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Weighted Banach spaces of harmonic functions

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Los directores:
José Antonio Bonet Solves y Enrique Jordá Mora

To Fernando,
Júlia, Meritxell,
and to all my family
and friends.

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Resumen

La presente memoria, “Espacios de Banach ponderados de funciones armónicas”, trata diversos tópicos del análisis funcional, como son las funciones peso, los operadores de composición, la diferenciabilidad Fréchet y Gâteaux de la norma y las clases de isomorfismos. El trabajo está dividido en cuatro capítulos precedidos de uno inicial en el que introducimos la notación y las propiedades conocidas que usamos en las demostraciones del resto de capítulos.

En el primer capítulo estudiamos espacios de Banach de funciones armónicas en conjuntos abiertos de $\mathbb{R}^d$ dotados de normas del supremo ponderadas. Definimos el peso asociado armónico, explicamos sus propiedades, lo comparamos con el peso asociado holomorfo introducido por Bierstedt, Bonet y Taskinen, y encontramos diferencias y condiciones para que sean exactamente iguales y condiciones para que sean equivalentes.

El capítulo segundo está dedicado al análisis de los operadores de composición con símbolo holomorfo entre espacios de Banach ponderados de funciones pluriarmónicas. Caracterizamos la continuidad, la compacidad y la norma esencial de operadores de composición entre estos espacios en términos de los pesos, extendiendo los resultados de Bonet, Taskinen, Lindström, Wolf, Contreras, Montes y otros para operadores de composición entre espacios de funciones holomorfas. Probamos que para todo valor del intervalo $[0, 1]$ existe un operador de composición sobre espacios ponderados de funciones armónicas tal que su norma esencial alcanza dicho valor.

La mayoría de los contenidos de los capítulos 1 y 2 han sido publicados por E. Jordá y la autora en [48].

El capítulo tercero está relacionado con el estudio de la diferenciabilidad Gâteaux y Fréchet de la norma. El criterio de Šmulyan establece que la norma de un espacio de Banach real X es Gâteaux diferenciable en $x \in X$ si y sólo si existe x^* en la bola unidad del dual de X débil expuesto por x y la norma es Fréchet diferenciable en x si y sólo si x^* es débil fuertemente expuesto en la bola unidad del dual de X por x. Mostramos que en este criterio la bola del dual de X puede ser reemplazada por un conjunto conveniente más pequeño, y aplicamos este criterio extendido para caracterizar los puntos de diferenciabilidad

Gâteaux y Fréchet de la norma de algunos espacios de funciones armónicas y continuas con valores vectoriales. A partir de estos resultados conseguimos una prueba sencilla del teorema sobre la diferenciabilidad Gâteaux de la norma de espacios de operadores lineales compactos enunciado por Heinrich y publicado sin la prueba. Además, éstos nos permiten obtener aplicaciones para espacios de Banach clásicos como H^∞ de funciones holomorfas acotadas en el disco y $A(\overline{\mathbb{D}})$ de funciones continuas en $\overline{\mathbb{D}}$ que son holomorfas en $\mathbb{D}$. Los contenidos de este capítulo han sido incluidos por E. Jordá y la autora en [47].

Finalmente, en el capítulo cuarto mostramos que para cualquier abierto U contenido en $\mathbb{R}^d$ y cualquier peso v en U, el espacio $h_{v_0}(U)$, de funciones armónicas tales que multiplicadas por el peso desaparecen en el infinito de U, es casi isométrico a un subespacio cerrado de c_0, extendiendo un teorema debido a Bonet y Wolf para los espacios de funciones holomorfas $H_{v_0}(U)$ en abiertos U de $\mathbb{C}^d$. Así mismo, inspirados por un trabajo de Boyd y Rueda también estudiamos la geometría de estos espacios ponderados examinando tópicos como la v-frontera y los puntos v-peak y damos las condiciones que proporcionan ejemplos donde $h_{v_0}(U)$ no puede ser isométrico a c_0. Para un conjunto abierto equilibrado U de $\mathbb{R}^d$, algunas condiciones geométricas en U y sobre convexidad en el peso v aseguran que $h_{v_0}(U)$ no es rotundo. Estos resultados han sido publicados por E. Jordá y la autora en [46].

Resum

La present memòria, "Espais de Banach ponderats de funcions harmòniques", tracta diversos tòpics de l'anàlisi funcional, com són les funcions pes, els operadors de composició, la diferenciabilitat Fréchet i Gâteaux de la norma i les classes d'isomorfismes. El treball està dividit en quatre capítols precedits d'un d'inicial en què introduïm la notació i les propietats conegudes que fem servir en les demostracions de la resta de capítols.

En el primer capítol estudiem espais de Banach de funcions harmòniques en conjunts oberts de $\mathbb{R}^d$ dotats de normes del suprem ponderades. Definim el pes associat harmònic, expliquem les seues propietats, el comparem amb el pes associat holomorf introduït per Bierstedt, Bonet i Taskinen, i trobem diferències i condicions perquè siguen exactament iguals i condicions perquè siguen equivalents.

El capítol segon està dedicat a l'anàlisi dels operadors de composició amb símbol holomorf entre espais de Banach ponderats de funcions pluriharmòniques. Caracteritzem la continuïtat, la compacitat i la norma essencial d'operadors de composició entre aquests espais en termes dels pesos, estenent els resultats de Bonet, Taskinen, Lindström, Wolf, Contreras, Montes i altres per a operadors de composició entre espais de funcions holomorfes. Provem que per a tot valor de l'interval $[0, 1]$ hi ha un operador de composició sobre espais ponderats de funcions harmòniques tal que la seua norma essencial arriba aquest valor.

La majoria dels continguts dels capítols 1 i 2 han estat publicats per E. Jordá i l'autora en [48].

El capítol tercer està relacionat amb l'estudi de la diferenciabilitat Gâteaux y Fréchet de la norma. El criteri de Šmulyan estableix que la norma d'un espai de Banach real X és Gâteaux diferenciable en x si i només si existeix x^* a la bola unitat del dual de X feble exposat per x i la norma és Fréchet diferenciable en x si i només si x^* és feble fortament exposat a la bola unitat del dual de X per x. Mostrem que en aquest criteri la bola del dual de X pot ser substituïda per un conjunt convenient més petit, i apliquem aquest criteri estès per caracteritzar els punts de diferenciabilitat Gâteaux i Fréchet de la norma d'alguns espais de funcions harmòniques i contínues amb valors vectorials. A partir d'aquests resultats aconseguim una prova senzilla del teorema sobre la diferenciabilitat Gâteaux

de la norma d'espais d'operadors lineals compactes enunciat per Heinrich i publicat sense la prova. A més, aquests ens permeten obtenir aplicacions per a espais de Banach clàssics com l'espai H^∞ de funcions holomorfes acotades en el disc i l'àlgebra $A(\overline{\mathbb{D}})$ de funcions contínues en $\overline{\mathbb{D}}$ que són holomorfes en $\mathbb{D}$. Els continguts d'aquest capítol han estat inclosos per E. Jordá i l'autora en [47].

Finalment, en el capítol quart mostrem que per a qualsevol conjunt obert U de $\mathbb{R}^d$ i qualsevol pes v en U, l'espai $h_{v_0}(U)$, de funcions harmòniques tals que multiplicades pel pes desapareixen en el infinit d'U, és gairebé isomètric a un subespai tancat de c_0, estenent un teorema degut a Bonet y Wolf per als espais de funcions holomorfes $H_{v_0}(U)$ en oberts U de $\mathbb{C}^d$. Així mateix, inspirats per un treball de Boyd i Rueda també estudiem la geometria d'aquests espais ponderats examinant tòpics com la v-frontera i els punts v-peak i donem les condicions que proporcionen exemples on $h_{v_0}(U)$ no pot ser isomètric a c_0. Per a un conjunt obert equilibrat U de $\mathbb{R}^d$, algunes condicions geomètriques en U i sobre convexitat en el pes v asseguren que $h_{v_0}(U)$ no és rotund. Aquests resultats han estat publicats per E. Jordá i l'autora en [46].

Summary

The Ph.D. thesis “Weighted Banach Spaces of harmonic functions” presented here, treats several topics of functional analysis such as weights, composition operators, Fréchet and Gâteaux differentiability of the norm and isomorphism classes. The work is divided into four chapters that are preceded by one in which we introduce the notation and the well-known properties that we use in the proofs in the rest of the chapters.

In the first chapter we study Banach spaces of harmonic functions on open sets of $\mathbb{R}^d$ endowed with weighted supremun norms. We define the harmonic associated weight, we explain its properties, we compare it with the holomorphic associated weight introduced by Bierstedt, Bonet and Taskinen, and we find differences and conditions under which they are exactly the same and conditions under which they are equivalent.

The second chapter is devoted to the analysis of composition operators with holomorphic symbol between weighted Banach spaces of pluriharmonic functions. We characterize the continuity, the compactness and the essential norm of composition operators among these spaces in terms of their weights, thus extending the results of Bonet, Taskinen, Lindström, Wolf, Contreras, Montes and others for composition operators between spaces of holomorphic functions. We prove that for each value of the interval $[0, 1]$ there is a composition operator between weighted spaces of harmonic functions such that its essential norm attains this value. Most of the contents of Chapters 1 and 2 have been published by E. Jordá and the author in [48].

The third chapter is related with the study of Gâteaux and Fréchet differentiability of the norm. The Šmulyan criterion states that the norm of a real Banach space X is Gâteaux differentiable at $x \in X$ if and only if there exists x^* in the unit ball of the dual of X weak* exposed by x and the norm is Fréchet differentiable at x if and only if x^* is weak* strongly exposed in the unit ball of the dual of X by x. We show that in this criterion the unit ball of the dual of X can be replaced by a smaller convenient set, and we apply this extended criterion to characterize the points of Gâteaux and Fréchet differentiability of the norm of some spaces of harmonic functions and continuous functions with vector values. Starting from

these results we get an easy proof of the theorem about the Gâteaux differentiability of the norm for spaces of compact linear operators announced by Heinrich and published without proof. Moreover, these results allow us to obtain applications to classical Banach spaces as the space H^∞ of bounded holomorphic functions in the disc and the algebra $A(\overline{\mathbb{D}})$ of continuous functions on $\overline{\mathbb{D}}$ which are holomorphic on $\mathbb{D}$. The content of this chapter has been included by E. Jordá and the author in [47].

Finally, in the forth chapter we show that for any open set U of $\mathbb{R}^d$ and weight v on U, the space $h_{v_0}(U)$ of harmonic functions such that multiplied by the weight vanishes at the boundary on U is almost isometric to a closed subspace of c_0, extending a theorem due to Bonet and Wolf for the spaces of holomorphic functions $H_{v_0}(U)$ on open sets U of $\mathbb{C}^d$. Likewise, we also study the geometry of these weighted spaces inspired by a work of Boyd and Rueda, examining topics such as the v-boundary and v-peak points and we give the conditions that provide examples where $h_{v_0}(U)$ cannot be isometric to c_0. For a balanced open set U of $\mathbb{R}^d$, some geometrical conditions in U and convexity in the weight v ensure that $h_{v_0}(U)$ is not rotund. These results have been published by E. Jordá and the author in [46].

Contents

Introduction

This work is devoted to the study of weighted Banach spaces of harmonic functions.

In the first chapter we start with an open and connected set U of $\mathbb{R}^d$. A weight on U is a continuous function $v : U \to]0, \infty[$. We introduce the harmonic associated weight $\widetilde{v}_h$ analogously as in the holomorphic case by Bierstedt, Bonet and Taskinen. The problem consists in explaining its properties, comparing the harmonic and holomorphic associated weights, finding differences and conditions for which they are the same or equivalent and studying the behavior with changes in the norm of $\mathbb{R}^d$ and in the dimension.

The second chapter is formulated in the following context. If G_1 and G_2 are open and connected subsets in $\mathbb{C}^N$ and $\mathbb{C}^M$ and $\varphi : G_2 \to G_1$ is a holomorphic function, then we can consider the composition operator

$$C_\varphi : ph(G_1) \to ph(G_2),$$

$$C_\varphi(f) := f \circ \varphi.$$

where $ph(G_1)$ and $ph(G_2)$ are spaces of pluriharmonic functions. The aim is to characterize the continuity, the compactness and the essential norm of composition operators among weighted spaces of pluriharmonic functions $ph_{v_1}(G_1)$ and $ph_{v_2}(G_2)$.

In the third chapter we study Gâteaux and Fréchet differentiability of the norm of $h_v(U)$. If X is a real Banach space and X^* denotes its topological dual, then Šmulyan criterion states that the norm of X is Gâteaux differentiable at $x \in X$ if and only if there exists x^* in the unit ball of X^* weak* exposed by x, and that the norm is Fréchet differentiable at x if and only if x^* is weak* strongly exposed in the unit ball of X^* by x. Our goal is to find a good approach of differentiability and to get applications for spaces of harmonic and continuous functions.

In the forth chapter we show that there is an isomorphism between $h_{v_0}(U)$ and a closed subspace of c_0, thus extending a theorem due to Bonet and Wolf for spaces of holomorphic functions $H_{v_0}(U)$, and study under which conditions there is an isometry and whether these properties have some connection with the rotundity of the space. Thus, another purpose wrapped in this question is to look for conditions on the weight to know the rotundity of the space $h_{v_0}(U)$.

In connection to these problems, investigations carried out by Bierstedt, Bonet, Boyd, Contreras, Domański, Galbis, García, Hernández-Díaz, Jordá, Lindström, Maestre, Rueda, Sevilla-Peris and Taskinen and others about weighted Banach spaces of holomorphic functions with weighted supremum norms have been published in [6, 10, 11, 12, 24, 32, 33, 45] and the references therein. Moreover, Laitila and Tylli have discussed the difference between strong and weak definitions for vector-valued functions in [52] and also, Banach spaces of vector-valued holomorphic functions have been recently studied by Bonet, Gómez-Collado, Jornet and Wolf in [15]. Spaces of harmonic functions are investigated by Shields and Williams [65], in connection with the growth of the harmonic conjugate of a function. In [64], they prove results of duality for weighted spaces of harmonic functions on the open unit disc.

Concerning the differentiability it is known that Fréchet's notion of differentiability was presented for the first time in [29]. Taylor refers to this work in [68] and [69, Part III, pp. 41-53]. Fréchet uses the concept of "little oh" which was previously introduced and worked by Stolz, Perpont and Young and appeared in Stolz's book [66]. The focus of differentiability notion used in [30] justifies that today we refer to Fréchet differentiability. In [34] the naming of Gâteaux derivative of f appears. The rest of Gâteaux's papers, written before spring 1914, are posthumously published by Lèvy in 1919, [35] and 1922, [36]. Šmulyan states results about the differentiability of the norm function in [71]. Banach in [4] proves that if K is a compact metric space, then $C(K) = C(K, \mathbb{R})$ is smooth at $f \neq 0$ if and only if f is a peaking function, i.e., there exists a point $t_0 \in K$ such that $\|f\| = |f(t_0)| > |f(t)|$ for all $t \in K$, $t \neq t_0$. Sundaresan in [67] and Cox and Nadler in [25] characterize the points of Gâteaux and Fréchet differentiability of the norm in $C(K, X)$ when K is compact Hausdorff. In the same paper, Cox and Nadler give a characterization of the points of Fréchet differentiability of the norm in $C_b(K, \mathbb{R})$ when K is locally compact and $C_b(K, \mathbb{R})$ denotes the Banach space of all bounded continuous functions on K with the supremum norm. Holub in [44] deals with this question of the geometry of $K(l_2, l_2)$ the space of compact linear operators on l_2. Heinrich states a theorem about the differentiability of the norm for spaces of compact linear operators which extends the result of Holub in [44]. Later on Hennefeld proves the theorem of Heinrich in a particular case. Leonard and Taylor in [53] obtain the points of Gâteaux and Fréchet differentiability of the norm in $C_b(K, X)$ where K is a locally compact Hausdorff space and X is a real Banach space and $C_b(K, X)$ denotes the Banach space of all bounded X-valued continuous functions on K with the supremum norm. Contreras studies the strong subdifferentiability of the norm in [23]. As an application of this property, he obtains a characterization of the points of Fréchet differentiability of the norm in $C_b(K,X)$ and $C_0(K,X)$ (the space of X-valued continuous functions vanishing at infinity on K with the supremum norm) where K is a locally compact Hausdorff space.

Boyd and Rueda have recently published a work that includes a result about the differentiability of the norm in weighted Banach spaces of holomorphic functions defined on bounded sets of $\mathbb{C}^d$, [19]. Also, in [18, 19] they study isometric classification of weighted spaces of holomorphic functions on bounded open sets of $\mathbb{C}^d$, examining topics such as the v-boundary and v-peak points. Moreover, they investigate the isometries between weighted spaces of harmonic functions on open sets of $\mathbb{C}$ in [20]. Lusky considers weighted spaces of harmonic functions in [55, 56, 57], where the isomorphism classes in the case of radial weights on the disc are determined. More precisely, Lusky [57, Proposition 6.3] proves that if v is a radial weight on the unit disc then the space of weighted harmonic functions $h_{v_0}(\mathbb{D})$ is isomorphic to a complemented subspace of $(\sum_{n=1}^{\infty} \bigoplus H_n)_{c_0}$, where H_n is the space of degree n polynomials on $\mathbb{C}$ endowed with the supremum norm, $\|P\| = sup_{\|z\|\leq 1}|P(z)|$. Bonet and Wolf, inspired by the proof of Kalton and Werner in [49, Corollary 4.9] get an almost isometry between the space H_{v_0} and a closed subspace of c_0 in [17].

Based on the mentioned studies and with a standard notation about locally convex spaces, complex and functional analysis and Banach space theory as in [5, 28, 38, 42, 51, 58, 61, 62] and the initial chapter we distribute the results obtained in four chapters. Most of the contents of chapter 1, 2 and 4 have been published by E. Jordá and A.-M. Zarco in [48, 46]. The results of chapter 3 are included in [47].

In the first chapter, we introduce the harmonic associated weight $\tilde{v}_h$ in these spaces in a natural way extending the work on the corresponding spaces of holomorphic functions due to Bierstedt, Bonet and Taskinen in [7, 1.A]. Thus, $v \leq \widetilde{v}_h \leq \widetilde{v}_H$. Associated weights constitute a very important tool for the study of these spaces. It is better linked to the space than the original weight, for example it has been used to study the most important properties about composition operators between weighted Banach spaces of holomorphic functions with holomorphic symbol. We extend most of the results in that paper to the harmonic case, and we check that in general the holomorphic and the harmonic associated weight are different. We also give some conditions under which they coincide and conditions under which they are equivalent.

In the second chapter, composition operators on weighted spaces of pluriharmonic functions are analyzed. This is the convenient context to consider the composition operator with holomorphic symbols. We characterize the norm and the essential norm for some important weights, extending results from [10, 24, 59] to these spaces. If v and w are weights on G_1 and G_2 respectively, then the continuity of composition operators $C_\varphi : ph_v(G_1) \to ph_w(G_2)$ is equivalent to $C_\varphi\left(ph_v(G_1)\right) \subset ph_w(G_2)$, and also $\sup_{z\in G_2} \frac{w(z)}{\widetilde{v}_{ph}(\varphi(z))} < \infty$. Moreover, if these equivalences hold, then $\|C_\varphi\| = \sup_{z\in G_2} \frac{w(z)}{\widetilde{v}_{ph}(\varphi(z))}$.

In case of v being a decreasing unitary weight on the unit ball $B_{\mathbb{C}^N}$ of $\mathbb{C}^N$ and convergent to 0 on the boundary, such that $\log(\frac{1}{v})$ is convex, G an open and connected in $\mathbb{C}^N$, w a weight on G and $\varphi : G \to B_{\mathbb{C}^N}$ a holomorphic function, we prove that $C_\varphi : H_v(B_{\mathbb{C}^N}) \to H_w(G)$ is compact if and only if $C_\varphi : ph_v(B_{\mathbb{C}^N}) \to ph_w(G)$ is compact if and only if $C_\varphi : ph_v(B_{\mathbb{C}^N}) \to ph_{w_0}(G)$ is compact.

We get a sequence of compact operators in Proposition 2.4.1 which is the key to the proofs of the results about the essential norm which extend the main theorems in [10, 59] to our context. If G is a balanced, bounded and open subset of $\mathbb{C}^N$ and v is a weight on G which vanishes at infinity and such that there exists $M > 0$ such that

$$\sup_{z\in G, 0<r<1} \frac{v(z)}{\widetilde{v}_h(rz)} \leq M,$$

then there exists a sequence of operators (T_n) on $(h(G), \tau_0)$ such that $T_n : h_v(G) \to h_v(G)$ is compact for each $n \in \mathbb{N}$ and the following conditions are fulfilled: $H(G)$ and $ph(G)$ are invariant subspaces of T_n for each $n \in \mathbb{N}$, $\tau_0 - \lim\limits_{n\to\infty} T_n = I$ and $\limsup\limits_{n\to\infty} \|I - T_n\| \leq 1$.

Under the same conditions on the weight v, Theorem 2.4.2 states as upper bound of essential norm of composition operator $C_\varphi : ph_v(G_1) \to ph_w(G_2)$:

$$\lim_{n\to\infty} \sup_{\varphi(z)\in G_1\setminus K_n} \frac{w(z)}{\widetilde{v}_{ph}(\varphi(z))},$$

where w is a weight on G_2 and $(K_n)_n$ is a fundamental sequence of compact subsets of G_1. Here also, we give a criterion to determine when a composition operator is compact. More precisely, C_φ is compact if and only if $\|C_\varphi\|_e = \lim\limits_{n\to\infty} \sup\limits_{\varphi(z)\in G_1\setminus K_n} \dfrac{w(z)}{\widetilde{v}_{ph}(\varphi(z))} = 0$.

In Theorem 2.4.4 we get $\|C_\varphi\|_e = \limsup_{|\varphi(z)|\to 1} \frac{w(z)}{\widetilde{v}(\varphi(z))}$, where B is the unit ball of $(\mathbb{C}^N, |\cdot|)$, $\varphi : G \to B$ a holomorphic function on an open and connected set G in $\mathbb{C}^M$, $g : [0, 1[\to \mathbb{R}^+$ a continuous function with $g(1^-) = 0$, $v(z) = g(|z|)$ a weight on B such that $\widetilde{v}_H = \widetilde{v}_{ph}$ and w a weight on G which vanishes at ∞ and supposing that the operator $C_\varphi : ph_v(B) \to ph_w(G)$ is continuous.

Zheng proved in [74] that for a holomorphic function $\varphi : \mathbb{D} \to \mathbb{D}$, the essential norm of the composition operator $C_\varphi : H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D})$ is either 0 or 1. In the weighted Banach spaces of holomorphic and harmonic functions the situation differs. For the typical weight $v(z) = 1 - |z|$ and each value of the interval $[0, 1]$ there is a composition operator $C_\varphi : H_v(\mathbb{D}) \to H_v(\mathbb{D})$ such that its essential norm attains this value.

In the third chapter we prove that if $(X, \|\cdot\|)$ is a real Banach space, $C \subseteq B_{X^*}$ is a weak* closed James boundary and $x_0 \in S_X$ weak* exposes x_0^* in C, then $\|\cdot\|$ is Gâteaux differentiable at x_0. In this case, if x_0^* is the differential of $\|\cdot\|$ at x_0 (i.e. $x_0^* \in S_{X^*}$ and $x_0^*(x_0) = 1$) and if it holds that given a sequence $(x_n^*)_n$ in $C \cap S_{X^*}$ which is weak* convergent to x_0^* then $\|x_n^* - x_0^*\|$ tends to 0, we can conclude that the norm $\|\cdot\|$ is Fréchet differentiable at x_0.

We include a vector-valued version of [26, Example I.1.6 (b)]. This result appears in [53, Theorem 3.1 and Corollary 3.2], [67, Thereom 1 and Theorem 2] and [23, Corollary 4] with subtle changes. More precisely, if $Y \subseteq C(K, X)$ is a closed subspace separating points of K, K being a compact Hausdorff space, then the norm of Y is Gâteaux differentiable at $f \in Y$ if and only if there is $k \in K$ peaking f such that the norm of X is Gâteaux differentiable at $f(k)$. If $Y = C(K, X)$, then the norm is Fréchet differentiable at f if and only if there is an isolated point $k \in K$ peaking f such that the norm of X is Fréchet differentiable at $f(k)$. The fact that the proof works for a subspace of $C(K, X)$ allows us to obtain as an application an easy proof of the characterization of Gâteaux differentiability of $K(X, Y)$, X being a reflexive Banach space, i.e. the norm of $K(X, Y)$ is Gâteaux differentiable at $T \in S_{K(X,Y)}$ if and only if there is $x_0 \in S_X$ such that $\|T(x_0)\| = 1$, the norm of Y is Gâteaux differentiable at $T(x_0)$ and $\|T(x)\| < \|x\|$ for all $x \in X \setminus span\,\{x_0\}$. This is the result announced by Heinrich in [40] without proof.

In Section 3.4, for a real Banach space X we denote by $h(U, X)$ the space of harmonic functions on U an open and connected set of $\mathbb{R}^d$ with values in X, $h(U)$ denotes the space of real valued harmonic functions. With this notation, the space of complex valued harmonic functions observed as a real topological vector space is denoted by $h(U, \mathbb{R}^2)$. A function $f : U \to X$ is harmonic in a strong sense, i.e. it is a C^∞ function which is in the kernel of the vector-valued Laplacian, if and only if $x^* \circ f \in h(U)$ for all $x^* \in X^*$ (e.g. [13, Corollary 10]). Here and in the following sections a weight on U is a bounded continuous function $v : U \to]0, \infty[$.

If $h_{v_0}(U)$ contains the polynomials of a degree smaller or equal than 1, then the norm $\|\cdot\|_v$ is Gâteaux differentiable at $f \in h_{v_0}(U, X)$ if and only if there exists $z_0 \in P_v$ peaking f with the norm of X being Gâteaux differentiable at $v(z_0)f(z_0)$. In this case $\|\cdot\|_v$ is also Gâteaux differentiable at f in $h_v^c(U, X)$ and the norm $\|\cdot\|_v$ is Fréchet differentiable at f considered as a function in $h_v^c(U, X)$ if and only $\|\cdot\|$ is Fréchet differentiable at $v(z_0)f(z_0)$, where $h_v^c(U, X) := \{f \in h_v(U, X) : (vf)(U)$ is relatively compact$\}$. We deduce that the norm of $A(\overline{\mathbb{D}})$ is Gâteaux differentiable at f if and only if there is $z \in \partial\overline{\mathbb{D}}$ such that $|f(z)| = 1$ and $|f(y)| < 1$ for each $z \neq y$ and although there are functions in $A(\overline{\mathbb{D}})$ for which the norm is Gâteaux differentiable, however, there is no function $f \in S_{A(\overline{\mathbb{D}})}$ such that the norm in $A(\overline{\mathbb{D}})$ is Fréchet differentiable at f. Moreover, there is no function $f \in S_{H^\infty(\mathbb{D})}$ such that the norm in $H^\infty(\mathbb{D})$ is Gâteaux differentiable at f. Therefore, $h_{v_0}(U,\mathbb{R}^2)$ and $h_v(U,\mathbb{R}^2)$, and the corresponding subspaces of holomorphic functions $H_{v_0}(U)$ and $H_v(U)$ behave differently from $A(\overline{\mathbb{D}})$ and $H^\infty(\mathbb{D})$ with respect to differentiability of the norm.

In the last chapter, the motivation of the proposed problem is given by the fact that for an open set $U \subseteq \mathbb{R}^d$, the space $h_v(U)$ is isometric to a subspace of l_∞ and for the proof it is enough to take $z_n)_n$ a countable and dense set of U and define $T : h_v(U) \to l_\infty$ by $T(f) = (v(z_n)f(z_n))_n$, but this argument does not work for $h_{v_0}(U)$. We show that if U is a connected open subset of $\mathbb{R}^d$, v is a weight on U and $0 < \varepsilon < 1$, then, there exists a continuous linear injective map with closed range $T : h_{v_0}(U) \to c_0$ such that $(1-\varepsilon)\left\|f\right\|_v \leq \left\|T(f)\right\|_{c_0} \leq \left\|f\right\|_v$ for each $f \in h_{v_0}(U)$. Hence, the space $h_{v_0}(U)$ is isomorphic to a closed subspace of c_0. The argument of the proof given by Bonet and Wolf in [17] for the corresponding spaces of holomorphic functions works here introducing Cauchy type inequalities for the derivatives of harmonic functions.

Some geometrical conditions in U and convexity in the weight v ensure that neither $H_{v_0}(U)$ nor $h_{v_0}(U)$ are rotund. These conditions also imply that neither $H_{v_0}(U)$ nor $h_{v_0}(U)$ can be isometric to any subspace of c_0.

Chapter 0

Preliminaries

0.1 Basic notation

We denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^d$ or $\mathbb{C}^d$, i.e. for $z = (x_1+iy_1, \cdots, x_d+iy_d)$, $\|z\| = \sqrt{x_1^2 + \cdots + x_d^2 + y_1^2 + \cdots + y_d^2}$. For $x, y \in \mathbb{R}^d$ (or $\mathbb{C}^d$) we denote by $< x, y >$ its canonical scalar product.

We use $|\cdot|$ for the modulus of a real or a complex number. Sometimes we make an abuse of notation and $|\cdot|$ denotes an arbitrary norm in $\mathbb{R}^d$ or in $\mathbb{C}^d$ and the modulus in $\mathbb{R}$ or $\mathbb{C}$.

A subset C of a vector space V over either $\mathbb{R}$ or $\mathbb{C}$ is circled if $\lambda C \subset C$ whenever $|\lambda| \leq 1$. We denote by $\mathbb{N} = \{1, 2, \cdots\}$ the set of natural numbers and $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$.

If $a \in \mathbb{C}$ and $r > 0$, then $D(a, r)$ denotes the *disc* of center a and radius r, i.e. $\{z \in \mathbb{C} : |z - a| < r\}$. For the unit disc $D(0, 1)$ we use the symbol $\mathbb{D}$. If $a = (a_1, \cdots, a_d) \in \mathbb{C}^d$ and $r > 0$ then, $B(a, r) := \{z \in \mathbb{C}^d : \|z - a\| < r\}$ and $D^d(a, r) := \{z \in \mathbb{C}^d : \max_j\{|z_j - a_j|\} < r\}$ denote the *open Euclidean ball* and the *open polydisc* respectively. We also consider *polydiscs* of the form $D(a_1, r_1) \times \cdots \times D(a_d, r_d)$.

0.2 Harmonic and Holomorphic functions

Let $U \subseteq \mathbb{C}^d$. There are many possible equivalent definitions of holomorphic function. Four of them are offered in [51, Chapter 0] relating the classical one-variable sense in each variable separately with the Cauchy-Riemann conditions, the power series and the Cauchy formula.

Definition 0.2.1. A function $f : U \to \mathbb{C}$ is holomorphic if it satisfies one of the following equivalent conditions:

1. For each $j = 1, \cdots, d$ and each fixed $z_1, \cdots, z_{j-1}, z_{j+1}, \cdots, z_d$ the function
$$\zeta \mapsto f(z_1, \cdots, z_{j-1}, \zeta, z_{j+1}, \cdots, z_d)$$
is holomorphic, in the classical one-variable sense, on the set
$$U(z_1, \cdots, z_{j-1}, z_{j+1}, \cdots, z_d) := \{\zeta \in \mathbb{C} : (z_1, \cdots, z_{j-1}, \zeta, z_{j+1}, \cdots, z_d) \in U\}.$$

2. f is continuously differentiable in each complex variable separately on U and f satisfies the Cauchy-Riemann equations in each variable separately.

3. For each $z_0 \in U$ there is an $r = r(z_0) > 0$ such that the closed polydisc $\overline{D}^d(z_0, r) \subseteq U$ and f can be written as an absolutely and uniformly convergent power series
$$f(z) = \sum_\alpha a_\alpha (z - z_0)^\alpha$$
for all $z \in \overline{D}^d(z_0, r)$.

4. f is continuous in each variable separately and locally bounded and for each $w \in U$ there is an $r = r(w) > 0$ such that $\overline{D}^d(w, r) \subseteq U$ and
$$f(z) = \frac{1}{(2\pi i)^d} \oint_{|\zeta_d - w_d| = r} \cdots \oint_{|\zeta_1 - w_1| = r} \frac{f(\zeta_1, \cdots, \zeta_d)}{(\zeta_1 - z_1) \cdots (\zeta_d - z_d)} d\zeta_1 \cdots d\zeta_d$$
for all $z \in D^d(w, r)$.

We denote by $H(U)$ the space of complex valued holomorphic functions on U.

Definition 0.2.2. Let U be an open and connected set of $\mathbb{R}^d$. A twice continuously differentiable, complex-valued function f defined on U is *harmonic* on U if $\triangle f = 0$, where $\triangle$ is the Laplacian operator, i.e., $\triangle = D_1^2 + \cdots + D_d^2$ and D_j^2 denotes the second partial derivative with respect to the $j - th$ coordinate variable.

We denote by $h(U)$ the space of complex valued harmonic functions on U.

Definition 0.2.3. If $U \subseteq \mathbb{R}^d$ and $\varphi \colon U \to \mathbb{R} \cup \{-\infty\}$ is a continuous function then, φ is called *subharmonic* if for each $a \in U$, there exists a closed ball $\overline{B(a, r)} \subseteq U$ such that
$$\varphi(a) \leq \int_S \varphi(a + r\zeta) d\sigma(\zeta)$$
whenever $0 < r \leq R$, where S is the unit sphere and σ is the unique Borel probability measure on S that is rotation invariant.

By identifying $\mathbb{C}$ with $\mathbb{R}^2$ we can consider U an open and connected subset of $\mathbb{C}^d$ and $h(U)$ the space of complex valued harmonic functions on U of $2d$ real variables.

Although the situation in several complex variables is different and other problems are present, some properties continue staying because of the inclusion $H(U) \subseteq h(U)$ which is deduced from the definitions. For instance, by [39] if h is a separately harmonic function on an open neighbourhood of an $(m-1)-$dimensional compact submanifold $\sum$ in $\mathbb{R}^m$ with $m \geq 2$ then, h can be extended to a separately harmonic function on the bounded component of $\mathbb{R}^m \setminus \Sigma$. However, in [72] an example of a separately subharmonic function which is not subharmonic is given.

In one complex variable, harmonic functions are (locally) the real parts of holomorphic functions. The analogous behaviour in several variables is given by pluriharmornic functions (see for instance [51, 2.2]).

Definition 0.2.4. Let $U \subseteq \mathbb{C}^d$ be an open and connected subset. A function $f : U \to \mathbb{C}$ of class C^2 is said to be *pluriharmonic* if for every complex line $l = \{a + b\lambda\}$ the function $\lambda \to f(a + b\lambda)$ is harmonic on the set $U_l = \{\lambda \in \mathbb{C} : a + b\lambda \in U\}$.

Proposition 0.2.5. *Let $D^d(P, r) \subseteq \mathbb{C}^d$ be a polydisc and assume that*

$$f : D^d(P, r) \to \mathbb{R}$$

is C^2. Then f is pluriharmonic on $D^d(P, r)$ if and only if f is the real part of a holomorphic function on $D^d(P, r)$.

We denote by $ph(U)$ the space of complex valued pluriharmonic functions on U.

All these concepts can be extended to functions which take vectorial values in the following way. A function $f : U \subseteq \mathbb{R}^d \to X$, X being a locally convex space, is said to be of class C^1 if, for all $1 \leq i \leq d$, there is a continuous function $\frac{\partial f}{\partial x_i} : U \to X$ such that

$$\frac{\partial f}{\partial x_i}(x) = \lim_{t \to 0} \frac{1}{t} \left(f(x + te_i) - f(x)\right), x \in U.$$

Here e_i denotes the $i-th$ vector of the canonical basis of $\mathbb{R}^d$. If $(\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$, then

$$\partial^\alpha f = \frac{\partial^{|\alpha|}}{\partial x^\alpha} f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} f.$$

The space of all functions $f : U \to X$ such that $\frac{\partial^{|\alpha|}}{\partial x^\alpha} f : U \to X$ is a well defined continuous function for $|\alpha| \leq k$ is denoted by $C^k(U, X)$. Whenever f is infinitely differentiable and $P(\partial, x) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha$ is a linear partial differential operator with smooth coefficients, then $P(\partial, x)f = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha f$ is also an infinitely differentiable function.

A function $f : U \subseteq \mathbb{C}^d \to X$ is said to be holomorphic if, for each $z_0 \in U$, there exists $r > 0$ and a sequence $(a_n)_n \subseteq X$ such that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for each $z \in B(z_0, r)$.

The space of holomorphic functions with values in X is denoted by $H(U, X)$. In the same way, the space of vector-valued harmonic functions $h(U, X)$ is defined as the set of twice continuously differentiable functions $f : U \to X$ such that $\triangle f \equiv 0$. If X is locally complete then a function $f \in C^\infty(U, X)$ is holomorphic if and only if f satisfies the Cauchy-Riemann equations, see [13].

If (Y, τ) is a topological space and X a locally complete locally convex Hausdorff space, we can consider Y' (topological dual of Y) endowed with the topology of uniform convergence on all convex compact sets. The set of all continuous linear operators from (Y', co) into X denoted by $Y \varepsilon X$ is called L. Schwartz' *$\varepsilon-$product of Y and X.*

As a consequence of results due to Bonet, Frerick, Jordá and Maestre (see [16, prop. 2] and [13, 2]) we obtain the following: If X is locally complete then these definitions are consistent with the following different definitions based on L. Schwartz' $\varepsilon-$product:

$$\begin{aligned} C^\infty(U, X) &: = \{x \to T(\delta_x) : T \in C^\infty(G)\varepsilon X\}, \\ H(U, X) &: = \{x \to T(\delta_x) : T \in H(U)\varepsilon X\}, \\ h(U, X) &: = \{x \to T(\delta_x) : T \in h(U)\varepsilon X\}. \end{aligned}$$

0.3 Banach spaces. Definitions and general theory

0.3.1 Notations and classical theorems

For a Banach space $(X, \|\cdot\|)$, the dual of X is denoted by X^*. (X, w) and $(X^*, w*)$ are X and X^* endowed with the weak $(\sigma(X, X^*))$ and the weak* $(\sigma(X^*, X))$ topology, respectively. We write B_X for the unit ball of X and B_{X^*} for the unit ball of X^*. We use S_X and S_{X^*} for the unit spheres of X and X^*, respectively. Let $M \subseteq X$ and $N \subseteq X^*$ both be nonempty. Define

$$M^\circ := \{x^* \in X^* : |x^*(x)| \leq 1 \text{ for all } x \in M\}$$

$$N^\circ := \{x \in X : |x^*(x)| \leq 1 \text{ for all } x^* \in N\}$$

M° is called the *polar* of M in X^*, N° is called the *polar* of N in X.

The well known extension and separation theorems due to Hahn and Banach are frequently used in the proofs of the results that we present in this work. We state below other important theorems which we also use.

Theorem 0.3.1 (**Krein-Milman**). *Every convex, compact subset $K \neq \emptyset$ of a locally convex space X is the closure of the convex hull of the extreme points of K.*

Theorem 0.3.2 (**Josefson-Nissenzweig**). *If X is an infinite dimensional Banach space then there is a sequence (x_n^*) of elements of the dual unit sphere S_{X^*} which converges to 0 in the weak* topology.*

Theorem 0.3.3 (**Alaoglu-Bourbaki**). *For each neighbourhood U of zero in a locally convex space X, its polar U° is absolutely convex and $\sigma(X^*, X)$-compact, being X^* the topological dual of X.*

Theorem 0.3.4 (**Mazur**). *The norm closure and the weak closure of a convex set A in a locally convex space X coincide.*

Theorem 0.3.5 (**James**). *A Banach space X is reflexive if and only if every bounded linear functional on X attains its maximum on the closed unit ball in X.*

Theorem 0.3.6 (**Eberlein-Šmulian-Grothendieck**). *Let X be a Banach space. A set in X is weakly relatively compact if and only if it is weakly sequentially relatively compact if and only if it is weakly countably relatively compact.*

Theorem 0.3.7 (**Krein-Šmulian**). *If C is a weakly compact set in a Banach space X, then $\overline{co}(C)$ is weakly compact.*

Theorem 0.3.8 (**Milman's converse to the Krein-Milman theorem**). *Let X be a locally convex space and F a nonempty subset of X such that $\overline{co}(F)$ is compact. Then $Ext(\overline{co}(F)) \subseteq \overline{F}$.*

0.3.2 Operators

Terminology about topologies on $L(X,Y)$, which denotes the normed space of all bounded linear mappings from the normed space X to the normed space Y, can be seen in [50, p.161–164] for instance. The norm on $L(X,Y)$ is the *operator norm*

$$\|T\| := sup_{\|x\|<1}\|T(x)\|.$$

It is also called the *uniform operator topology* (u.o.t.). We can consider two additional topologies on $L(X,Y)$, weaker than the uniform operator topology. A net $\{T_j : j \in J\}$ converges to T in the *strong operator topology* (s.o.t.) of $L(X,Y)$ if $T_j(x) \to T(x)$ strongly in Y, for each $x \in X$. This convergence is clearly weaker than convergence in the u.o.t. If one requires that $(T_j(x))_j$ converges weakly to $T(x)$, for each $x \in X$, one gets a still weaker convergence concept called convergence in the *weak operator topology* (w.o.t.). The s.o.t. and the w.o.t. may be defined by giving bases as follows:

1. A basis for the strong operator topology $L(X,Y)$ consists of all the sets of the form

$N(T,F,\varepsilon) := \{S \in L(X,Y) : \|(S-T)(x)\| < \varepsilon, x \in F\}$, where $T \in L(X,Y)$, $F \subseteq X$ is finite, and $\varepsilon > 0$.

2. A basis for the weak operator topology on $L(X,Y)$ consists of all sets of the form

$N(T,F,\bigwedge,\varepsilon) := \{S \in L(X,Y) : |y^*((S-T)(x))| < \epsilon, x \in F, y^* \in \bigwedge\}$, where $T \in L(X,Y)$, $F \subseteq X$ and $\bigwedge \subseteq Y^*$ are finite sets, and $\varepsilon > 0$.

Since the bases consist of convex sets it is clear that $L(X,Y)$ is a locally convex topological vector space for each of the above topologies. By $L(X,Y)_{s.o.}$ we denote L(X,Y) endowed with the strong operator topology. Analogously we write $L(X,Y)_{w.o.}$ if it is equipped with the weak operator topology. It can be shown that $L(X,Y)^*_{s.o.} = L(X,Y)^*_{w.o.}$. Moreover, the general form of an element g of this (common) dual is $g(T) = \sum_k y_k^*(T(x_k))$, $(T \in L(X,Y))$, where the sum is finite, $x_k \in X$ and $y_k^* \in Y^*$, (see [50, Theorem 6.19]). A convex subset of $L(X,Y)$ has the same closure in the w.o.t. and in the s.o.t.

0.3.3 Biduality

Theorem 0.3.9 (Dixmier-Ng). *Let E be a Banach space and*

$$B = \{x \in E : \|x\| \leq 1\}.$$

We suppose that there is a locally convex and separated topology t such that B is compact in (E,t). Then:

(i) $F := \{u \in E^* : u|_{(B,t)}$ is continuous$\}$ endowed with the norm $p(u) := \sup_{x \in B} |u(x)|$ is a Banach space.

(ii) $J : E \to F'$ defined by $J(x)(u) := u(x)$, $x \in E$, $u \in F$, defines a linear and surjective isometry.

Theorem 0.3.10 (Bierstedt-Bonet-Summers). *Let E be a Banach space and*

$$B = \{x \in E : \|x\| \leq 1\}.$$

Suppose that a topology t locally convex and separated such that B is compact in (E,t) exists. Let $F := \{u \in E^* : u|_{(B,t)}$ is continuous $\}$ endowed with the norm $p(u) := \sup_{x\in B} |u(x)|$. Let H be a closed vectorial subspace in $(E, \|\cdot\|)$. Then, for the map $R : F \to H'$, $Ru := u|_H$ the following properties hold:

(a) R is well defined, linear and continuous.

(b) R is a monomorphism (injective and the range is open) if and only if there exists $M > 0$ such that $B \subset M\overline{B \cap H}^{(E,t)}$.

(c) R is a isometric monomorphism if and only if $B \subset \overline{B \cap H}^{(E,t)}$.

(d) If R is a monomorphism, then R is surjective if and only if for all v in H' holds $v|_{(B\cap H)}$ is t-continuous.

0.3.4 Extreme points

Definition 0.3.11. Let K be a subset of a real or complex linear vector space X. A non void subset $A \subseteq K$ is said to be an *extremal subset* of K if a proper convex combination $ak_1 + (1-a)k_2$, $0 < a < 1$, of two points of K is in A only if both k_1 and k_2 are in A. An extremal subset of K consisting of just one point is called an *extreme point* of K.

We denote by $Ext(K)$ the set of the extreme points of K. If A and B are subsets of X and $A \subseteq B$ then $Ext(B) \bigcap A \subseteq Ext(A)$.

If K is convex, then x is an extreme point if, and only if there are no $y, z \in K$ such that $y \neq z$ and $x = \frac{1}{2}(y+z)$.

The set of extreme points is not necessarily closed, even in spaces of finite dimension. For instance, if we take a circle in the plane $z = 0$ and a segment vertical in $\mathbb{R}^3$ that cuts to the circle in an interior point, then the set of extreme points is not closed.

Since B_X is convex and $S_X \subseteq B_X$ it holds that every extreme point of B_X is an extreme point of S_X and reciprocally. Therefore, we can take the following definition which is equivalent.

Definition 0.3.12. A vector x in B_X of a Banach space X is called *extreme point* if $x \in S_X$ and we can not find $y, z \in S_X$ with $y \neq z$ and $x = \frac{1}{2}(y+z)$.

Definition 0.3.13. A Banach space X is said to be *rotund* if $\left\|\frac{x+y}{2}\right\| < 1$ for $\|x\| = \|y\| = 1$ and $x \neq y, x, y \in X$.

Definition 0.3.14. A vector $x \in S_X$ is called *exposed* if there is $x^* \in X^*$ such that $\|x^*\| = 1, x^*(x) = 1$ and $x^*(y) \neq 1$ for each $y \in S_X \setminus \{x\}$.

Proposition 0.3.15. *A vector $x \in S_X$ is exposed if and only if there is $x^* \in X^*$ such that $\|x^*\| = 1, x^*(x) = 1$ and $\mathrm{Re}(x^*(y)) < 1$ for each $y \in B_X \setminus \{x\}$, where $\mathrm{Re}(z)$ denotes the real part of a complex number z.*

Proof. Indeed, $\operatorname{Re}(x^*(y)) \leq |\operatorname{Re}(x^*(y))| \leq |x^*(y)| \leq \|x^*\| \|y\| \leq 1$, as $x^*(y) \neq 1$, we have the strict inequality.

Conversely, if there exists $y \in S_X \setminus \{x\}, x^*(y) = 1$, then $\operatorname{Re}(x^*(y)) = 1$, a contradiction. □

Example 0.3.16. If we consider $\mathbb{C}$ as a $\mathbb{C}$-vector space, then every point $z_0 \in \partial\overline{\mathbb{D}}$ is exposed. It is enough to take $u : \mathbb{C} \to \mathbb{C}$ defined by $u(z) = \overline{z_0}z$.

Proposition 0.3.17. *If X is a real Banach space on $\mathbb{R}$ then a vector $x \in S_X$ is exposed if and only if there is $x^* \in X^*$ such that $x^*(x) > x^*(y)$ for each $y \in S_X \setminus \{x\}$.*

Proof. First, we assume that $x \in S_X$ is exposed. Then we can find $x^* \in X^*$ with $\|x^*\| = 1, x^*(x) = 1$ and $x^*(y) \neq 1$ for each $y \in S_X \setminus \{x\}$. Thus, $1 = x^*(x) = \|x^*\| = \sup_{\|z\|\leq 1} |x^*(z)| \geq |x^*(y)| \geq x^*(y)$ for each $y \in S_X \setminus \{x\}$.

The inequality is strict since $x^*(y) \neq 1$.

Conversely, if there is $x^* \in X^*$ such that $x^*(x) > x^*(y)$ for each $y \in S_X \setminus \{x\}$, then, $1=\|x^*\| = x^*(x)$ and $x^*(y) \neq 1$ for each $y \in S_X \setminus \{x\}$. □

Proposition 0.3.18. *If $x \in S_X$ is exposed, then x is extreme.*

Proof. Let us assume that there are $y, z \in S_X$ such that $y \neq z$ and $x = \frac{1}{2}(y + z)$. Hence $y \neq x, z \neq x$.

Since x is exposed, there is $x^* \in X^*$ such that $\|x^*\| = 1, x^*(x) = 1$ and $\operatorname{Re}(x^*(y)) < 1$. But for each $y \in B_X \setminus \{x\}$ we have

$$1 = \operatorname{Re}(x^*(x)) = \frac{1}{2}(\operatorname{Re}(x^*(y)) + \operatorname{Re}(x^*(z))) < 1$$

which is a contradiction. □

Proposition 0.3.19. *If X is a rotund Banach space, then every point in S_X is exposed.*

Proof. Let $x \in S_X$. By the Hahn-Banach theorem there exists $x^* \in X^*$ such that $x^*(x) = 1, \|x^*\| = 1$.

If for some $y \in S_X \setminus \{x\}$ we have $x^*(y) = 1$, then

$$1 = \left|\frac{1}{2}x^*(x + y)\right| \leq \|x^*\| \left\|\frac{1}{2}(x + y)\right\| < 1.$$

In the first equality we have used the linearity of x^* and the last inequality is deduced since the space is rotund. □

We can also consider exposed points of X^*.

Definition 0.3.20. A functional $x^* \in S_{X^*}$ is called *weak*-exposed* if there is $x \in X$ such that $\|x\| = 1, x^*(x) = 1$ and $y^*(x) \neq 1$ for each $y^* \in S_{X^*} \setminus \{x^*\}$.

Definition 0.3.21. A functional $x^* \in S_{X^*}$ is called *strongly weak*-exposed* if there is a unit vector $x \in X$ such that $x^*(x) = 1$ and given any sequence $(x_k^*)_k$ in the unit ball of X^* with $x_k^*(x) \to 1$, $(x_k^*)_k$ converges to x^* in norm. In this case, we say that x *exposes strongly-weak** the unit ball of X^* at x^*.

Lemma 0.3.22. *Let $U \subseteq \mathbb{C}^d$ be the unit ball for a norm $|\cdot|$. Then there exists $z_0 \in \partial U$ which is exposed.*

The key of the proof of the following proposition is in the use of Straszewicz's theorem and the theorem of Krein-Milman applied to the finite dimensional space $\mathbb{C}^d$.

Theorem 0.3.23 (Straszewicz). *[1, Theorem 7.89]*

In a finite dimensional vector space, the set of exposed points (and hence the set of strongly exposed points) of a nonempty closed subset is dense in the set of its extreme points.

Proof. (Of Lemma 0.3.22)

Seeking a contradiction, if $\partial\overline{U}$ has no exposed points then by an application of Straszewicz's theorem it follows that $\partial\overline{U}$ has no extreme points.

Now the theorem of Krein-Milman implies that $\overline{U}$ is the closure of the convex hull of its extreme points. Thus $\partial\overline{U} = \emptyset$, which is a contradiction.

□

Lemma 0.3.24. *[27, Lemma V.8.6] Let K be a compact and Hausdorff set. Then, $A = \{\alpha\delta_k : |\alpha| = 1, \alpha \in \mathbb{C}, k \in K\}$ is the set of extreme points of the closed unit ball of $C(K)^*$.*

Proposition 0.3.25. *Let X and Y be Banach spaces and $T : X \to Y$ a linear continuous and surjective map. Let $T^* : Y^* \to X^*$, $T^*(y^*) = y^* \circ T$.*

If T is an isometry then T^* is also an isometry and y^* is an extreme point of the unit ball of Y^* if and only if $T^*(y^*)$ is an extreme point of the unit ball of X^*.

0.3.5 Smoothness

Definition 0.3.26. Let f be a real valued function defined on a Banach space X. We say that f is *Gâteaux differentiable* or *Gâteaux smooth* at $x \in X$, if for each $h \in X$,

$$f'(x)(h) := \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}$$

exists and is a linear continuous function in h (i.e. $f'(x) \in X^*$).

If, in addition, the limit above is uniform in $h \in S_X$, we say that f is *Fréchet differentiable* at x. Equivalently, f is Fréchet differentiable at x if there exists $f'(x) \in X^*$ such that

$$\lim_{y \to 0} \frac{f(x+y) - f(x) - f'(x)(y)}{\|y\|} = 0.$$

If f is Fréchet differentiable at x, then f is Gâteaux differentiable at x. Let us remark that the Gâteaux differential $f'(x)$, assigns to h its directional derivative. But, there exist functions with directional derivatives in all directions that are not Gâteaux differentiable. In [70] Šmulyan shows that x^* is a weak*-exposed vector in B_{X^*} by $x \in B_X$ if and only if the norm in X (a real Banach space) is Gâteaux differentiable at x with derivative x^* while in [71] he shows that x^* is a strongly weak*-exposed vector in B_{X^*} by $x \in B_X$ if and only if the norm in X is Fréchet differentiable at x with derivative x^*.

Lemma 0.3.27. *(**Šmulyan**)[26, Lemma 1.2] [70, 71] If $\|\cdot\|$ denotes the norm of a Banach space X and $x \in S_X$, then the following are equivalent:*

(i) The norm $\|\cdot\|$ is Gâteaux differentiable at x.

(ii) $\lim_{t\to 0} \frac{\|x+th\|-\|x\|}{t}$ exists for every $h \in X$.

(iii) $\lim_{t\to 0} \frac{\|x+th\|+\|x-th\|-2\|x\|}{t} = 0$ for every $h \in X$.

Lemma 0.3.28. *(**Šmulyan**)[26, Lemma 1.3] [70, 71] If $\|\cdot\|$ denotes the norm of a Banach space X and $x \in S_X$, then the following are equivalent:*

(i) The norm $\|\cdot\|$ is Fréchet differentiable at x.

(ii) $\lim_{t\to 0} \frac{\|x+th\|-\|x\|}{t}$ exists for every $h \in X$ and is uniform in $h \in S_X$.

(iii) $\lim_{\|y\|\to 0} \frac{\|x+y\|+\|x-y\|-2\|x\|}{\|y\|} = 0$.

Proposition 0.3.29. *(**Šmulyan**)[26, Theorem 1.4] [70, 71] Suppose that $\|\cdot\|$ is a norm on a Banach space X with dual norm $\|\cdot\|^*$. Then*

1. *The norm $\|\cdot\|$ is Fréchet differentiable at $x \in S_x$ if and only if whenever $f_n, g_n \in S_X^*$, $f_n(x) \to 1$ and $g_n(x) \to 1$, then $\|f_n - g_n\|^* \to 0$.*

2. *The norm $\|\cdot\|^*$ is Fréchet differentiable at $f \in S_{X^*}$ if and only if whenever x_n, $y_n \in S_X$, $f(x_n) \to 1$ and $f(y_n) \to 1$, then $\|x_n - y_n\| \to 0$.*

3. *The norm $\|\cdot\|$ is Gâteaux differentiable at $x \in S_X$ if and only if whenever f_n, $g_n \in S_{X^*}$, $f_n(x) \to 1$ and $g_n(x) \to 1$, then $f_n - g_n \to^{w^*} 0$.*

4. *The norm $\|\cdot\|^*$ is Gâteaux differentiable at $f \in S_{X^*}$ if and only if whenever x_n, $y_n \in S_X$, $f(x_n) \to 1$ and $f(y_n) \to 1$, then $x_n - y_n \to^w 0$.*

Proposition 0.3.30. ***(Šmulyan)*** *[26, Corollary 1.5] [70, 71]*
Let X be a real Banach space and $\|\cdot\|$ be a norm on X.
(i) The norm $\|\cdot\|$ is Gâteaux differentiable at $x \in S_X$ with derivative x^* if and only if there is a unique $x^* \in S_{X^*}$ such that $x^*(x) = 1$
(i.e. x^* is weak-exposed in S_{X^*}).*
(ii) The norm $\|\cdot\|$ is Fréchet differentiable at $x \in S_X$ with derivative x^* if and only if there is a unique $x^* \in S_{X^*}$ satisfying:
for every $\varepsilon > 0$ there exists $\delta > 0$ such that $y^* \in B_{X^*}$ and $y^*(x) > 1 - \delta$ imply $\|y^* - x^*\| < \varepsilon$ (i.e. u is strongly weak-exposed in S_{X^*}).*

0.4 Classical Banach spaces of holomorphic and harmonic functions: H^∞ and h^∞

We denote by H^∞ the space of bounded holomorphic functions on $\mathbb{D}$ and h^∞ the space of bounded harmonic functions on $\mathbb{D}$ endowed with the supremum norm. In this section we include some of the results on these spaces and the composition operators on them (see [22, 74]).

Proposition 0.4.1. *[22, Proposition 3.2] Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Then the following are equivalent:*
(i) C_φ is compact on h^∞.
(ii) C_φ is compact on H^∞.
(iii) $\|\varphi\| < 1$.

If we define $\rho(z,w) = \left|\frac{z-w}{1-\bar{z}w}\right|$, the pseudo-hyperbolic distance between z and w in $\mathbb{D}$, then

$$\sup_{f \in B_{H^\infty}} |f(z) - f(w)| = \frac{2\rho(z,w)}{1 + \sqrt{1 - \rho(z,w)^2}}.$$

Proposition 0.4.2. *[22, Proposition 3.4] Let $\varphi, \psi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function, $\varphi \neq \psi$ and $\|\varphi\| = \|\psi\| = 1$. Then the following are equivalent:*
(i) $C_\varphi - C_\psi$ is compact on h^∞.
(ii) $C_\varphi - C_\psi$ is compact on H^∞.
(iii) $\lim_{|\varphi(z)| \to 1} \rho(\varphi(z), \psi(z)) = \lim_{|\psi(z)| \to 1} \rho(\varphi(z), \psi(z)) = 0$.

The essential norm of C_φ on H^∞ is defined to be

$$\|C_\varphi\|_e = \inf\{\|C_\varphi - K\| : K \text{ is compact operator on } H^\infty\}.$$

Theorem 0.4.3. *[74, Theorem 1] If C_φ is not compact on H^∞, then its essential norm is* 1.

The results on interpolating sequences constitute an important tool to study the differentiability of the norm function for the space H^∞. Blaschke products is the key for the proof of well-known theorems due to Carleson, Hayman and Newman. We shall call $(z_k)_k$ an *interpolating sequence* if, for each bounded sequence of complex numbers $(w_k)_k$, there exists a function $f \in H^\infty$ such that $f(z_k) = w_k$.

Theorem 0.4.4. ***(Carleson; Newman)*** *[43, p. 197] Let $(z_k)_k$ be a sequence of points in the open unit disc. Then $(z_k)_k$ is an interpolating sequence if and only if the following condition is satisfied:*

(1) (Carleson's condition)

$$\prod_{j\neq k}\left|\frac{z_k - z_j}{1-\overline{z_j}z_k}\right| \geq \delta > 0, k = 1, 2, 3, \cdots$$

Theorem 0.4.5. ***(Hayman; Newman)****[43, p. 203] Suppose $(z_k)_k$ is a sequence of points in the open unit disc which approaches the boundary exponentially, i.e., $\frac{1-|z_n|}{1-|z_{n-1}|} < c < 1$. Then $(z_k)_k$ is an interpolating sequence.*

Corollary 0.4.6. *[43, p. 204] Any $(z_k)_k$ such that $\overline{lim}|z_k| = 1$ contains a subsequence which is an interpolating sequence.*

Corollary 0.4.7. ***(Hayman; Newman)*** *[43, p. 204] If $(z_k)_k$ is an increasing sequence of points on the positive axis, then $(z_k)_k$ is an interpolating sequence if and only if $\frac{1-z_n}{1-z_{n-1}} < c < 1$.*

0.5 Weighted Banach spaces of harmonic functions

Definition 0.5.1. Let U be an open and connected subset of $\mathbb{C}^d$ or $\mathbb{R}^d$. A *weight* on U is a function $v : G \to \mathbb{R}$ which is strictly positive and continuous.

Let U be an open and connected subset of $\mathbb{C}^d$. For a weight v the *weighted Banach spaces of holomorphic functions* with weight v are defined by:

$$\begin{aligned} H_v(U) \ &: \ = \{f \in H(U) : \|f\|_v := \sup_{z\in U} v(z)\,|f(z)| < \infty\}, \text{ and} \\ H_{v_0}(U) \ &: \ = \{f \in H(U) : vf \text{ vanishes at infinity on } U\}. \end{aligned}$$

A function $g : U \to \mathbb{C}$ is said to *vanish at infinity on* U if for each $\varepsilon > 0$ there exists a compact set $K \subset U$ such that $|g(z)| < \varepsilon$ for each $z \in U \setminus K$.

In the same way, for a weight v the *weighted Banach spaces of pluriharmonic functions* with weight v are defined by:

$$\begin{aligned} ph_v(U) \ &: \ = \{f \in ph(U) : \|f\|_v := \sup_{z\in U} v(z)\,|f(z)| < \infty\}, \text{ and} \\ ph_{v_0}(U) \ &: \ = \{f \in ph(U) : vf \text{ vanishes at infinity on } U\}. \end{aligned}$$

Let U be an open and connected subset of $\mathbb{R}^d$. For a weight v the *weighted Banach spaces of harmonic functions* with weight v are defined by:

$$\begin{aligned} h_v(U) \ &: \ = \{f \in h(U) : \|f\|_v := \sup_{z\in U} v(z)\,|f(z)| < \infty\}, \text{ and} \\ h_{v_0}(U) \ &: \ = \{f \in h(U) : vf \text{ vanishes at infinity on } U\}. \end{aligned}$$

The unit balls in these spaces are denoted by B_v, B_{v_0}, b_v^p, $b_{v_0}^p$, b_v and b_{v_0}. The balls B_v, b_v^p and b_v are compact if we endow them with the compact open topology τ_0.

Definition 0.5.2. A weight $v : U \to \mathbb{R}$ on a balanced domain U is *radial* if $v(z) = v(\lambda z)$ for all $z \in U$ and for all $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

0.5.1 Biduality

In this section we establish the relation between the spaces $ph_v(G)$ and $(ph_{v_0}(G))^{**}$. To do this, we use the Theorems 0.3.9 and 0.3.10 which are due to Dixmier and Bierstedt-Bonet-Summers respectively. When $G = B_{\mathbb{C}^N}$ we see that under certain conditions on the weight v, these spaces are canonically isometric and we also characterize the compact sets of $ph_{v_0}(B_{\mathbb{C}^N})$. Applying these results in the section 2.4, we give a generalization for the upper and lower estimates of the essential norm of composition operators in the case of pluriharmonic functions, following the way of [10] and [59] for holomorphic functions.

Proposition 0.5.3. *Let v be a weight on a connected and open set G of $\mathbb{C}^N$. Then $ph_v(G)$ is canonically isometric to $(ph_{v_0}(G))^{**}$ if and only if $b_v^p \subset \overline{b_{v_0}^p}^{\tau_0}$.*

Proof. We proceed in the same way as in the case of holomorphic functions. Let $E = ph_v(G)$, $H = ph_{v_0}(G)$ and $t = \tau_0$.

The condition is necessary, according to Theorem 0.3.10 c),

$$b_v^p \subset \overline{b_v^p \cap ph_{v_0}(G)}^{\tau_0} = \overline{b_{v_0}^p}^{\tau_0}.$$

To prove that the condition is sufficient, first we check that for all maps $\psi \in ph_{v_0}(G)^*$ we have that $\psi|_{b_{v_0}^p}$ is τ_0-continuous. It is enough to prove that it is τ_0-continuous in 0. We consider the normed space

$$C_{v_0}(G) = \{f : G \to \mathbb{C}, f \text{ continuous}, vf \text{ vanishes at infinity on } G\}$$

with the norm $\|\cdot\|_v$ and $ph_{v_0}(G)$ is a subspace. Let $\psi \in ph_{v_0}(G)^*$. By the theorem of Hahn-Banach, there exists $\widetilde{\psi} \in C_{v_0}(G)^*$ such that $\widetilde{\psi}_{|ph_{v_0}(G)} = \psi$, $\|\psi\| = \left\|\widetilde{\psi}\right\|$. By means of the map $\varphi : f \to vf$ we establish an isometric isomorphism between $C_{v_0}(G)$ and $C_0(G)$. Then, $\widetilde{\psi} \circ \varphi^{-1} \in C_0(G)^*$. By Riesz representation theorem applied to $C_0(G)$, a regular measure of μ exists such that $|\mu| < \infty$ and $\widetilde{\psi} \circ \varphi^{-1}(\varphi(f)) = \int_G \varphi(f) d\mu$, for all $f \in b_{v_0}^p$, therefore $\psi(f) = \int_G vf d\mu$, for every $f \in b_{v_0}^p$. Given $\varepsilon > 0$, since μ is regular there exists a compact set K of G such that $|\mu|(G \setminus K) < \frac{\varepsilon}{2}$. Then

$$|\psi(f)| \leq \int_{G\setminus K} v\,|f|\,d\mu + \int_K v\,|f|\,d\mu \leq$$

$$\|f\|_v\,|\mu|\,(G \setminus K) + \sup_{z\in K} v(z) \sup_{z\in K} |f(z)|\,|\mu|\,(K) <$$

$$\frac{\varepsilon}{2} + C\,\|f\|_K\,,$$

where

$$C = \sup_{z\in K} v(z)\,|\mu|\,(K).$$

Taking $\delta = \frac{\varepsilon}{2C} > 0$, and f in the neighborhood of 0,

$V_K(\delta) = \left\{f \in b_{v_0}^p : \|f\|_K < \delta\right\}$, we have: $|\psi(f)| < \varepsilon$. □

Let G be an open and balanced set of $\mathbb{C}^N$ and $f \in ph(G)$. Then, $f = u + iv$, for certain functions u, $v \in ph^{\mathbb{R}}(G)$. Hence, there exist f_1 and f_2 in $H(G)$ such that $u = \mathrm{Re} f_1$ and $v = \mathrm{Re} f_2$. Therefore, $f = \frac{1}{2}\left(f_1 + \overline{f_1}\right) + i\frac{1}{2}\left(f_2 + \overline{f_2}\right) = \frac{f_1 + if_2}{2} + \overline{\left(\frac{f_1 - if_2}{2}\right)}$.

Let $\sum_{n=0}^{\infty} \sum_{|\alpha|=n} a_\alpha z^\alpha$, $\sum_{n=0}^{\infty} \sum_{|\alpha|=n} b_\alpha z^\alpha$ be the developments in a power series in $\mathbb{D}$ of f_1 and f_2 respectively, where

$$\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{N}_0 \times \cdots \times \mathbb{N}_0,$$

$$|\alpha| = \alpha_1 + \cdots + \alpha_N,$$
$$z^\alpha = z_1^{\alpha_1} \cdot \cdots \cdot z_N^{\alpha_N}.$$

From the continuity of the function $z \to \overline{z}$, it follows:
$f(z) = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \frac{a_\alpha + ib_\alpha}{2} z^\alpha + \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \overline{\frac{a_\alpha - ib_\alpha}{2}} \overline{z}^\alpha$, for all $z \in \mathbb{D}$.

We denote by $C_n(f)$ the Cesàro means of degree n associated to f defined by

$$C_n(f)(z) := \frac{1}{n+1} \sum_{k=0}^{n} (p_k(z) + \overline{q_k(z)}),$$

where $p_k(z) = \sum_{j=0}^{k} \sum_{|\alpha|=j} \frac{a_\alpha + ib_\alpha}{2} z^\alpha$, $q_k(z) = \sum_{j=0}^{k} \sum_{|\alpha|=j} \frac{a_\alpha - ib_\alpha}{2} z^\alpha$.

Moreover, $p_k(z) + \overline{q_k(z)} = \mathrm{Re}\left(\sum_{j=0}^{k} \sum_{|\alpha|=j} a_\alpha z^\alpha\right) + i\mathrm{Re}\left(\sum_{j=0}^{k} \sum_{|\alpha|=j} b_\alpha z^\alpha\right)$. Hence, $C_n(f) \in ph(G)$.

Lemma 0.5.4. *If f is pluriharmonic then,*

$$max_{\theta \in [0,2\pi]} |C_n f(e^{i\theta} z_0)| \leq max_{\theta \in [0,2\pi]} |f(e^{i\theta} z_0)|$$

for all $z_0 \in \mathbb{D}$.

Proof. For fixed $z_0 \in \mathbb{D}$, we consider the compact set $K = \{\lambda z_0 : |\lambda| \leq 1\}$. By applying the maximum modulus principle to the pluriharmonic function $C_n(f) - f$ we obtain

$$\sup_{z \in K} |C_n(f)(z) - f(z)| = \sup_{|z|=|z_0|} |C_n(f)(z) - f(z)| =$$

$$\sup_{\theta \in [0,2\pi]} \left| C_n(f)(e^{i\theta} z_0) - f(e^{i\theta} z_0) \right|.$$

Now,

$$g(\theta) = f(e^{i\theta} z_0) = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \mathrm{Re}(a_\alpha z_0^\alpha e^{i|\alpha|\theta}) + i \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \mathrm{Re}(b_\alpha z_0^\alpha e^{i|\alpha|\theta}) =$$

$$\mathrm{Re}(a_0) + i\mathrm{Re}(b_0) + \sum_{n=1}^{\infty} \sum_{|\alpha|=n} (\mathrm{Re}(a_\alpha z_0^\alpha) + i\mathrm{Re}(b_\alpha z_0^\alpha)) \cos(n\theta)$$

$$- \sum_{n=1}^{\infty} \sum_{|\alpha|=n} (\mathrm{Im}(a_\alpha z_0^\alpha) + i\mathrm{Im}(b_\alpha z_0^\alpha)) \sin(n\theta)$$

is continuous and $2\pi-$periodic.

Let s_n be the $n-th$ partial sum of the series of Fourier generated by g and $\sigma_n(\theta) = \frac{s_0(\theta)+\cdots+s_n(\theta)}{n+1} = C_n(f)(e^{i\theta}z_0)$. From Fejér's Theorem, [2, Theorem 11.15], the sequence $(\sigma_n)_n$ converges uniformly to g in $[0, 2\pi]$, together with (0.5.1) implies that $C_n(f)$ converges uniformly to f in K, so the convergence is pointwise. Moreover, from [2, Theorem 11.14],

$$\sigma_n(\theta) = \frac{1}{(n+1)\pi}\int_0^\pi \frac{g(\theta+t)+g(\theta-t)}{2}\frac{\sin^2\left(\frac{n+1}{2}t\right)}{\sin^2\left(\frac{1}{2}t\right)}dt$$

and

$$\frac{1}{(n+1)\pi}\int_0^\pi \frac{\sin^2\left(\frac{n+1}{2}t\right)}{\sin^2\left(\frac{1}{2}t\right)}dt = 1.$$

Since g is continuous and periodic it follows:

$$\left|\frac{g(\theta+t)+g(\theta-t)}{2}\right| \leq \max_{\theta\in[0,2\pi]}|g(\theta)|.$$

Hence,

$$\left|C_n(f)(e^{i\theta}z_0)\right| \leq \frac{1}{(n+1)\pi}\int_0^\pi \left|\frac{g(\theta+t)+g(\theta-t)}{2}\right|\frac{\sin^2\left(\frac{n+1}{2}t\right)}{\sin^2\left(\frac{1}{2}t\right)}dt \leq$$

$$\max_{\theta\in[0,2\pi]}|g(\theta)| = \max_{\theta\in[0,2\pi]}\left|f(e^{i\theta}z_0)\right|.$$

Then,

$$\max_{\theta\in[0,2\pi]}\left|C_n(f)(e^{i\theta}z_0)\right| \leq \max_{\theta\in[0,2\pi]}\left|f(e^{i\theta}z_0)\right|. \tag{1}$$

□

Proposition 0.5.5. *Let v be a radial weight on an open and balanced set G of $\mathbb{C}^N$. If $ph_{v_0}(G)$ contains every function $f \in ph(G)$ such that $f(z) = p_n(z)+q_m(\overline{z})$, where p_n and q_m are polynomials of degree n and m respectively, then $b_v^p = \overline{b_{v_0}^p}^{\tau_0}$.*

Proof. As $b_{v_0}^p \subset b_v^p$ and $\overline{b_v^p}^{\tau_0} = b_v^p$, then it suffices to check one inclusion. Let $f \in b_v^p$ and $C_n(f)$ be the Cesàro means of degree n associated to f. The sequence $(C_n(f))_n$ converges to f in $(ph(/D), \tau_0)$. By hypothesis, $(C_n(f))_n \subset ph_{v_0}(G)$. Now, we fix $z_0 \in G$. Since (1) holds, v is radial and $\|f\|_v \leq 1$ it follows:

$$v(z_0)\left|C_n(f)(z_0)\right| \leq v(z_0)\max_{\theta\in[0,2\pi]}\left|C_n(f)(e^{i\theta}z_0)\right| \leq v(z_0)\max_{\theta\in[0,2\pi]}\left|f(e^{i\theta}z_0)\right| =$$

$$v(z_0)\left|f(e^{i\theta_0}z_0)\right| = v(e^{i\theta_0}z_0)\left|f(e^{i\theta_0}z_0)\right| \leq 1.$$

Therefore, $\|C_n(f)\|_v \leq 1$, for all n and as the pointwise topology on b_v^p coincides with the topology τ_0 then we have that $f \in \overline{b_{v_0}^p}^{\tau_0}$. □

The following results are immediately deduced from the continuity of the polynomials.

Proposition 0.5.6. *Let v be a radial weight on an open balanced and bounded set G of $\mathbb{C}^N$, which vanishes at infinity on G. Then, $ph_{v_0}(G)$ contains the functions $f \in ph(G)$ such that $f(z) = p_n(z) + q_m(\overline{z})$, where p_n, q_m are polynomials of degree n and m respectively.*

Proposition 0.5.7. *Let v be a radial weight on $\mathbb{C}^N$, such that*

$$\lim_{|z|\to+\infty} |z|^m v(z) = 0$$

for all $m \in \mathbb{N}$. Then, $ph_{v_0}(\mathbb{C}^N)$ contains all $f \in ph(\mathbb{C}^N)$ with $f(z) = p_n(z)+q_m(\overline{z})$, where p_n, q_m are polynomials of degree n and m respectively.

Corollary 0.5.8. *If v is a radial weight on an open balanced and bounded set G of $\mathbb{C}^N$ which vanishes at the boundary of G then $b_v^p = \overline{b_{v_0}^p}^{\tau_0}$ and therefore $ph_v(G)$ is canonically isometric to $(ph_{v_0}(G))^{**}$.*

Corollary 0.5.9. *Let v be a radial weight on $\mathbb{C}^N$, such that*

$$\lim_{|z|\to+\infty} |z|^m v(z) = 0$$

for all $m \in \mathbb{N}$. Then, $b_v^p = \overline{b_{v_0}^p}^{\tau_0}$ and therefore $ph_v(G)$ is canonically isometric to $(ph_{v_0}(G))^{**}$.

Proposition 0.5.10. *If G is balanced and bounded, v vanishes at the boundary of G and $sup_{z\in G}\frac{v(z)}{\tilde{v}_{ph}(rz)} \leq 1$ then, $b_v^p = \overline{b_{v_0}^p}^{\tau_0}$.*

Proof. If $f \in b_v^p$ then, for $0 < r < 1$, $C_r(f) := f(r\cdot) \in (b_p)_{v_0}$, since $v(z)|f(rz)| \leq \frac{v(z)}{\tilde{v}_{ph}(rz)}\|f\|_v$ for all $z \in G$. Now, if we take a sequence $(r_n)_n$ such that $0 < r_n < 1$ and $r_n \to 1$ then, $C_r(f)$ converges to f in compact open topology.

□

The next result is proved by Montes in an analogous way in [59, Lemma 2.1] for holomorphic functions on $\mathbb{D}$.

Lemma 0.5.11. *Let v be a radial decreasing weight on $B_{\mathbb{C}^N}$, which vanishes at the boundary of $B_{\mathbb{C}^N}$, $K \subset ph_{v_0}(B_{\mathbb{C}^N})$ a closed subset. Then K is compact if and only if K is bounded and for each $\varepsilon > 0$ there exists $0 < \delta < 1$ such that:*

$$\sup_{f \in K} \sup_{|z|>\delta} v(z)\,|f(z)| < \varepsilon. \tag{2}$$

Proof. We first prove that the condition is necessary. Let $\varepsilon > 0$ be fixed. As K is compact, there exist $f_1, \cdots f_n$ in K such that for every $f \in K$, we can find $1 \leq i \leq n$ such that $\|f - f_i\|_v < \frac{\varepsilon}{2}$. On the other hand, from $f_i \in ph_{v_0}(B_{\mathbb{C}^N})$ it follows that for $\frac{\varepsilon}{2}$ there is $\delta_0 \in]0, 1[$ such that $\sup_{|z|>\delta_i} v(z)\,|f_i(z)| < \frac{\varepsilon}{2}$ for every $1 \leq i \leq n$.

Therefore, if $f \in K$, we take f_i such that $\|f - f_i\|_v < \frac{\varepsilon}{2}$. Now, let $1 > s > \delta$

$$\sup_{|z|>s} v(z)\,|f(z)| \leq \sup_{|z|>\delta} v(z)\,|f(z) - f_i(z) + f_i(z)| \leq$$
$$\sup_{|z|>\delta} v(z)|f(z) - f_i(z)| + \sup_{|z|>\delta} v(z)\,|f_i(z)| \leq$$
$$\|f - f_i\|_v + \sup_{|z|>\delta} v(z)\,|f_i(z)| < \varepsilon.$$

Conversely, let us suppose that K is a closed and bounded subset of $ph_{v_0}(B_{\mathbb{C}^N})$ satisfying (2). Let $(f_n)_n$ be a sequence in K. As K is bounded, a constant $M > 0$ exists such that $K \subset M(b_v^p)$ and as b_v^p is compact in $(ph(B_{\mathbb{C}^N}), \tau_0)$ we get a subsequence $(f_{n_k})_k$ which converges to a function $f \in ph(B_{\mathbb{C}^N})$ in τ_0. Next, we fix $\varepsilon > 0$. Now take $\delta > 0$ given by (2) and obtain

$$\sup_{|z|>\delta} v(z)\,|f_{n_k}(z)| < \varepsilon,$$

for every $k \in \mathbb{N}$. Therefore,

$$\sup_{|z|>\delta} v(z)\,|f(z)| \leq \varepsilon. \tag{3}$$

Considering $B_{\mathbb{C}^N}(0,1)$ and the uniform convergence on compact of $B_{\mathbb{C}^N}$, for $\frac{\varepsilon}{\sup_{|z|<1} v(z)} > 0$ there exists k_0 such that $k \geq k_0$ implies

$$\sup_{|z|\leq\delta} |f_{n_k}(z) - f(z)| < \frac{\varepsilon}{\sup_{|z|<1} v(z)}.$$

Hence, for $k \geq k_0$,

$$\sup_{|z|\leq\delta} v(z)\,|f_{n_k}(z) - f(z)| < \varepsilon. \tag{4}$$

From (3) and (4), for $k \geq k_0$,

$$\|f_{n_k} - f\|_v \leq$$

$$\max\left\{\sup_{|z|\leq\delta} v(z)\,|f_{n_k}(z) - f(z)|\,, \sup_{|z|>\delta} v(z)\,|f_{n_k}(z) - f(z)|\right\} < 2\varepsilon.$$

This proves that $(f_{n_k})_k$ converges to f in $ph_v(B_{\mathbb{C}^N})$, and as $ph_{v_0}(B_{\mathbb{C}^N})$ is a closed subspace of $ph_v(B_{\mathbb{C}^N})$ and K is closed in $ph_{v_0}(B_{\mathbb{C}^N})$, we conclude $f \in K$. □

0.5.2 Compact operators

Let X and Y be normed spaces and let $U := \{x \in X : \|x\| \leq 1\}$ be the closed unit ball in X. A linear map $A : X \to Y$ is called *compact*, if $A(U)$ is relatively compact in Y. If X is a Banach space, then $A(U)$ is relatively compact in Y if and only if, for every sequence $(x_n)_n$ in U, the sequence $(Ax_n)_n$ has a convergent subsequence in Y, [58, Corollary 4.10]. The weak topology on X is denoted by $\sigma(X, X^*)$, and it is the topology induced by the seminorm system $(p_M)_{M\in\varepsilon(X^*)}$, $p_M : x \to \sup_{y\in M} |y(x)|$, $x \in E$, where $\varepsilon(X^*)$ is the set of finite subsets in X^* (See [58, 23]).

Proposition 0.5.12. *Let X be a Banach space and let $H \subseteq X^*$ be a $\sigma(X^*, X^{**})$ dense subspace. In B_X, the weak topology $\sigma(X, X^*)$ coincides with $\sigma(X, H)$.*

Proof. (B_X, w) is a subspace of the compact topological space $(B_{X^{**}}, w^*)$. The hypothesis on H implies that $\sigma(X^{**}, H)$ is a Hausdorff topology, which is obviously weaker than $w^* = \sigma(X^{**}, X^*)$. Thus, compactness of (B_{X^*}, w^*) implies the result. □

Corollary 0.5.13. *If $h_{v_0}^{**}(U) = h_v(U)$ then in b_{v_0} the weak topology coincides with the pointwise convergence in U.*

Theorem 0.5.14. *Let X, Y be normed spaces. A compact operator $T : X \to Y$ sends weakly convergent sequences to sequences which converge in norm. Precisely, if $(x_n)_n \to x$ weakly then:* 1) $\sup_n \|x_n\| < \infty$, 2) $\|T(x_n)\| \leq \|T\|\,\|x_n\| < \infty$, 3) *$(T(x_n))_n$ converges to Tx weakly in Y,* 4) *$(T(x_n))_n$ converges to Tx in norm .*

Corollary 0.5.15. *Let G be an open and connected subset of $\mathbb{C}^N$, w a weight on G, v a radial weight on $B_{\mathbb{C}^N}$ which vanishes at the boundary of $B_{\mathbb{C}^N}$ and $K : ph_v(B_{\mathbb{C}^N}) \to ph_w(G)$ a compact operator. If $(g_n)_n \subset b_{v_0}^p$ converges pointwise to* 0 *then,*

$$\lim_{n\to\infty} \|K(g_n)\| = 0.$$

Proof. By Proposition 0.5.12,

$$g_n \to 0 \text{ in } \sigma(H, H^*)_{|b^p_{v_0}}.$$

As K is a compact operator $\overline{K(b^p_{v_0})}^{ph_w(G)} \subset \overline{K(b^p_v)}^{ph_w(G)}$ we have $K_{|H} : H \to ph_w(G)$ is also a compact operator. Now, by applying Theorem 0.5.14 we arrive at

$$\|K(g_n) - 0\| = \|K(g_n) - K(0)\| \to 0.$$

□

Chapter 1

Harmonic associated weights

1.1 Definitions and notation

Let G be an open and connected subset of $\mathbb{R}^N$. A *weight* on G is a function $v : G \to \mathbb{R}$ which is strictly positive and continuous.

For a weight v the *weighted Banach spaces of harmonic functions* with weight v are defined by:

$$\begin{aligned} h_v(G) \quad &: \quad = \{f \in h(G) : \|f\|_v := \sup_{z \in G} v(z)\,|f(z)| < \infty\}, \text{ and} \\ h_{v_0}(G) \quad &: \quad = \{f \in h(G) : vf \text{ vanishes at infinity on } G\}. \end{aligned}$$

A function $g : G \to \mathbb{C}$ is said to *vanish at infinity on* G if for each $\varepsilon > 0$ there exists a compact set $K \subset G$ such that $|g(z)| < \varepsilon$ for each $z \in G \setminus K$. If $G \subseteq \mathbb{C}^N$, the corresponding spaces of holomorphic functions $H_v(G)$ and $H_{v_0}(G)$ have been deeply studied as we mentioned in the introduction. The unit balls in these spaces are denoted by b_v, b_{v_0}, B_v and B_{v_0}. The balls B_v and b_v are compact if we endow them with the compact open topology τ_0.

An open and connected set (domain) G of $\mathbb{R}^N$ (or $\mathbb{C}^N$) is called *balanced* if for each $z \in G$ and for each $\lambda \in \mathbb{R}$ ($\lambda \in \mathbb{C}$) with $|\lambda| \leq 1$ we have $\lambda z \in G$. A weight $v : G \to \mathbb{R}$ on a balanced domain G is *radial* if $v(z) = v(\lambda z)$ for all $z \in G$ and for all $|\lambda| = 1$. It is called *unitary* for a norm $|\cdot|$ if there exists a positive continuous function $g : [0, \infty[\to]0, \infty[$ such that $v(x) = g(|x|)$, and it is said to be unitary whenever it is unitary for the Euclidean norm $\|\cdot\|$.

At first glance, since all the norms are equivalent in $\mathbb{C}^N$ and $\mathbb{R}^N$, one might expect that the results in weighted Banach spaces of harmonic functions do not depend on the norm. But, as we shall see below, there are some differences. Most of them are due to the fact that the Euclidean norm has special properties related with orthogonal transformations which allow us to obtain optimal results.

Specifically, if T is orthogonal and $f \in C^2(G)$ then $\triangle f \circ T = (\triangle f) \circ T$. This calculation appears in [3, Chapter 1].

The function $f \circ T$ is called a *rotation* of f. It follows that rotations of harmonic functions are harmonic.

Definition 1.1.1. Let v be a *weight* on G and let $w = \frac{1}{v}$. By $\widetilde{w}_h$, : $G \to \mathbb{R}$ we denote the function

$$\widetilde{w}_h(z) := \sup\{|f(z)| : f \in b_v\},$$

w is called the growth condition of the weight v and $\widetilde{w}_h(z)$ the associated growth condition. The *harmonic associated weight* with v is defined as $\widetilde{v}_h := \frac{1}{\widetilde{w}_h}$.

For each $z \in G$ we consider the function $\delta_z : (h(G), \tau_0) \to \mathbb{C}$ defined by $\delta_z(f) := f(z)$. This function is called *the evaluation* at z and it is linear and continuous. The compactness of the unit ball b_v in $(h(G), \tau_0)$ implies that the supremum in the definition is a maximum. Since the norm topology on $h_v(G)$ is stronger than the one induced by τ_0, we have that the restriction $\delta_z : h_v(G) \to \mathbb{C}$ is also linear and continuous. We denote by $h_v(G)^*$ the topological dual space of $h_v(G)$. From the very definition we have $\widetilde{w}_h(z) = \|\delta_z\|_{h_v(G)^*}$. In case $G \subseteq \mathbb{C}^N$, the holomorphic associated weight is denoted by $\widetilde{v}_H$ and its corresponding associated growth condition by $\widetilde{w}_H$.

In the same way, by $h^{\mathbb{R}}(G)$ we mean the corresponding space of real valued harmonic functions. We can consider the weighted space $h_v^{\mathbb{R}}(G)$ of real valued harmonic functions and define over this space the growth condition

$$\widetilde{w}_h^{\mathbb{R}}(z) := \sup\left\{|f(z)| : f \in b_v \cap h^{\mathbb{R}}(G)\right\}.$$

Since $(b)_v \cap h^{\mathbb{R}}(G)$ is a compact set of $\left(h^{\mathbb{R}}(G), \tau_0\right)$ then the supremum of the definition of $\widetilde{w}_h^{\mathbb{R}}$ is a maximum, just as it happens with the complex case.

1.2 Properties of weights

In this section we present a list of properties satisfied by the growth conditions w, w_1 and w_2 and the weight v defined on G. Many of them constitute a generalization of the well known properties for the holomorphic case. The first proposition is an extension to the harmonic case of results for the holomorphic case which are due to Bierstedt, Bonet and Taskinen (see [7, 1.A]) and the proofs are analogous. All these properties can be stated in terms of the weights instead of the growth conditions. In particular, the first one is used during the rest of this work mainly in the form

$$v \leq \widetilde{v}_h \leq \widetilde{v}_H.$$

Proposition 1.2.1. *Let w, w_1 and w_2 be growth conditions defined on $G \subseteq \mathbb{R}^N$. Then the following assertions are true:*

(1) $0 \leq \widetilde{w}_H \leq \widetilde{w}_h \leq w$ *for* $N = 2k$.

(2) Let $f \in h(G)$. *Then* $|f| \leq w \Leftrightarrow |f| \leq \widetilde{w}_h$.

(3) For each $z_0 \in G$ *there exists* $f_{z_0} \in h(G)$ *such that* $|f_{z_0}(z_0)| = \widetilde{w}_h(z_0)$ *and* $|f_{z_0}(z)| \leq w(z)$ *for all* $z \in G$.

(4) $\widetilde{w}_h$ *is continuous and subharmonic.*

(5) $w_1 \leq w_2 \Rightarrow \widetilde{w}_{1h} \leq \widetilde{w}_{2h}$.

(6) $(Cw)\widetilde{}_h = C\widetilde{w}_h$ *for every constant* $C > 0$.

(7) $(\widetilde{w}_h)\widetilde{}_h = \widetilde{w}_h$.

(8) $(\min(w_1, w_2))\widetilde{}_h = (\min(\widetilde{w}_{1h}, \widetilde{w}_{2h}))\widetilde{}_h$.

(9) $(\max(\widetilde{w_{1h}}, \widetilde{w_{2h}}))\widetilde{}_h \leq (\max(w_1, w_2)\widetilde{}_h$.

Proof. (1) This follows from $H(G) \subset h(G)$ and the definitions of $\widetilde{w}_H$ and $\widetilde{w}_h$.

(2) This is trivial.

(3) Since $\widetilde{w}_h(z_0) = \sup_{f \in b_v} |\delta_{z_0}(f)|$, b_v is $\tau_0 -$ *compact* and $\delta_{z_0} : b_v \to \mathbb{C}$ is continuous.

(4) First, we show that $\widetilde{w}_h : G \to \mathbb{R}$ is upper semicontinuous, that is, for all $\alpha \in [0, +\infty[$ the set $\{z \in G : \widetilde{w}_h(z) \geq \alpha\}$ is closed. To do this, we fix $\alpha \geq 0$ and consider a convergent sequence $(z_n)_n$ to z_0 such that $\widetilde{w}_h(z_n) \geq \alpha$. By (3), for each $n \in \mathbb{N}$ there exists $f_n \in b_v$ such that $|f_n(z_n)| = \widetilde{w}_h(z_n) \geq \alpha$. Since b_v is a compact subset of $(h(G), \tau_0)$, there exists a subsequence $(f_{n_j})_j \subset b_v$ that converges to a certain $f \in b_v$ with τ_0 topology. Hence, for the compact set $K = \{z_{n_j} : j \in \mathbb{N}\} \cup \{z_0\}$, and for each $\varepsilon > 0$ there exists $j_1 \in \mathbb{N}$ such that $j \geq j_1$ implies $\sup_{z \in K} \left|f(z) - f_{n_j}(z)\right| < \frac{\varepsilon}{2}$. Since f is continuous in z_0, there exists j_2 such that if $j \geq j_2$ then $\left|f(z_0) - f(z_{n_j})\right| < \frac{\varepsilon}{2}$. Therefore, by taking $j_0 = \max(j_1, j_2)$, for all $j \geq j_0$ we have

$$\left|f(z_0) - f_{n_j}(z_{n_j})\right| \leq \left|f(z_0) - f(z_{n_j})\right| + \left|f(z_{n_j}) - f_{n_j}(z_{n_j})\right| \leq$$

$$\left|f(z_0) - f(z_{n_j})\right| + \sup_{z \in K} \left|f(z) - f_{n_j}(z)\right| < \varepsilon.$$

Thus,

$$\lim_{j\to\infty} f_{n_j}(z_{n_j}) = f(z_0),$$

together with $\left|f_{n_j}(z_{n_j})\right| = \widetilde{w}_h(z_{n_j}) \geq \alpha$ for all $j \in \mathbb{N}$, and 1) yields $\widetilde{w}_h(z_0) \geq |f(z_0)| \geq \alpha$, that is $z_0 \in \{z \in G : \widetilde{w}_h(z) \geq \alpha\}$.

Now, we show that $\widetilde{w}_h : G \to \mathbb{R}$ is lower semicontinuous, that is, for all $\alpha \in [0, +\infty[$ the set $\{z \in G : \widetilde{w}_h(z) > \alpha\}$ is open. Thus, we fix $\alpha \geq 0$ and consider $z_0 \in G$ such that $\widetilde{w}_h(z_0) > \alpha$. From the definition of $\widetilde{w}_h$ we can deduce that there is $f \in b_v$ such that $\alpha < |f(z_0)| \leq \widetilde{w}_h(z_0)$. The continuity of the function $|f|$ in G yields the existence of an open $U \subset G$, with $z_0 \in U$ such that $|f(z)| > \alpha$, for all $z \in U$. Therefore, $\widetilde{w}_h(z) \geq |f(z)| > \alpha$, for all $z \in U$. Then, U is contained in $\{z \in G : \widetilde{w}_h(z) > \alpha\}$.

Finally, we check that $\widetilde{w}_h$ is subharmonic. Let $a \in G$ and $r > 0$ such that the closed ball $\overline{B(a,r)}$ of center a and radius r is contained in G. By (3), we can find $f \in b_v$ such that $|f(a)| = \widetilde{w}_h(a)$. The Mean-Value Property [3, 1.4] implies that f equals the average of f over $\partial B(a,r)$, i.e.

$$f(a) = \int_S f(a + r\zeta) d\sigma(\zeta),$$

where S denotes the unit sphere and σ is the unique Borel probability measure on S that is rotation invariant. Therefore,

$$|f(a)| \leq \int_S |f(a + r\zeta)| d\sigma(\zeta)$$

and $|f(a + r\zeta)| \leq \widetilde{w}_h(a + r\zeta)$ for all $\zeta \in S$. We deduce

$$\widetilde{w}_h(a) \leq \int_S \widetilde{w}_h(a + r\zeta) d\sigma(\zeta).$$

Since we have already shown that $\widetilde{w}_h$ is continuous and $0 \leq \widetilde{w}_h(z) \leq w(z) < \infty$, for all $z \in G$, we can conclude that $\widetilde{w}_h$ is subharmonic.

(5) This follows directly from the definition.

(6) Let $a \in G$ and $C > 0$. By (3) there exists $f \in h(G)$, such that $|f| \leq w, |f(a)| = \widetilde{w}_h(a)$. Therefore, $Cf \in h(G), |Cf| = C\,|f| \leq Cw$. Now, by (1) we have, $|Cf(a)| \leq \widetilde{(Cw)}_h(a)$. Finally we can conclude

$$C\widetilde{w}_h(a) \leq \widetilde{(Cw)}_h(a).$$

(3) yields that there is $g \in h(G)$, such that $|g| \leq Cw, |g(a)| = \widetilde{(Cw)}_h(a)$.

Since $C > 0$ it follows $\frac{1}{C}g \in h(G), \left|\frac{1}{C}g\right| \leq w$. By (1) we have $\left|\frac{1}{C}g(a)\right| \leq \widetilde{w_h}(a)$, which implies

$$\frac{1}{C}\,(Cw)\widetilde{\ }_h(a) \leq \widetilde{w}_h(a).$$

(7) Let $a \in G$. From $\widetilde{w}_h \leq w$ and (4), we get $(\widetilde{w}_h)\widetilde{\ }_h \leq \widetilde{w}_h$.

By (3) there exists $f \in h(G)$, such that $|f| \leq w, |f(a)| = \widetilde{w}_h(a)$. By (1), $|f| \leq \widetilde{w}_h$, and applying the definition of $(\widetilde{w}_h)\widetilde{\ }_h$, we obtain $|f(a)| \leq (\widetilde{w}_h)\widetilde{\ }_h(a)$. Hence, $\widetilde{w}_h \leq (\widetilde{w}_h)\widetilde{\ }_h$.

(8) Obviously, $\min(\widetilde{w_{1h}}, \widetilde{w_{2h}}) \leq \widetilde{w_{jh}} \leq w_j$, $j = 1, 2$. Then,

$$\min(\widetilde{w_{1h}}, \widetilde{w_{2h}}) \leq \min(w_1, w_2).$$

An applyication of (5) gives

$$(\min(\widetilde{w_{1h}}, \widetilde{w_{2h}}))\widetilde{\ }_h \leq (\min(w_1, w_2))\widetilde{\ }_h.$$

On the other hand, $\min(w_1, w_2) \leq w_j$, $j = 1, 2$. By (5), $(\min(w_1, w_2))\widetilde{\ }_h \leq \widetilde{w_{jh}}$, $j = 1, 2$. Hence, $(\min(w_1, w_2))\widetilde{\ }_h \leq \min(\widetilde{w_{1h}}, \widetilde{w_{2h}})$. Now, using (5) and (7) we arrive at

$$(\min(w_1, w_2))\widetilde{\ }_h = ((\min(w_1, w_2))\widetilde{\ }_h)\widetilde{\ }_h \leq (\min(\widetilde{w_{1h}}, \widetilde{w_{2h}}))\widetilde{\ }_h.$$

(9) We have $w_j \leq \max(w_1, w_2)$, $j = 1, 2$. Then, by (5) $\widetilde{w_{jh}} \leq (\max(w_1, w_2)\widetilde{)_h}$, $j = 1, 2$. Thus, $\max(\widetilde{w_{1h}}, \widetilde{w_{2h}}) \leq (\max(w_1, w_2)\widetilde{)_h}$. Now, (5) and (7) yield that

$$(\max(\widetilde{w_{1h}}, \widetilde{w_{2h}}))\widetilde{\ }_h \leq (\max(w_1, w_2)\widetilde{)_h})\widetilde{\ }_h = \max(w_1, w_2)\widetilde{_h}.$$

$\square$

Proposition 1.2.2. *For a growth condition w defined on $G \subseteq \mathbb{C}^N$, the following results hold:*

(i) $\widetilde{w}_H \leq (\widetilde{w}_H)\widetilde{\ }_h \leq \widetilde{w}_h$,

(ii) $(\widetilde{w}_h)\widetilde{\ }_H = \widetilde{w}_H$.

Proof. By Properties 1.2.1 and [7, Section 1A]:

i) $\widetilde{w}_H = (\widetilde{w}_H)\widetilde{\ }_H \leq (\widetilde{w}_H)\widetilde{\ }_h \leq (\widetilde{w}_h)\widetilde{\ }_h = \widetilde{w}_h$,

ii) $\widetilde{w}_H = (\widetilde{w}_H)\widetilde{\ }_H \leq (\widetilde{w}_h)\widetilde{\ }_H \leq \widetilde{w}_H$. $\square$

Definition 1.2.3. Two weights $v_1, v_2 : G \to \mathbb{R}$ are called *equivalent* if there exist two positive constants C and D such that $Cv_1 \leq v_2 \leq Dv_1$. We write $v_1 \sim v_2$.

Proposition 1.2.4. *Let v be a weight on $G \subseteq \mathbb{C}^N$. If v and $\widetilde{v}_H$ are equivalent then v and $\widetilde{v}_h$ are equivalent.*

Proof. This is a direct consequence from $0 < v \leq v_h \leq v_H$. □

Definition 1.2.5. A weight v on $G \subseteq \mathbb{C}^N$ is said to be *essential* if $v \sim \widetilde{v}_H$.

Proposition 1.2.6. *If v_1 and v_2 are equivalent weights then:*

(i) $\widetilde{v}_{1h}$, $\widetilde{v}_{2h}$ are equivalent,

(ii) $\widetilde{v}_{1H}$, $\widetilde{v}_{2H}$ are equivalent.

Proof. This follows from the definition and Property 1.2.1 (5). □

Proposition 1.2.7. *If w is a growth condition on $G \subseteq \mathbb{R}^N$, then $\widetilde{w}_h^{\mathbb{R}} = \widetilde{w}_h$.*

Proof. From $h^{\mathbb{R}}(G) \subset h(G)$, we obtain $\widetilde{w}_h^{\mathbb{R}} \leq \widetilde{w}_h$. Let $z_0 \in G$ and $f \in h(G)$ such that $|f(z_0| = \widetilde{w}_h(z_0)$ and $|f(z)| \leq w(z)$ for all $z \in G$. Let $\alpha = \frac{\overline{f(z_0)}}{|f(z_0)|}$. Then,
$\text{Re}(\alpha f) \in h^{\mathbb{R}}(G)$, and, moreover,

$$|\text{Re}(\alpha f(z)| \leq |f(z)| \leq w(z),$$

for all $z \in G$. Therefore,

$$|\text{Re}(\alpha f)(z_0)| \leq \widetilde{w}_h^{\mathbb{R}}(z_0).$$

Since

$$\widetilde{w}_h(z_0) = |f(z_0)| = |\text{Re}(\alpha f)(z_0)|$$

then

$$\widetilde{w}_h(z_0) \leq \widetilde{w}_h^{\mathbb{R}}(z_0).$$

□

Corollary 1.2.8. *For each $z_0 \in G \subseteq \mathbb{R}^N$ there exists $f_{z_0} \in h^{\mathbb{R}}(G)$ such that $f_{z_0}(z_0) = \widetilde{w}_h(z_0)$ and $|f_{z_0}(z)| \leq w(z)$ for all $z \in G$.*

Proposition 1.2.9. *If G is an open, bounded and connected subset of $\mathbb{R}^N$, and w has a continuous extension on $\overline{G}$ with $w|_{\partial G} = 0$, then $\widetilde{w}_h = 0$.*

Proof. We proceed by contradiction. We suppose that there exists $z_0 \in G$ such that $\widetilde{w}_h(z_0) > 0$. By Corollary 1.2.8, we can find $f \in h^{\mathbb{R}}(G)$ such that $f(z_0) = \widetilde{w}_h(z_0)$ and $|f(z)| \leq w(z)$ for all $z \in G$. From the continuity of w in $\overline{G}$ (which is compact), and from $w|_{\partial G} = 0$, we obtain that for $\varepsilon = \frac{f(z_0)}{2}$, there exists $n \in \mathbb{N}$ sufficiently large so that $d(z, \partial G) \leq \frac{1}{n}$ implies $w(z) < \frac{f(z_0)}{2}$. Let $\Omega := \left\{z \in G : d(z, \partial G) > \frac{1}{n}\right\}$. Clearly $z_0 \in \Omega$ and $\overline{\Omega} \subset G$. By applying the Maximum Modulus Principle of harmonic functions [3, 1.8] to f in Ω, we get $a \in \partial\Omega$ such that $|f(z_0)| \leq \max_{\varsigma \in \partial\Omega} \{|f(\varsigma)|\} = |f(a)| \leq \frac{f(z_0)}{2}$, which is a contradiction. □

Proposition 1.2.10. *If* $\overline{b_{v_0}}^{\tau_0} = b_v$ *then* $\sup\{|f(z)| : f \in b_v\} = \sup\{|f(z)| : f \in b_{v_0}\}$ *for all* $z \in G$.

Proof. Since $b_v \supset b_{v_0}$ we obtain

$$\sup\{|f(z)| : f \in b_v\} \geq \sup\{|f(z)| : f \in b_{v_0}\}$$

for each $z \in G$.

To show the reverse inequality we fix $z \in G$. By Proposition 1.2.1(3) there exists $f \in b_v$ such that $|f(z)| = \sup\{|f(z)| : f \in b_v\}$. By hypothesis, $\overline{b_{v_0}}^{\tau_0} = b_v$. Thus, $f \in \overline{b_{v_0}}^{\tau_0}$, that implies the existence of a sequence $(f_n)_n \subset b_{v_0}$, which converges uniformly on compacts to f. Hence it also converges pointwise. Now, $|f_n(z)| \leq \sup\{|f(z)| : f \in b_{v_0}\}$ for all $n \in \mathbb{N}$. Then $|f(z)| = \lim_{n\to+\infty} |f_n(z)| \leq \sup\{|f(z)| : f \in b_{v_0}\}$.

□

In general, $\sup\{|f(z)| : f \in b_{v_0}\}$ is not a maximum since b_{v_0} is not τ_0-compact.

Proposition 1.2.11. *(i) Let* v *be a weight on* $G \subset \mathbb{R}^N$. *If* $\widetilde{v}_h(z) < \infty$, *for all* $z \in G$ *then* $h_v(G) = h_{\widetilde{v}_h}(G)$ *and the norms* $\|.\|_v$ *and* $\|.\|_{\widetilde{v}_h}$ *coincide.*

(ii) Let v *be a weight on* $G \subset \mathbb{C}^N$. *If* $\widetilde{v}_H(z) < \infty$, *for all* $z \in G$ *then* $H_v(G) = H_{\widetilde{v}_h}(G) = H_{\widetilde{v}_H}(G)$ *and the norms* $\|.\|_v$, $\|.\|_{\widetilde{v}_H}$ *and* $\|.\|_{\widetilde{v}_h}$ *coincide.*

Proof. (i) From $v \leq \widetilde{v}_h$, it is deduced that $\|.\|_v \leq \|.\|_{\widetilde{v}_h}$ and $h_{\widetilde{v}_h}(G) \subset h_v(G)$.

If $f \in h_v(G)$, $f \neq 0$ then $\frac{f}{\|f\|_v} \in b_v$. By proposition 1.2.1(1), $\frac{f}{\|f\|_v} \in b_{\widetilde{v}_h}$. Hence, $\|f\|_{\widetilde{v}_h} \leq \|f\|_v$, and $f \in h_{\widetilde{v}_h}(G)$.

(ii) It is a consequence of $v \leq \widetilde{v}_h \leq \widetilde{v}_H$ and the above argument.

□

Remark 1.2.12. By [7, Observation 1.12] we know that $H_v(G) = H_{\widetilde{v}_H}(G)$ and the norms $\|.\|_v$ and $\|.\|_{\widetilde{v}_H}$ coincide.

If v is a bounded weight in G then $\sup_{z\in G} v(z) < \infty$. Therefore, $1 \in H_v(G) \subset h_v(G)$ and hence $\{0\} \neq H_v(G)$, $\{0\} \neq h_v(G)$.

In [14, Ob.1], it is shown that the condition $\widetilde{v}_H(z) < \infty$, for all $z \in G$ is equivalent to $H_v(G) \neq \{0\}$ for $G \subset \mathbb{C}$.

Condition $\widetilde{v}_H < \infty$ implies $\widetilde{v}_h < \infty$, but the converse is not in general true as we will see in Example 1.2.23.

Proposition 1.2.13. *For the growth conditions* w, w_1 *and* w_2 *defined on* G, *the following results hold:*

(1) If G *is balanced (*G *is a ball or* $G = \mathbb{R}^N$*) then* $\widetilde{w}_h$ *is unitary whenever* w *is.*

(2) *If* $\widetilde{w}_h$ *is unitary in a ball* B *(or the whole space) for a norm* $|\cdot|$ *then* $\widetilde{w}_h$ *is non decreasing.*

(3) *If* G *bounded and* v *is bounded or* $G = \mathbb{R}^n$ *and* $v(z)$ *tends to zero when* $\|z\|$ *tends to infinity then* $h_v(G) \neq 0$*, and then* $\widetilde{v}_h$ *is bounded.*

(4) *If* G *is the unit ball for a norm* $|\cdot|$ *of* $\mathbb{R}^N$ *or* $G = \mathbb{R}^N$*,* v *is unitary for* $|\cdot|$ *and* $h_v(G) \neq 0$ *then* $\sup_{z\in G} v(z) < \infty$*. If* G *is the Euclidean ball (or* $G = \mathbb{R}^N$*) and* v *is unitary then* $\sup_{z\in G} \widetilde{v}_h(z) < \infty$*. In case* $G = \mathbb{R}^N$ *we have even* $\lim_{|z|\to\infty} v(z) = 0$ *and if* v *is unitary with respect to the Euclidean norm we also have* $\lim_{\|z\|\to\infty} \widetilde{v}_h(z) = 0$*, whenever there is a non-constant function in* $h_v(G)$*.*

Proof. (1) Assume that G is the Euclidean unit ball of $\mathbb{R}^N$ and v is unitary. Then, there exists a positive continuous function $g : [0,\infty[\to \mathbb{R}$ such that $v(x) = g(\|x\|)$. We show that for every $0 < r < 1$ and every $\|x\| = \|y\| = r$ we have $\widetilde{w}_h(x) = \widetilde{w}_h(y)$ which implies that $\widetilde{w}_h(z) = \sup\{M(f,z) : f \in b_v\}$, where $M(f,z) = \sup\{|f \circ T(z) :$ T orthogonal tranformation$\}$. For each orthogonal transformation T we have that $f \circ T$ is harmonic (cf. [3, Chapter 1]). Moreover, since v is unitary, then $v(x)|f(T(x))| = v(T(x))|f(T(x))|$ and from this it follows that composition of functions in $h_v(G)$ with orthogonal transformations gives an isometry in $h_v(G)$.

Let $0 < r < 1$ be fixed. There exists $f \in h_v(G)$ with $\|f\|_v = 1$ such that $f(r,0,\dots,0) = \widetilde{w}_h(r,0,\dots,0)$. Let $x_0 \in G$ such that $\|x_0\| = r$. Consider an orthogonal transformation T such that $T(x_0) = (r,0,\dots,0)$. Since $f \circ T$ is in b_v we get that $\widetilde{w}_h(x_0) \geq f \circ T(x_0) = \widetilde{w}_h(r,0,\dots,0)$. If there is $f_1 \in b_v$ such that $f_1(x_0) > f\circ T(x_0)$, then $f_1(x_0) = f_1 \circ T^{-1} \circ T(x_0) = f_1 \circ T^{-1}(r,0,\dots,0) \leq w(x_0) = w(r,0,\dots,0)$ and $g\circ T^{-1}$ is in b_v because T^{-1} is an orthogonal transformation. This contradicts $f(r,0,\dots,0) = \widetilde{w}_h(r,0,\dots,0)$. Therefore, $\widetilde{w}(x_0) = \widetilde{w}(\|x_0\|,0,\cdots,0)$.

(2) We consider a weight v on a ball B with respect to a norm $|\cdot|$ which is unitary. Let $0 < r_1 < r_2$ and $z_1 \in B$ with $|z_1| = r_1$. We take $f_0 \in b_v$ such that $|f_0(z_1)| = \widetilde{w}_h(z_1)$. By the maximum modulus principle for harmonic functions [3, 1.8] there exists z_2 such that $|f_0(z_2)| \geq |f_0(z_1)|$. Hence $\widetilde{w}_h(z_2) = \sup_{f\in b_v} |f(z_2)| \geq |f_0(z_1)| = \widetilde{w}_h(z_1)$. Since $\widetilde{w}_h$ is unitary with respect to $|\cdot|$ we conclude that $\widetilde{w}_h$ is not decreasing.

(3) In both cases we have $1 \in h_v(G)$ and therefore $\widetilde{v}_h \leq \|1\|_v$.

(4) Seeking a contradiction, we suppose that v is not bounded. We consider $v(x) = g(|x|)$. Then, there exists a strictly increasing sequence $(r_n)_n$ of positive numbers such that $(g(r_n))_n \to \infty$, $r_n \to 1$ when G is the unit ball for $|\cdot|$, and $r_n \to \infty$ when $G = \mathbb{R}^N$. From $h_v(G) \neq 0$, we get $h_v^{\mathbb{R}}(G) \neq 0$. Let $u \in h_v^{\mathbb{R}}(G)$,

$u \neq 0$ and $M(r) = \sup_{|z|=r} |u(z)|$. Then

$$\sup_{n \in \mathbb{N}} g(r_n) M(r_n) \leq \sup_{z \in G} v(z) |u(z)| = \|u\|_v < \infty$$

and furthermore, the sequence $(M(r_n))_n$ is increasing by the Maximum Modulus Principle of harmonic functions [3, 1.8]. Hence,

$$M(r_n) \leq \frac{\|u\|_v}{g(r_n)}, \text{ for all } n \in \mathbb{N}.$$

Taking limit when n tends to ∞ we get a contradiction since $u \neq 0$. If $G = \mathbb{R}^N$ and we assume that there exist c and a sequence $(r_n)_n$ with $r_n \to \infty$ such that $v(r_n) > c$, then we obtain that, for each non-constant $f \in b_v$ and $|z| = r_n$ we have $|f(z)| \leq 1/c$. Applying again the Maximum Modulus Principle for harmonic functions we get a contradiction with the Liouville Theorem for harmonic functions [3, 2.1]. Properties (1) and (2) together mean that if v is a unitary weight in the unit ball B (or B is the whole space) of $(\mathbb{C}^N, |\cdot|)$, then $\widetilde{v}_h$ is unitary and non increasing. The corresponding statements for $\widetilde{v}_h$ when $v(x) = g(\|x\|)$ follow from the equality $h_v(G) = h_{\widetilde{v}_h}(G)$ and the fact that $\widetilde{v}_h$ is unitary (Proposition 1.2.13 (1)).

□

Proposition 1.2.14. *Let $G \subset \mathbb{C}$ be balanced, v a radial weight on G and w the corresponding growth condition. For $f \in h^{\mathbb{R}}(G)$, let*
$M(f, z) := \max\{|f(\lambda z)| : |\lambda| = 1\}$.
Then $\widetilde{w}_h(z) = \sup\{M(f, z) : f \in b_v^{\mathbb{R}}\}$ for all $z \in G$ and the supremum is a maximum, $\widetilde{w}_h$ is radial and increasing in r.

Proof. If $f \in b_v^{\mathbb{R}}$, $z \in G$ and $|\lambda| = 1$ then $|f(\lambda z)| \leq w(\lambda z)$. Since w is radial and G balanced, we have $w(\lambda z) = w(|\lambda z|) = w(|z|) = w(z)$. The function $z \to f(\lambda z)$ is in $b_v^{\mathbb{R}}$. By Proposition 1.2.1(2), $|f(\lambda z)| \leq \widetilde{w}_h(z)$. Thus, $\sup\{M(f, z) : f \in b_v^{\mathbb{R}}\} \leq \widetilde{w_h}(z)$. On the other hand, by Corollary 1.2.8, for each $z_0 \in G$, there exists $f_{z_0} \in b_v^{\mathbb{R}}$ and $f_{z_0}(z_0) = \widetilde{w}_h(z_0)$. Now,
$f_{z_0}(z_0) = |f_{z_0}(z_0)| \leq M(f_{z_0}, z_0) \leq \sup\{M(f, z_0) : f \in b_v^{\mathbb{R}}\}$. So, the supremum is a maximum. From $M(f, \mu z) = M(f \circ R_\mu, z)$, for $R_\mu(z) = \mu z$, $|\mu| = 1$, $z \in G$ and $f \in b_v^{\mathbb{R}}$, it follows $\widetilde{w_h}(\mu z) \leq \widetilde{w_h}(z)$, $z \in G$ and $|\mu| = 1$. Therefore, $\widetilde{w}_h(\frac{1}{\mu}\mu z) \leq \widetilde{w}_h(\mu z)$, $z \in G$ and $|\mu| = 1$. Thus, $\widetilde{w}_h$ is radial. If $0 \leq r_1 \leq r_2$ then by Maximum Modulus Principle of harmonic functions, $M(f, r_1) \leq M(f, r_2)$, for all $f \in b_v^{\mathbb{R}}$ and taking supremum, we get $\widetilde{w}_h(r_1) \leq \widetilde{w}_h(r_2)$. Hence, $\widetilde{w}_h$ is increasing in r. □

Corollary 1.2.15. *Let $G \subset \mathbb{C}$ balanced, v a radial weight on G and w the corresponding growth condition. If there exists $f \in h^{\mathbb{R}}(G)$, such that $w(z) = M(f, z)$, for all $z \in G$, then $\widetilde{w}_h = w$.*

Proof. $\widetilde{w}_h(z) \le w(z) = M(f,z) \le \sup\left\{M(f,z) : f \in b_v^{\mathbb{R}}\right\} = \widetilde{w}_h(z)$. □

To state the following result, we remark that if G is the unit ball of $\mathbb{R}^n$ (or $G = \mathbb{R}^n$) and v is a unitary weight then Proposition 1.2.13 (1) proved above implies the existence of a positive function g such that $\widetilde{v}_h(x) = g(\|x\|)$. Moreover, g is a non increasing function by Proposition 1.2.13 (2).

Proposition 1.2.16. *Let $g : [0,1[\to \mathbb{R}^+$ (or $g : [0,\infty[\to \mathbb{R}^+$) be a continuous function and let G_n be the unit ball of $\mathbb{R}^n$ (or $G_n = \mathbb{R}^n$) for the Euclidean norm. Define $v_n(x) = g(\|x\|)$ for $x \in \mathbb{R}^n$. Let $g_n : [0,1[\to \mathbb{R}^+$ be the function such that $(\widetilde{v}_n)_h(x) = g_n(\|x\|)$ for each $n \ge 2$. Then $g_{n+1} \le g_n$ for each $n \ge 2$.*

Proof. We only give the proof for G_n being the unit ball. Let $n \ge 2$ and $0 < r < 1$. Denote $r_n = (r, 0\ldots, 0) \in \mathbb{R}^n$, and let f be in the unit ball of $h_{v_n}(G_n)$ such that $1/g_n(r) = f(r_n)$. Such an f exists because Proposition 1.2.1 (3). Consider $\tilde{f}(x_1,\ldots,x_n,x_{n+1}) = f(x_1,x_2,\ldots,x_n)$. It is immediate that $\tilde{f} \in b_{v_{n+1}}$. We have $\tilde{f}(r_{n+1}) = f(r_n) = 1/g_n(r) \le 1/g_{n+1}(r)$. □

Our aim is to connect this harmonic associated weight with the holomorphic one. First we observe that Proposition 1.2.13 (1) above is valid for the holomorphic case in the ball (or the whole space) with respect to any norm in $\mathbb{C}^N$. Further, Bonet proved in [21, Proposition 2] that, for unitary weights, the corresponding associated weight does not depend either on the norm or on the dimension. We state this result below as we need it, the original result is written in terms of entire functions on Banach spaces.

Proposition 1.2.17 (**Bonet**). *Let g be a non-increasing positive continuous function defined on positive numbers. If we consider a norm $|\cdot|$ in $\mathbb{C}^n$ and a weight $v = g(|z|)$ for $z \in G_n \subset \mathbb{C}^n$, G_n being the unit ball of $\mathbb{C}^n$ for $|\cdot|$ or $G_n = \mathbb{C}^n$, then $\widetilde{v}_H(z) = \widetilde{g}(|z|)$, where*

$$\widetilde{g}(t) = 1/\{\sup|h(z)| : h \in H(G_1),\ |h(z)| \le 1/g(|z|) \text{ for all } z \in G_1, |z| = t\},$$

for $G_1 = \mathbb{D}$ in case G_n the ball of $(\mathbb{C}^n, |\cdot|_n)$ or $G_1 = \mathbb{C}$ if we are considering $G_n = \mathbb{C}^n$.

The argument given there does not work for the harmonic case since $f : \mathbb{C}^n \to \mathbb{C}$ being harmonic does not imply that $\lambda \to f(\lambda z)$ ($\lambda \in \mathbb{C}$) is harmonic, for instance, $f(z_1,z_2) = z_1\overline{z_2}$. Next, we look for conditions which ensure that $\widetilde{v}_H = \widetilde{v}_h$.

Definition 1.2.18. A function $f\ :]a,b[\subset]0,+\infty[\to \mathbb{R}$ is called *convex in* $\log r$ with $r \in]a,b[$, when the function ψ defined by $\psi(t) := f(e^t)$ is a convex function and is called *log-convex* when the function ψ defined by $\psi(t) := \log f(e^t)$, for $t \in \log(]a,b[)$ is a convex function (i.e. $\log f$ is convex in $\log r$).

The following lemma is [10, Lemma 5].

Lemma 1.2.19 (Bonet, Domański, Lindström). *Let* $v : [0,1[\to]0,+\infty[$ *a decreasing and continuous function with* $\lim_{r\to 1^-} v(r) = 0$. *Let* $v : \mathbb{D} \to [0,+\infty[$, *and consider the radial extension* ($v(z) = v(|z|)$). *If* $\frac{1}{v}$ *is log-convex, then* v *is equivalent to* $\widetilde{v}_H$ *(and consequently also to* $\widetilde{v}_h$*).*

As a clear consequence we have that for a radial weight v in $\mathbb{D}$ if $\lim_{r\to 1^-} v(r) = 0$ and $\frac{1}{\widetilde{v}_h}$ is log-convex, then $\widetilde{v}_h$ is equivalent to $\widetilde{v}_H$. In view of [12, Proposition 1.1], the hypothesis of $\lim_{r\to 1^-} v(r) = 0$ is equivalent to $\lim_{r\to 1^-} \widetilde{v}_H(r) = 0$. Moreover, because of Proposition 1.2.17, if $v = g(|z|)$ is equivalent to $\widetilde{v}_H$ in $\mathbb{D}$, and we define $v_n(z) = g(|z|)$ for z in the unit ball B_n of $(\mathbb{C}^n, |\cdot|)$, $|\cdot|$ being a norm in $\mathbb{C}^n$, Proposition 1.2.17 implies that there is $\widetilde{g}$ not depending neither on n nor on $|\cdot|$ such that $(\widetilde{v}_n)_H(z) = \widetilde{g}(|z|)$. This implies the equivalence between v_n and $(\widetilde{v}_n)_H$ for each $n \in \mathbb{N}$. The order relation $v_n \leq (\widetilde{v}_n)_h \leq (\widetilde{v}_n)_H$ between the associated weights implies that v_n and $(\widetilde{v}_n)_h$ are also equivalent. It seems remarkable that we do not know if $(\widetilde{v}_n)_h$ is unitary with respect to an arbitrary norm $|\cdot|$ if it is not the Euclidean.

By [10, Remark 2] there exist radial and continuous weights on $\mathbb{C}$ which are decreasing in $[0,+\infty[$, $\lim_{r\to\infty} r^n v(r) = 0$ for all $n \in \mathbb{N}$ and for which $1/v$ is log-convex, but still v is not an essential weight. An inspection of the proof of [7, Example 3.3] shows that v is not equivalent to $\widetilde{v}_h$.

Proposition 1.2.20. *Let* $G \subset \mathbb{C}$ *be balanced,* v *a radial weight on* G *and* w *the corresponding growth condition. Then* $\widetilde{w}_h$ *is radial, subharmonic, continuous, increasing in* r *and convex in* $\log r$.

Proof. It has been proved in the Proposition 1.2.14 that for a radial growth condition w the corresponding $\widetilde{w}_h$ is radial and increasing in r. By Proposition 1.2.1 (4), $\widetilde{w}_h$ is subharmonic and continuous. Applying Hadamard's Three Circles Theorem [5, Proposition 4.4.32], and using the fact that $\widetilde{w}_h$ is radial, we deduce that $r \to \sup_{|\zeta|=r} \widetilde{w}_h(\zeta) = \widetilde{w}_h(r)$ is convex in $\log r$. □

The relation between holomorphic functions and subharmonic functions is given by the fact that if f is a holomorphic function, then $\log|f|$ and $|f|^p$ for $p > 0$ are subharmonic functions. Furthermore, for a subharmonic function $u \geq 0$ not identically null one has that $\log u$ is subharmonic if and only if u^p is subharmonic for each $p > 0$ (See [5, 4]). These properties are used to show that $\log \widetilde{w}_H$ is subharmonic for any growth condition w. Indeed, let $p > 0$, $z_0 \in \mathbb{D}$, $\overline{D(z_0, r)} \subset \mathbb{D}$, $\widetilde{w}_H(z_0) = |f(z_0)|$, f holomorphic, $f \leq w$. By [5, 4.4.19 (4)], $|f|^p$ is subharmonic for each $p > 0$. Then, $|f(z_0)|^p \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|^p\, dt \leq \frac{1}{2\pi}\int_0^{2\pi} (\widetilde{w}_H(z_0 + re^{it}))^p dt$. The function $\widetilde{w}_H^p$ is continuous. Therefore, $\widetilde{w}_H^p$ is subharmonic, for each $p > 0$.

Thus, $\log \widetilde{w}_H$ is subharmonic. If we add that w is radial, we have that $\widetilde{w}_H$ is radial and now, from Hadamard's Three Circle Theorem [5, Proposition 4.4.32], it follows $\widetilde{w}_H$ is log-convex, i.e. the function $t \to \log \widetilde{w}_H(e^t)$ is convex.

In general, the function $|\mathrm{Re} f|^p$ is not subharmonic for all $p > 0$, as well as the function $M :]0,1[\to \mathbb{R}$ defined by $M(r) := \max_{|z|=r} |U(z)|$, where U is a real harmonic function, is not always log-convex, and finally, the function $t \to \log \widetilde{w}_h(e^t)$ is not convex for all t, as it happens for $U = \mathrm{Re}(z^3 - z)$, for which $M(r) = r - r^3$ if $0 < r < \frac{1}{3}$ and $M(r) = \frac{1}{9}\sqrt{9r^2+3}(3r^2+1)$ if $\frac{1}{3} \leq r < 1$.

Although we do not know if for unitary weights in the unit disc the associated and the harmonic weights are equivalent, we have a negative answer for the equivalence of associated weights in the disc with non radial weights. This example is inspired by [7, Proposition 3.6].

Example 1.2.21. We define the function g in the following way: $g : \partial\mathbb{D} \to \mathbb{R}$, $g(e^{it}) = |1 - e^{it}|^2$, $t \in [-\pi, \pi]$. Then g belongs to $L^1(\partial\mathbb{D})$, is continuous, $g(1) = 0$ and $g \geq 0$. Let $w : \mathbb{D} \to \mathbb{R}$ be the Poisson Kernel of the function g, that is $w(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right) g(e^{it})dt$. By [61, Theorem 11.7], w is harmonic in $\mathbb{D}$. Furthermore, w is positive since $\mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right) = \frac{1-|z|^2}{|e^{it}-z|^2} > 0$ and $g > 0$ in $\partial\mathbb{D} \setminus \{1\}$. Therefore, w is a growth condition in $\mathbb{D}$ and $\widetilde{w}_h = w$.

On the other hand, $\log g \in L^1(\partial\mathbb{D})$, since

$$\int_{-\pi}^{\pi} \left|\log \left|1 - e^{it}\right|^2\right| dt = \int_{-\pi}^{\pi} \left|\log\left((1-\cos t)^2 + \sin^2(t)\right)\right| dt =$$

$$\int_{-\pi}^{\pi} \left|\log\left(2\left(1-\cos t\right)\right)\right| dt < \infty$$

By applying [7, Corollary 3.7], $\widetilde{w}_H = |Q_g|$, where Q_g is the outer function of g, that is, $Q_g(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log g\left(e^{it}\right) dt\right)$. For every $\varepsilon > 0$, the solution of Dirichlet problem in the disc $D(0, 1-\varepsilon)$ with boundary condition $t \to \log|1-(1-\varepsilon)e^{it}|^2$, $t \in [-\pi, \pi]$, is the function $z \to \log|1-z|^2$. Since this function is harmonic on $\mathbb{D}$ and $\int_0^{\pi} \log(2(1-cos(t))dt = 0$, we conclude $|Q_g(z)| = |1-z|^2$.

Let $r \in]0,1[$,

$$\frac{\widetilde{v}_H(r)}{\widetilde{v}_h(r)} = \frac{\widetilde{w}_h(r)}{\widetilde{w}_H(r)} = \frac{1}{(1-r)^2}\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Re}\left(\frac{e^{it}+r}{e^{it}-r}\right)\left|1-e^{it}\right|^2 dt =$$

$$\frac{1}{(1-r)^2}\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{\left|e^{it}-r\right|^2}\left|1-e^{it}\right|^2 dt \geq \frac{1}{(1-r)}\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1+r}{(1+r)^2}\left|1-e^{it}\right|^2 dt \geq$$

$$\geq \frac{1}{1-r^2}\frac{1}{2\pi}\int_{-\pi}^{\pi} 2(1-\cos t)dt = \frac{2}{1-r^2}$$

Hence,

$$\sup_{r\in]0,1[} \frac{\widetilde{v}_H(r)}{\widetilde{v}_h(r)} = \infty.$$

Therefore, the corresponding associated weights $\widetilde{v}_H, \widetilde{v}_h$ are not equivalent.

Example 1.2.22. This example shows that there are non-radial weights $v : \mathbb{D} \to \mathbb{R}$ such that $\widetilde{v}_H$ and $\widetilde{v}_h$ are equivalent, but $\widetilde{v}_h < \widetilde{v}_H$ for each $z \in \mathbb{D}$.

Let m and M be two different positive real numbers. We define the function g by $g : \partial\mathbb{D} \to \mathbb{R}$, $g(e^{it}) = m$ if $t \in [-\pi, 0]$ and $g(e^{it}) = M$ if $t \in]0, \pi]$, which is in $L^1(\partial\mathbb{D})$. Now, we consider the function $w : \mathbb{D} \to \mathbb{R}$, $w(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right) g(e^{it})dt$. This is in fact the Poisson Kernel of the function g and by [61, Theorem 11.7], w is harmonic in $\mathbb{D}$. Furthermore, w is positive since $\operatorname{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right) g(e^{it}) = \frac{1-|z|^2}{|e^{it}-z|^2} g(e^{it}) > 0$. Thus, w is a growth condition in $\mathbb{D}$. Since w is harmonic and positive, it follows $\widetilde{w}_h = w$. In [7, Corollary 3.7], it is shown that with these conditions $\widetilde{w}_H = |Q_g|$, where Q_g is the outer function of g, i.e., $Q_g(z) = \exp\left(\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log g\left(e^{it}\right) dt\right)$. Now, as we have taken $m \neq M$, and by applying that the geometric mean is smaller than the arithmetic mean, we get the inequality:

$m^a M^{1-a} < am + (1-a)M$, for all $0 < a < 1$.

In particular, for $a = \frac{1}{2\pi}\int_{-\pi}^{0} \operatorname{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right) dt$:

$\widetilde{w}_H(z) = |Q_g(z)| = \exp(\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \operatorname{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right) \log g\left(e^{it}\right) dt) =$

$m^a M^{1-a} < am + (1-a)M = w(z) = \widetilde{w}_h(z)$.

Therefore, the corresponding associated weights $\widetilde{v}_H$, $\widetilde{v}_h$ satisfy $\widetilde{v}_H > \widetilde{v}_h$.

To check that $\widetilde{v}_H$ and $\widetilde{v}_h$ are equivalent, suppose that $M > m$, then,

$$0 < \frac{(1-a)\frac{M}{m} + a}{\left(\frac{M}{m}\right)^{1-a}} \leq (1-a)\frac{M}{m} + a \leq \frac{M}{m}.$$

Hence, $am + (1-a)M \leq \frac{M}{m} m^a M^{1-a}$. So, $\widetilde{w}_H(z) < \widetilde{w}_h(z) \leq \frac{M}{m}\widetilde{w}_H(z)$ for each $z \in \mathbb{D}$.

We can also obtain an example with a radial weight considering the punctured disc where $\widetilde{v}_H$ and $\widetilde{w}_h$ are not equivalent.

Example 1.2.23. Let $G = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Let consider the weight $v : G \to \mathbb{R}$ defined by $v(z) = \frac{1}{-\log|z|}$, and the corresponding growth condition $w := \frac{1}{v}$.

Let $f(z) = \log|z|$, $z \in G$. From $f \in h^{\mathbb{R}}(G)$ and $|f(z)| = w(z) = -\log|z|$ we obtain $\widetilde{w}_h = \widetilde{w}_h^{\mathbb{R}} = w$.

If $f \in H(G)$, $|f(z)| \leq w(z)$, for all $z \in G$ then $\lim_{z\to a,|a|=1} f(z) = 0$. Moreover $e^{|f(z)|} \leq e^{-\log|z|} = \frac{1}{|z|}$, for all $z \in G$, and then $\left|ze^{f(z)}\right| \leq 1$, for all $z \in G$. This implies that $ze^{f(z)}$ can be holomorphically extended to a function g on $\mathbb{D}$ with $g(z) \neq 0$ if $z \neq 0$. Hence there exists $k \in \mathbb{N}_0$ such that $\hat{g}(z) := \frac{g(z)}{z^k}$ is holomorphic on $\mathbb{D}$ and satisfies $\hat{g}(0) \neq 0$. Now $\log|\hat{g}|$ is a harmonic function that can be extended continuously to $\partial\mathbb{D}$ as 0. This yields that $\log|\hat{g}|$ is identically null and consequently $|e^{f(z)}| = |z|^{k-1}$ for each $z \in \mathbb{D} \setminus \{0\}$. The unique $k \in \mathbb{N}_0$ which does not give contradiction is $k = 1$. Thus f has to be constantly 0 because of the behaviour when going to the boundary. This means $\widetilde{w}_H = 0$.

The next result is inspired by the results of [19, Section 6]. Besides putting the more general context, we remove from the weight the condition of being twice differentiable. For the proof we use the following property:

Let $\Psi :]0,1[\to \mathbb{R}$ be an increasing and convex function. Then, for each $r_0 \in [0,1[$ there exists $\alpha_0 \geq 0$ depending on r_0 such that $\Psi(r) \geq \alpha_0(r - r_0) + \Psi(r_0)$.

If, in addition Ψ is a strictly convex function, then, for each $r_0 \in [0,1[$ there exists $\alpha_0 \geq 0$ depending on r_0 such that $\Psi(r) > \alpha_0(r - r_0) + \Psi(r_0)$ for all $r \in [0,1[\backslash r_0$.

Theorem 1.2.24. *Let $g : [0,1[\to \mathbb{R}^+$ be a non increasing continuous function such that $g(1^-) = 0$ and $\log(\frac{1}{g})|_{]0,1[}$ is convex. Let $N \geq 2$. Consider the unitary weight $v : B_{\mathbb{R}^N} \to \mathbb{R}^+$ defined by $v(x) := g(\|x\|)$. Then $v = \widetilde{v}_h$. Moreover if $N = 2k$ is even and we consider $v(z) = g(|z|)$ for a norm $|\cdot|$ in $\mathbb{C}^k$ and z in the corresponding unit ball, then we have $v = \widetilde{v}_H$.*

Proof. We restrict ourselves to the case $v : \mathbb{D} \to \mathbb{R}^+$, considering $v(z) = g(|z|)$. Proving the equality $v = \widetilde{v}_H$ in this case, the statement is a consequence of Proposition 1.2.16, Proposition 1.2.17 and the order relation $v \leq \widetilde{v}_h \leq \widetilde{v}_H$.

We fix $r_0 \in [0,1[$. Define $\Psi := log(\frac{1}{v}|_{[0,1[})$. As Ψ is increasing and convex we can get $\alpha_0 \geq 0$ which depends on r_0 such that $\Psi(r) \geq \alpha_0(r - r_0) + \Psi(r_0)$, for all $r \in [0,1[$. Now we compute

$$\sup_{0<r<1} v(r)\exp(\alpha_0 r) = \exp\left(\sup_{0<r<1} (\log v(r) + \alpha_0 r)\right) =$$
$$\exp\left(\sup_{0<r<1} (-\Psi(r_0) + \alpha_0 r)\right) \leq$$
$$\exp\left(\sup_{0<r<1} (-\alpha_0(r - r_0) - \Psi(r_0) + \alpha_0 r)\right) = v(r_0)\exp(\alpha_0 r_0).$$

Now we consider $f_0(z) := \frac{1}{v(r_0)\exp(\alpha_0 r_0)} \exp(\alpha_0 z)$ for $z \in \mathbb{D}$. We have

$$v(r_0)\,|f_0(r_0)| = 1.$$

Let $z \in \mathbb{D}$,

$$v(|z|)\,|f_0(z)| \le \frac{v(|z|)\exp(\alpha_0|z|)}{v(r_0)\exp(\alpha_0 r_0)} \le 1,$$

and this means $f_0 \in H_v(\mathbb{D})$ and $\|f_0\|_v = 1$. (a) Let $|\cdot|$ be a norm in $\mathbb{C}^N$ for $N \ge 2$ fixed. Now we denote $v_N(z) := g\left(|z|\right)$ for $z \in \mathbb{C}^N$. We apply Proposition 1.2.17 and $v = \widetilde{v}_H$ to get $\left(\widetilde{v}_N\right)_H(z) = g\left(|z|\right)$ for each $z \in \mathbb{C}^N$. Hence, we have from the order relation (Proposition 1.2.1) in the weights that $v_N = \left(\widetilde{v}_N\right)_h = \left(\widetilde{v}_N\right)_H$ holds.

Finally, if we consider now $v_N(x) := g\left(\|x\|\right)$ for $x \in \mathbb{R}^N$, then 1.2.16 implies also that, for $N \ge 1$ there exists $g_{2N+1} : [0,1[\to]0,\infty[$ such that

$$v_{2N+1}(x) = g\left(\|x\|\right) \le \left(\widetilde{v}_{2N+1}(x)\right)_h = g_{2N+1}\left(\|x\|\right) \le \left(\widetilde{v}_{2N}(x)\right)_h = g\left(\|x\|\right),$$

the last equality being a consequence from (a) since the Euclidean norm is the same in $\mathbb{C}^N$ as in $\mathbb{R}^{2N}$.

□

Remark 1.2.25. (a) From the proof given above we can deduced that for a weight $v(z) = g(|z|)$ defined on $\mathbb{C}$ for $g : [0,\infty[\to \mathbb{R}^+$ continuous and such that $f(z) := \exp(\alpha z) \in H_v(\mathbb{C})$ for all $\alpha \ge 0$ we have $v = \widetilde{v}_H$. This implies that, for such a function g there exists $\widetilde{g} : [0,\infty[\to \mathbb{R}^+$ such that if we define v_n in $\mathbb{C}^n$ as $v_n(z) = g(|z|)$ then $(\widetilde{v}_n)_H(z) = \widetilde{g}(|z|)$ for each $n \in \mathbb{N}$ and for each norm $|\cdot|$ defined in $\mathbb{C}^n$.

(b) For any increasing function $h : [0,1[\to [0,\infty[$ which is strictly convex and satisfies $\lim_{t\to 1} h(t) = \infty$ (or $h : [0,\infty[\to [0,\infty[$ with $e^{at} = o(e^{h(t)})$ when t goes to infinity for each $a > 0$), the function $g(t) = e^{-h(t)}$ satisfies the hypothesis of Theorem 1.2.24. Thus, we obtain examples of weights where $v = \widetilde{v}_H$.

Example 1.2.26. In [7, Examples 1.7] and in [19, Example 13] there are examples of weights $v(z) = g(\|z\|)$ defined on the Euclidean unit ball of $\mathbb{C}^n$ which satisfy that $v = \widetilde{v}_H$. We write below some of them for which we have $v = g(|z|)$ with $\log \frac{1}{g}$ convex and $|\cdot|$ being any norm in $\mathbb{C}^n$. The function g can be obtained using the general method given in Remark 1.2.25.

(a) $v(z) = \exp(C/(1-|z|^\beta)$, $C > 0, \beta > 1$,
(b) $v(z) = (1-|z|)^\alpha$, $\alpha > 0$,
(c) $v(z) = \arccos(|z|)$,
(d) $v(z) = \cos(\frac{\pi}{2}|z|)$,
(e) $v(z) = 1/\max\left(1, -C\log(1-|z|)\right)$.

Chapter 2

Composition operators

2.1 Pluriharmonic functions

Let $G \subseteq \mathbb{C}^N$ be an open and connected set. A function $f : G \to \mathbb{C}$ of class C^2 is said to be pluriharmonic (see for instance [51, 2.2]) if for every complex line $l = \{a + b\lambda\}$ the function $\lambda \to f(a + b\lambda)$ is harmonic on the set $G_l \equiv \{\lambda \in \mathbb{C} : a + b\lambda \in G\}$. This condition is equivalent to:

$$\frac{\partial^2 f}{\partial z_j \partial \overline{z_k}} = 0, \forall j, k = 1, \cdots, n.$$

Let $ph(G)$ denote the set of pluriharmonic functions on G. In this case we have the inclusions

$$H(G) \subset ph(G) \subset h(G).$$

For a weight v the *weighted Banach spaces of pluriharmonic functions* with weight v are defined by:

$$\begin{aligned} ph_v(G) \ &: \ = \{f \in ph(G) : \|f\|_v := \sup_{z \in G} v(z) \, |f(z)| < \infty\}, \text{ and} \\ ph_{v_0}(G) \ &: \ = \{f \in ph(G) : vf \text{ vanishes at infinity on } G\}. \end{aligned}$$

The unit balls in these spaces are denoted by b_v^p and $b_{v_0}^p$.

Definition 2.1.1. Let v be a *weight* on G and let $w = \frac{1}{v}$. By $\widetilde{w}_{ph}, : G \to \mathbb{R}$ we denote the function

$$\widetilde{w}_{ph}(z) := \sup \{|f(z)| : f \in b_v^p\},$$

w is called the *growth condition* of the weight v and $\widetilde{w}_{ph}(z)$ the associated growth condition. The *pluriharmonic associated weight* with v is defined as $\widetilde{v}_{ph} := \frac{1}{\widetilde{w}_{ph}}$.

The inclusions $H_v(G) \subseteq ph_v(G) \subseteq h_v(G)$ imply $v \leq \tilde{v}_h \leq \tilde{v}_{ph} \leq \tilde{v}_H$. The unit ball b_v^p is compact for the compact open topology τ_0. Hence, these pluriharmonic associated weights share all the properties of the corresponding harmonic associated weights given in Chapter 1.

If $f \in ph(G)$, then $f = u + iv$ with u, v in the space $ph(G)^{\mathbb{R}}$ of real valued pluriharmonic functions. By [51, Proposition 2.2.3], u and v are locally real parts of holomorphic functions. Thus, the argument of Proposition 1.2.17, which is not valid for the harmonic case, works for the pluriharmonic associated weight.

If G_1 and G_2 are open and connected subsets of $\mathbb{C}^N$ and $\mathbb{C}^M$ and $\varphi : G_2 \to G_1$ is a holomorphic function, then we can consider the composition operator

$$C_\varphi : ph(G_1) \to ph(G_2),$$

$$C_\varphi(f) := f \circ \varphi.$$

It is in fact well defined, since if $f \in ph(G_1)$, then $f \circ \varphi \in C^2(G_2)$ and $f = u + iv$ with u, v in the space $ph(G_1)^{\mathbb{R}}$ of real valued pluriharmonic functions, being locally real parts of holomorphic functions. Now, the composition of holomorphic functions is a holomorphic function (see [38, Theorem 5. Chap 1]). From this, it is concluded that $u \circ \varphi$ and $v \circ \varphi$ are also locally real parts of holomorphic functions. Thus, by [51, Proposition 2.2.3], $u \circ \varphi, v \circ \varphi \in ph(G_1)^{\mathbb{R}}$. Also, $C_\varphi : (ph(G_1), \tau_0) \to (ph(G_2), \tau_0)$ is a continuous and linear map.

2.2 Continuity characterization

Proposition 2.2.1. *Let v and w be weights on G_1 and G_2 respectively. The following conditions are equivalent for the composition operator C_φ:*

(a) $C_\varphi : ph_v(G_1) \to ph_w(G_2)$ is continuous,

(b) $C_\varphi\left(ph_v(G_1)\right) \subset ph_w(G_2)$,

(c) $\sup_{z \in G_2} \frac{w(z)}{\tilde{v}_{ph}(\varphi(z))} < \infty$.

Moreover, if these equivalences hold, then $\|C_\varphi\| = \sup_{z \in G_2} \frac{w(z)}{\tilde{v}_{ph}(\varphi(z))}$.

Proof. The arguments are an adaptation of those used in [14] for spaces of holomorphic functions. (a) and (b) are equivalent by the Closed Graph Theorem. We assume that (a) holds. If (c) does not hold then there exists a sequence $(z_n)_n \subseteq G_2$ such that $\frac{w(z_n)}{\tilde{v}_{ph}(\varphi(z_n))}$ tends to ∞. For each $n \in \mathbb{N}$ we take $f_n \in b_v^p$ such that $|f_n(\varphi(z_n))| = \frac{1}{\tilde{v}_{ph}(\varphi(z_n))}$. This implies that, for all $n \in \mathbb{N}$, $\|C_\varphi\| \geq |C_\varphi(f_n)| \geq w(z_n)|f(\varphi(z_n))| = \frac{w(z_n)}{\tilde{v}_{ph}(\varphi(z_n))}$, a contradiction. If (c) is true then for all $f \in b_v^p$ we have $\|C_\varphi(f)(z)\|_w = \sup_{z \in G_2} w(z)|f \circ \varphi(z)| \leq \sup_{z \in G_2} \frac{w(z)}{\tilde{v}_{ph}(\varphi(z))}\|f\|_v$.

To estimate the norm of C_φ when it is continuous, we proceed in a similar way as in [10] the holomorphic case, obtaining a slight improvement. From the above argument we have that $\|C_\varphi\| \leq \sup_{z\in G} \frac{w(z)}{\widetilde{v}_{ph}(\varphi(z))}$. The transpose C_φ^t is also continuous and for each $z \in G_2$ we have $w(z)\delta_z$ is in the unit ball of $ph_w(G_2)^*$. Now, for each $z \in G_2$,

$$C_\varphi^t(w(z)\delta_z)(f) = w(z)C_\varphi^t(\delta_z)(f) = w(z)\delta_z \circ C_\varphi(f) = w(z)f(\varphi(z)) = w(z)\delta_{\varphi(z)}(f),$$

$$\|C_\varphi\| = \|C_\varphi^t\| \geq \|C_\varphi^t(w(z)\delta_z)\| = w(z)\|\delta_{\varphi(z)}\| = \frac{w(z)}{\widetilde{v}_{ph}(z)},$$

when $\|\cdot\|$ denotes both the operator norm and the dual norm. □

For essential weights, i.e. those for which $v \sim \widetilde{v}_H$, the continuity in the weighted space of holomorphic functions is equivalent to the continuity in the space of pluriharmonic functions. For weights on the unit ball of $\mathbb{C}^N$ endowed with an arbitrary norm $|\cdot|$, defined by $v(x) = g(|x|)$ with $g : [0, 1[\to \mathbb{R}^+$ being continuous and non increasing this happens when $t \mapsto \log(1/g)(e^t)$ is convex by [10] and Proposition 1.2.16.

2.3 Compactness characterization

Proposition 2.3.1. *The following conditions are equivalent:*

(i) $C_\varphi : ph_{v_1}(G_1) \to ph_{(v_2)_0}(G_2)$ *is compact.*

(ii) $C_\varphi(ph_{v_1}(G_1)) \subset ph_{(v_2)_0}(G_2)$ *and, for every sequence* $(f_n)_n \subset b^p_{v_1}$ *and convergent to* 0 *in* τ_0*, the sequence* $(C_\varphi(f_n))_n$ *has a convergent subsequence to* 0 *in the norm topology* $\|\cdot\|_{v_2}$.

Proof. (i)⟹(ii) Let $(f_n)_n \subset b^p_{v_1}$ be such that it converges to 0 in τ_0. Since $(f_n)_n \subset b^p_{v_1}$, using (i) we get a subsequence $(C_\varphi(f_{n_t}))_t$ which is convergent in $ph_{v_2}(G_2)$, and therefore, convergent to the same function in τ_0. As $(f_n)_n$ converges to 0 in τ_0 and $C_\varphi : (ph(G_1), \tau_0) \to (ph(G_2), \tau_0)$ is continuous, then $(C_\varphi(f_n))_n$ converges to 0 in τ_0 and we obtain $(C_\varphi(f_{n_t}))_t$ is convergent to 0 in $ph_{v_2}(G_2)$.

(ii)⟹(i) By the first part of (ii), $C_\varphi : ph_{v_1}(G_1) \to ph_{(v_2)_0}(G_2)$ is a well defined linear map. Let $(f_n)_n \subset b^p_{v_1}$. Since $b^p_{v_1}$ is compact in τ_0, the sequence $(f_n)_n$ has a convergent subsequence $(f_{n_t})_t$ to some $f \in b^p_{v_1}$ in τ_0. By the second part of (ii), the sequence $(C_\varphi(\frac{f_{n_t}-f}{2}))_t$ has a convergent subsequence to 0 in the norm $\|\cdot\|_{v_2}$, and since C_φ is $\mathbb{C}$-linear, $(C_\varphi(f_{n_t}))_t$ has a convergent subsequence in norm $\|\cdot\|_{v_2}$. So, $C_\varphi : ph_{v_1}(G_1) \to ph_{(v_2)_0}(G_2)$ is compact. □

Theorem 2.3.2. *Consider the following assertions:*

(i) $C_\varphi : ph_{v_1}(G_1) \to ph_{(v_2)_0}(G_2)$ *is compact.*

(ii) $C_\varphi : ph_{v_1}(G_1) \to ph_{v_2}(G_2)$ *is compact and* $C_\varphi\left(ph_{(v_1)_0}(G_1)\right) \subset ph_{(v_2)_0}(G_2)$.

(iii) $C_\varphi : ph_{(v_1)_0}(G_1) \to ph_{(v_2)_0}(G_2)$ *is compact.*

(iv) For each $\varepsilon > 0$ *there exists a compact subset* $K_2 \subset G_2$ *such that* $\frac{v_2(z)}{\widetilde{v_1}_{ph}(\varphi(z))} < \varepsilon$ *for every* $z \in G_2 \setminus K_2$.

Then (i)⟹(ii), (ii)⟹ (iii) and (iv)⟹(i). If we assume $\overline{b^p_{(v_1)_0}}^{\tau_0} = b^p_{v_1}$, *then (iii)⟹ (iv) and all the conditions are equivalent.*

Proof. (i)⟹(ii) and (ii)⟹ (iii) are trivial.

(iv)⟹(i) We show that (ii) of Proposition 2.3.1 is satisfied. We fix $\varepsilon > 0$. Let $f \in ph_{v_1}(G_1)$ satisfy $\|f\|_{v_1} \neq 0$. For $\frac{\varepsilon}{\|f\|_{v_1}}$ we select a compact set $K_2 \subset G_2$ as in (iv). For each $z \in G_2 \setminus K_2$,

$$v_2(z)\,|C_\varphi(f)(z)| = v_2(z)\,|f(\varphi(z))| = \frac{v_2(z)}{\widetilde{v_1}_{ph}(\varphi(z))}\widetilde{v_1}_{ph}(\varphi(z))\,|f(\varphi(z))|$$

$$\leq \frac{v_2(z)}{\widetilde{v_1}_{ph}(\varphi(z))}\,\|f\|_{\widetilde{v_1}_{ph}} < \varepsilon.$$

This implies $C_\varphi(ph_{v_1}(G_1)) \subset ph_{(v_2)_0}(G_2)$.

Let $\varepsilon > 0$ and $(f_n)_n$ be a sequence in $(b_p)_{v_1}$ which tends to 0 in τ_0. By (iv), there exists a compact subset $K_2 \subset G_2$ such that $v_2(z) < \frac{\varepsilon}{2}\widetilde{v_1}_{ph}(\varphi(z))$, whenever $z \in G_2 \setminus K_2$. Hence,

$$v_2(z)\,|f_n(\varphi(z))| \leq \frac{v_2(z)}{\widetilde{v_1}_{ph}(\varphi(z))}\widetilde{v_1}_{ph}(\varphi(z))\,|f_n(\varphi(z))| < \frac{\varepsilon}{2}\,\|f_n\|_{\widetilde{v_1}_{ph}} \leq \frac{\varepsilon}{2} \tag{2.1}$$

for each $z \in G_2 \setminus K_2$ and for each $n \in \mathbb{N}$. Since $(f_n)_n$ converges to 0 in τ_0, for the compact set $\varphi(K_2)$ and $\frac{\varepsilon}{2\max_{z\in K_2} v_2(z)} > 0$ there exists n_0 such that $n \geq n_0$ implies

$$\sup_{\zeta\in\varphi(K_2)} |f_n(\zeta)| < \frac{\varepsilon}{2\max_{z\in K_2} v_2(z)}. \tag{2.2}$$

From (2.1) and (2.2) it follows that, for each $n \geq n_0$,

$$\|C_\varphi(f_n)\|_{v_2} \leq \sup_{z\in G_2\setminus K_2} v_2(z)\,|f_n(\varphi(z))| + \sup_{z\in K_2} v_2(z)\,|f_n(\varphi(z))| < \varepsilon.$$

We assume $\overline{b^p_{(v_1)_0}}^{\tau_0} = b^p_{v_1}$ to show (iii)⟹ (iv).

We suppose that

$$C_\varphi : ph_{(v_1)_0}(G_1) \to ph_{(v_2)_0}(G_2)$$

is compact. By the Schauder Theorem [58, 15.3], the transpose operator

$$C_\varphi^t : ph_{(v_2)_0}(G_1)^* \to ph_{(v_1)_0}(G_2)^*$$

($C_\varphi^t(y) = y \circ C_\varphi$) is compact. Therefore, $C_\varphi^t(U)$ is relatively compact in $ph_{(v_1)_0}(G_2)^*$, where U is the unit closed ball in $ph_{(v_2)_0}(G_2)^*$, i.e.,

$$U = \left\{ y \in ph_{(v_2)_0}(G_2)^* : \|y\|_{ph_{(v_2)_0}(G_2)^*} \leq 1 \right\}.$$

Now, let $A := \{v_2(z)\delta_z : z \in G_2\}$. For each $z \in G_2$,

$$\|v_2(z)\delta_z\|_{ph_{(v_2)_0}(G_2)^*} =$$

$$\sup_{f \in ph_{(v_2)_0}(G_2), \|f\|_{v_2} \leq 1} |v_2(z)\delta_z(f)| \leq \sup_{f \in ph_{(v_2)_0}(G_2), \|f\|_{v_2} \leq 1} \|f\|_{v_2} \leq 1.$$

Hence, $A \subset U$ and $C_\varphi^t(A) \subset C_\varphi^t(U)$. For each $z \in G_2$ and every $f \in ph_{(v_1)_0}(G_1)$,

$$C_\varphi^t(v_2(z)\delta_z)(f) = v_2(z)\delta_{\varphi(z)}(f).$$

Thus, $C_\varphi^t(A) = \left\{ v_2(z)\delta_{\varphi(z)} : z \in G_2 \right\}$. Let B be the closing of $C_\varphi^t(U)$ in $ph_{(v_1)_0}(G_1)^*$. Since B is compact, the identity map $(B, \|\cdot\|_{ph_{(v_1)_0}(G_1)^*}) \to (B, \sigma^*)$ is continuous and closed, where σ^* denotes the weak topology $\sigma(ph_{(v_1)_0}(G_1)^*, ph_{(v_1)_0}(G_1))$. Therefore, the norm topology on B coincides with the one induced by σ^*. This implies that, for every $\varepsilon > 0$ there exists a finite set $\{f_1, \cdots f_s\} \subset ph_{(v_1)_0}(G_1)$ such that

$$B \cap V(\{f_1, \cdots f_s\}, \varepsilon) \subset \left\{ \psi \in B : \|\psi\|_{ph_{(v_1)_0}(G_1)^*} < \varepsilon \right\}$$

where $V(\{f_1, \cdots f_s\}, \varepsilon) = \{\psi \in ph_{(v_2)_0}(G_2)^* : |\psi(f_i) \leq \varepsilon, \text{ for all } i = 1 \cdots s\}$.

Hypothesis (iii) implies $C_\varphi(f_j) \in ph_{(v_2)_0}(G_2)$, $j = 1, \cdots, s$, so that there exists a compact subset K_2 of G_2 such that

$\left|v_2(z)\delta_{\varphi(z)}(f_j)\right| = v_2(z)\,|f_j \circ \varphi(z)| < \varepsilon$, $j = 1, \cdots, s$, $\quad z \in G_2 \setminus K_2$.

Thus, for each $z \in G_2 \setminus K_2$ we have $v_2(z)\delta_{\varphi(z)} \in B \cap V(\{f_1, \cdots f_s\}, \varepsilon)$. Then,

$$\left\|v_2(z)\delta_{\varphi(z)}\right\|_{ph_{(v_1)_0}(G_1)^*} < \varepsilon, \text{ for all } z \in G_2 \setminus K_2. \tag{2.3}$$

As $(b_p)_{v_1} = \overline{(b_p)_{(v_1)_0}}^{\tau_0}$, we can apply Proposition 1.2.10 to get

$\left\|\delta_{\varphi(z)}\right\|_{ph_{(v_1)_0}(G_1)^*} = \left\|\delta_{\varphi(z)}\right\|_{ph_{v_1}(G_1)^*} = \frac{1}{\widetilde{v_1}_{ph}(\varphi(z))}$,

which together with (2.3) gives us the condition (iv). □

Corollary 2.3.3. *Let v be a decreasing unitary weight on $B_{\mathbb{C}^N}$, which converges to 0 at the boundary, such that $\log(\frac{1}{v})$ is convex, G an open and connected set in $\mathbb{C}^N$, w a weight on G and $\varphi : G \to B_{\mathbb{C}^N}$ a holomorphic function. The following conditions are equivalent:*

(i) $C_\varphi : H_v(B_{\mathbb{C}^N}) \to H_w(G)$ is compact.

(ii) $C_\varphi : ph_v(B_{\mathbb{C}^N}) \to ph_w(G)$ is compact.

(iii) $C_\varphi : ph_v(B_{\mathbb{C}^N}) \to ph_{w_0}(G)$ is compact.

Proof. By Theorem 1.2.24 we have $\widetilde{v}_{ph} = \widetilde{v}_H$. Now we apply [13, Theorem 8] and Theorem 2.3.2. □

2.4 Essential norm

Given normed spaces E and F and a continuous linear map $A : E \to F$, the *essential norm* of A is defined by $\|A\|_e = \inf\{\|A - K\| : K \text{ is compact }\}$. It is clear from the definition that A is compact if and only if $\|A\|_e = 0$. Our purpose in the rest of the chapter is to extend the results in [10, 59] and calculate the essential norm of the composition operator between spaces of pluriharmonic functions.

We present below a generalization of [59, Proposition 2.1] to a wider context. We remark that in the statement $\mathbb{C}^N$ could be replaced by $\mathbb{R}^N$, but we have preferred to restrict to the complex variables case because our natural examples of spaces of functions satisfying the hypotheses are the spaces of pluriharmonic and holomorphic functions.

Proposition 2.4.1. *Let G be a balanced, bounded and open subset of $\mathbb{C}^N$ and let v be a weight on G which vanishes at infinity and such that there exists $M > 0$ such that*

$$\sup_{z \in G, 0<r<1} \frac{v(z)}{\widetilde{v}_h(rz)} \leq M.$$

Then there exists a sequence of operators $(T_n)_n$ on $(h(G), \tau_0)$ such that $T_n : h_v(G) \mapsto h_v(G)$ is compact for each $n \in \mathbb{N}$ and the following conditions are fulfilled:

(i) $H(G)$ and $ph(G)$ are invariant subspaces of T_n for each $n \in \mathbb{N}$

(ii) $\tau_0 - \lim_{n\to\infty} T_n = I$

(iii) $\limsup_{n\to\infty} \|I - T_n\| \leq 1$.

Proof. We assume without loss of generality $v(x) \leq 1$ for all $x \in G$. For $0 < r < 1$, let $C_r(f)(\cdot) = f(r\cdot)$. The same proof of Proposition 2.2.1 shows that the hypothesis on the weight is equivalent to the existence of an $M > 0$ such that $\|C_r\| \leq M$ for each $0 < r < 1$, the norm of the operators taken in $L(h_v(G))$. For each $0 < r < 1$ the operator C_r is compact on $h_v(G)$. This can be checked observing that the image of a bounded sequence pointwise (compact open) convergent to zero is norm convergent to zero. This compactness and the fact that $C_r(h_v(G)) \subseteq h_{v_0}(G)$ implies that for each $0 < r < 1$ there exists $L \subset G$ compact such that $v(z)|C_r(f)(z)| < \varepsilon$ for all $z \in G \setminus L$ and for all $f \in b_v$. The standard compactness argument necessary to prove this is the same used in [59, Lemma 2.1] for the space of one variable holomorphic functions on the unit disc and a radial weight on it. Moreover, since G has a fundamental sequence of compact sets which are balanced, we have $\tau_0 - \lim_n C_{r_n}(f) = f$ for each $f \in h(G)$ and for each sequence $(r_n)_n \subset]0,1[$ tending to 1. Hence the sequence $(C_{r_n})_n$ tends uniformly on b_v to the identity for the compact open topology [58, Proposition 23.27], since this subset is relatively compact in this topology.

Let $(\varepsilon_n)_n$ be a decreasing sequence of positive numbers tending to zero. The facts established above permit us to choose an increasing sequence of positive numbers $(r_n)_n$ tending to 1 and a fundamental sequence $(L_n)_n$ of compact subsets of G such that

$$\sup_{f \in b_v, z \in L_n} |(I - C_{r_n})(f)(z)| \leq \varepsilon_n$$

and

$$\sup_{f \in b_v, z \in G \setminus L_{n+1}} v(z)|C_{r_n} f(z)| \leq \varepsilon_n.$$

For each $n \in \mathbb{N}$ we choose $m(n) \in \mathbb{N}$ satisfying $(1 + M)/m(n) < \varepsilon_n$. We define

$$T_n = \frac{1}{m(n)} \sum_{j=n}^{n+m(n)-1} C_{r_j}.$$

The construction implies that $\|T_n\| \leq M$ for each $n \in \mathbb{N}$ and $(T_n)_n$ is a sequence convergent to the identity for τ_0. We observe

$$I - T_n = \frac{1}{m(n)} \sum_{j=n}^{n+m(n)-1} (I - C_{r_j}).$$

For $z \in G$, let $j_0 \geq n$ be the minimum natural number with $z \in L_{j_0}$. Then, for each $f \in b_v$, we compute

(a) $v(z)|f(z) - C_{r_j}(f)(z)| \leq |f(z) - C_{r_j}(f)(z)| \leq \varepsilon_j$ for $j_0 \leq j$.

(b) $v(z)|f(z) - C_{r_{j_0-1}}(f)(z)| \leq \|I - C_{r_{j_0-1}}\| \leq 1 + M$.

(c) $v(z)|f(z) - C_{r_j} f(z)| \leq 1 + \sup_{z \in G \setminus L_{j+1}} v(z)|C_{r_j} f(z)| \leq 1 + \varepsilon_j$ if $n \leq j \leq j_0 - 2$.

And we achieve,

$$\|I - T_n\| \leq \frac{m(n)-1}{m(n)}(1+\varepsilon_n) + \frac{1+M}{m(n)} \leq 1 + 2\varepsilon_n,$$

which clearly implies (ii).

□

The next results extend the main theorems in [10, 59] to our context.

Theorem 2.4.2. *Let $G_1 \subset \mathbb{C}^N$ be a balanced, bounded and open subset of $\mathbb{C}^N$ and let v be a weight on G_1 which vanishes at infinity and such that there exists $M > 0$ such that $\sup_{z \in G_1, 0<r<1} \frac{v(z)}{\widetilde{v}_h(rz)} \leq M$. Let $G_2 \subset \mathbb{C}^M$ a connected and open set, let w be a weight on G_2 and let $\varphi : G_2 \to G_1$ be a holomorphic function. For the composition operator $C_\varphi : ph_v(G_1) \to ph_w(G_2)$ and for any fundamental sequence $(K_n)_n$ of compact subsets of G_1 we have*

(a) $\|C_\varphi\|_e \leq \lim_{n\to\infty} \sup_{\varphi(z) \in G_1 \setminus K_n} \frac{w(z)}{\widetilde{v}_{ph}(\varphi(z))}$, *and*

(b) C_φ *is compact if and only if* $\|C_\varphi\|_e = \lim_{n\to\infty} \sup_{\varphi(z) \in G_1 \setminus K_n} \frac{w(z)}{\widetilde{v}_{ph}(\varphi(z))} = 0.$

The same result is true for the corresponding operator considered between the spaces of holomorphic functions.

Proof. To prove (a) we consider a sequence $(T_n)_n$ of compact operators given by Proposition 2.4.1. The invariance of $ph(G_1)$ and $T_n(h_v(G_1)) \subseteq h_v(G_1)$ implies that $T_n|_{ph_v(G_1)} \in K(ph_v(G_1))$ for each $n \in \mathbb{N}$. Therefore, $C_\varphi \circ T_n$ (we are denoting by T_n the restriction) is also compact for each n. Thus,

$$\|C_\varphi\|_e \leq \|C_\varphi - C_\varphi(T_n)\| = \|C_\varphi(I - T_n)\|.$$

On the other hand, we have that for each $j \in \mathbb{N}$:

$$\|C_\varphi(I - T_n)\| = \sup_{f \in b_v^p} \|C_\varphi(I - T_n)f\| = \sup_{f \in b_v^p} \sup_{z \in G_2} w(z)\,|(I - T_n)f(\varphi(z))| \leq$$

$$\leq \sup_{f \in b_v^p} \sup_{\varphi(z) \in K_j} w(z)\,|(I - T_n)f(\varphi(z))| + \sup_{f \in b_v^p} \sup_{\varphi(z) \in G_1 \setminus K_j} w(z)\,|(I - T_n)f(\varphi(z))|$$

The first term above goes to 0 as n tends to ∞ because w is bounded, b_v^p is τ_0-compact and $T_n - I$ is τ_0 convergent to 0 [58, Proposition 23.27]. For the second term we have the estimate

$$\sup_{f\in b_v^p}\sup_{\varphi(z)\in G_1\setminus K_j} w(z)\,|(I-T_n)f(\varphi(z))| \leq \sup_{\varphi(z)\in G_1\setminus K_j} \frac{w(z)}{\widetilde{v}_{ph}(\varphi(z))}\|I-T_n\|.$$

Finally taking the limit for $n\to\infty$ and using the fact that the inequlity is valid for each $j\in\mathbb{N}$, we see that the claim follows.

To show (b) we assume that there is $c>0$ and a sequence $(z_j)_j$ such that $\varphi(z_j)\in G_1\setminus K_j$ and such that $\frac{w(z_j)}{\widetilde{v}_{ph}(\varphi(z_j))}\geq c$. Since $C_r(f)\to f$ pointwise as $r\to 1$ and $\|C_r\|\leq M$ by hypothesis, taking a subsequence of $(z_j)_j$ if necessary we can find a sequence $(f_j)_j \in Mb_v^p \cap ph_{v_0}(G_1)$ such that $|\widetilde{v}_{ph}(z_j)f_j(z_j)-1|\leq 1/4$ and $v(\varphi(z_k))|f_j(\varphi(z_k))|<1/4$ for each $k>j$. This last condition is possible since each f_j is in $h_{v_0}(G_1)$ and $(\varphi(z_j))_j$ tends to infinity at the boundary of G_1. This yields that $(f_j)_j$ is a bounded sequence in $ph_v(G_1)$ such that $\|C_\varphi(f_j)-C_\varphi(f_k)\|\geq 1/2$ for $j\neq k$, and then C_φ is not compact. □

Remark 2.4.3. If in Proposition 2.4.1 (and then also Theorem 2.4.2) we consider weights v defined on $\mathbb{C}^N$ and satisfying the additional assumption

$$\lim_{\|z\|\to\infty}\frac{v(z)}{\widetilde{v}(rz)}=0$$

for each $0<r<1$, then the corresponding operators $C_r : ph_v(\mathbb{C}^N)\to ph_v(\mathbb{C}^N)$ are compact and $C_r\left(ph_v(\mathbb{C}^N)\right)\subset ph_{v_0}(\mathbb{C}^N)$. This can be checked using standard arguments, more precisely seeing that bounded sequences pointwise convergent to zero are mapped to sequences which converge to zero in norm. Accordingly both 2.4.1 and 2.4.2 remain valid for this kind of weights. An example of a weight satisfying the hypothesis is $v(z)=e^{-\|z\|}$.

Theorem 2.4.4. *Let B be the unit ball of $(\mathbb{C}^N,|\cdot|)$ and let $\varphi : G\to B$ be a holomorphic function on an open and connected set G in $\mathbb{C}^M$. Let $g:[0,1[\to\mathbb{R}^+$ be a continuous function with $g(1^-)=0$ and $v(z)=g(|z|)$ be a weight on B such that $\widetilde{v}_H=\widetilde{v}_{ph}$. Let w be a weight on G which vanishes at ∞. Suppose that the operator $C_\varphi : ph_v(B)\to ph_w(G)$ is continuous. Then*

$$\|C_\varphi\|_e = \limsup_{|\varphi(z)|\to 1}\frac{w(z)}{\widetilde{v}(\varphi(z))}$$

where $\widetilde{v}$ denotes the common associated weight for the spaces of holomorphic and pluriharmonic functions.

Proof. By Theorem 2.4.2 we only have to show the lower bound of the essential norm. Let us consider a compact operator $K : ph_v(B) \to ph_w(G)$. We show that the restriction $C_\varphi - K$ to $H_v(B)$ has norm not smaller than $\limsup_{|\varphi(z)|\to 1} \frac{w(z)}{\widetilde{v}(\varphi(z))}$, concluding from this that the norm of the operator $C_\varphi - K$ on $ph_v(B)$ has the same lower bound.

We can find a sequence $(z_n)_n$ in G with $|\varphi(z_n)| > 1 - 1/n$ and

$$\lim_n \frac{w(z_n)}{\widetilde{v}(\varphi(z_n))} = \limsup_{|\varphi(z)|\to 1} \frac{w(z)}{\widetilde{v}(\varphi(z))}.$$

Now, taking a subsequence if necessary, we can assume that there is $a \in \mathbb{C}^N$ such that $|a| = 1$ and $\lim \varphi(z_n) = a$. We apply the theorem of Hahn-Banach to get $b \in \mathbb{C}^N$ satisfying $\langle a, b\rangle = 1$ and $|\langle x, b\rangle| \le |x|$ for each $x \in \mathbb{C}^N$. Take a sequence $(\alpha(n))_n$ of natural numbers tending to ∞ such that

$$\lim_{n\to\infty} \langle \varphi(z_n), b\rangle^{\alpha(n)} = 1.$$

Let $(\varepsilon_n)_n$ be a decreasing sequence of positive numbers tending to 0. Since $B_v = \overline{B_{v_0}}^{\tau_0}$ [8, Example 2.1 (ii)], for each $n \in \mathbb{N}$, we can find $f_n \in B_{v_0}$ such that

$$|f_n(\varphi(z_n))| \ge \frac{1}{\widetilde{v}(\varphi(z_n))} - \varepsilon_n.$$

We set $h_n(z) := \langle z, b\rangle^{\alpha(n)} f_n(z)$, $z \in B$. The sequence $(h_n)_n$ is in the unit ball of $H_{v_0}(B)$ and converges to 0 in τ_0. As we have $H_{v_0}^{**}(B) = H_v(B)$ as a consequence of [8, Corollary 1.2, Example 2.1 (ii)], the Ng construction of the predual of $H_v(B)$ [60] implies that the compact open (pointwise) topology τ_0 agrees with the weak* topology on $B_{v_0} \subset B_v$, since it is Hausdorff. Then the pointwise convergence of $(h_n)_n$ implies the weak convergence of this sequence in $H_{v_0}(B)$. This argument is well-known. Therefore, since K is a compact operator, we get

$$\lim_{n\to\infty} \|K(h_n)\|_w = 0.$$

Now we compute

$$\|C_\varphi(h_n) - K(h_n)\|_w \ge \|h_n \circ \varphi\|_w - \|K(h_n)\|_w =$$

$$\sup_{z\in G} w(z)\,|h_n(\varphi(z))| - \|K(h_n)\|_w \ge$$

$$\ge \frac{w(z_n)}{\widetilde{v}(\varphi(z_n))}\,|\langle \varphi(z_n), b\rangle|^{\alpha(n)} - w(z_n)\,|T(\varphi(z_n))|^{\alpha(n)}\,\varepsilon_n - \|K(h_n)\|_w.$$

Taking the limit as n tends to ∞, we get the desired inequality.

□

Concluding remarks. (a) The hypotheses of the above theorem are satisfied by every weight $v = g(|z|)$ for $g : [0,1[\to \mathbb{R}^+$ continuous and non-increasing with $g(1^-) = 0$ and $\log(1/g)$ convex in $]0,1[$, by Theorem 1.2.24.
(b) If $v = g(|z|)$ with g non-increasing and $g(1) = 0$ and no additional assumptions, then Proposition 2.4.1 can be applied for $G = H_v(B)$. For $\varphi : G \to B$ holomorphic and w being a weight on B vanishing at infinity, if we consider the composition operator $C_\varphi : H_v(B) \to H_w(G)$, then the proof of Theorem 2.4.4 shows

$$\|C_\varphi\|_e = \limsup_{|\varphi(z)|\to 1} \frac{w(z)}{\widetilde{v}_H(\varphi(z))}.$$

(c) In the same situation, if we consider $C_\varphi : ph_v(B) \to ph_w(G)$, then we have

$$\limsup_{|\varphi(z)|\to 1} \frac{w(z)}{\widetilde{v}_H(\varphi(z))} \le \|C_\varphi\|_e \le \limsup_{|\varphi(z)|\to 1} \frac{w(z)}{\widetilde{v}_{ph}(\varphi(z))}.$$

(d) Zheng proved in [74] that for a holomorphic function $\varphi : \mathbb{D} \to \mathbb{D}$, the essential norm of the composition operator $C_\varphi : H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D})$ is either 0 or 1. In the case of weighted Banach spaces of holomorphic and harmonic functions the situation differs. Consider the typical weight $v(z) = 1 - |z|$ in $\mathbb{D}$ and the family of symbols $\varphi_{a,n}(z) = z^n\left(\frac{z+a}{1+a}\right)$, $n \in \mathbb{N}$, $a \ge 0$. Since $\varphi_{a,n}(0) = 0$, each $\varphi_{a,n}$ is contractive by the Schwarz lemma, we obtain

$$\|C_{\varphi_{a,n}}\| = \sup_{z\in\mathbb{D}} \frac{v(z)}{v(\varphi_{a,n}(z))} = 1,$$

where the maximum is attained at $z = 0$. To calculate the essential norm, we observe that $|\varphi_{a,n}(z)| \le \varphi_{a,n}(|z|)$ to compute

$$\|C_{\varphi_{a,n}}\|_e = \limsup_{|\varphi_{a,n}(z)|\to 1} \frac{v(z)}{v(\varphi_{a,n}(z))} = \lim_{r\to 1^-} \frac{1-r}{1-\varphi_{a,n}(r)} = \frac{1+a}{1+n+an}.$$

All the values in $]0,1]$ are attained considering all $a \ge 0$ and all $n \in \mathbb{N}$.

Chapter 3

Smoothness of the norm

The starting point of this chapter is the study of a Šmulyan type criterion for the smoothness with the purpose of characterizing the points where the norm function of a real Banach space X is Fréchet and Gâteaux differentiable and in which cases the two concepts are equivalent. We show that in this criterion B_{X^*} can be replaced by a convenient smaller set, and we apply this extended criterion to some spaces of vector-valued continuous and harmonic functions.

3.1 A Šmulyan type criterion

If $(X, \|\cdot\|)$ is a real Banach space, we say that $\|\cdot\|$ is *Gâteaux differentiable at* x, if for each $h \in X$, the limit

$$u(x)(h) = \lim_{t\to 0} \frac{\|x+th\| - \|x\|}{t}$$

exists and is a linear continuous function in h (i.e. $u(x) \in X^*$). If, in addition, the limit above is uniform in $h \in S_X$, we say that $\|\cdot\|$ is *Fréchet differentiable at* x. Equivalently, $\|\cdot\|$ is Fréchet differentiable at x if there exists $u(x) \in X^*$ such that

$$\lim_{y\to 0} \frac{\|x+y\| - \|x\| - u(x)\,(y)}{\|y\|} = 0.$$

Definition 3.1.1. A subset $C \subseteq B_{X^*}$ is called a *James boundary* if $\|x\| = \max\{x^*(x) : x^* \in C\}$ for all $x \in X$, i.e., if for every $x \in X$ there exists $c^* \in C$ such that $c^*(x) = \|x\|$.

This definition is a particular case of James boundary of a weak* compact set, see, e.g.,[28], Definition 3.118. Notice that if C is weak* closed in the above definition then the maximum is always attained since the function $y \to y(x)$ defined on (X^*, w^*) is continuous and B_{X^*} is weak* compact. It always holds that $\|x^*\| = \sup\{x^{**}(x^*) : x^{**} \in J(B_X)\}$, where $J : X \to X^{**}$ is the canonical embedding. However, this supremum is not a maximum unless X is reflexive by James theorem, [28, Corollary 3.131]. Further, if C is circled, then $\sup\{y(x) : y \in C\} = \sup\{|y(x)| : y \in C\}$.

Definition 3.1.2. Let C be a bounded subset of X^*. A functional $x_0^* \in X^*$ is said to be *weak* exposed* in $C \subseteq X^*$ by $x_0 \in X$ if x_0^* is the unique functional in C such that $x_0^*(x_0) = \max\{|x^*(x_0)| : x^* \in C\}$.

Therefore, a functional $x_0^* \in S_{X^*}$ is weak* exposed in B_{X^*} by $x_0 \in S_X$ if and only if x_0^* is the unique functional in S_{X^*} such that $x_0^*(x_0) = 1$.

Definition 3.1.3. If $x_0^* \in X^*$ is weak* exposed in C by x_0 and $(x_n^*) \to x_0^*$ in norm whenever $x_n^*(x_0) \to x_0^*(x_0)$ then x_0^* is said to be *weak* strongly exposed* in C.

Šmulyan's theorem [26, Theorem I.1.4, Corollary I.1.5] asserts that the norm in X is Gâteaux (Fréchet) differentiable at x_0 if and only if x_0 weak* (strongly) exposes a functional x_0^* in B_{X^*}.

The following result is well known. We provide here a proof for the sake of completeness.

Lemma 3.1.4. *Assume that $\|\cdot\|$ is Gâteaux differentiable at $x_0 \in S_X$, and $x_0^*(x_0) = 1$, $x_0^* \in S_{X^*}$. Let $(x_n^*)_n$ be a sequence in B_{X^*} such that $(x_n^*(x_0))_n$ tends to 1. Then $(x_n^*)_n$ weak* converges to x_0^*.*

Proof. Assume not. Then we can find $x_1 \in S_X$ such that

$$|x_{n_k}^*(x_1) - x_0^*(x_1)| \geq \varepsilon,$$

for a certain subsequence $(x_{n_k}^*)_k$ of $(x_n^*)_n$ and some $\varepsilon > 0$. The sequence $(x_{n_k}^*)_k$ has a weak* cluster point, say y_0^*. Observe that $y_0^* \in B_{X^*}$, $|y_0^*(x_1) - x_0^*(x_1)| \geq \varepsilon$, and $y_0^*(x_0) = 1$, a contradiction with the fact that $\|\cdot\|$ is Gâteaux differentiable at x_0. □

Remark 3.1.5. Let $C \subset B_{X^*}$ such that C is a James boundary. Then $\overline{\text{co}}^{w^*} C = B_{X^*}$. Indeed, if not we can find, by the Hahn–Banach separation theorem, $x \in S_X$ such that $\sup\{f(x) : f \in \overline{\text{co}}^{w^*} C\} < f_0(x)$ for some $f_0 \in B_{X^*}$. This contradicts that C is a James boundary. Since $\overline{\text{co}}^{w^*} C = B_{X^*}$, Milman's theorem [28, 3.66] gives $Ext(B_{X^*}) \subset \overline{C}^{w^*}$. Assume $C \subset B_{X^*}$ such that C is a James boundary and w^*-closed. Then $Ext(B_{X^*}) \subset C$. If C is w^*-closed and a James boundary, then every $x \in X$ attains its norm at a point in C. This is obvious since the supremum in the definition of James boundary becomes a maximum. If C is w^*-closed and a James boundary and $\|\cdot\|$ is Gâteaux differentiable at $x_0 \in S_X$, then x_0 attains its norm on B_{X^*} exactly at one point f_0, and $f_0 \in C$.

Proposition 3.1.6. *If $C \subseteq B_{X^*}$ is a weak* closed James boundary and $x_0 \in S_X$ weak* exposes x_0^* in C, then $\|\cdot\|$ is Gâteaux differentiable at x_0.*

Proof. The element $x_0 \in S_X$ attains its supremum on B_{X^*} at an extremal point of B_{X^*}. This point is in C by Remark 3.1.5, hence it is unique by the assumption. It follows that x_0 exposes x_0^* in B_{X^*}, hence $\|\cdot\|$ is Gâteaux differentiable at x_0. □

Proposition 3.1.7. *Let $(X, \|\cdot\|)$ be a real Banach space and let $C \subseteq B_{X^*}$ be a weak* closed James boundary. Assume that $\|\cdot\|$ is Gâteaux differentiable at $x_0 \in S_X$. Let x_0^* be the differential of $\|\cdot\|$ at x_0 (i.e. $x_0^* \in S_{X^*}$ and $x_0^*(x_0) = 1$). Assume that given a sequence $(x_n^*)_n$ in $C \cap S_{X^*}$ which is weak* convergent to x_0^* then $\|x_n^* - x_0^*\|$ tends to 0. Then we can conclude that the norm $\|\cdot\|$ is Fréchet differentiable at x_0.*

Proof. Assume not. Then we can find by [26, Lemma I.1.3] a sequence $(h_n)_n$ of non-zero vectors in X which is norm convergent to 0 and $\varepsilon > 0$ such that

$$\|x_0 + h_n\| + \|x_0 - h_n\| \geq 2 + \varepsilon\|h_n\|.$$

Since C is a James boundary we can get $x_n^*, y_n^* \in C$ such that $x_n^*(x_0 + h_n) = \|x_0 + h_n\|$ and $y_n^*(x_0 - h_n) = \|x_0 - h_n\|$. Both x_n^* and y_n^* belong to S_{X^*} and also both $x_n^*(x_0)$ and $y_n^*(x_0)$ tend to 1, since $x_n^*(x_0) = \|x_0 + h_n\| - x_n^*(h_n)$ and the analogous for y_n^*. By Lemma 3.1.4 it follows that both $(x_n^*)_n$ and $(y_n^*)_n$ tend weakly to x_0^*, and then $\|x_n^* - y_n^*\|$ tends to 0 by the hypothesis. But

$$(x_n^* - y_n^*)(h_n) = \|x_0 + h_n\| + \|x_0 - h_n\| - x_n^*(x_0) - y_n^*(x_0)$$

$$\geq \|x_0 + h_n\| + \|x_0 - h_n\| - 2 \geq \varepsilon\|h_n\|.$$

Hence $\|x_n^* - y_n^*\| \geq \varepsilon$ for all n, which contradicts that $\|x_n^* - y_n^*\|$ tends to 0.

□

3.2 Spaces of continuous functions

Definition 3.2.1. Let K be a compact topological space and let X be a Banach space.

(a) For a continuous function $f : K \to X$ we say that $k_0 \in K$ *peaks* f if $\|f(k_0)\| = 1$ and $\|f(k)\| < 1$ for each $k \in K \setminus \{k_0\}$.

(b) We say that a subspace Y of $C(K, X)$ *separates* points of K if for every two different $k_1, k_2 \in K$ there is an $f \in Y$ such that $f(k_1) \neq f(k_2)$.

We include a vector-valued version of [26, Example I.1.6 (b)] and provide here a proof for the sake of completeness. This result appears in [67, Thereom 1 and Theorem 2], [53, Theorem 3.1 and Corollary 3.2], and [23, Corollary 4]. For $k \in K$ we denote by δ_k the evaluations on $C(K, X)$, i.e. $\delta_k : C(K, X) \ni f \to f(k) =: \delta_k(f) \in X$.

Theorem 3.2.2. *Let K be a compact Hausdorff space and let X be a Banach space. Let $Y \subseteq C(K, X)$ be a closed subspace separating points of K and let $f \in Y$ be given. The norm of Y is Gâteaux differentiable at f if and only if there is $k \in K$ peaking f, such that the norm of X is Gâteaux differentiable at $f(k)$. If $Y = C(K, X)$, then the norm is Fréchet differentiable at f if and only if there is an isolated point $k \in K$ peaking f, such that the norm of X is Fréchet differentiable at $f(k)$.*

Proof. First we show that the map

$$\begin{array}{rrcl} T: & K \times (B_{X^*}, w^*) & \to & (B_{Y^*}, w^*) \\ & (k, x^*) & \mapsto & x^* \circ \delta_k \restriction_Y \end{array}$$

is continuous. Let $(k_0, x_0^*) \in (K \times B_{X^*})$. A neighbourhood of $(x_0^* \circ \delta_{k_0})$ in the image of T is given by

$$\{x^* \circ \delta_k \restriction_Y : \ |(x^* \circ \delta_k - x_0^* \circ \delta_{k_0})(f_i)| =$$

$$|x^*(f_i(k)) - x_0^*(f_i(k_0))| < \varepsilon, \ x^* \in B_{X^*}, \ k \in K, \ 1 \leq i \leq n\},$$

for some finite subset $\{f_1, \ldots, f_n\} \subseteq Y$. Let U be a neighbourhood of k_0 such that $f_i(k) \in B(f_i(k_0), \varepsilon/2)$ for all $k \in U$ and for all $1 \leq i \leq n$. We define

$$V := \{x^* \in X^* : \ |x^*(f_i(k_0))| \leq \frac{\varepsilon}{2}, \ 1 \leq i \leq n\}.$$

The subset V is a zero neighbourhood of (X^*, w^*). For $(k, x^*) \in (U, (x_0^* + V) \cap B_{X^*})$ and for all $1 \leq i \leq n$ we get

$$|x^*(f_i(k)) - x_0^*(f_i(k_0))| \leq |x^*(f_i(k) - f_i(k_0))| + |(x^* - x_0^*)(f_i(k_0))| \leq \varepsilon.$$

This proves the continuity of T. From this we get that

$$C = \{x^* \circ \delta_k \restriction_Y : \ k \in K, x^* \in B_{X^*}\}$$

is a compact subset of (B_{Y^*}, w^*). $C \subseteq B_{Y^*}$ is easily seen to be a James boundary. Since Y separates points of K it follows that T restricted to $K \times \{B_{X^*} \setminus \{0\}\}$ is injective. Apply Proposition 3.1.6 and the Hahn–Banach theorem to show that the condition is sufficient. To check that the condition is necessary, notice that the Gâteaux differentiabily of the norm of Y at $f \in S_Y$ implies that there are $x_0^* \in B_{X^*}$ and $k_0 \in K$ such that $x_0^*(f(k_0)) = 1$ and $x^*(f(k)) \neq 1$ for $(k, x^*) \in K \times B_{X^*} \setminus \{(k_0, x_0^*)\}$. From this, it is easily deduced that $f(k_0) \in S_X$ exposes x_0^* in B_{X^*} (i.e. the norm of X is Gâteaux differentiable at $f(k_0)$) and $\|f(k)\| < 1$ for all $k \in K \setminus \{k_0\}$ (i.e. k_0 peaks f).

To show the assertion about Fréchet differentiability, we assume that the norm of $C(K, X)$ is Gâteaux differentiable at f. Suppose that $k_0 \in K$ peaks f and k_0 is an accumulation point of K. Then there exists $(k_n)_n \subset K \setminus \{k_0\}$ such that $(f(k_n))_n$ tends to $f(k_0)$. Let $x_0^* \in S_{X^*}$ such that $x_0^*(f(k_0)) = 1$. Then $x_0^* \circ \delta_{k_n}$ tends weak* to $x_0^* \circ \delta_{k_0}$ by Lemma 3.1.4. By Uryshon's Lemma there are functions $g_n \in C(K)$ with $0 \leq g_n \leq 1$, $g_n(k_n) = 1$ and $g_n(k_0) = 0$. For $f \in C(K)$ and $x \in X$, the tensor $f \otimes x \in C(K, X)$ is the function defined as $(f \otimes X)(k) = f(k) \cdot x$. Considering functions of the form $f_n = g_n \otimes f(k_0) \in C(K) \otimes X$ we get that $\|x_0^* \circ \delta_{k_n} - x_0^* \circ \delta_{k_0}\| \geq 1$. Hence the norm of $C(K, X)$ is not Fréchet differentiable at f by the Šmulyan criterion.

If k_0 is isolated and the norm of X is Gâteaux differentiable at $f(k_0)$ with differential x_0^* but not Fréchet differentiable, then there exists a sequence $(x_n^*)_n \in B_{X^*}$ and $\varepsilon > 0$ such that $(x_n^*(f(k_0)))_n$ tends to 1 but $\|x_n^* - x_0^*\| > \varepsilon$ for each $n \in \mathbb{N}$. This implies that for every $n \in \mathbb{N}$ there exists $x_n \in S_X$ such that $|x_n^*(x_n) - x_0^*(x_n)| > \varepsilon$. Then $x_n^* \circ \delta_{k_0}(f)$ tends to 1, but for $f_n = 1 \otimes x_n$ we have that $\|f_n\| = 1$ and $|x_n^* \circ \delta_{k_0}(f_n) - x_0^* \circ \delta_{k_0}(f_n)| > \varepsilon$, and hence $\|x_n^* \circ \delta_{k_0} - x_0^* \circ \delta_{k_0}\| > \varepsilon$. Hence the norm of $C(K, X)$ is not Fréchet differentiable at f by the Šmulyan criterion. This proves the necessity of the condition.

we assume now that $f \in C(K, X)$ is peaked at an isolated point k_0 of K with the norm of X being Fréchet differentiable at $f(k_0)$ (i.e there is $x_0^* \in B_{X^*}$ which is strongly exposed by $f(k_0)$). These conditions imply by the first part of the Theorem that the norm of $C(K, X)$ is Gâteaux differentiable at f. Assume there exist sequences $(x_n^*)_n \subset B_{X^*}$ and $(k_n)_n \subset K$ such that $x_n^* \circ \delta_{k_n}$ tends weak* to $x_0^* \circ \delta_{k_0}$, the continuity of the function g defined on K and with values in X as $g(k_0) = f(k_0)$ and $g(k) = 0$ for $k \neq k_0$ yields that the sequence $(k_n)_n$ is eventually constant, i.e. we can assume $k_n = k_0$ for all $n \in \mathbb{N}$. Since this implies that $(x_n^*(f(k_0)))_n$ tends to 1, the hypothesis of Fréchet differentiability yields that $(x^*)_n$ tends to x_0^* in norm, i.e. uniformly on B_X. From this it follows that $x_n^* \circ \delta_{k_0}$ tends to $x_0^* \circ \delta_{k_0}$ uniformly on $B_{C(K,X)}$, and Proposition 3.1.7 and the fact already proved that C is weak* closed gives the conclusion.

□

Remark 3.2.3. The space $C(K) \otimes X$ formed by the span of the elements of the form $f \otimes x$, $f \in C(K)$, $x \in X$, is dense in $C(K, X)$ [63, 3.2]. Hence the proof of Fréchet differentiability in Theorem 3.2.2 does not work for a proper subspace $Y \subseteq C(K, X)$.

3.3 Applications in $K(X, Y)$

In this section we give an application of Proposition 3.1.6 to get an easy proof of a Theorem stated without proof by Heinrich in [40]. Heinrich's theorem contains results due to Holub and Hennefeld in ([41, 44]).

Theorem 3.3.1. *If X and Y are Banach spaces and X is reflexive, then the norm of $K(X, Y)$ is Gâteaux differentiable at $T \in S_{K(X,Y)}$ if and only if there is $x_0 \in S_X$ such that $\|T(x_0)\| = 1$, the norm of Y is Gâteaux differentiable at $T(x_0)$ and $\|T(x)\| < \|x\|$ for all $x \in X \setminus span\{x_0\}$.*

Proof. Since every bounded linear operator is weakly-weakly continuous and in any relatively compact set of Y the norm topology agrees with the weak topology, we can regard $K(X, Y)$ as a subspace of $C((B_X, w), Y)$. Now, we show that the map

$$\begin{array}{rccc} \mathcal{T}: & (B_X, w) \times (B_{Y^*}, w^*) & \to & (B_{K(X,Y)^*}, w^*) \\ & (x, y^*) & \mapsto & y^* \circ \delta_x \restriction_{K(X,Y)} \end{array}$$

is continuous. Let $(x_0, y_0^*) \in (B_X \times B_{Y^*})$. A neighbourhood of $(y_0^* \circ \delta_{x_0})$ in the image of $\mathcal{T}$ is given by

$$\{y^* \circ \delta_x \restriction_{K(X,Y)}: \ |(y^* \circ \delta_x - y_0^* \circ \delta_{x_0})(f_i)| =$$

$$|y^*(f_i(x)) - y_0^*(f_i(x_0))| < \varepsilon, \ y^* \in B_{Y^*}, \ x \in B_X, \ 1 \le i \le n\},$$

for some finite subset $\{f_1, \dots, f_n\} \subseteq K(X, Y)$ and so $\{f_1, \dots, f_n\} \subseteq C((B_X, w), Y)$. Let U be a neighbourhood of x_0 such that $f_i(x) \in B(f_i(x_0), \varepsilon/2)$ for all $x \in U$ and for all $1 \le i \le n$. We define

$$V := \{y^* \in Y^* : \ |y^*(f_i(x_0))| \le \frac{\varepsilon}{2}, \ 1 \le i \le n\}.$$

The subset V is a zero neighbourhood of (Y^*, w^*). For $(x, y^*) \in (U, (y_0^* + V) \cap B_{Y^*})$ and for all $1 \le i \le n$ we get

$$|y^*(f_i(x)) - y_0^*(f_i(x_0))| \le |y^*(f_i(x) - f_i(x_0))| + |(y^* - y_0^*)(f_i(x_0))| \le \varepsilon.$$

This proves the continuity of $\mathcal{T}$. From this we get that

$$C = \{y^* \circ \delta_x \restriction_{K(X,Y)}\colon\ x \in B_X, y^* \in B_{Y^*}\}$$

is a compact subset of $(B_{K(X,Y)^*}, w^*)$. $C \subseteq B_{K(X,Y)^*}$ is easily seen to be a James boundary.

The norm in $K(X,Y)$ is Gâteaux differentiable at $T \in S_{K(X,Y)}$ if and only if there is a unique $T^* \in K(X,Y)^*$ such that $T^*(T) = 1$, $\|T^*\| = 1$. So, T^* is extreme and therefore $T^* \in C$.

Now, let $x_1, x_2 \in S_X$ and $y_1^*, y_2^* \in S_{Y^*}$. We show that $y_1^* \circ \delta_{x_1} \restriction_{K(X,Y)}= y_2^* \circ \delta_{x_2} \restriction_{K(X,Y)}$ if and only if either $y_1^* = y_2^*$, $x_1 = x_2$ or $y_1^* = -y_2^*$, $x_1 = -x_2$. Indeed, we assume that $y_1^*\circ\delta_{x_1} \restriction_{K(X,Y)}= y_2^*\circ\delta_{x_2} \restriction_{K(X,Y)}$ and $x_1 \neq x_2$. The Theorem of Hahn–Banach provides us $x^* \in X^*$ such that $x^*(x_1) = 1$ and $x^*(x_2) = 0$. We choose $y_1 \in Y$ such that $y_1^*(y_1) \neq 0$ and define $g : X \to Y$ by $g(x) = x^*(x) \cdot y_1$. We have $y_1^* \circ \delta_{x_1}(g) \neq 0$ and $y_2^* \circ \delta_{x_2}(g) = 0$. If $x_1 = x_2$, it is clear that $y_1^* \circ \delta_{x_1} \restriction_{K(X,Y)}= y_2^* \circ \delta_{x_2} \restriction_{K(X,Y)}$ implies $y_1^* = y_2^*$. Conversely, by the linearity $y_1^* \circ \delta_{x_1} \restriction_{K(X,Y)}= (-y_1^*) \circ \delta_{-x_1} \restriction_{K(X,Y)}$. From this and Proposition 3.1.6 we get the desired conclusion.

□

3.4 Spaces of harmonic functions

Let $U \subseteq \mathbb{R}^d$ be an open and connected set. For a real Banach space X we denote by $h(U,X)$ the space of harmonic functions on U with values in X, $h(U)$ denotes the space of real valued harmonic functions. With this notation, the space of complex valued harmonic functions regarded as a real topological vector space is written $h(U,\mathbb{R}^2)$. A function $f : U \to X$ is harmonic in a strong sense, i.e. it is a C^∞-function which is in the kernel of the vector-valued Laplacian, if and only if $x^* \circ f \in h(U)$ for all $x^* \in X^*$ (e.g. [13, Corollary 10]). Here and at the following sections a weight on U is a bounded continuous function $v : U \to]0,\infty[$. We define the weighted spaces of vector-valued functions as

$$h_v(U,X)\colon = \{f \in h(U,X) :\ \|f\|_v = \sup_{z\in U} v(z)\|f(z)\| < \infty\}$$

$$h_{v_0}(U,X)\colon = \{f \in h_v(U,X) :\ v\|f\| \text{ vanishes at infinity on } U\}.$$

$$h_v^c(U,X)\colon = \{f \in h_v(U,X) : (vf)(U) \text{ is relatively compact}\}.$$

A function $g : U \to \mathbb{R}$ is said to *vanish at infinity* on U if for each $\varepsilon > 0$ there exists a compact set $K \subset U$ such that $|g(z)| \leq \varepsilon$ for each $z \in U \setminus K$. Hence,

$h_{v_0}(U,X)$ can be identified by the isometry $f \to vf$ with a subspace of $C(\widehat{U},X)$, $\widehat{U}$ being the Alexandroff compactification of U. This gives the inclusions

$$h_{v_0}(U,X) \subseteq h_v^c(U,X) \subseteq h_v(U,X).$$

Obviously $h_v^c(U,X) = h_v(U,X)$ when X is finite dimensional.

As $\mathbb{R}^d$ is locally compact and Hausdorff, so it is Tychonoff, i.e. T_1 and completely regular, we can consider the Stone-Čech compactification $(\beta U, e)$. For each $f \in h_v^c(U,X)$, we have a continuous map with values in a compact Hausdorff space, i.e. $(vf) : U \to \overline{(vf)(U)}$. If we take the continuous extension that the Stone-Čech Theorem provides, we can consider $h_v^c(U,X)$ as a subspace of $C(\beta U,X)$. For $X = \mathbb{R}$ we write $h_{v_0}(U)$ and $h_v(U)$. The unit balls of these spaces are denoted by b_v and b_{v_0}. The following definition is taken from [19].

Definition 3.4.1. Let $U \subseteq \mathbb{R}^d$ be an open subset and let v be a weight on U. We define *the set of peak points of* $h_{v_0}(U)$ as

$$P_v = \{z \in U : \exists f \in b_{v_0} \text{ such that } v(z)f(z) = 1 \text{ and } v(y)|f(y)| < 1 \ \forall y \in U \setminus \{z\}\}.$$

The following basic lemma shows that being a peak point in spaces of vector-valued harmonic functions does not depend on the range space.

Lemma 3.4.2. *Let $U \subseteq \mathbb{R}^d$ be an open subset, let v be a weight on U and let X be a Banach space. Let $z \in U$. The following are equivalent*

(i) $z \in P_v$.

(ii) There exists $f \in h_{v_0}(U,X)$ such that $v(z)\|f(z)\| = 1$ and $v(y)\|f(y)\| < 1$ for all $y \in U \setminus \{z\}$.

Proof. If z is a peak point, let $g \in h_{v_0}(U)$ be such that $v(z)g(z) = 1$ and $v(y)|g(y)| < 1$ for $y \neq z$. Let $x_0 \in S_X$. The function $f(z) = g(z)x_0$, $z \in U$ satisfies (ii). Conversely, if $f \in h_{v_0}(U,X)$ satisfies the conditions on (ii) and $x_0^* \in S_{X^*}$ is such that $x_0^*(f(z)) = \|f(z)\|$, then $x_0^* \circ f \in h_{v_0}(U)$ and z is a peak point by $x_0^* \circ f$. □

If $f \in h_{v_0}(U,X)$ satisfies condition [(ii)] of the previous lemma we say that z *peaks* f. This lemma permits to use [19, Example 13] and [46, Proposition 10] to construct examples of functions in weighted Banach spaces of harmonic functions satisfying the conditions of the theorem below.

Theorem 3.4.3. *Let X be a real Banach space. Let U be an open subset of $\mathbb{R}^d$, let v be a weight on U such that $h_{v_0}(U)$ contains the polynomials of degree smaller or equal than 1 and let f be an element of $h_{v_0}(U,X)$, with $\|f\|_v = 1$. Then*

(a) *The norm* $\|\cdot\|_v$*, when is restricted to the space* $h_{v_0}(U)$*, even to* $h_v^c(U,X)$*, is Gâteaux differentiable at* $f \in h_{v_0}(U,X)$ *if and only if there exists a* $z_0 \in U$ *that peaks the function* f *and such that the norm* $\|\cdot\|$ *on* X *is Gâteaux differentiable at* $v(z_0)f(z_0)$*.*

(b) *The norm* $\|\cdot\|_v$*, when is restricted to the space* $h_{v_0}(U)$*, even to* $h_v^c(U,X)$*, is Fréchet differentiable at* $f \in h_{v_0}(U,X)$ *if and only if there exists a* $z_0 \in U$ *that peaks the function* f *and such that the norm* $\|\cdot\|$ *on* X *is Fréchet differentiable at* $v(z_0)f(z_0)$*.*

Proof. Let $\widehat{U}$ be the Alexandroff compactification of U. By the isometric embedding $h_{v_0}(U,X) \to C(\widehat{U},X), f \mapsto vf$ we apply the Theorem 3.2.2 to get that the norm of $h_{v_0}(U,X)$ is Gâteaux differentiable at $f \in h_{v_0}(U,X)$ if and only if there exists $z_0 \in P_v$ peaking f such that the norm of X is Gâteaux differentiable at $v(z_0)f(z_0)$. The same argument is applied to f considered as a function of $h_v^c(U,X)$, isometrically embedded also by $f \mapsto vf$ in $C(\beta U,X)$. For each $z \in \beta U \setminus U$, we define $v(z)\delta_z : h_v^c(U,X) \to X$, $v(z)\delta_z(f) = \widetilde{vf}(z)$, i.e. the continuous extension of vf at z. This is an abuse of notation since $f(z)$ is not necessarily defined on $\beta U \setminus U$. With this definition, $v(z)\delta_z$ is identically null on $h_{v_0}(U,X)$ for each $z \in \beta U \setminus U$. Hence, the Gâteaux differential of $\|\cdot\|_v$ at f must be a functional in the subset of $h_v^c(U,X)^*$

$$C := \{x^* \circ v(z)\delta_z :\ x^* \in B_{X^*},\ z \in U\}.$$

Let us assume that z_0 peaks f, $f \in h_{v_0}(U,X)$ and the norm of X is Fréchet differentiable at $v(z_0)f(z_0)$. This means that there is $x_0^* \in S_{X^*}$ such that $v(z_0)f(z_0)$ strongly exposes x_0^* in B_{X^*}. By the first part, we have that the norm of $h_{v_0}(U,X)$ is Gâteaux differentiable at f. Necessarily the exposed functional is $x_0^* \circ v(z_0)\delta_{z_0}$ since $v(z_0)x_0^* \circ \delta_{z_0}(f) = x_0^*(v(z_0)f(z_0)) = 1$. We consider the following set

$$C_1 := \{x^* \circ v(z)\delta_z :\ x^* \in B_{X^*},\ z \in \beta U\}.$$

By Proposition 3.1.7 we have to show that each sequence in $C_1 \cap S_{h_{v_0}(U,X)^*}$ which is weak* convergent to $x_0^* \circ v(z_0)\delta_{z_0}$ is also norm convergent, since C_1 is a weak* compact James boundary of $h_{v_0}(U,X)^*$ by the isometry $f \mapsto vf$ into $C(\beta U,X)$ and the proof of Theorem 3.2.2. Let $(z_n)_n$ and $(x_n^*)_n$ be two sequences in βU and B_{X^*} respectively such that $(x_n^* \circ v(z_n)\delta_{z_n})$ is weak* convergent to $x_0^* \circ v(z_0)\delta_{z_0}$. The sequence $(z_n)_n$ is relatively compact in U, otherwise $(x_n^* \circ v(z_n)\delta_{z_n})_n$ would contain a subsequence $\sigma(h_v^c(U,X)^*, h_{v_0}(U,X))$ convergent to 0. The hypothesis on $h_{v_0}(U)$ yields that the functions $P(z)x$, P being a polynomial of degree 1 and $x \in X$ are in $h_{v_0}(U,X)$. If we suppose that $(z_n)_n$ does not tend to z_0 then we would get $w_0 \in U$ a cluster point for $(z_n)_n$. We consider the function $f(z) =$

$P(z)x_0$ where P is a polynomial of degree one with $P(z_0) = 1$ and $P(w_0) = 0$, and $x_0 \in X$ such that $x_0^*(x_0) > 0$. By the hypothesis $f \in h_{v_0}(U, X)$ and 0 is a cluster point of $(\langle x_n^* \circ v(z_n)\delta_{z_n}), f\rangle)_n$. This contradicts $x_0^*(v(z_0)f(z_0)) > 0$. Analyzing constant functions we also get that $(x_n^*)_n$ weak* converges to x_0^*. By the hypothesis $v(z_0)f(z_0)$ strongly exposes x_0^* in B_{X^*}. Hence we have that $(x_n^*)_n$ is norm convergent to x_0^*. The map

$$\begin{array}{rccl} F: & U & \to & L(h_v^c(U,X),X) \\ & z & \mapsto & \delta_z \end{array}$$

is harmonic for the Weak Operator Topology, i.e. $z \mapsto x^* \circ F(f)(z) = x^* \circ f(z) \in h(U)$ for each $x^* \in X^*$ and $f \in h_v^c(U, X)$. Hence F is harmonic (and then continuous) for the norm operator topology by [13, Corollary 10, Remark 11]. Thus $(v(z_n)\delta_{z_n})_n$ converges uniformly on $B_{h_{v_0}(U,X)}$ to $v(z_0)\delta_{z_0}$. We use this and the norm convergence from $(x_n^*)_n$ to x_0^* to obtain the desired conclusion. Let $\varepsilon > 0$. Getting n_0 such that $\|v(z_n)g(z_n) - v(z_0)g(z_0)\| < \varepsilon/2$ and $\|x_n^* - x_0^*\| < \varepsilon/2$ for each $n \geq n_0$ and each $g \in B_{h_{v_0}(U,X)}$ we get

$$|x_n^*(v(z_n)g(z_n)) - x_0^*(v(z_0)g(z_0))| \leq$$
$$|x_n^*(v(z_n)g(z_n) - v(z_0)g(z_0))| + |(x_n^* - x_0^*)(v(z_0)g(z_0))| \leq \varepsilon$$

for each $f \in B_{h_{v_0}(U,X)}$, i.e. $x_n^* \circ v(z_n)\delta_{z_n}$ converges uniformly on $B_{h_{v_0}(U,X)}$ to $x_0^* \circ v(z_0)\delta_{z_0}$.

Conversely, if there is $z_0 \in U$ such that $x_0^* \circ v(z_0)\delta_{z_0}$ is strongly exposed by $f \in B_{h_{v_0}(U,X)}$ it follows that $z_0 \in P_v(U)$ by Lemma 3.4.2. Suppose that x_0^* is weak* exposed but not strongly exposed by $v(z_0)f(z_0)$. Then there exists a sequence $(x_n^*)_n$ which is weak* convergent to x_0^* but there exists $\varepsilon > 0$ such that $\|x_n^* - x_0^*\| > \varepsilon$ for all $n \in \mathbb{N}$. Let $g \in h_{v_0}(U)$ be peaked by z_0, i.e $v(z_0)g(z_0) = 1$ and $v(z)|g(z)| < 1$ for all $z \in U \setminus \{z_0\}$. For all $x \in B_X$, the function $f_x(z) = g(z)x$ is in $B_{h_v^c(U,X)}$. The sequence $(x_n^* \circ v(z_0)\delta_{z_0})_n$ converges weak* to $x_0^* \circ v(z_0)\delta_{z_0}$, but the hypothesis on $(x_n^*)_n$ implies that, for all $n \in \mathbb{N}$ there exists $x_n \in B_X$ such that

$$|v(z_0)x_0^* \circ \delta_{z_0}(g_{x_n}) - v(z_0)x_n^* \circ \delta_{z_0}(g_{x_n})| = |x_0^*(x_n) - x_n^*(x_n)| > \varepsilon,$$

a contradiction. □

We close this section with the following corollary that extends a result of Boyd and Rueda, see [19, Theorem 1].

Corollary 3.4.4. *Let $U \subseteq \mathbb{R}^d$ be an open set and let $\mathbb{R}^n$ be endowed with the Euclidean norm. Let v be a weight on U such that $h_{v_0}(U)$ contains the polynomials of degree smaller or equal than 1. The norm $\|\cdot\|_v$ is Gâteaux differentiable at $f \in h_{v_0}(U, \mathbb{R}^n)$ if and only if $\|\cdot\|_v$ is Fréchet differentiable at f in $h_{v_0}(U, \mathbb{R}^n)$ if and only if there exists $z \in U$ such that $v(z)\|f(z)\| = 1$ and $v(y)\|f(y)\| < 1$ for any $y \neq z$.*

3.5 Applications in $A(\overline{\mathbb{D}})$ and $H^\infty(\mathbb{D})$

We consider the space $A(\overline{\mathbb{D}})$ of continuous functions on $\overline{\mathbb{D}}$ which are holomorphic on $\mathbb{D}$ and regarde it as a real Banach space of functions $f : \overline{\mathbb{D}} \to \mathbb{R}^2$.

Proposition 3.5.1. *The following properties hold:*

(i) $ExtS_{A(\overline{\mathbb{D}})^*} \subset \left\{\lambda\delta_z : \lambda \in \partial\overline{\mathbb{D}}, z \in \overline{\mathbb{D}}\right\}$.

(ii) The set of peak point of $A(\overline{\mathbb{D}})$ *is contained in* $\partial\overline{\mathbb{D}}$.

(iii) The norm of $A(\overline{\mathbb{D}})$ *is Gâteaux differentiable at* f *if and only if there is* $z \in \partial\overline{\mathbb{D}}$ *such that* $|f(z)| = 1$ *and* $|f(y)| < 1$ *for each* $z \neq y$.

(iv) There are functions in $A(\overline{\mathbb{D}})$ *for which the norm is Gâteaux differentiable.*

Proof. The first three properties are a consequence of Theorem 3.2.2 and the Maximum Modulus Principle. To prove (iv), notice that if $a \in \partial\overline{\mathbb{D}}$ for $f(z) = \frac{z+a}{2}$, $z \in \mathbb{D}$, we have

$$\sup_{z\in\mathbb{D}} |f(z)| = f(a) = 1, \ |f(z)| < 1, \text{ for all } z \neq a, \, z \in \overline{\mathbb{D}}.$$

We reach the same conclusion with the function $f(z) = e^{\langle z,\overline{a}\rangle - 1}$. □

Proposition 3.5.2. *There is no function* $f \in S_{A(\overline{\mathbb{D}})}$ *such that the norm in* $A(\overline{\mathbb{D}})$ *is Fréchet differentiable at* f.

Proof. Take any $f \in S_{A(\overline{\mathbb{D}})}$. If $\|\cdot\|$ is not Gâteaux differentiable at f, we are done. Further assume that the opposite holds. By Proposition 3.5.1 ii) we can find $z \in \partial\mathbb{D}$ such that $|f(z)| = 1$ and $|f(y)| < 1$ for $y \in \overline{\mathbb{D}} \setminus z$. Select $\alpha \in \partial\mathbb{D}$ so that $\alpha f(z) = 1$. Then for every $y \in \overline{\mathbb{D}} \setminus \{z\}$ and every $n \in \mathbb{N}$ we have

$$\|\delta_z - \delta_y\| \geq \langle \delta_z - \delta_y, (\alpha f)^n \rangle = 1 - \alpha^n f^n(y) \geq 1 - |f(y)|^n \to 1;$$

thus $\|\delta_z - \delta_y\| \geq 1$. But, clearly, $\alpha\delta_y \to \alpha\delta_z$ weak* as $\overline{\mathbb{D}} \ni y \to z$. Hence the Šmulyan Criterion [26, Theorem 1.4] yields that $\|\cdot\|$ is not Fréchet differentiable at f. □

The space H^∞ can be regarded as a real Banach space generated by functions defined on $\mathbb{D}$ and with values in $\mathbb{R}^2$ which are continuous in the Stone-Čech compactification $\beta\mathbb{D}$ of the disc (or in the spectrum $\mathcal{M} \subseteq (H^\infty)^*$) and which are holomorphic in $\mathbb{D}$. A sequence $(z_n)_n \subset \mathbb{D}$ is said to be an *interpolating sequence* for H^∞, if for each bounded sequence of complex numbers $(y_n)_n$ there exists a function $f \in H^\infty$ such that $f(z_n) = y_n$ for every $n \in \mathbb{N}$.

Lemma 3.5.3. *If* $(z_n)_n$ *is an interpolating sequence for* $H = H^\infty(\mathbb{D})$ *and we consider* $F = \{\delta_{z_n} : n \in \mathbb{N}\} \subseteq H^*$ *endowed with weak* topology* $w^* = \sigma(H^*, H)$ *then* $\overline{F}^{w^*} \setminus F$ *is homeomorphic to* $\beta\mathbb{N} \setminus \mathbb{N}$*. Moreover,* $card(\overline{F}^{w^*}) = 2^{\mathfrak{c}}$*,* $\mathfrak{c}$ *being the cardinality of the continuum.*

Proof. If we fix $j \in \mathbb{N}$ then there exists a function $f \in H$ such that $f(z_j) = 1$ and $f(z_n) = 0$ for all $n \neq j$, since $(z_n)_n$ is an interpolating sequence for H. So,

$$\left\{\delta_{z_n} \in F : \left|\delta_{z_n}(f) - \delta_{z_j}(f)\right| < \frac{1}{2}\right\} = \left\{\delta_{z_j}\right\}$$

Therefore, F is a discrete set in the inherited weak* topology. Taking in $\mathbb{N}$ the discrete topology we get and homeomorphism between F and $\mathbb{N}$. Hence, their Stone-Čech compactification $(\beta F, e)$ is homeomorphic to $\beta\mathbb{N}$ which has the cardinal $2^{\mathfrak{c}}$ [73, Examples 19.13.d].

Since $(z_n)_n$ is interpolating and F discrete, for each $g : F \to [0, 1]$ continuous there is $f \in H^\infty(\mathbb{D})$ interpolating $(g(z_n))_n$. Thus, $f : \overline{F}^{w^*} \to [0, 1]$ is a continuous extension of g. This implies that $\overline{F}^{w^*}$ is homeomorphic to $(\beta F, w^*)$. □

Corollary 3.5.4. *There is no function* $f \in S_{H^\infty(\mathbb{D})}$ *such that the norm in* $H^\infty(\mathbb{D})$ *is Gâteaux differentiable at* f*.*

Proof. Let $f \in S_{H^\infty(\mathbb{D})}$. The Maximum Modulus Principle yields a sequence $(z_n)_n$ with $(z_n)_n \to a \in \partial\overline{\mathbb{D}}$ such that $f(z_n) \to \alpha \in \partial\overline{\mathbb{D}}$ and by a corollary of a theorem of Hayman and Newman [42, p.204] the sequence can be chosen to be interpolating. Without loss of generality we may suppose that $\alpha = 1$ since if the norm in $H^\infty(\mathbb{D})$ is Gâteaux differentiable at $f \in S_{H^\infty(\mathbb{D})}$ then, the norm in $H^\infty(\mathbb{D})$ is Gâteaux differentiable at $\frac{f}{\alpha} \in S_{H^\infty(\mathbb{D})}$. Seeking a contradiction, if the norm in $H^\infty(\mathbb{D})$ is Gâteaux differentiable at $f \in S_{H^\infty(\mathbb{D})}$ then by [26, Corollary 1.5] there is $u \in S_{H^\infty(\mathbb{D})^*}$ weak* exposed at f, i.e., $\|u\| = 1$, $u(f) = 1$ and $v(f) \neq 1$ for $v \in S_{H^\infty(\mathbb{D})^*} \setminus \{u\}$, but this is not possible because $\{u \in H^\infty(\mathbb{D})^* : \|u\| = 1, u(f) = 1\}$ has the cardinality $2^{\mathfrak{c}}$ by Lemma 3.5.3.

□

Corollary 3.4.4 and these examples show that $h_{v_0}(U, \mathbb{R}^2)$ and $h_v(U, \mathbb{R}^2)$, and the corresponding subspaces of holomorphic functions $H_{v_0}(U)$ and $H_v(U)$ behave differently from $A(\overline{\mathbb{D}})$ and $H^\infty(\mathbb{D})$ with respect to the differentiability of the norm.

Chapter 4

Isomorphism classes

This chapter is devoted to solve the problem of finding the conditions to ensure the existence of an isometric isomorphism between $h_{v_0}(U)$ and a closed subspace of c_0. The results obtained are used to deduce properties about the rotundity of the spaces $h_{v_0}(U)$ and $H_{v_0}(U)$, which are considered here as weighted spaces of harmonic and holomorphic functions with values in $\mathbb{C}$ defined on an open set $U \subseteq \mathbb{R}^d$ or $\mathbb{C}^d$, being $v : U \to]0, \infty[$ a bounded continuous function.

4.1 Isomorphisms on $h_{v_0}(U)$ and $h_v(U)$.

For an open set $U \subseteq \mathbb{R}^d$, the space $h_v(U)$ is isometric to a subspace of l_∞. To show that it is enough to take $(z_n)_n$ a countable and dense set of U and define $T : h_v(U) \to l_\infty$ by $T(f) = (v(z_n)f(z_n))_n$. But this argument does not work for $h_{v_0}(U)$.

In this case we show that we can find an almost isometry, and that in the general case the result is sharp, because the isometry is not possible by geometric reasons. The argument of the next theorem is an adaptation of [17, Theorem 1] where it is proved the statement for spaces of holomorphic functions.

Theorem 4.1.1. *Let U be a connected open subset of $\mathbb{R}^d$ and let v be a weight on U. Then, the space $h_{v_0}(U)$ is isomorphic to a closed subspace of c_0. More precisely, for each ε there exists a continuous linear injective map with closed range*

$T : h_{v_0}(U) \to c_0$ such that

$$(1 - \varepsilon) \|f\|_v \leq \|T(f)\|_{c_0} \leq \|f\|_v$$

for each $f \in h_{v_0}(U)$.

Proof. Let $(K_j)_j$ be a fundamental sequence of compact subsets of U and let $\varepsilon > 0$. We claim that there exists a sequence of pairwise disjoint finite subsets $(F_j)_j$ such that $F_j \subset A_j := K_j \setminus K_{j-1}^{\circ}$, and satisfying that for all $f \in h_{v_0}(U)$ with $\|f\|_v = 1$ we have

$$\sup_{x \in A_j} v(x)|f(x)| \leq \varepsilon + \sup_{y \in F_j} v(y)|f(y)|.$$

We take $K_0^{\circ} = \emptyset$. These sets $(F_j)_j$ do not depend on the function f and then, if we denote $F := \cup_j F_j$, we have

$$\sup_{x \in F} v(x)|f(x)| \leq \sup_j \sup_{x \in A_j} v(x)|f(x)| = \|f\|_v \leq \varepsilon + \sup_{x \in F} v(x)|f(x)|, \tag{4.1}$$

and consequently, since F is a countable set, we can write F as a sequence $(w_j)_j$. Moreover, $(w_j)_j \to \partial U$ and since vf vanishes at infinity on U it follows $(v(w_j)f(w_j))_j \to 0$. We define

$$T : h_{v_0}(U) \to c_0$$

$$f \mapsto (v(w_j)f(w_j))_j.$$

Now $\sup_{x \in F} v(x)|f(x)| = \|T(f)\|_{c_0}$, and by (4.1)

$$\|T(f)\|_{c_0} \leq \|f\|_v \leq \varepsilon + \|T(f)\|_{c_0}.$$

If $0 \neq \|f\|_v$ then,

$$\left\| T(\frac{f}{\|f\|_v}) \right\|_{c_0} \leq \left\| \frac{f}{\|f\|_v} \right\|_v \leq \varepsilon + \left\| T(\frac{f}{\|f\|_v}) \right\|_{c_0}.$$

We get from this

$$\|f\|_v \leq \varepsilon \|f\|_v + \|T(f)\|_{c_0}.$$

Hence,

$$(1 - \varepsilon) \|f\|_v \leq \|T(f)\|_{c_0} \leq \|f\|_v$$

for each $f \in h_{v_0}(U)$. Thus T is continuous, injective and the range of T is a closed subspace of c_0.

We proceed to show the claim. For each $j \in \mathbb{N}$ we define

$$M_j \ : \ = \sup_{x \in K_j} v(x),$$

$$a_j \ : \ = \min \left(\frac{1}{2} d \left(K_j, \mathbb{R}^d \setminus K_{j+1}^{\circ} \right), 1 \right)$$

and we select $x^{(j)} \in K_j$ such that $v\left(x^{(j)}\right) = \min_{x \in K_j} v(x)$. For each $f \in h_{v_0}(U)$ with $\|f\|_v = 1$ and each $j \in \mathbb{N}$ we have

$$\sup_{x \in K_j} |f(x)| \leq \frac{1}{v\left(x^{(j)}\right)}.$$

If $x \in K_j$ and $\zeta \in \overline{D}(x, a_j)$ (here $\overline{D}(x, a_j)$ denotes the closed ball with center x and radius a_j) then, from the definition of a_j we deduce that $\zeta \in K_{j+1}$, and hence, together with the previous inequality, we get

$$|f(\zeta)| \leq \frac{1}{v\left(x^{(j+1)}\right)}.$$

By Cauchy's inequalities for harmonic functions given in [3, 2.4], there exists a positive number C (independent on x) such that, for each $\zeta \in \overline{D}(x, a_j)$

$$|D_i f(\zeta)| \leq \frac{C}{a_j v\left(x^{(j+1)}\right)}, i = 1, 2, \ldots, d. \tag{4.2}$$

where D_i denotes la i-th derivative of f. Let δ_j such that
$\delta_j < a_j$ and

$$\left(\frac{1}{v\left(x^{(j)}\right)} + \frac{M_{j+1} d C}{a_j v\left(x^{(j+1)}\right)}\right) \delta_j < \varepsilon.$$

The compactness of A_j permits to get $F_j \subset A_j$ finite such that

$$A_j \subset \bigcup_{x \in F_j} \{x' \in U : \|x' - x\| < \delta_j, |v(x') - v(x)| < \delta_j\},$$

Consequently, for each $x \in A_j$, there exists $w \in F_j$ with $\|w - x\| < \delta_j$ and $|v(w) - v(x)| < \delta_j$. On the other hand, $\overline{D}(x, \delta_j) \subset \overline{D}(x, a_j) \subset K_{j+1}$ for each $x = (x_1, x_2, \ldots, x_d) \in A_j$. We write $w = (w_1, w_2, \ldots, w_d)$ to get

$$|f(x)| = |f(x_1, \ldots, x_d)| \leq |f(x_1, \ldots, x_d) - f(w_1, x_2, \ldots, x_d)|$$

$$+ |f(w_1, x_2, \ldots, x_d) - f(w_1, w_2, x_3 \ldots, x_d)| + \cdots + |f(w_1, w_2, \ldots, w_d)|.$$

We apply the mean value theorem to each difference and (4.2) to get

$$|f(x)| \leq \frac{C}{a_j v\left(x^{(j+1)}\right)} d \|x - w\| + |f(w)| \leq \frac{d \delta_j C}{a_j v\left(x^{(j+1)}\right)} + |f(w)|.$$

Now, since $w \in K_{j+1}$:

$$v(x) |f(x)| \leq |v(x) - v(w)| |f(x)| + v(w) |f(x)| \leq$$

$$\delta_j \frac{1}{v\left(x^{(j)}\right)} + v(w)\left(\frac{d\delta_j C}{a_j v\left(x^{(j+1)}\right)} + |f(w)|\right) \leq$$
$$\frac{\delta_j}{v\left(x^{(j)}\right)} + M_{j+1}\frac{d\delta_j C}{a_j v\left(x^{(j+1)}\right)} + v(w)\,|f(w)|$$
$$\leq \varepsilon + v(w)\,|f(w)| \leq \varepsilon + \max_{w\in F_j} v(w)\,|f(w)|\,.$$

Therefore,

$$\sup_{x\in A_j} v(x)\,|f(x)| \leq \varepsilon + \max_{w\in F_j} v(w)\,|f(w)|\,.$$

□

The application T in general is not a surjection. In [31] the sliding-hump technique is used to construct rapidly decreasing continuous and radial weights v on $\mathbb{C}$ such that $H_{v_0}(\mathbb{C})$ contains the polynomials and is not isomorphic to c_0.

Corollary 4.1.2. *There are not infinite dimensional subspaces of $h_{v_0}(U)$ which are reflexive.*

Proof. The conclusion follows from the isomorphism of $h_{v_0}(U)$ to a subspace of c_0 and by [54, Proposition 2a.2] each infinite dimensional subspace of c_0 contains a complemented subspace which is isomorphic to c_0, and then it is not reflexive.

□

In order to get examples where the isomorphism above cannot be an isometry, we consider a v-boundary condition which is inspired by Boyd and Rueda in [18, Theorem 16, Corollary 17] for weighted spaces defined on bounded open sets of $\mathbb{C}^d$. First, we observe that for any weight v for which $h_{v_0}(U)$ contains the polynomials then

$$\{\delta_z : z \in U\} \subset h_{v_0}(U)^*$$

is linearly independent.

If U is bounded and v tends to 0 at the boundary then h_{v_0} contains the polynomials and if $v(x) = g\left(|x|\right)$, with $|\cdot\|$ being a norm in $\mathbb{R}^d$ and $g : [0,\infty[\to]0,\infty[$ being a continuous function rapidly decreasing, (i.e with $\lim_{t\to\infty} t^n g(t) = 0$ for any $n \in \mathbb{N}$), we have that $h_{v_0}(/R^d)$ contains the polynomials.

Next, we extend [18, Lemma 10] to unbounded domains.

Lemma 4.1.3. *Let $U \subset \mathbb{R}^d$ be a non-empty open set and let v be a continuous strictly positive weight which vanishes at infinity on U. If $h_{v_0}(U)$ contains the polynomials then the map $\mu : U \to (h_{v_0}(U)^*, \sigma(h_{v_0}(U)^*, h_{v_0}(U)))$, $z \to v(z)\delta_z$ is a homeomorphism onto its image.*

Proof. Since the polynomials belong to $h_{v_0}(U)$, we get that $\{\delta_z : z \in U\} \subset h_{v_0}(U)^*$ are linearly independent functionals. We consider the map $\widehat{\mu} : \widehat{U} \to (h_{v_0}(U)^*, \sigma(h_{v_0}(U)^*, h_{v_0}(U)))$, $z \to v(z)\delta_z$ if $z \in U$ and $\widehat{\mu}(\infty) = 0$.

$(\widehat{U}, \tau')$ is the Alexandroff Compactification of U. Let z_0 be in $\widehat{U}$. A neighbourhood basis of $\widehat{\mu}(z_0)$ is given by finite intersections of sets of the form

$$V(\widehat{\mu}(z_0), f, \varepsilon) := \{u \in h_{v_0}(U)^* : |u(f) - \widehat{\mu}(z_0)(f)| < \varepsilon\},$$

where $f \in h_{v_0}(U)$ and $\varepsilon > 0$.

From the very definition of $h_{v_0}(U)$ it follows that for each $f \in h_{v_0}(U)$, the map $v \cdot f : (\widehat{U}, \tau') \to \mathbb{C}$,$(v \cdot f)(z) = v(z)f(z)$ if $z \in U$, $(v \cdot f)(\infty) = 0$ is continuous. Therefore, if we fix $\varepsilon > 0$, $f \in h_{v_0}(U)$ and $z_0 \in \widehat{U}$, then there exists $U_{z_0} \in \tau'$ such that $|(v \cdot f)(z) - (v \cdot f)(z_0)| < \varepsilon$ for each $z \in U_{z_0}$. Hence,

$$\widehat{\mu}(U_{z_0}) \subset V(\widehat{\mu}(z_0), f, \varepsilon),$$

and consequently $\widehat{\mu}$ is continuous. By the linear independence of the evaluations we have that $\widehat{\mu}$ is injective. The compactness of $\widehat{U}$ implies that $\widehat{\mu}$ is a homeomorphism onto its image, and hence also its restriction μ to U which is a topological subspace of $\widehat{U}$.

□

The *harmonic v-boundary* is defined as follows

$$b_v(U) := \{z \in U : \ v(z)\delta_z \text{ is extreme in the unit sphere of } h_{v_0}(U)^*\}.$$

The *set of harmonic v-peak points* is defined as

$$p_v(U) := \{z_0 \in U : \text{ there is } f \text{ in the unit sphere of } h_{v_0}(U)$$
$$\text{such that } v(z_0)f(z_0) = 1 \text{ and } v(z)|f(z)| < 1 \text{ for all } z \neq z_0\}.$$

According to Chapter 3 the extreme points in the unit sphere of $h_{v_0}(U)^*$ can be written as $\lambda v(z)\delta_z$, for some $\lambda \in \partial\overline{\mathbb{D}}$ and $z \in U$. From Corollary 3.4.4 it follows

$$p_v(U) := \{z \in U : \ v(z)\delta_z \text{ is weak* exposed in the unit sphere of } h_{v_0}(U)^*\}.$$

If $U \subseteq \mathbb{C}^d$ then the holomorphic v-boundary and the set of holomorphic v-peak points are defined analogously and they are denoted by $B_v(U)$ and $P_v(U)$ respectively. Since $H_{v_0}(U) \subset h_{v_0}(U)$ in this case we have

$$P_v(U) \subseteq B_v(U) \subseteq U$$

and also

$$P_v(U) \subseteq p_v(U) \subseteq b_v(U) \subseteq U. \tag{4.3}$$

Lemma 4.1.4 ([18, 19]). *Let $U \subseteq \mathbb{R}^d$ ($U \subseteq \mathbb{C}^d$) be a non empty open set and let v be a weight which vanishes at infinity on U.*

(a) For any $\lambda \in \partial\overline{\mathbb{D}}$ and $z \in U$ the functional $\lambda v(z)\delta_z$ is extreme in the unit sphere of $h_{v_0}(U)^*$ (resp. $H_{v_0}(U)^*$) if and only if $z \in b_v(U)$ (resp. $B_v(U)$).

(b) If $U \subseteq \mathbb{C}^d$ is balanced, v is radial and $\lambda \in \partial\overline{\mathbb{D}}$ then $\lambda z \in b_v(U)$ for each $z \in b_v(U)$ (resp. $\lambda z \in B_v(U)$ for each $z \in B_v(U)$).

The following result is proved by Boyd and Rueda for a bounded domain U, v vanishing at infinity on U and $B_v(U) = U$ or U balanced and v radial (See [18, Theorem 16 and Corollary 17]).

Proposition 4.1.5. *Let $U \subset \mathbb{R}^d$ ($U \subset \mathbb{C}^d$) be a non-empty open set and let v a weight for which $h_{v_0}(U)$ (resp. $H_{v_0}(U)$) contains the polynomials. Assume that $b_v(U)$ (resp. $B_v(U)$) is not discrete. Then $h_{v_0}(U)$ ($H_{v_0}(U)$) cannot be isometric to any subspace of c_0.*

Proof. We only prove the harmonic case. We denote by $\widehat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ the Alexandroff Compactification of $\mathbb{N}$. Let $T : h_{v_0}(U) \to c_0$ be an isometry onto its image. Let $F := T(h_{v_0}(U)) \subseteq c_0$. Since c_0 is the subspace of $C(\widehat{\mathbb{N}})$ formed by the functions which vanish at ∞, the extreme points of the unit sphere of F^* are contained in $\{\lambda\delta_n : \lambda \in \partial\overline{\mathbb{D}},\ n \in \mathbb{N}\}$ ([27, Lemma V.8.6]). Let $Q : F \to h_{v_0}(U)$ be the inverse isometry of T. The transpose linear mapping $Q^t : h_{v_0}(U)^* \to F^*$ is also an isometry and it maps extreme points to extreme points. Thus for each $z \in b_v(U)$ there exists $\lambda_z \in \partial\overline{\mathbb{D}}$ and $n(z) \in \mathbb{N}$ such that $Q^t(v(z)\delta_z) = \lambda_z\delta_{n(z)}$. From the linear independence of $\{\delta_z : z \in U\}$ in $h^*_{v_0}(U)$ we conclude that $n(z) \neq n(w)$ for $z, w \in U$ with $z \neq w$, and then $b_v(U)$ is countable and $\{\lambda_z\delta_{n(z)} :\ z \in b_v(U)\}$ is discrete for the weak* topology, since it is a sequence in the sphere of F^* which is $\sigma(F^*, F)$ convergent to 0. But Q^t is a weak*-weak* homeomorphism. Hence we conclude from Lemma 4.1.3 that $b_v(U)$ is discrete. □

Since the set of extreme points of the unit ball of a dual Banach space is never empty by The Krein–Milman Theorem and Lemma 4.1.4, we deduce the following result.

Corollary 4.1.6. *If $U \subseteq \mathbb{C}^d$ is a balanced open set and v is radial weight which vanishes at infinity on U then neither $H_{v_0}(U)$ nor $h_{v_0}(U)$ is isometric to a subspace of c_0.*

Despite that we do not have an example of a weight v on U such that $b_v(U)$ is a discrete subset of U, we see below that we can reformulate the problem of finding an example of space $h_{v_0}(U)$ which can be isometrically embedded in c_0 in terms of the possible existence of an open set $U \subseteq \mathbb{R}^d$ and a weight v such that $b_v(U)$ is not discrete. Of course the analogous result is also true for the holomorphic case. In view of Lemma 4.1.4 this is not possible for radial weights in balanced domains.

Proposition 4.1.7. *If $U \subseteq \mathbb{R}^d$ is an open set and v is a weight on U such that $h_{v_0}(U)$ contains the polynomials and $b_v(U)$ is discrete then $h_{v_0}(U)$ embeds isometrically in c_0.*

Proof. Since the closed unit ball D of $h_{v_0}(U)^*$ is the weak*-closure of the absolutely convex hull of the extreme points of the sphere, Lemma 4.1.4 implies that D is in fact the closure of the absolutely convex hull of $\{v(z)\delta_z : z \in U\}$. Thus this implies that $b_v(U)$ cannot be finite, and since it is discrete we have that $b_v(U) = (x_n)_n$ with $(x_n)_n$ tending to infinity on U when n goes to infinity. We get now that D is the weak*-closure of the absolutely convex hull of $\{v(x_n)\delta_{x_n} : n \in \mathbb{N}\}$, and hence, we have

$$\|f\|_v := \sup_{n\in\mathbb{N}} v(x_n)|f(x_n)|.$$

for each $f \in h_{v_0}(U)$. Indeed, $\|f\|_v \geq \sup_{n\in\mathbb{N}} v(x_n)|f(x_n)| := M$ is obvious. If $u \in acx\{v(x_n)\delta_{x_n}\}$ then, $u = \Sigma_i^m \lambda_i v(x_i)\delta_{x_i}$, being $\lambda_i \in \mathbb{R}$, $1 \leq i \leq m$, $\Sigma_i^m|\lambda_i| \leq 1$. So, $u(f) = \Sigma_i^m \lambda_i v(x_i) f(x_i) \leq \Sigma_i^m|\lambda_i|M \leq M$. Thus, $\|f\|_v = sup\{u(f) : u \in \overline{acx}\{v(x_n)\delta_{x_n} : n \in \mathbb{N}\}\} = sup\{u(f) : u \in acx\{v(x_n)\delta_{x_n} : n \in \mathbb{N}\}\} \leq M$.

Since $(x_n)_n$ tends to infinity on U, we have that the linear map

$$T : h_{v_0}(U) \to c_0, \; f \mapsto (v(x_n)f(x_n))_n$$

is an isometry.

□

4.2 Geometry of $h_{v_0}(U)$, $H_{v_0}(U)$ and their duals

In [19, Theorem 29] Boyd and Rueda showed that if $U \subseteq \mathbb{C}^d$ is a balanced bounded open set and v is a radial weight vanishing at infinity on U such that $P_v(U) = B_v(U)$ then $H_{v_0}(U)$ is not rotund, and then neither $h_{v_0}(U)$ is. This condition is trivially satisfied when $P_v(U) = U$. To the best of our knowledge, so far there are no concrete examples of open subsets $U \subset \mathbb{C}^d$ and weights v on U such that $B_v(U) \setminus P_v(U) \neq \emptyset$. In the concrete examples of spaces $H_{v_0}(U)$ given in [19] where $B_v(U)$ is calculated the equality $B_v(U) = P_v(U)$ is always satisfied. With a similar proof to the one of [19, Theorem 29], we present below a new condition for spaces $H_{v_0}(U)$ with U balanced and v radial depending only on the size of $P_v(U)$ which ensures that $H_{v_0}(U)$ is not rotund.

Proposition 4.2.1. *(a) Let $U \subseteq \mathbb{C}^d$ be the open unit ball for a norm $|\cdot|$ in $\mathbb{C}^d$ and let $v(x) = g(|x|)$ for $g : [0,1[\to]0,\infty[$, being a non-increasing continuous functions with $g(1^-) = 0$. If there exists $z_0 \in \partial U$ such that the set $\{tz_0 : 0 < t < 1\} \subseteq P_v(U)$ then $H_{v_0}(U)$ is not rotund, and then neither $h_{v_0}(U)$ is.*

(b) Let $|\cdot|$ be a norm in $\mathbb{C}^d$ and let $v(x) = g(|x|)$ for $g : [0,\infty[\to]0,\infty[$, being a non-increasing continuous function with $\lim_{t\to\infty} t^n g(t) = 0$ for all $n = 0,1,2,\ldots$ If there exists $z_0 \in \partial U$ such that the set $\{tz_0 : 0 < t\} \subseteq P_v(\mathbb{C}^d)$ then $H_{v_0}(\mathbb{C}^d)$ is not rotund, and then neither $h_{v_0}(\mathbb{C}^d)$ is.

Proof. (a) We choose a linear functional on $\mathbb{C}^d$ such that $\|\varphi^2\|_v = 1$. We take a non-null linear map φ. Since the polynomials are contained in $H_{v_0}(U)$ it follows that any power of the functional is in $H_{v_0}(U)$. If $\|\varphi^2\|_v \neq 1$ then we take $\overline{\varphi} = \frac{\varphi}{\sqrt{\|\varphi^2\|_v}}$ whose norm equals one, is in $H_{v_0}(U)$ and is a continuous linear map.

Since $v\varphi^2$ is continuous on $\widehat{U}$ and vanishes at infinity there exists $z_1 \in U$ such that

$$1 = \left\|\varphi^2\right\|_v = v(z_1)\left|\varphi^2(z_1)\right| = v(-z_1)\left|\varphi^2(-z_1)\right|.$$

This z_1 is always different to zero since φ is linear.

We may assume that

$$1 = \left\|\varphi^2\right\|_v = v(z_1)\varphi^2(z_1) = v(-z_1)\varphi^2(-z_1),$$

since we can multiply by a suitable scalar.

As $|z_0| = 1$, by applying the Theorem of Hahn–Banach we get $y \in \mathbb{C}^d$ such that $\langle y, z_0\rangle = 1$ and $|\langle y, z\rangle| \leq |z|$ for each $z \in \mathbb{C}^d$. Let $r_0 := |z_1|$. We define

$$\begin{aligned} h &: U \to \mathbb{C} \\ h(z) &= \varphi^2\left(\frac{|\langle y, z\rangle|}{r_0} z_1\right). \end{aligned}$$

We have

$$v(r_0 z_0)h(r_0 z_0) = g(r_0)\varphi^2\left(\frac{|\langle y, r_0 z_0\rangle|}{r_0} z_1\right) = g\left(|z_1|\right)\varphi^2\left(z_1\right) = v(z_1)\varphi^2\left(z_1\right) = 1.$$

Since φ is linear we also get

$$v(-r_0 z_0)h(-r_0 z_0) = g(r_0)\varphi^2\left(-z_1\right) = v(-z_1)\varphi^2\left(-z_1\right) = 1.$$

For $z \in U$ arbitrary, since g is non increasing and $\left|\frac{z_1}{r_0}\right| = 1$, it follows,

$$\begin{aligned} v(z)\,|h(z)| &= v(z)\,|h(z)| = g\left(|z|\right)|h(z)| \leq \\ g\left(|\langle y,z\rangle|\right)\left|\varphi^2\left(\frac{|\langle y,z\rangle|}{r_0}z_1\right)\right| &= v\left(\frac{|\langle y,z\rangle|}{r_0}z_1\right)\left|\varphi^2\left(\frac{|\langle y,z\rangle|}{r_0}z_1\right)\right|. \end{aligned}$$

From $1 = \|\varphi^2\|_v$ and $\frac{|\langle y,z\rangle|}{r_0}z_1 \in U$ we get

$$v\left(\frac{|\langle y,z\rangle|}{r_0}z_1\right)\left|\varphi^2\left(\frac{|\langle y,z\rangle|}{r_0}z_1\right)\right| \leq 1.$$

Therefore,

$$v(z)\,|h(z)| \leq 1.$$

Thus, $h \in H_{v_0}(U)$. However,

$$v(r_0z_0)\delta_{r_0z_0}(h) = 1 = v(-r_0z_0)\delta_{-r_0z_0}(h).$$

Now, as r_0z_0 is assumed to be a peak point we can find f in the unit ball of $H_{v_0}(U)$ different from h such that $v(r_0z_0)f(r_0z_0) = 1$ and $v(z)\mathrm{Re}(f(z)) < 1$ for each $z \in U \setminus \{r_0z_0\}$. Hence we conclude from

$$\|(f+h)/2\|_v = v(r_0z_0)(f(r_0z_0) + h(r_0z_0))/2 = 1$$

that $H_{v_0}(U)$ is not rotund.

(b) The condition $\lim_{t\to\infty} t^n g(t) = 0$ for all $n = 0, 1, 2, ..$ implies that $H_{v_0}(\mathbb{C}^d)$ contains the polynomials. So, any power of a linear functional is in $H_{v_0}(\mathbb{C}^d)$. Now, we proceed as in a). □

Let $g : [0,1] \to [0,\infty[$ be a function such that $g(1) = 0$, g is decreasing, g is twice differentiable in $]0,1[$ and $\log(1/g(t))$ is convex. In [19, Corollary 12] the authors showed that for a weight v defined in the Euclidean unit ball U of $\mathbb{C}^d$ by $v(x) = g(\|x\|)$ the equality $P_v(U) = U$ holds, and then $H_{v_0}(U)$ cannot be rotund by [19, Theorem 29]. Also if $U_1 \subseteq \mathbb{C}^d$ and $U_2 \subseteq \mathbb{C}^k$ are the Euclidean unit balls, and v_1 and v_2 satisfy this last condition for certain g_1 and g_2, then $v : U_1 \times U_2 \to]0,\infty[$, $v(x,y) = v_1(x)v_2(y)$ also satisfies $P_v(U) = U$ (cf. [19, Proposition 26]). We present below new examples of spaces $H_{v_0}(U)$ which cannot be rotund. In the required assumptions, besides considering a wider class of open sets, we remove from the weight the condition of differentiability given by Boyd and Rueda as we did in Theorem 1.2.24 of Chapter 1, to give conditions under which $v = \tilde{v}$, $\widetilde{v}$ being the associated weight introduced by Bierstedt, Bonet and Taskinen in [7]. The same technique permits us to get concrete radial weights $v(x) = g(\|x\|)$ defined in the Euclidean ball U of $\mathbb{R}^d$ with d odd such that the spaces $h_{v_0}(U)$ are not isometric to any subspace of c_0. If $d = 2k$ then the assertion is a consequence of Corollary 4.1.6 since U is the Euclidean unit ball of $\mathbb{C}^k$ and $H_{v_0}(U) \subset h_{v_0}(U)$.

Proposition 4.2.2. *Let $g : [0,1] \to [0,\infty[$ be a continuous decreasing function such that $g(1) = 0$ and $\log(1/g)$ is strictly convex in $[0,1[$. Let $U \subseteq \mathbb{C}^d$ be the unit ball for a norm $|\cdot|$ and let $v : U \to]0,\infty[$, $x \mapsto g(|x|)$. Then $H_{v_0}(U)$ is not rotund (and then neither $h_{v_0}(U)$ is).*

Proof. Let $h(r) := \log(1/g(r))$, $0 \le r < 1$. Since h is increasing and strictly convex, for each $0 \le r_0 < 1$ the map

$$[0,1[\backslash\{r_0\} \to]0,\infty[,\ r \mapsto (h(r) - h(r_0))/(r - r_0)$$

is increasing. Hence for each $0 \le r_0 < 1$ there exists $\alpha_0 \ge 0$ which depends on r_0 such that

$$h(r) - h(r_0) > \alpha_0(r - r_0)$$

for all $r \in [0,1[\backslash\{r_0\}$. Now we compute

$$\sup_{0\le r<1, r\ne r_0} g(r)\exp(\alpha_0 r) = \exp\left(\sup_{0<r<1, r\ne r_0}(-h(r) + \alpha_0 r)\right) <$$

$$< \exp\left(\sup_{0\le r<1, r\ne r_0}(-\alpha_0(r - r_0) - h(r_0) + \alpha_0 r)\right) = g(r_0)\exp(\alpha_0 r_0). \qquad (4.4)$$

Straszewicz's Theorem [1, Theorem 7.89] together with the Krein–Milman Theorem implies that there exists $z_0 \in \partial U$ which is exposed. Let $y \in \mathbb{C}^d$ such that $\langle z_0, y\rangle = 1$ and $\mathrm{Re}(\langle z, y\rangle) < |z|$ if $z \in U \setminus \{rz_0 :\ 0 \le r < 1\}$. Let $0 \le r_0 < 1$ be arbitrary, and let α_0 be as above. The function $f_0 : U \to \mathbb{C}$ defined by $f_0(z) := \frac{\exp(\alpha_0\langle z,y\rangle)}{g(r_0)\exp(\alpha_0 r_0)}$ is holomorphic and $v(r_0 z_0)f_0(r_0 z_0) = 1$. It follows immediately from (4.4) that if $0 \le r < 1$, $r \ne r_0$

$$v(rz_0)|f_0(rz_0)| = \frac{g(r)\exp(\alpha_0 r)}{g(r_0)\exp(\alpha_0 r_0)} < 1.$$

If $z \in U \setminus \{rz_0 : 0 \le r < 1\}$ then we use (4.4) and $\mathrm{Re}(< z, y >) < |z|$ to get

$$v(z)|f_0(z)| = g(|z|)\frac{\exp(\alpha_0 \mathrm{Re}(< z, y >))}{g(r_0)\exp(\alpha_0 r_0)} < g(|z|)\frac{\exp(\alpha_0|z|)}{g(r_0)\exp(\alpha_0 r_0)} \le 1.$$

$f_0 \in H_{v_0}(U)$, since f_0 is bounded in U and v vanishes at infinity on U. Hence f_0 peaks $v(r_0 z_0)\delta_{r_0 z_0}$. We conclude that $\{rz_0 : 0 \le r < 1\} \subseteq P_v(U)$ since r_0 is arbitrary. We apply Proposition 4.2.1 to get that $H_{v_0}(U)$ is not rotund. □

Proposition 4.2.3. *Let $U \subset \mathbb{R}^d$ be the Euclidean unit ball and $g : [0,1[\to]0,\infty[$ be a non-increasing continuous function such that $g(1^-) = 0$ and $\log(\frac{1}{g})|_{]0,1[}$ is strictly convex. Let $d \ge 2$. Let us consider the unitary weight $v : U \to]0,\infty[$ defined by $v(x) := g(\|x\|)$. Then $p_v(U) = U$.*

Proof. We fix $r_0 \in]0,1[$ and we define $\Psi := log(\frac{1}{v}|_{]0,1[})$. As Ψ is increasing and strictly convex we can find $\alpha_0 \geq 0$ dependent on r_0 such that $\Psi(r) > \alpha_0(r-r_0) + \Psi(r_0)$, for all $r \in [0,1[$, $r \neq r_0$. We proceed again as in Proposition 4.2.2

$$\sup_{0<r<1} v(r)\exp(\alpha_0 r) = \exp\left\{\sup_{0<r<1}\{\log v(r) + \alpha_0 r\}\right\} <$$
$$\exp\left\{\sup_{0<r<1}\{-\alpha_0(r-r_0) - \Psi(r_0) + \alpha_0 r\}\right\} = v(r_0)\exp(\alpha_0 r_0).$$

Let $x_0 \in B_{\mathbb{R}^d}$ such that $\|x_0\| = r_0$. Let us suppose without loss of generality that the first component of x_0 is not zero. Let T be the linear map $T : \mathbb{R}^d \to \mathbb{R}^d$ with $\|T(x)\| = \|x\|$ for all x, such that $T(x_0) = (r_0, 0, \cdots, 0)$, $T = (T_1, T_2, \cdots, T_d)$ and we take

$$f_0(x_1, ..., x_d) := \frac{1}{v(r_0)\exp(\alpha_0 r_0)}(\exp(\alpha_0 x_1))\cos(\alpha_0 x_2)$$

for $x \in B_{\mathbb{R}^d}$, which is harmonic. Also $f_0 \circ T$ is harmonic by [3, Chapter 1]. Since $f \circ T$ is bounded in U and $g(1^-) = 0$ it follows that $f_0 \circ T \in hv_0^{\mathbb{R}}$. Moreover, we have $v(r_0)f_0(T(x_0)) = 1$.

Let $x \in B_{\mathbb{R}^d}$, $x \neq x_0$, $\|x\| = r$.

If $r \neq r_0$ then

$$v(x)\,|f_0(T(x))| \;=\; \frac{v(r)\,|Re(\exp(\alpha_0(T_1(x) + iT_2(x)))|}{v(r_0)\exp(\alpha_0 r_0)} \leq$$

$$\leq \frac{v(r)\exp(\alpha_0\,|T_1(x) + iT_2(x)|)}{v(r_0)\exp(\alpha_0 r_0)} \leq \frac{v(r)\exp(\alpha_0\,\|T(x)\|)}{v(r_0)\exp(\alpha_0 r_0)} \leq$$
$$\leq \frac{v(r)\exp(\alpha_0 r)}{v(r_0)\exp(\alpha_0 r_0)} \;<\; 1.$$

If $r = r_0$ then $T_1(x)^2 + ... + T_d(x)^2 = r_0^2$.

If $T_2(x)^2 + ... + T_d(x)^2 \neq 0$ then $|T_1(x)| < r_0 = r$ and by applying that the exponential function is strictly increasing,

$$v(x)\,|f_0(T(x))| = \frac{v(r)\,|Re(\exp(\alpha_0(T_1(x) + iT_2(x)))|}{v(r_0)\exp(\alpha_0 r_0)} = \frac{v(r)exp(\alpha_0 T_1(x))}{v(r_0)exp(\alpha_0 r_0)}$$

$$\leq \frac{\exp(\alpha_0\,|T_1(x)|)}{\exp(\alpha_0 r_0)} < 1.$$

If $T_2(x)^2+...+T_d(x)^2 = 0$ then injectivity yields $x \neq -x_0$ and $T(x) = (-r_0, 0, ..., 0)$.

$$v(x)\,|f_0(T(x))| = \frac{v(r)\,|Re(\exp(\alpha_0(T_1(x) + iT_2(x)))|}{v(r_0)\exp(\alpha_0 r_0)} =$$

$$= \frac{\exp(\alpha_0(-r_0))}{\exp(\alpha_0 r_0)} < 1.$$

Finally, if $r_0 = 0$ then we take the constant function $f_0 \equiv \frac{1}{v(0)}$.

Thus, in all cases we have $v(x_0)f_0(x_0) = 1$ and $v(x)f_0(x) < 1$ for each $x \in U \setminus \{x_0\}$. Hence $p_v(U) = U$. □

Corollary 4.2.4. *Let $U \subseteq \mathbb{R}^d$ be the Euclidean unit ball, $g : [0,1[\to]0,\infty[$ be a non-increasing continuous function such that $g(1^-) = 0$ and $\log(\frac{1}{g})|_{]0,1[}$ is strictly convex and let $v(x) = g(\|x\|)$ for $x \in U$. Then $h_{v_0}(U)$ is not isometric to any subspace of c_0.*

Proof. By Proposition 4.2.3 $p_v(U) = U$. This implies that $h_{v_0}(U)$ is not isometrically embedded in c_0 by Proposition 4.1.5. □

Proposition 4.2.5. *Let $g : [0,\infty[\to]0,\infty[$ be a continuous decreasing function such that $\log(1/g)$ is strictly convex in $[0,\infty[$ such that $g(t) = o(e^{-at})$ for each $a > 0$.*

(a) Let $|\cdot|$ be a norm on $\mathbb{C}^d$ and let $v : \mathbb{C}^d \to]0,\infty[$, $x \mapsto g(|x|)$. Then $H_{v_0}(\mathbb{C}^d)$ is not rotund (and then neither $h_{v_0}(\mathbb{C}^d)$ is).

(b) Let $v(x) = g(\|x\|)$ for $x \in \mathbb{R}^d$. Then $h_{v_0}(\mathbb{R}^d)$ is not isometric to any subspace of c_0.

Proof. (a) We repeat the argument of the proof of Proposition 4.2.2 which is valid because the condition $g(t) = o(e^{-at})$ for each $a > 0$ implies that

$$f_0(z) = \frac{exp(\alpha_0\langle y, z\rangle)}{g(r_0)exp(\alpha_0 r_0)} \in h_{v_0}(\mathbb{C}^d).$$

Thus, we can apply Proposition 4.2.1.

Also (b) is obtained from this observation and an inspection of the proof of Proposition 4.2.3 and Corollary 4.2.4. □

The condition $g(t) = o(e^{-at})$ for each $a > 0$ can be improved because according to the proof it would be enough to take $g(t) = o(e^{-ant})$ for all $n \in \mathbb{N}$ and $a = \alpha_0$ being α_0 the right first derivative in 0 of the function $log(1/g)$.

For this kind of weights, if the differentiation operator D on $H_{v_0}(\mathbb{C})$ is continuous then D is hypercyclic by [9, Theorem 2.3] since $e^t v(t) \to 0$ where $t \to \infty$. Furthermore, $g(t) = o(e^{-at})$ with $a > 0$ implies that

$$\lim_{t\to\infty} \frac{\log(1/g)(t)}{\log(t)} = +\infty$$

and by [37] there are harmonic functions h on $\mathbb{R}^d$ satisfying the inequality $|h(x)| \leq \frac{1}{g(\|x\|)}$ for every $x \in \mathbb{R}^d$, which are universal concerning to translations. This means that h is in $h_v(\mathbb{C})$.

Remark 4.2.6. For any increasing continuous function $h : [0,1[\to [0,\infty[$ which is strictly convex and satisfies $\lim_{t\to 1} h(t) = \infty$ (or $h : [0,\infty[\to]0,\infty[$ with $e^{at} = o(e^{h(t)})$ when t goes to infinity for each $a > 0$), the function $g(t) = e^{-h(t)}$ satisfies the hypotheses of Proposition 4.2.1 and Proposition 4.2.2 (or Proposition 4.2.5). The examples given in [19, Example 13] of weights in the Euclidean unit ball can be obtained using this general method.

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