Three-step iterative methods with optimal eighth-order convergence ⋆

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Abstract

In this paper, based on Ostrowski's method, a new family of eighth-order methods for solving nonlinear equations is derived. In terms of computational cost, each iteration of these methods requires three evaluations of the function and one evaluation of its first derivative, so that their efficiency indices are 1.682, which is optimal according to Kung and Traub's conjecture. Numerical comparisons are made to show the performance of the new family.

Key words: Nonlinear equations, Iterative methods, Convergence order, Efficiency index, Ostrowski's method, Optimal order

1 Introduction

In this work, we consider iterative methods to find a simple root α of the nonlinear equation $f(x) = 0$, where $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a scalar function on an open interval I.

In the last years, many modified methods have been proposed to improve the local order of convergence of the Newton's method, see for example [1–7] and references therein. King in [3]

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developed a one-parameter family of fourth-order methods, which is written as

$$\begin{aligned} y_n &= x_n - \tfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \tfrac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)} \tfrac{f(y_n)}{f'(x_n)}, \end{aligned} \tag{1}$$

where $\beta \in \mathbb{R}$ is a parameter. In particular, Ostrowski's method [8] is a member of this family when $\beta = 0$, and it can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)},$$

being y_n the step of Newton's method.

As the order of an iterative method increases, so does the number of functional evaluations per step. The efficiency index (see [8]) gives a measure of the balance between those quantities, according to the formula $I = p^{1/d}$, where p is the order of convergence of the method and d the number of functional evaluations per step. Kung and Traub conjectured in [9] that the order of convergence of any multipoint method cannot exceed the bound 2^{d-1}, (called the optimal order). Thus, the optimal order for a method with 3 functional evaluations per step would be 4. King's method [3], Ostrowski's method and Jarrat's method [10] are some of optimal fourth-order methods, because they only perform three functional evaluations per step. In order to compare the different methods, we also use the operational index, defined in [11] as $IO = p^{1/op}$, where *op* is the total number of products and quotients per iteration.

Recently, based on Ostrowski's or King's methods, some higher order multipoint methods have been proposed for solving nonlinear equations. For example, Liu and Wang developed in [5] a family of variants of Ostrowski's method with eighth-order convergence by weight function methods. This family, which we will refer as LW8, is written as

$$\begin{aligned} y_n &= x_n - \tfrac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \tfrac{f(x_n)}{f'(x_n)} \tfrac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}, \\ x_{n+1} &= z_n - \tfrac{f(z_n)}{f'(x_n)} \left[\left(\tfrac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)} \right)^2 + \tfrac{f(z_n)}{f(y_n)-\alpha f(z_n)} + G(\mu_n) \right], \end{aligned} \tag{2}$$

where α is constant, $\mu_n = f(z_n)/f(x_n)$ and $G(\mu_n)$ denotes a real-valued function. The same strategy is used by Sharma et al. in [6] and Kou et al. in [4]. On the other hand, Bi et al. [1] also presented a new family of eighth-order methods based on King's methods and the family of sixth-order iteration methods developed by Chun et al. [12]. This family, denoted BRW8, has the following expression:

$$\begin{aligned} y_n &= x_n - \tfrac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \tfrac{f(x_n)-(1/2)f(y_n)}{f(x_n)-(5/2)f(y_n)} \tfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - H(\mu_n) \tfrac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}, \end{aligned} \tag{3}$$

where $\mu_n = f(z_n)/f(x_n)$, $H(\mu_n)$ represents a real-valued function and divided differences are denoted by $f[,]$.

In this paper, based on Ostrowski's method, we present a new family of optimal eighth-order of convergence, without using other derivatives than the first. The rest of the paper is organized as follows: in Section 2 we describe our family of variants of Ostrowski's method and we show its optimal order of convergence. In Section 3, different numerical tests confirm the theoretical results and allow us to compare these variants with other known methods mentioned in the Introduction. Concluding remarks are given in the last section.

2 The methods and analysis of convergence

We consider the iteration scheme consisting of three steps. The first step is the Ostrowski's iteration to get z_n from x_n, that is

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)} \frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)},$$

where y_n is the iteration of Newton's method. The second and third steps calculate x_{n+1} from the new point z_n by the family of methods given by

$$\begin{aligned} u_n &= z_n - \tfrac{f(z_n)}{f'(x_n)} \left(\tfrac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)} + \tfrac{1}{2} \tfrac{f(z_n)}{f(y_n)-2f(z_n)} \right)^2, \\ x_{n+1} &= u_n - \tfrac{f(z_n)}{f'(x_n)} \tfrac{\alpha_1(u_n-z_n)+\alpha_2(y_n-x_n)+\alpha_3(z_n-x_n)}{\beta_1(u_n-z_n)+\beta_2(y_n-x_n)+\beta_3(z_n-x_n)}, \end{aligned} \tag{4}$$

where $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2, 3$.

The order of convergence of this family is analyzed in the following result.

Theorem 1 *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ in an open interval I. If x_0 is sufficiently close to α, then the iterative schemes described by (4) have optimal eight convergence order, for $\alpha_2 = \alpha_3 = 0$, $\alpha_1 = 3(\beta_2 + \beta_3)$ and $\beta_2 + \beta_3 \neq 0$.*

Proof: Let e_n be the error in x_n, that is $e_n = x_n - \alpha$. By using Taylor's expansion around $x = \alpha$ and taking into account $f(\alpha) = 0$, we have

$$f(x_n) = f'(\alpha) \left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^8 \right] + O(e_n^9), \tag{5}$$

and

$$f'(x_n) = f'(\alpha) \left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + 8c_8e_n^7 \right] + O(e_n^8), \tag{6}$$

where $c_k = \dfrac{f^{(k)}(\alpha)}{k!} f'(\alpha)$, $k = 2, 3, \ldots$.

So, using (5) and (6), we obtain

$$
\begin{aligned}
&y_n-\alpha = x_n-\alpha-\frac{f(x_n)}{f'(x_n)}\\
&= c_2e_n^2+(-2c_2^2+2c_3)e_n^3+(4c_2^3-7c_2c_3+3c_4)e_n^4+(-8c_2^4+20c_2^2c_3-6c_3^2-10c_2c_4+4c_5)e_n^5\\
&+(16c_2^5-52c_2^3c_3+28c_2^2c_4-17c_3c_4+c_2(33c_3^2-13c_5)+5c_6)e_n^6\\
&-2(16c_2^6-64c_2^4c_3-9c_3^3+36c_2^3c_4+6c_4^2+9c_2^2(7c_3^2-2c_5)+11c_3c_5\\
&+c_2(-46c_3c_4+8c_6)-3c_7)e_n^7+O(e_n^8).
\end{aligned}
$$

Again expanding $f(y_n)$ around α, we have

$$
\begin{aligned}
&f(y_n) = f'(\alpha)\,[c_2e_n^2-2(c_2^2-c_3)e_n^3+(5c_2^3-7c_2c_3+3c_4)e_n^4-2(6c_2^4-12c_2^2c_3+3c_3^2+5c_2c_4-2c_5)e_n^5\\
&+(28c_2^5-73c_2^3c_3+34c_2^2c_4-17c_3c_4+c_2(37c_3^2-13c_5)+5c_6)e_n^6\\
&-2(32c_2^6-103c_2^4c_3-9c_3^3+52c_2^3c_4+6c_4^2+c_2^2(80c_3^2-22c_5)\\
&+11c_3c_5+c_2(-52c_3c_4+8c_6)-3c_7)e_n^7]+O(e_n^8).
\end{aligned} \tag{7}
$$

Now, from (5), (6) and (7), we obtain

$$
\begin{aligned}
&z_n-\alpha = x_n-\alpha-\frac{f(x_n)}{f'(x_n)}\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}\\
&= (c_2^3-c_2c_3)e_n^4-2(2c_2^4-4c_2^2c_3+c_3^2+c_2c_4)e_n^5\\
&+(10c_2^5-30c_2^3c_3+12c_2^2c_4-7c_3c_4+3c_2(6c_3^2-c_5))e_n^6\\
&-2(10c_2^6-40c_2^4c_3-6c_3^3+20c_2^3c_4+3c_4^2+8c_2^2(5c_3^2-c_5)+5c_3c_5+c_2(-26c_3c_4+2c_6))e_n^7+O(e_n^8).
\end{aligned}
$$

Taylor expansion of $f(z_n)$ around α is

$$
\begin{aligned}
&f(z_n) = f'(\alpha)\,[(c_2^3-c_2c_3)e_n^4-2(2c_2^4-4c_2^2c_3+c_3^2+c_2c_4)e_n^5\\
&+(10c_2^5-30c_2^3c_3+12c_2^2c_4-7c_3c_4+3c_2(6c_3^2-c_5))e_n^6\\
&-2(10c_2^6-40c_2^4c_3-6c_3^3+20c_2^3c_4+3c_4^2+8c_2^2(5c_3^2-c_5)+5c_3c_5+c_2(-26c_3c_4+2c_6))e_n^7]+O(e_n^8).
\end{aligned} \tag{8}
$$

So, from (5), (6), (7) and (8) we have

$$
\begin{aligned}
&u_n-\alpha = z_n-\alpha-\frac{f(z_n)}{f'(x_n)}\left(\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}+\frac{1}{2}\frac{f(z_n)}{f(y_n)-2f(z_n)}\right)^2\\
&= 3(c_2^3-c_2c_3)^2e_n^7-(1/4)(c_2(c_2^2-c_3)(97c_2^4-194c_2^2c_3+53c_3^2+44c_2c_4))e_n^8+O(e_n^9).
\end{aligned}
$$

Finally, the expression of the asymptotic error is

$$
\begin{aligned}
x_{n+1}-\alpha &= u_n-\alpha-\frac{f(z_n)}{f'(x_n)}\frac{\alpha_1(u_n-z_n)+\alpha_2(y_n-x_n)+\alpha_3(z_n-x_n)}{\beta_1(u_n-z_n)+\beta_2(y_n-x_n)+\beta_3(z_n-x_n)}\\
&= -\frac{1}{\beta_2+\beta_3}(\alpha_2+\alpha_3)(c_2^3-c_2c_3)e_n^4+\frac{1}{(\beta_2+\beta_3)^2}\left[(6\alpha_2\beta_2+5\alpha_3\beta_2+7\alpha_2\beta_3+6\alpha_3\beta_3)c_2^4\right.\\
&\quad -(10\alpha_2\beta_2+9\alpha_3\beta_2+11\alpha_2\beta_3+10\alpha_3\beta_3)c_2^2c_3++2(\alpha_2+\alpha_3)(\beta_2+\beta_3)c_3^2\\
&\quad \left.+\,2(\alpha_2+\alpha_3)(\beta_2+\beta_3)c_2c_4\right]e_n^5+O(e_n^6),
\end{aligned}
$$

and, in order to get sixth-order, it is necessary to fix the value of some parameters, specifically $\alpha_2=\alpha_3=0$ and $\beta_2+\beta_3\neq 0$. Then, the error equation becomes:

$$
\begin{aligned}
x_{n+1}-\alpha &= -\frac{1}{\beta_2+\beta_3}(\alpha_1-3(\beta_2+\beta_3))(c_2^3-c_2c_3)^2e_n^7\\
&+\frac{1}{4(\beta_2+\beta_3)^2}c_2(c_2^2-c_3)\left[(-97(\beta_2+\beta_3)^2+4\alpha_1(9\beta_2+10\beta_3))c_2^4\right.\\
&+2(97(\beta_2+\beta_3)^2-2\alpha_1(17\beta_2+18\beta_3))c_2^2c_3+(\beta_2+\beta_3)(16\alpha_1-53(\beta_2+\beta_3)c_3^2)\\
&\left.+4(\beta_2+\beta_3)(4\alpha_1-11(\beta_2+\beta_3))c_2c_4\right]e_n^8+O(e_n^9).
\end{aligned}
$$

Finally, if $\alpha_1=3(\beta_2+\beta_3)$ the convergence order of any method of the family (4) arrives to eight, and the error equation is

$$
\begin{aligned}
e_{n+1} &= \frac{1}{4(\beta_2+\beta_3)}c_2(c_2^2-c_3)\left[(11\beta_2+23\beta_3)c_2^4-2(5\beta_2+11\beta_3)c_2^2c_3\right.\\
&\left.-5(\beta_2+\beta_3)c_3^2+4(\beta_2+\beta_3)c_2c_4\right]e_n^8+O(e_n^9).
\end{aligned}
$$

Note that there is no restriction on the value of β_1, and the values of β_2 and β_3 must satisfy $\beta_2+\beta_3\neq 0$. Moreover, it is easy to prove that it is not possible to reach the ninth-order of convergence. □

So, we have obtained an eight-order convergence family of methods with two degrees of freedom:

$$
\begin{aligned}
y_n &= x_n-\frac{f(x_n)}{f'(x_n)},\\
z_n &= x_n-\frac{f(x_n)}{f'(x_n)}\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}\\
x_{n+1} &= u_n-\frac{f(z_n)}{f'(x_n)}\frac{3(\beta_2+\beta_3)(u_n-z_n)}{\beta_1(u_n-z_n)+\beta_2(y_n-x_n)+\beta_3(z_n-x_n)},
\end{aligned}
\tag{9}
$$

where $u_n=z_n-\frac{f(z_n)}{f'(x_n)}\left(\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}+\frac{1}{2}\frac{f(z_n)}{f(y_n)-2f(z_n)}\right)^2$ and $\beta_2+\beta_3\neq 0$.

In terms of computational cost, the developed methods require only four functional evaluations. So, their efficiency indices are $8^{1/4}=1.682$, that is, the new family of methods reach the optimal order of convergence eight, conjectured by Kung and Traub.

3 Numerical results

In this section we check the effectiveness of the new optimal eighth order family of methods (4), taking $\alpha_1 = 3$, $\beta_1 = \beta_3 = 0$ and $\beta_2 = 1$, which is denoted by M8 and the last step of its iterative expression is

$$x_{n+1} = u_n - 3\frac{f(z_n)}{f'(x_n)}\frac{u_n - z_n}{y_n - x_n},$$

compared with the classical Newton's (N2) and Ostrowski's (O4) methods, the optimal eighth order methods, BRW8, with $H(t) = 1 + \frac{2t}{1+\alpha t}$ and $\alpha = 1$; and LW8 with $\alpha = 1$ and $G(t) = 4t$. In order to get this aim, let us consider the following nonlinear test functions, which are the same as in [2,4].

- $f_1(x) = x^3 + 4x^2 - 15, \ \ \alpha \approx 1.6319808055661,$
- $f_2(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5, \ \ \alpha \approx -1.2076478271309,$
- $f_3(x) = \sin(x) - \frac{x}{2}, \ \ \alpha \approx 1.8954942670339,$
- $f_4(x) = 10xe^{-x^2} - 1, \ \ \alpha \approx 1.6796306104285,$
- $f_5(x) = \cos(x) - x, \ \ \alpha \approx 0.73908513321516,$
- $f_6(x) = \sin^2(x) - x^2 + 1, \ \ \alpha \approx 1.4044916482153,$
- $f_7(x) = e^{-x} + \cos(x), \ \ \alpha \approx 1.7461395304080.$

Nowadays, high-order methods are important because numerical applications use high precision in their computations; for this reason numerical computations have been carried out using variable precision arithmetic in Matlab 7.1 with 2000 significant digits. The computer specifications are: Intel(R) Core(TM)2 Quad CPU, Q9550 @ 2.83GHz with 4.00GB of RAM.

Table 1 shows, for some initial estimations, the number of iterations required to obtain $|x_{n+1} - x_n| < 10^{-200}$ or $|f(x_{n+1})| < 10^{-200}$, the value of function f in the last iteration, the distance between the two last iterations, the mean elapsed time (e-time) after 100 performances of the program, calculated by means of the command "cputime" of Matlab, and the computational order of convergence (COC), ρ, introduced in [13]:

$$\rho \approx \frac{\ln(|x_{n+1} - \alpha| \,/\, |x_n - \alpha|)}{\ln(|x_n - \alpha| \,/\, |x_{n-1} - \alpha|)}. \tag{10}$$

The value of α used in (10) have been calculated by Newton's method with 500 exact digits. Moreover, in Table 1, denoted by COC, appears the last coordinate of vector ρ when the variation between its coordinates is small.

Numerical results are in concordance with the theory developed in this paper. In all the cases, the results obtained with our new methods are similar than the other optimal methods. In fact, the elapsed time that appears in Table 1 is completely understood by means of the operational index of the different methods:

$$IO_{BRW8} = 8^{\frac{1}{11}} < IO_{M8} = 8^{\frac{1}{9}} < IO_{LW8} = 8^{\frac{1}{8}} < IO_{O4} = 4^{\frac{1}{3}} < IO_{N2} = 2^1,$$

that is, methods that have the same optimal order of convergence only differ in the mean elapsed time if the number of products and quotients per iteration are different.

4 Conclusions

We have obtained a new family of variants of Ostrowski's method. The convergence order of these methods is eight, and consist of three evaluations of the function and one evaluation of the first derivative per iteration, so they have an efficiency index equal to $8^{1/4} = 1.682$. Therefore, the family of methods agrees with the conjecture of Kung-Traub for $n = 4$ and its operational index is similar than the corresponding one of other known methods.

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Table 1
Numerical results

		$N2$	$O4$	$BRW8$	$LW8$	$M8$
f_1, $x_0 = 2$	$\|x_{n+1} - x_n\|$	6.4650e-110	9.6816e-058	7.9134e-059	7.5148e-049	7.1376e-054
	$\|f(x_{n+1})\|$	3.7181e-218	1.0251e-228	0	0	0
	COC	2.0000	4.0000	7.8747	8.0000	8.0000
	iter	8	4	3	3	3
	e-time	0.0881	0.0902	0.1356	0.1103	0.1242
f_2, $x_0 = -1$	$\|x_{n+1} - x_n\|$	1.8805e-128	1.8368e-056	4.0748e-028	3.9269e-043	1.0709e-050
	$\|f(x_{n+1})\|$	1.0787e-254	8.8236e-223	9.7125e-217	0	0
	COC	2.0000	4.0000	8.0047	8.0000	8.0000
	iter	9	4	3	3	3
	e-time	0.5759	0.5479	0.2736	0.2442	0.2594
f_3, $x_0 = 1.9$	$\|x_{n+1} - x_n\|$	6.0762e-166	2.5639e-164	3.5525e-168	7.0879e-155	4.8032e-161
	$\|f(x_{n+1})\|$	0	0	0	0	0
	COC	2.0000	4.0000	7.7670	7.7606	7.5698
	iter	7	4	3	3	3
	e-time	0.6759	0.5633	0.5627	0.5620	0.5622
f_4, $x_0 = 1.5$	$\|x_{n+1} - x_n\|$	2.0290e-108	3.0429e-053	6.6497e-055	3.5595e-045	5.3098e-52
	$\|f(x_{n+1})\|$	1.0878e-215	1.9108e-210	0	0	0
	COC	2.0000	3.9999	7.9314	8.0000	8.0000
	iter	8	4	3	3	3
	e-time	0.5000	0.3448	0.3445	0.3440	0.3440
f_5, $x_0 = 1$	$\|x_{n+1} - x_n\|$	7.1182e-167	3.5827e-074	3.3062e-083	1.6619e-066	5.2538e-082
	$\|f(x_{n+1})\|$	0	7.0526e-296	0	0	0
	COC	2.0000	4.0000	7.9999	8.0000	8.0000
	iter	8	4	3	3	3
	e-time	0.6687	0.5633	0.5788	0.5630	0.5780
f_6, $x_0 = 1.5$	$\|x_{n+1} - x_n\|$	2.6094e-148	1.6166e-075	6.2434e-086	2.3305e-066	3.8163e-072
	$\|f(x_{n+1})\|$	1.3245e-295	6.9915e-300	0	0	0
	COC	2.0000	4.0000	7.7689	8.0000	8.0000
	iter	8	4	3	3	3
	e-time	0.9068	0.6725	0.6751	0.6413	0.6461
f_7, $x_0 = 2$	$\|x_{n+1} - x_n\|$	9.5606e-170	4.5563e-070	2.6708e-080	2.8428e-061	5.3453e-078
	$\|f(x_{n+1})\|$	0	1.0461e-279	0	0	0
	COC	2.0000	4.0000	7.9460	8.0988	7.9182
	iter	8	4	3	3	3
	e-time	0.8758	0.7199	0.7137	0.7020	0.7090