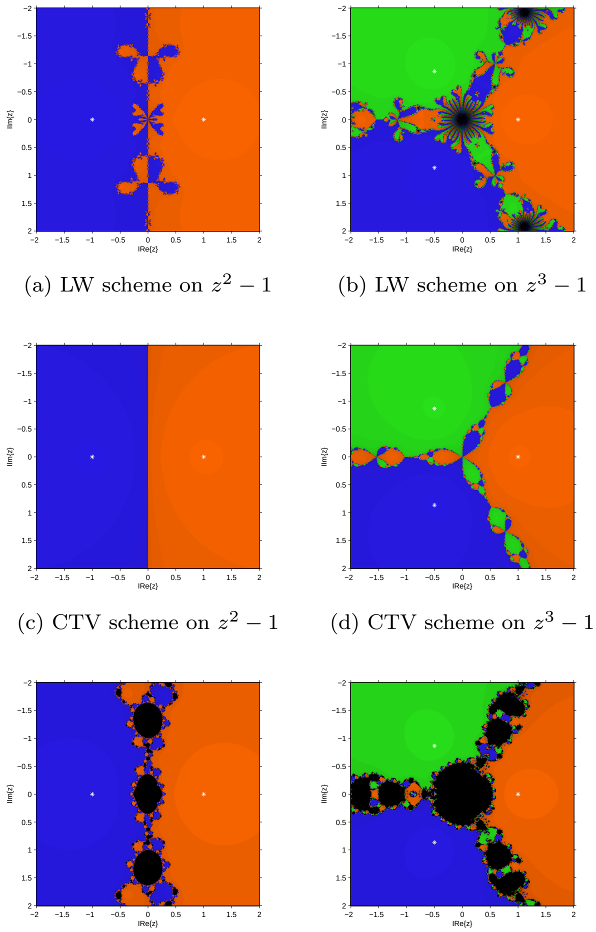
Fig. 2: Dynamical planes of the new methods on cubic polynomials



(a) Diagonal orbit (b) Horizontal orbit

Fig. 3: Some orbits of period 4 of FA1

Finally, we can compare the results obtained for the presented methods with the same schemes that have been used in numerical tests. All the methods have a good behavior on quadratic polynomials, but in case of cubic ones, the differences between them are clear. It can be observed (see Figure 4) that CTV is the most stable method on quadratic and cubic polynomials, meanwhile LW scheme shows a similar behavior than the methods proposed in this paper. However, SK procedure has big regions of convergence to the infinity (in black in Figures 4e and 4f) in quadratic and cubic polynomials. In this case, no periodic orbits appear but starting points in these black regions lead to infinity after few iterations.



(a) LW scheme on $z^2 - 1$ (b) LW scheme on $z^3 - 1$

(c) CTV scheme on $z^2 - 1$ (d) CTV scheme on $z^3 - 1$

(e) SK scheme on $z^2 - 1$ (f) SK scheme on $z^3 - 1$

Fig. 4: Dynamical planes of known methods on polynomials

5 Conclusions

We have designed and studied a family of optimal iterative method (in the sense of Kung-Traub's conjecture) of eighth-order. We have tested some elements of the family and compared them with other known schemes. Finally, a dynamical analysis of the particular methods has been made on quadratic and cubic polynomials, showing the dynamical richness of the family.

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