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CONTRACTION MAPPINGS IN FUZZY QUASI-METRIC SPACES AND [0,1]-FUZZY POSETS

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Abstract. It is well known that each bounded ultraquasi-metric on a set induces, in a natural way, an [0,1]-fuzzy poset. On the other hand, each [0,1]-fuzzy poset can be seen as a stationary fuzzy ultraquasi-metric space for the continuous t-norm Min. By extending this construction to any continuous t-norm, a stationary fuzzy quasi-metric space is obtained. Motivated by these facts, we present several contraction principles on fuzzy quasi-metric spaces that are applied to the class of spaces described above. Some illustrative examples are also given. Finally, we use our approach to deduce in an easy fashion the existence and uniqueness of solution for the recurrence equations typically associated to the analysis of Probabilistic Divide and Conquer Algorithms.
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1. Introduction

Throughout this paper the letters $\mathbb{R}$ and $\mathbb{N}$ will denote the set of all real numbers and the set of all positive integer numbers, respectively.

Our basic reference for quasi-metric spaces is [15], for general topology it is [1], and for lattices it is [6].

Following the modern terminology, by a quasi-metric on a nonempty set X we mean a nonnegative real valued function d on $X \times X$ such that for all $x, y, z \in X$:

(i) $x = y$ if and only if $d(x, y) = d(y, x) = 0$;

(ii) $d(x, z) \leqslant d(x, y) + d(y, z)$.

If d satisfies condition (i) above and

(ii') $d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$ for all $x, y, z \in X$,

then, d is called an ultraquasi-metric (or a non-Archimedean quasi-metric) on X.

A quasi-metric d on a set X is said to be bounded if there is $K > 0$ such that $d(x, y) \leqslant K$ for all $x, y \in X$. In this case we say that K is a bound for d.

A quasi-metric space (respectively, an ultraquasi-metric space) is a pair (X, d) such that X is a nonempty set and d is a quasi-metric (respectively, an ultraquasi-metric) on X.

Each quasi-metric d on X generates a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x,r) : x \in X, r > 0\}$, where $B_d(x,r) = \{y \in X : d(x,y) < r\}$ for all $x \in X$ and $r > 0$.

Given a quasi-metric (an ultra quasi-metric) d on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x,y) = d(y,x)$, is also a quasi-metric (an ultra quasi-metric) on X, called the conjugate of d, and the function d^s defined on $X \times X$ by $d^s(x,y) = \max\{d(x,y), d^{-1}(x,y)\}$ is a metric (an ultra metric) on X.

Next we recall some fuzzy concepts and facts which will be useful later on.

According to [22], a binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $*$ is continuous; (iii) $a * 1 = a$ for every $a \in [0,1]$; (iv) $a * b \leqslant c * d$ whenever $a \leqslant c$ and $b \leqslant d$, with $a, b, c, d \in [0,1]$.

Recall that continuous t-norms are closely related to basic fuzzy logic BL as investigated in [12]; in particular, each continuous t-norm induces a (linearly ordered) BL-algebra ([13]).

Paradigmatic examples of continuous t-norm are Min, Prod, and T_L (the Lukasiewicz t-norm).

In the following Min will be denoted by $\wedge$, Prod by $\cdot$ and T_L by $*_L$. In particular, we have $a \wedge b = \min\{a,b\}$, and $a *_L b = \max\{a + b - 1, 0\}$ for all $a, b \in [0,1]$. We also recall that the following relations hold: $\wedge \geqslant \cdot \geqslant *_L$. In fact $\wedge \geqslant *$ for any continuous t-norm $*$.

Definition 1.1 ([10]). A KM-fuzzy quasi-metric on a nonempty set X is a pair $(M,*)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0,\infty)$ such that for all $x, y, z \in X$:

(i) $M(x,y,0) = 0$;

(ii) $x = y$ if and only if $M(x,y,t) = M(y,x,t) = 1$ for all $t > 0$;

(iii) $M(x,z,t+s) \geqslant M(x,y,t) * M(y,z,s)$ for all $t, s \geqslant 0$;

(iv) $M(x,y,\_) : [0,\infty) \to [0,1]$ is left continuous.

Note that a KM-fuzzy quasi-metric $(M,*)$ satisfying for all $x, y \in X$ and $t > 0$ the symmetry axiom $M(x,y,t) = M(y,x,t)$, is a fuzzy metric in the sense of Kramosil and Michalek [14] (see also [4, 5], where a slight but interesting modification of this notion of fuzzy metric space is discussed).

In the following, KM-fuzzy quasi-metrics will we simply called fuzzy quasi-metrics. In particular, by a fuzzy metric we will mean a fuzzy quasi-metric satisfying the symmetry axiom; thus, fuzzy metrics are taken in the sense of [14].

A triple $(X,M,*)$ where X is a nonempty set and $(M,*)$ is a fuzzy (quasi-)metric on X, is said to be a fuzzy (quasi-)metric space.

A fuzzy (quasi-)metric $(M,*)$ on X is called stationary if for each $x, y \in X$, $M(x,y,\_)$ is a constant function on $(0,\infty)$. In this case we say that $(X,M,*)$ is a stationary fuzzy (quasi-)metric space.

Stationary fuzzy metrics have been recently considered in questions concerning convergence in fuzzy metric spaces [8], and completion of fuzzy metric spaces [9].

It was shown in Proposition 1 of [10] that if $(M,*)$ is a fuzzy quasi-metric on X, then for each $x, y \in X$, $M(x,y,\_)$ is nondecreasing, i.e. $M(x,y,t) \leqslant M(x,y,s)$ whenever $t \leqslant s$.

Definition 1.2. A fuzzy quasi-metric space $(X, M, *)$ such that $M(x, y, t) \geqslant \min\{M(x, z, t), M(z, y, t)\}$ for all $x, y, z, \in X$, $t > 0$, is called a fuzzy ultraquasi-metric space, and $(M, *)$ is called a fuzzy ultraquasi-metric.

The notion of a fuzzy ultrametric space is defined in the obvious manner. Fuzzy ultrametric spaces are called non-Archimedean fuzzy metric spaces in [21], and fuzzy ultraquasi-metric spaces are called non-Archimedean fuzzy quasi-metric spaces in [19].

If $(M, *)$ is a fuzzy (ultra)quasi-metric on X, then $(M^{-1}, *)$ is also a fuzzy (ultra)quasi-metric on X, where M^{-1} is the fuzzy set in $X \times X \times [0, \infty)$ defined by $M^{-1}(x, y, t) = M(y, x, t)$. Moreover, if we denote by M^i the fuzzy set in $X \times X \times [0, \infty)$ given by $M^i(x, y, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\}$, then $(M^i, *)$ is a fuzzy (ultra)metric on X [10].

Similarly to the fuzzy metric case (compare [10]), each fuzzy quasi-metric $(M, *)$ on X generates a T_0 topology τ_M on X which has as a base the family of open balls $\{B_M(x, r, t) : x \in X,\ 0 < r < 1,\ t > 0\}$, where $B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$.

Therefore, a sequence $(x_n)_n$ in $(X, M, *)$ converges to $x \in X$ with respect to τ_M if and only if $\lim_n M(x, x_n, t) = 1$ for all $t > 0$.

Example 1.3. Let (X, d) be a quasi-metric (an ultra quasi-metric) space and let M_d be the function defined on $X \times X \times [0, \infty)$ by $M_d(x, y, 0) = 0$ and

$$M_d(x, y, t) = \frac{t}{t + d(x, y)} \qquad \text{whenever} \quad t > 0.$$

It is easily seen [10] that $(M_d, *)$ is a fuzzy (ultra)quasi-metric on X for any continuous t-norm $*$. Moreover $\tau_d = \tau_{M_d}$ and $\tau_{d^{-1}} = \tau_{(M_d)^{-1}}$, and hence $\tau_{d^s} = \tau_{(M_d)^i}$ on X. If d is a metric (an ultra metric), then $(M_d, *)$ is obviously a fuzzy (ultra)metric on X (compare [4]).

Definition 1.4 ([2],[26]). Let X be a nonempty set and L be a complete lattice. A mapping $e : X \times X \to L$ is called an L-fuzzy partial order on X if it satisfies the following conditions for all $x, y, z \in X$:

(i) $e(x, y) = e(y, x) = 1$ if and only if $x = y$;

(ii) $e(x, z) \wedge e(z, y) \leqslant e(x, y)$.

An L-fuzzy partial ordered set (in short, an L-fuzzy poset) is a pair (X, e) such that X is a (nonempty) set and e is an L-fuzzy partial order on X. If $L = [0, 1]$, we say that (X, e) is an [0,1]-fuzzy poset.

It is well known ([2]) that each bounded ultraquasi-metric on a set X induces, in a natural way, an [0,1]-fuzzy poset. On the other hand, each [0,1]-fuzzy poset can be seen as a stationary fuzzy ultraquasi-metric space for the continuous t-norm $\wedge$. In Section 2 we shall extend this construction to any continuous t-norm and observe that, in this case, stationary fuzzy quasi-metric spaces are induced. Motivated by these facts, we present, in Section 3, several contraction principles on fuzzy quasi-metric spaces that are applied to the class of spaces described above. Some illustrative examples are also given. Finally, in Section 4 we use our approach to deduce in an easy fashion the existence and uniqueness of solution for the recurrence equations typically associated to the analysis of Probabilistic Divide and Conquer Algorithms.

Several results in this paper were presented by the author at the VII Iberoamerican Conference on Topology and its Applications, Valencia, June 2008 ([24]).

2. Fuzzy quasi-metric spaces and generalized [0,1]-fuzzy posets

Let d be a bounded ultraquasi-metric on a set X, with bound $K > 0$, then the fuzzy set e in $X \times X$ given by $e(x, y) = 1 - d(x, y)/K$, is a [0,1]-fuzzy partial order on X and hence (X, e) is an [0,1]-fuzzy poset ([2]).

Conversely, if (X, e) is an [0,1]-fuzzy poset, then it is clear that $(X, e, \wedge)$ is a stationary fuzzy ultraquasi-metric space.

In the light of these facts it seems natural to extend the notion of a [0,1]-fuzzy poset to any continuous t-norm and establish the relationship between this structure and the notion of a stationary fuzzy quasi-metric space. Thus Definition 1.4 suggests the following notion.

Definition 2.1. A generalized [0,1]-fuzzy partial order on a (nonempty) set X is a pair $(e, *)$ such that $*$ is a continuous t-norm and e is a fuzzy set in $X \times X$ such that for all $x, y, z \in X$:

(i) $e(x, y) = e(y, x) = 1$ if and only if $x = y$;

(ii) $e(x, z) * e(z, y) \leqslant e(x, y)$.

By a generalized [0,1]-poset we mean a triple $(X, e, *)$ such that X is a (nonempty) set and $(e, *)$ is a generalized [0,1]-fuzzy partial order on X.

Then, the following facts are immediate consequences of Definition 2.1.

Proposition 2.2. *Let d be a bounded quasi-metric on a set X, with bound $K > 0$. Then for each continuous t-norm $*$ such that $* \leqslant *_L$, the pair $(e, *)$ is a stationary fuzzy quasi-metric on X, where e is the fuzzy set in $X \times X$ given by*

$$e(x, y) = 1 - \frac{d(x, y)}{K},$$

for all $x, y \in X$.

Proposition 2.3. *Let $(X, e, *)$ be a generalized $[0, 1]$-fuzzy poset. Then $(X, e, *)$ is a stationary fuzzy quasi-metric space.*

3. Contraction mappings and fixed points

Fixed point theory of fuzzy metric spaces, in the sense of Kramosil and Michalek, began with the well-known theorem of M. Grabiec [7], which is a nice fuzzy metric version of the Banach contraction principle endowed with a nice contraction condition. Recent results related to Grabiec's theorem may be found in [16], [11], [19] and [20]. In particular, in [19] and [20] are obtained extensions of this theorem to fuzzy quasi-metric spaces and intuitionistic fuzzy metric spaces respectively.

In order to obtain his theorem, Grabiec ([7]) introduced the following notions: A sequence $(x_n)_n$ in a fuzzy metric space $(X, M, *)$ is Cauchy provided that $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for each $t > 0$ and $p \in \mathbb{N}$. A fuzzy metric space $(X, M, *)$ is complete provided that every Cauchy sequence in X is convergent. In this case, $(M, *)$ is called a complete fuzzy metric on X.

In the sequel, complete fuzzy metric spaces in Grabiec's sense will be called G-complete.

Theorem 3.1 ([7]). *Let $(X, M, *)$ be a G-complete fuzzy metric space such that $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$. If f is a self-mapping on X such that there is $k \in (0, 1)$ satisfying*

$$M(f(x), f(y), kt) \geqslant M(x, y, t)$$

for all $x, y \in X$ and $t > 0$, then f has a unique fixed point.

G-completeness has the disadvantage that is a very strong notion of completeness; in fact, if d is the Euclidean metric on $\mathbb{R}$, then the induced fuzzy metric $(M_d, *)$ of Example 1.3 is not G-complete ([25]). This fact, motivated the following well-known alternative notion of fuzzy (quasi-)metric completeness: A sequence $(x_n)_n$ in a fuzzy metric space $(X, M, *)$ is a Cauchy sequence provided that for each $\varepsilon \in (0, 1)$ and each $t > 0$ there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geqslant n_0$. The fuzzy metric space $(X, M, *)$ is complete provided that every Cauchy sequence in X is convergent [4]. In this case, $(M, *)$ is said to be a complete fuzzy metric on X.

On the other hand, condition $\lim_{t\to\infty} M(x, y, t) = 1$ cause Grabiec's theorem cannot be applied to stationary fuzzy (quasi-)metric spaces, and hence to generalized [0,1]-fuzzy posets.

In our next results we shall obtain some contraction mapping results which avoid these inconveniences of Grabiec's theorem. To this end, the following notions of a Cauchy sequence and of a complete fuzzy quasi-metric space will be suitable.

Definition 3.2. Let $(X, M, *)$ be a fuzzy quasi-metric space. A sequence $(x_n)_n$ in X is called forward Cauchy if for each $\varepsilon \in (0, 1)$ and each $t > 0$ there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1-\varepsilon$ whenever $n_0 \leqslant n \leqslant m$. We say that $(X, M, *)$ is complete if every forward Cauchy sequence is convergent with respect to the topology $\tau_{M^{-1}}$, i.e. if there exists $y \in X$ such that $\lim_{n\to\infty} M(x_n, y, t) = 1$ for all $t > 0$.

A generalized [0,1]-fuzzy poset $(X, e, *)$ is called complete if it is complete as a stationary fuzzy quasi-metric space (recall Proposition 2.3).

Theorem 3.3. *Let $(X, M, *)$ be a complete fuzzy quasi-metric space such that $* \geqslant *_L$. If f is a self-mapping on X such that there exist $x_0 \in X$ and $k \in (0, 1)$ satisfying*

$$M(f^n x_0, f^{n+1}x_0, t) \geqslant 1 - k + kM(f^{n-1}x_0, f^n x_0, t)$$

for all $n \in \mathbb{N}$ and $t > 0$, then the sequence $(f^n x_0)_n$ converges to some $y \in X$ with respect to $\tau_{M^{-1}}$.

If, in addition, $(X, \tau_{M^{-1}})$ is a Hausdorff topological space and f is continuous from $(X, \tau_{M^{-1}})$ into itself, then y is a fixed point of f.

Proof. We first show that $M(f^n x_0, f^{n+1}x_0, t) \geqslant 1 - k^n$ for all $n \in \mathbb{N}$ and $t > 0$.

Indeed, for $n = 1$ we have

$$M(fx_0, f^2x_0, t) \geq 1 - k + kM(x_0, fx_0) \geq 1 - k.$$

So

$$\begin{aligned} M(f^2x_0, f^3x_0, t) &\geq 1 - k + kM(fx_0, f^2x_0, t) \\ &\geq 1 - k + k(1 - k) \\ &= 1 - k^2. \end{aligned}$$

Now assume that the inequality holds for $n-1$, with $n>3$. Then

$$\begin{aligned} M(f^n x_0, f^{n+1}x_0, t) &\geqslant 1-k+kM(f^{n-1}x_0, f^n x_0, t) \\ &\geqslant 1-k+k(1-k^{n-1}) \\ &= 1-k^n. \end{aligned}$$

Next we show that $(f^n x_0)_n$ is a forward Cauchy sequence in $(X, M, *)$.

Indeed, for each $n, m \in \mathbb{N}$ (we assume without loss of generality that $m = n+j$ for some $j \in \mathbb{N}$)), we obtain

$$\begin{gathered} M(f^n x_0, f^m x_0, t) = M(f^n x_0, f^{n+j} x_0, t) \\ \geqslant M(f^n x, f^{n+1} x, t/j) * M(f^{n+1} x, f^{n+2} x, t/j) * ... * M(f^{n+j-1} x, f^{n+j} x, t/j) \\ \geqslant (1-k^n) * (1-k^{n+1}) * ... * (1-k^{n+j-1}) \\ \geqslant (1-k^n) *_L (1-k^{n+1}) *_L ... *_L (1-k^{n+j-1}). \end{gathered}$$

Given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty} k^n < \varepsilon.$$

Therefore, for $n, m \geqslant n_0$, with $n = m+j$, it follows that $k^n + k^{n+1} + ... + k^{n+j-1} < \varepsilon$, and hence

$$\begin{aligned} M(f^n x_0, f^m x_0, t) &\geqslant (1-k^n) *_L (1-k^{n+1}) *_L ... *_L (1-k^{n+j-1}) \\ &= 1-(k^n + k^{n+1} + ... + k^{n+j-1}) \\ &> 1-\varepsilon. \end{aligned}$$

Consequently $(f^n x_0)_n$ is a forward Cauchy sequence in $(X, M, *)$. Then, there is $y \in X$ such that $(f^n x_0)_n$ converges to y with respect to $\tau_{M^{-1}}$.

Finally, if $(X, \tau_{M^{-1}})$ is a Hausdorff topological space and f is continuous from $(X, \tau_{M^{-1}})$ into itself, then $(f^n x_0)_n$ converges to $f(y)$ with respect to $\tau_{M^{-1}}$, and by Hausdorffness, $y = f(y)$. The proof is complete. □

Corollary 3.4. *Let $(X, M, *)$ be a complete fuzzy quasi-metric space such that $(X, \tau_{M^{-1}})$ is a Hausdorff topological space and $* \geqslant *_L$. If f is a self-mapping on X such that there is $k \in (0,1)$ satisfying*

$$M(fx, fy, t) \geqslant 1-k+kM(x, y, t)$$

for all $x, y \in X$ and $t > 0$, then f has a unique fixed point.

Proof. By Theorem 3.3 we only need to show uniqueness of the fixed point of f. Let $x, y \in X$ be such that $fx = x$ and $fy = y$. Then $M(x, y, t) = M(fx, fy, t) \geqslant 1-k+kM(x, y, t)$ for all $t > 0$, and thus $M(x, y, t) = 1$ for all $t > 0$. Hence $x = y$. □

Corollary 3.5. *Let $(X, M, *)$ be a complete fuzzy metric space such that $* \geqslant *_L$. If f is a self-mapping on X such that there is $k \in (0,1)$ satisfying*

$$M(fx, fy, t) \geqslant 1-k+kM(x, y, t)$$

for all $x, y \in X$ and $t > 0$, then f has a unique fixed point.

Next we shall obtain a "G-complete" version of Theorem 3.3 and its corollaries. In this way, a fixed point theorem for [0,1]-fuzzy posets will be derived by virtue of Lemma 3.12 below.

Remark 3.6. In [17], Mihet introduced and studied the so-called fuzzy ψ-contractive mappings in order to obtain fixed point theorems. A subclass of fuzzy ψ-contractive mappings is the set of self-mappings f satisfying:

$$M(fx, fy, kt) \geqslant 1 - k + kM(x, y, t)$$

for all $x, y \in X$ and $t > 0$, for some $k \in (0, 1)$, (Sherwood, [23]). Note that the class of self-mappings considered in Corollary 3.5 is more general than Sherwood´s class given above. Recently, Mihet has obtained in [18] a generalization of Theorem 3.3 for fuzzy metric spaces.

Definition 3.7. Let $(X, M, *)$ be a fuzzy quasi-metric space. A sequence $(x_n)_n$ in X is called forward G-Cauchy if for each $p \in \mathbb{N}$ and each $t > 0$, $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$. We say that $(X, M, *)$ is G-complete if every forward G-Cauchy sequence is convergent with respect to the topology $\tau_{M^{-1}}$.

A generalized [0,1]-fuzzy poset $(X, e, *)$ is called G-complete if it is G-complete as a stationary fuzzy quasi-metric space.

At this point it seems interesting to present some examples illustrating the differences between the notions of a forward Cauchy sequence and of a forward G-Cauchy sequence, and also between the notions of completeness and G-completeness for non-metrizable fuzzy quasi-metric spaces.

Example 3.8. Let d_S be the quasi-metric on $\mathbb{R}$ given by $d_S(x, y) = \min\{1, y - x\}$ if $x \leq y$ and $d_S(x, y) = 1$ if $x > y$. It is well known, and easy to see, that d_S generates the Sorgenfrey topology on $\mathbb{R}$ (for each $x \in \mathbb{R}$ basic open neighbourhoods of x are of the form $[x, x + r)$ where $r > 0$), which is not a metrizable topology. Moreover $(d_S)^{-1}$ generates the "conjugate" Sorgenfrey topology on $\mathbb{R}$ (for each $x \in \mathbb{R}$ basic open neighbourhoods of x are of the form $(x - r, x]$ where $r > 0$). Now define $M_{1,d_s} : \mathbb{R}R \times R{\times}[0, \infty) \to [0, 1]$ by $M_{1,d_S}(x, y, 0) = 0$ for all $x, y \in \mathbb{R}$, $M_{1,d_S}(x, y, t) = 1 - d_S(x, y)$ for all $x, y \in \mathbb{R}$ and $t \in (0, 1]$, and $M_{1,d_S}(x, y, t) = 1$ for all $x, y \in \mathbb{R}$ and $t > 1$. It is immediate to check that $(\mathbb{R}, M_{1,d_S}, *_L)$ is a complete fuzzy quasi-metric space. Now consider the sequence $(u_n)_n$ in $\mathbb{R}$ such that $u_n = \sum_{k=1}^{n} 1/k$ for all $n \in \mathbb{N}$. Since for $t \in (0, 1]$, $M_{1,d_S}(u_n, u_{n+1}, t) = 1-(u_{n+1}-u_n) = 1-1/(n+1)$, it follows that $(u_n)_n$ is a forward G-Cauchy sequence. However, it is obvious that it is not a forward Cauchy sequence (compare Example 2.1 in [25]). Since $(u_n)_n$ is not convergent with respect to the topology generated by $(M_{1,d_S})^{-1}$ we conclude that $(\mathbb{R}, M_{1,d_S}, *_L)$ is not G-complete.

Theorem 3.9. *Let $(X, M, *)$ be a G-complete fuzzy quasi-metric space. If f is a self-mapping on X such that there exist $x_0 \in X$ and $k \in (0, 1)$ satisfying*

$$M(f^n x_0, f^{n+1} x_0, t) \geqslant 1 - k + kM(f^{n-1} x_0, f^n x_0, t)$$

for all $n \in \mathbb{N}$ and $t > 0$, then the sequence $(f^n x_0)_n$ converges to some $y \in X$ with respect to $\tau_{M^{-1}}$.

If, in addition, $(X, \tau_{M^{-1}})$ *is a Hausdorff topological space and* f *is continuous from* $(X, \tau_{M^{-1}})$ *into itself, then* y *is a fixed point of* f.

Proof. As in the proof of Theorem 3.3 we obtain that $M(f^n x_0, f^{n+1} x_0, t) \geqslant 1 - k^n$ for all $n \in \mathbb{N}$ and $t > 0$.

Next we show that $(f^n x_0)_n$ is a forward G-Cauchy sequence in $(X, M, *)$.

Fix $p \in \mathbb{N}$. Thus we have:

$$\begin{aligned} M(f^n x_0, f^{n+p} x_0, t) &\geqslant M(f^n x_0, f^{n+1} x_0, t/p) * \overset{(p)}{\cdots} * M(f^{n+p-1} x_0, f^{n+p} x_0, t/p) \\ &\geqslant (1 - k^n) * \overset{(p)}{\cdots} * (1 - k^{n+p-1}) \\ &\geqslant (1 - k^n) * \overset{(p)}{\cdots} * (1 - k^n) \end{aligned}$$

Now given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$(1 - k^{n_0}) * \overset{(p)}{\cdots} * (1 - k^{n_0}) > 1 - \varepsilon$$

So for $n \geqslant n_0$ it follows that

$$M(f^n x_0, f^{n+p} x_0, t) \geqslant (1 - k^n) * \overset{(p)}{\cdots} * (1 - k^n) \geqslant (1 - k^{n_0}) * \overset{(p)}{\cdots} * (1 - k^{n_0}) > 1 - \varepsilon$$

Consequently $(f^n x_0)_n$ is a forward G-Cauchy sequence in $(X, M, *)$. Then, there is $y \in X$ such that $(f^n x_0)_n$ converges to y with respect to $\tau_{M^{-1}}$.

Finally, if $(X, \tau_{M^{-1}})$ is a Hausdorff topological space and f is continuous from $(X, \tau_{M^{-1}})$ into itself, then $(f^n x_0)_n$ converges to $f(y)$ with respect to $\tau_{M^{-1}}$, and by Hausdorffness, $y = f(y)$. The proof is complete. □

Corollary 3.10. *Let* $(X, M, *)$ *be a G-complete fuzzy quasi-metric space such that* $(X, \tau_{M^{-1}})$ *is a Hausdorff topological space. If* f *is a self-mapping on* X *such that there is* $k \in (0, 1)$ *satisfying*

$$M(fx, fy, t) \geqslant 1 - k + kM(x, y, t)$$

for all $x, y \in X$ *and* $t > 0$, *then* f *has a unique fixed point.*

Corollary 3.11. *Let* $(X, M, *)$ *be a G-complete fuzzy metric space. If* f *is a self-mapping on* X *such that there is* $k \in (0, 1)$ *satisfying*

$$M(fx, fy, t) \geqslant 1 - k + kM(x, y, t)$$

for all $x, y \in X$ *and* $t > 0$, *then* f *has a unique fixed point.*

Lemma 3.12. *Each G-Cauchy sequence in a fuzzy ultraquasi-metric space is a Cauchy sequence.*

Proof. Let $(x_n)_n$ be a G-Cauchy sequence in the fuzzy ultraquasi-metric space $(X, M, *)$. Fix $\varepsilon \in (0, 1)$ and $t > 0$. Since $\lim_{n \to \infty} M(x_n, x_{n+1}, t) = 1$, there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_{n+1}, t) > 1 - \varepsilon$ for all $n \geqslant n_0$.

Now let $m, n \geqslant n_0$ with $m > n$. Then $m = n + j$, for some $j \in \mathbb{N}$. So

$$\begin{aligned} M(x_n, x_m, t) &= M(x_n, x_{n+j}, t) \geq \min\{M(x_n, x_{n+1}, t), ..., M(x_{n+j-1}, x_{n+j}, t)\} \\ &> 1 - \varepsilon. \end{aligned}$$

We conclude that $(x_n)_n$ is a Cauchy sequence in $(X, M, *)$. □

Theorem 3.13. *Let (X, e) be a complete $[0, 1]$-fuzzy poset such that $(X, \tau_{e^{-1}})$ is a Hausdorff topological space. If f is a self-mapping on X such that there is $k \in (0, 1)$ satisfying*

$$e(fx, fy) \geqslant 1 - k + ke(x, y)$$

for all $x, y \in X$, then f has a unique fixed point.

Proof. We show that (X, e) is G-complete. Indeed, let $(x_n)_n$ be a G-Cauchy sequence in the fuzzy ultraquasi-metric space $(X, e, \wedge)$. By Lemma 3.12, $(x_n)_n$ is a Cauchy sequence. Since (X, e) is complete, $(x_n)_n$ converges with respect to $\tau_{e^{-1}}$. Therefore (X, e) is G-complete. Theorem 3.9 concludes the proof. □

We conclude this section with an example for which we can apply Theorem 3.3 but not Theorem 3.9, and an example for which we can apply Corollary 3.11, and hence Theorem 3.9, but not Theorem 3.1.

Example 3.14. Let $(\mathbb{R}, M_{1,d_S}, *_L)$ be the complete fuzzy quasi-metric space of Example 3.8. Let $f : \mathbb{R} \to \mathbb{R}$ given by $fx = x^2$ if $x > 0$ and $fx = -x^2$ if $x \leq 0$. It is easy to check that f is continuous from $(\mathbb{R}, \tau_{(M_{1,d_S})^{-1}})$ into itself and it is obvious that the contraction condition of Theorem 3.3 follows for $(f^n x_0)_n$ where $x_0 \in \{-1, 0, 1\}$ and $k \in (0, 1)$ (in fact, $-1, 0$ and 1 are the fixed points of f). So conditions of Theorem 3.3 are satisfied. However, can not be applied Theorem 3.9 because $(\mathbb{R}, M_{1,d_S}, *_L)$ is not G-complete.

Example 3.15. Let $X = \{a, b, c\}$ and let M be the fuzzy set on $X \times X \times [0, \infty)$ defined as $M(x, y, 0) = 0$ for all $x, y \in X$, $M(x, x, t) = 1$ for all $x \in X$ and $t > 0$, $M(a, b, t) = M(b, a, t) = 1/2$ for all $t \in (0, 1]$, $M(a, c, t) = M(c, a, t) = M(b, c, t) = M(c, b, t) = 0$ for all $t \in (0, 1]$, and $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 1$. It is immediate to see that $(X, M, *)$ is a G-complete fuzzy metric space for every continuous t-norm $*$. Now let $f : X \to X$ given by $fa = fb = a$ and $fc = b$. Then f satisfies the contraction condition of Corollary 3.11, and hence the ones of Theorem 3.9, for $k = 1/2$. However, f does not satisfy the contraction condition of Theorem 3.1, because given $k \in (0, 1)$ we can choose $\varepsilon > 0$ such that $k + \varepsilon < 1$, and then for $t = 1/(k + \varepsilon)$ we have $t > 1$ and $kt < 1$, so that

$$M(fa, fc, kt) = M(a, b, kt) = 1/2 < 1 = M(a, c, t).$$

4. Application to recurrence equations

We conclude the paper by applying the results obtained in Section 3 to show, in a direct and easy way, the existence and uniqueness of solution for the following general recurrence equation

$$T(n) = p(n) + \sum_{k=1}^{n-1} q(n, k)T(k) \tag{1}$$

for $n \geqslant 2$, where $T(1) \geqslant 0$, $p(n) > 0$, and $q(n, k) > 0$.

Equations of type (1) appear when discussing the analysis of Probabilistic Divide and Conquer Algorithms by means of recurrences (see, for instance, Section 4 of [3]):

Denote by Σ^∞ the set of all (nonempty) finite and infinite sequences of nonnegative real numbers. If w is a finite sequence, with $w := w_1, w_{2,...,}w_n$, we write $\ell(w) = n$, and we say that the length of w is n. If w is an infinite sequence we write $\ell(w) = \infty$.

Given $v, w \in \Sigma^\infty$, we say that v is a prefix of w, and we write $v \sqsubseteq w$, if $\ell(v) \leq \ell(w)$ and $v_k = w_k$ for all $k \in \{1, ..., \ell(v)\}$.

Now define a fuzzy set M in $\Sigma^\infty \times \Sigma^\infty$ by $M(v, w, t) = 1$ if $v = w$ and $t > 0$; $M(v, w, t) = 1 - 2^{-\ell(v)}$ if $v \sqsubseteq w$ and $t > 0$; and $M(v, w, t) = 0$, otherwise.

It is routine to show that $(M, \wedge)$ is a G-complete stationary fuzzy ultraquasi-metric on Σ^∞ with $(X, \tau_{M^{-1}})$ a Hausdorff topological space.

Let T be a recurrence equation of type (1). We associate to T the functional $\Phi_T : \Sigma^\infty \to \Sigma^\infty$ given by $(\Phi_T w)_1 = T(1)$ and

$$(\Phi_T w)_n = p(n) + \sum_{k=1}^{n-1} q(n, k) w_k$$

for all $n \geqslant 2$.

It is clear by the construction that if $\ell(w) = n$ then $\ell(\Phi_T w) = n + 1$ (in particular, $\ell(\Phi_T w) = \infty$ whenever $\ell(w) = \infty$).

Moreover, we have that $\Phi_T v \sqsubseteq \Phi_T w$ whenever $v \sqsubseteq w$. Hence, for $v \sqsubseteq w$ with $\ell(w) = \infty$ we deduce

$$M(\Phi_T v, \Phi_T w, t) = 1 - 2^{-\ell(\Phi_T v)} \geqslant 1 - 2^{-\ell(v)} = M(v, w, t),$$

which implies that Φ_T is continuous with respect to $\tau_{M^{-1}}$.

Now let w be the element of Σ^∞ given by $w := T(1)$. Then $\ell(w) = 1$. Since $w \sqsubseteq \Phi_T w$, it follows that $\Phi_T^n w \sqsubseteq \Phi_T^{n+1} w$ for each $n \in \mathbb{N}$, so

$$M(\Phi_T^n w, \Phi_T^{n+1} w, t) = 1 - 2^{-\ell(\Phi_T^n w)} = 2^{-1} M(\Phi_T^{n-1} w, \Phi_T^n w, t) + 2^{-1},$$

for all $n \in \mathbb{N}$. It follows from Theorem 3.3 (or Theorem 3.9), for $k = 1/2$, that Φ_T has a fixed point which is clearly unique by the construction of Φ_T. Hence it is the unique solution of the recurrence T.

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