Interpolation subspaces of L^1 of a vector measure and norm inequalities for the integration operator

J.M. Calabuig, J. Rodríguez, and E.A. Sánchez-Pérez

Abstract. Let m be a Banach space valued measure. We study some domination properties of the integration operator that are equivalent to the existence of Banach ideals of $L^1(m)$ that are interpolation spaces. These domination properties are closely connected with some interpolated versions of summing operators, like (p,θ)-absolutely continuous operators.

1. Introduction

Let (Ω, Σ) be a measurable space, X a Banach space and $m : \Sigma \to X$ a vector measure. For $1 \leq p < \infty$, let $L^p(m)$ be the Banach lattice of all p-integrable functions with respect to m. The domination properties (i.e. vector norm inequalities) of the integration operator $I : L^1(m) \to X$, $f \mapsto \int_\Omega f\, dm$, are directly related to the structure of $L^1(m)$ and determine the existence of some characteristic subspaces. From this point of view, the existence of Lebesgue subspaces of $L^1(m)$ has recently been studied in [**2**] (cf. [**10**, Section 3.4 and Chapter 6]): geometric or summability properties of I (namely, p-concavity on $L^p(m)$ or positive p-summability on $L^1(m)$) are shown to characterize either the inclusions $L^p(m) \hookrightarrow L^p(\nu) \hookrightarrow L^1(m)$ or the order isomorphism $L^1(m) \simeq L^1(\nu)$, for some control measure ν of m.

The aim of this paper is to continue this research by showing which vector norm inequalities for I characterize the inclusion of some special *Calderón-Lozanovskii lattice interpolation spaces* in $L^1(m)$. Our results can be applied to analyze the inclusion of such subspaces in a broad class of Banach lattices by means of the

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well-known representation technique via vector measures (cf. [**10**, Chapter 3]). In particular, we center our attention in the following problem (left open in [**2**, p.31]): find a domination property of I which is equivalent to the existence of a control measure ν of m and $0 \leq \theta < 1$ such that

$$(L^p(\nu), L^p(m))_\theta \hookrightarrow L^1(m),$$

where $(L^p(\nu), L^p(m))_\theta$ is the Calderón-Lozanovskii lattice interpolation space of $L^p(m)$ and $L^p(\nu)$. We will show that the requested domination property of I is a concavity-type property which we call (p,θ)-*concavity* (Theorem 2.3). At the end of the paper we analyze some summability properties related to (p,θ)-concavity, like the largely studied (p,θ)-*absolute continuity* (see [**7, 9**] and the references therein). Along this line, in Theorem 2.8 we prove that the positive (p,θ)-absolute continuity of I has the same structural consequences on $L^1(m)$ than its non-interpolated version (i.e. positive p-summability), namely: $L^1(m)$ is order isomorphic to the L^1 space of a non-negative scalar measure.

Terminology. Unexplained terminology can be found in our standard references [**3, 4, 6**]. All our linear spaces are real. Given a Banach space Z, the symbol Z' stands for the topological dual of Z and the duality is denoted by $\langle \cdot, \cdot \rangle$. We write B_Z to denote the closed unit ball of Z. The norm of Z is denoted by $\|\cdot\|_Z$ if needed explicitly. A Banach space E is called *Banach function space* over a finite measure space (Ω, Σ, μ) if E is a linear subspace of $L^0(\mu)$ such that: (i) if $f \in L^0(\mu)$ and $|f| \leq |g|$ μ-a.e. for some $g \in E$, then $f \in E$ and $\|f\|_E \leq \|g\|_E$; (ii) the characteristic function χ_A of each $A \in \Sigma$ belongs to E. Then E is a Banach lattice when endowed with the μ-a.e. order. We write B_E^+ to denote the intersection of B_E with the positive cone E^+ of E.

Let E and F be two Banach function spaces over a finite measure space (Ω, Σ, μ). Given $0 \leq \theta \leq 1$, the *Calderón-Lozanovskii lattice interpolation space* $(E,F)_\theta$ is the Banach function space over (Ω, Σ, μ) made up of all $h \in L^0(\mu)$ for which there are $e \in E$ and $f \in F$ such that $|h| = |e|^{1-\theta}|f|^\theta$, endowed with the norm

$$\|h\|_{(E,F)_\theta} = \inf\left\{\|e\|_E^{1-\theta}\|f\|_F^\theta : \ |h| = |e|^{1-\theta}|f|^\theta, \, e \in E, \, f \in F\right\}.$$

We write $F \hookrightarrow E$ if the 'identity' mapping is a well-defined operator (i.e. linear continuous map) from F to E. In this case, we have $F \hookrightarrow (E,F)_\theta \hookrightarrow E$. The space $(E,F)_\theta$ is sometimes denoted by $E^{1-\theta}F^\theta$ and coincides with the complex interpolation space $[F,E]_{1-\theta}$ under mild assumptions on E and F. For detailed information on Calderón-Lozanovskii spaces, we refer the reader to [**1**] and [**8**].

Throughout the paper (Ω, Σ) is a measurable space, X is a Banach space and $m : \Sigma \to X$ is a (countably additive) vector measure. A *control measure* of m is a non-negative scalar measure ν on (Ω, Σ) such that $\nu(A) = 0$ if and only if $\|m\|(A) = 0$, where $\|\cdot\|$ stands for the semivariation of m. We fix a *Rybakov* *control measure* μ of m, that is, a control measure of the form $\mu = |\langle m, x_0'\rangle|$ with $x_0' \in B_{X'}$, cf. [**4**, p. 268]. For each $x' \in X'$, we write $\langle m, x'\rangle$ to denote the scalar measure defined by $\langle m, x'\rangle(A) := \langle m(A), x'\rangle$ for all $A \in \Sigma$. A Σ-measurable function $f : \Omega \to \mathbb{R}$ is *m-integrable* if it is integrable with respect to $\langle m, x'\rangle$ for every $x' \in X'$ and, for each $A \in \Sigma$, there exists a vector $\int_A f\, dm \in X$ such that $\langle \int_A f\, dm, x'\rangle = \int_A f\, d\langle m, x'\rangle$ for all $x' \in X'$. Given $1 \le p < \infty$, the space $L^p(m)$ is the Banach function space over (Ω, Σ, μ) made up of all (equivalence classes of) functions f such that $|f|^p$ is m-integrable, endowed with the norm

$$\|f\|_{L^p(m)} := \sup_{x' \in B_{X'}} \left(\int_\Omega |f|^p d|\langle m, x'\rangle| \right)^{\frac{1}{p}}.$$

For the basic properties of this space, we refer the reader to [**5**] and [**10**, Chapter 3]. The mapping $I : L^1(m) \to X$ given by $I(f) := \int_\Omega f\, dm$ is an operator which is usually called *integration operator*.

We recall that each functional $\varphi \in L^1(m)'$ can be represented as $\varphi(f) = \int_\Omega fh\, d\mu$ for some $h \in L^1(m)^\times$. The Köthe dual $L^1(m)^\times$ of $L^1(m)$ is the Banach function space over (Ω, Σ, μ) made up of all $h \in L^0(\mu)$ such that $fh \in L^1(\mu)$ for every $f \in L^1(m)$. Given $h \in L^1(m)^\times$, if the scalar measure $h\, d\mu$ on (Ω, Σ) defined by $A \mapsto \int_A h\, d\mu$ is a control measure of m, then $L^1(h\, d\mu)$ is a Banach function space over (Ω, Σ, μ) and we have $L^1(m) \hookrightarrow L^1(h\, d\mu)$.

2. (p, θ)-concave integration operators

Definition 2.1. *Let E be a Banach function space over (Ω, Σ, μ) and let Y be a Banach space. We say that an operator $T : E \to Y$ is (p, θ)-concave (where $1 \le p < \infty$ and $0 \le \theta < 1$) if there is a constant $K > 0$ such that*

$$\left(\sum_{i=1}^n \big\| T(h_i) \big\|_Y^{\frac{p}{1-\theta}} \right)^{\frac{1}{p}} \le K \left\| \left(\sum_{i=1}^n |f_i|^p \|g_i\|^{\frac{\theta p}{1-\theta}} \right)^{\frac{1}{p}} \right\|_E$$

whenever $h_i, f_i, g_i \in E$ satisfy $|h_i| = |f_i|^{1-\theta} |g_i|^\theta$ for every $i = 1, 2, \ldots, n$.

Notice that $(p, 0)$-concavity is just the usual notion of p-concavity.

Remark 2.2. *Every (p, θ)-concave operator is p_θ-concave in the sense of* [**11**]. *We stress that an operator $T : E \to Y$ is p_θ-concave if and only if it factorizes through a specific real interpolation space, see* [**11**, Theorem 3.7].

Theorem 2.3. *Let $1 \le p < \infty$ and $0 \le \theta < 1$. The following statements are equivalent:*

(a) *The integration operator $I : L^p(m) \to X$ is (p, θ)-concave.*
(b) *There exist $C > 0$ and $h_0 \in B^+_{L^1(m)'}$ such that*

$$\left\| \int_\Omega v\, dm \right\|_X \le C \left(\int_\Omega |f|^p h_0\, d\mu \right)^{\frac{1-\theta}{p}} \|g\|^\theta_{L^p(m)}$$

whenever $v, f, g \in L^p(m)$ *satisfy* $|v| = |f|^{1-\theta}|g|^\theta$.

(c) *There is* $h_0 \in B^+_{L^1(m)'}$ *such that* $h_0\, d\mu$ *is a control measure of* m *and* $(L^p(h_0\, d\mu), L^p(m))_\theta \hookrightarrow L^1(m)$.

(d) *There is a control measure* ν *of* m *such that*

$$L^1(m) \hookrightarrow L^1(\nu) \quad \text{and} \quad (L^p(\nu), L^p(m))_\theta \hookrightarrow L^1(m).$$

Proof. (a)$\Rightarrow$(b). Let $K > 0$ be a constant like in Definition 2.1 applied to the integration operator $I : L^p(m) \to X$.

Given finitely many $v_i, f_i, g_i \in L^p(m)$, $i = 1, \ldots, n$, such that $|v_i| = |f_i|^{1-\theta}|g_i|^\theta$, let us consider the function $\Phi : B^+_{L^1(m)'} \to \mathbb{R}$ defined by

$$\Phi(h) := \sum_{i=1}^n \left\| \int_\Omega v_i\, dm \right\|^{\frac{p}{1-\theta}} - K^p \int_\Omega \left(\sum_{i=1}^n |f_i|^p \|g_i\|^{\frac{\theta p}{1-\theta}}_{L^p(m)} \right) h\, d\mu.$$

Clearly Φ is w^*-continuous on the w^*-compact set $B^+_{L^1(m)'}$, so it attains its infimum at some $h_\Phi \in B^+_{L^1(m)'}$. We claim that $\Phi(h_\Phi) \leq 0$. Indeed, for each $h \in B^+_{L^1(m)'}$, the inequality $\Phi(h_\Phi) \leq \Phi(h)$ implies

$$\int_\Omega \left(\sum_{i=1}^n |f_i|^p \|g_i\|^{\frac{\theta p}{1-\theta}}_{L^p(m)} \right) h\, d\mu \leq \int_\Omega \left(\sum_{i=1}^n |f_i|^p \|g_i\|^{\frac{\theta p}{1-\theta}}_{L^p(m)} \right) h_\Phi\, d\mu.$$

Therefore

$$\left\| \left(\sum_{i=1}^n |f_i|^p \|g_i\|^{\frac{\theta p}{1-\theta}}_{L^p(m)} \right)^{\frac{1}{p}} \right\|^p_{L^p(m)} = \left\| \sum_{i=1}^n |f_i|^p \|g_i\|^{\frac{\theta p}{1-\theta}}_{L^p(m)} \right\|_{L^1(m)} = \tag{2.1}$$

$$= \sup_{h \in B^+_{L^1(m)'}} \int_\Omega \left(\sum_{i=1}^n |f_i|^p \|g_i\|^{\frac{\theta p}{1-\theta}}_{L^p(m)} \right) h\, d\mu \leq \int_\Omega \left(\sum_{i=1}^n |f_i|^p \|g_i\|^{\frac{\theta p}{1-\theta}}_{L^p(m)} \right) h_\Phi\, d\mu.$$

On the other hand, since $I : L^p(m) \to X$ is (p, θ)-concave, we have

$$\left(\sum_{i=1}^n \left\| \int_\Omega v_i\, dm \right\|^{\frac{p}{1-\theta}}_X \right)^{\frac{1}{p}} \leq K \left\| \left(\sum_{i=1}^n |f_i|^p \|g_i\|^{\frac{\theta p}{1-\theta}} \right)^{\frac{1}{p}} \right\|_{L^p(m)},$$

which combined with (2.1) yields

$$\sum_{i=1}^n \left\| \int_\Omega v_i\, dm \right\|^{\frac{p}{1-\theta}}_X \leq K^p \int_\Omega \left(\sum_{i=1}^n |f_i|^p \|g_i\|^{\frac{\theta p}{1-\theta}}_{L^p(m)} \right) h_\Phi\, d\mu,$$

and so $\Phi(h_\Phi) \leq 0$, as claimed. Notice also that Φ is convex (in fact, it is affine).

It is easy to check that the collection of all Φ's as above is a convex cone in $\mathbb{R}^{B^+_{L^1(m)'}}$. An appeal to Ky Fan's Lemma (cf. [**3**, Lemma 9.10]) ensures the existence of $h_0 \in B^+_{L^1(m)'}$ such that $\Phi(h_0) \leq 0$ for every function Φ as above. In particular, if $v, f, g \in L^p(m)$ satisfy $|v| = |f|^{1-\theta}|g|^\theta$, then

$$\left\| \int_\Omega v\, dm \right\|^{\frac{p}{1-\theta}}_X \leq K^p \left(\int_\Omega |f|^p h_0\, d\mu \right) \|g\|^{\frac{\theta p}{1-\theta}}_{L^p(m)}$$

and taking $C := K^{1-\theta}$ we have

$$\left\| \int_\Omega v\, dm \right\|_X \leq C \left(\int_\Omega |f|^p h_0\, d\mu \right)^{\frac{1-\theta}{p}} \|g\|^\theta_{L^p(m)}.$$

This completes the proof of (a)⇒(b).

(b)⇒(c). Since $L^p(m) \hookrightarrow L^p(h_0\, d\mu)$, we have

$$L^p(m) \hookrightarrow (L^p(h_0\, d\mu), L^p(m))_\theta \hookrightarrow L^p(h_0\, d\mu).$$

We divide the proof of (b)⇒(c) into several steps.

STEP 1.- Condition (b) yields

$$\|m(B)\|_X \; \leq \; C \left(\int_B h_0\, d\mu \right)^{\frac{1-\theta}{p}} \|\chi_\Omega\|^\theta_{L^p(m)} \; \leq \; C \left(\int_A h_0\, d\mu \right)^{\frac{1-\theta}{p}} \|\chi_\Omega\|^\theta_{L^p(m)}$$

for every $B \subset A$ in Σ. Hence $h_0\, d\mu$ is a control measure of m.

STEP 2.- Fix an arbitrary simple function v. We claim that

$$\|v\|_{L^1(m)} \leq C \|v\|_{(L^p(h_0\, d\mu), L^p(m))_\theta}. \tag{2.2}$$

Let $f \in L^p(h_0\, d\mu)$ and $g \in L^p(m)$ such that $|v| = |f|^{1-\theta}|g|^\theta$. Choose sequences (f_n) and (g_n) of simple functions such that $|f_n| \nearrow |f|$ and $|g_n| \nearrow |g|$ μ-a.e. Define $v_n := |f_n|^{1-\theta}|g_n|^\theta$ for every $n \in \mathbb{N}$, so that $v_n \nearrow |v|$ μ-a.e. We next show that

$$\|v_n\|_{L^1(m)} \leq C \|f_n\|^{1-\theta}_{L^p(h_0\, d\mu)} \|g_n\|^\theta_{L^p(m)} \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

To this end, take any $\xi \in L^\infty(\mu)$. Since the functions $v_n\xi, f_n\xi, g_n\xi \in L^p(m)$ satisfy $|v_n\xi| = |f_n\xi|^{1-\theta}|g_n\xi|^\theta$, condition (b) yields

$$\left\| \int_\Omega v_n \xi\, dm \right\|_X \leq C \left(\int_\Omega |f_n\xi|^p h_0\, d\mu \right)^{\frac{1-\theta}{p}} \|g_n\xi\|^\theta_{L^p(m)} \leq$$

$$\leq C \left(\int_\Omega |f_n|^p h_0\, d\mu \right)^{\frac{1-\theta}{p}} \|g_n\|^\theta_{L^p(m)} = C \|f_n\|^{1-\theta}_{L^p(h_0\, d\mu)} \|g_n\|^\theta_{L^p(m)}.$$

Bearing in mind that

$$\|v_n\|_{L^1(m)} = \sup_{\xi \in B_{L^\infty(\mu)}} \left\| \int_\Omega v_n \xi\, dm \right\|_X,$$

cf. [**10**, (3.64)], inequality (2.3) follows at once. Now, since

$$\|v_n\|_{L^1(m)} \to \|v\|_{L^1(m)}, \quad \|f_n\|_{L^p(h_0\, d\mu)} \to \|f\|_{L^p(h_0\, d\mu)}, \quad \|g_n\|_{L^p(m)} \to \|g\|_{L^p(m)},$$

we can take limits in (2.3) to infer that $\|v\|_{L^1(m)} \leq C \|f\|^{1-\theta}_{L^p(h_0\, d\mu)} \|g\|^\theta_{L^p(m)}$. As $f \in L^p(h_0\, d\mu)$ and $g \in L^p(m)$ are arbitrary functions satisfying $|v| = |f|^{1-\theta}|g|^\theta$, inequality (2.2) holds true.

STEP 3.- The space $(L^p(h_0\, d\mu), L^p(m))_\theta$ is order continuous, cf. [**8**, Lemma 20], and so the subspace S made up of all simple functions is dense in $(L^p(h_0\, d\mu), L^p(m))_\theta$. Fix $v \in (L^p(h_0\, d\mu), L^p(m))_\theta$ and let (v_n) be a sequence in S such that

$$\|v_n - v\|_{(L^p(h_0\, d\mu), L^p(m))_\theta} \to 0.$$

Then $\|v_n - v\|_{L^p(h_0\,d\mu)} \to 0$ and so, by passing to a further subsequence, we can assume without loss of generality that $v_n \to v$ μ-a.e. (by Step 1, $h_0\,d\mu$ has the same null sets as m). On the other hand, by Step 2, the 'identity' mapping $S \to L^1(m)$ is continuous (with norm less than or equal to C). Thus, (v_n) is a Cauchy sequence in $L^1(m)$ and so there is $w \in L^1(m)$ such that $\|v_n - w\|_{L^1(m)} \to 0$ and, in particular, $\|v_n - w\|_{L^1(h_0 d\mu)} \to 0$. Hence $v = w \in L^1(m)$ and $\|v_n - v\|_{L^1(m)} \to 0$. Moreover, we have $\|v\|_{L^1(m)} \le C\|v\|_{(L^p(h_0\,d\mu), L^p(m))_\theta}$. This shows that

$$(L^p(h_0\,d\mu), L^p(m))_\theta \hookrightarrow L^1(m)$$

and the proof of (b)⇒(c) is finished.

(c)⇒(d) is obvious.

(d)⇒(c). Observe that if ν is a control measure of m such that $L^1(m) \hookrightarrow L^1(\nu)$, then the positive linear mapping $f \mapsto \int_\Omega f\,d\nu$ is continuous on $L^1(m)$ and so there is $0 < h \in L^1(m)'$ such that $\int_\Omega f\,d\nu = \int_\Omega fh\,d\mu$ for all $f \in L^1(m)$, hence $\nu = h\,d\mu$. Finally just consider $h_0 = h/\|h\|_{L^1(m)'} \in B^+_{L^1(m)'}$ in order to obtain the result since $h_0 d\mu$ is a control measure of m and $L^p(h_0 d\mu) = L^p(h d\mu) = L^p(\nu)$.

(c)⇒(a). Let $K > 0$ be a constant such that $\|v\|_{L^1(m)} \le K\|v\|_{(L^p(h_0\,d\mu), L^p(m))_\theta}$ for every $v \in (L^p(h_0\,d\mu), L^p(m))_\theta$. Take finitely many functions $v_i, f_i, g_i \in L^p(m)$, $i = 1, \ldots, n$, satisfying $|v_i| = |f_i|^{1-\theta}|g_i|^\theta$. Then each $v_i \in (L^p(h_0\,d\mu), L^p(m))_\theta$ and

$$\begin{aligned}
\sum_{i=1}^n \left\| \int_\Omega v_i\,dm \right\|^{\frac{p}{1-\theta}} &\le \sum_{i=1}^n \|v_i\|_{L^1(m)}^{\frac{p}{1-\theta}} \le K^{\frac{p}{1-\theta}} \sum_{i=1}^n \|v_i\|_{(L^p(h_0\,d\mu), L^p(m))_\theta}^{\frac{p}{1-\theta}} \le \\
&\le K^{\frac{p}{1-\theta}} \sum_{i=1}^n \left(\|f_i\|_{L^p(h_0\,d\mu)}^{1-\theta} \|g_i\|_{L^p(m)}^\theta \right)^{\frac{p}{1-\theta}} = K^{\frac{p}{1-\theta}} \sum_{i=1}^n \|f_i\|_{L^p(h_0\,d\mu)}^p \|g_i\|_{L^p(m)}^{\frac{\theta p}{1-\theta}} = \\
&= K^{\frac{p}{1-\theta}} \sum_{i=1}^n \left(\int_\Omega |f_i|^p h_0\,d\mu \right) \|g_i\|_{L^p(m)}^{\frac{\theta p}{1-\theta}} = K^{\frac{p}{1-\theta}} \int_\Omega \sum_{i=1}^n |f_i|^p \|g_i\|_{L^p(m)}^{\frac{\theta p}{1-\theta}} h_0\,d\mu \le \\
&\le K^{\frac{p}{1-\theta}} \left\| \sum_{i=1}^n |f_i|^p \|g_i\|_{L^p(m)}^{\frac{\theta p}{1-\theta}} \right\|_{L^1(m)} = K^{\frac{p}{1-\theta}} \left\| \left(\sum_{i=1}^n |f_i|^p \|g_i\|_{L^p(m)}^{\frac{\theta p}{1-\theta}} \right)^{\frac{1}{p}} \right\|_{L^p(m)}^p .
\end{aligned}$$

Therefore, the integration operator $I : L^p(m) \to X$ is (p,θ)-concave. □

REMARK 2.4. *Our previous theorem generalizes* [**2**, Theorem 2.3]*, where we proved that $I : L^p(m) \to X$ is p-concave if and only if there is a control measure ν of m such that $L^p(m) \hookrightarrow L^p(\nu) \hookrightarrow L^1(m)$. In this case, for each $0 \le \theta < 1$ we have*

$$L^p(m) \hookrightarrow (L^p(\nu), L^p(m))_\theta \hookrightarrow L^p(\nu) \hookrightarrow L^1(m).$$

However, there are cases where $L^p(\nu) \not\hookrightarrow L^1(m)$ and $(L^p(\nu), L^p(m))_\theta \hookrightarrow L^1(m)$ for some Rybakov control measure ν of m, as in the following example.

EXAMPLE 2.5. *Let $\Omega := [0,1]$ with the Lebesgue σ-algebra Σ and consider the vector measure $m : \Sigma \to L^2[0,1]$ given by $m(A) := \chi_A$. Then the Lebesgue* *measure λ is a Rybakov control measure of m and the 'identity' mapping is an isometric isomorphism between $L^1(m)$ and $L^2[0,1]$. Then:*

(i) $(L^{3/2}[0,1], L^{3/2}(m))_{1/2} \hookrightarrow L^1(m)$.

(ii) $L^{3/2}(\nu) \not\hookrightarrow L^1(m)$ *for any Rybakov control measure ν of m.*

Proof. (i) Fix $v \in (L^{3/2}[0,1], L^{3/2}(m))_{1/2}$ arbitrary. Take functions $f \in L^{3/2}[0,1]$ and $g \in L^{3/2}(m) = L^3[0,1]$ satisfying $|v| = |f|^{1/2}|g|^{1/2}$. Hölder's inequality yields

$$\int_\Omega |v|^2\,d\lambda = \int_\Omega |f||g|\,d\lambda \le$$
$$\le \left(\int_\Omega |f|^{3/2}\,d\lambda\right)^{\frac{2}{3}} \left(\int_\Omega |g|^3\,d\lambda\right)^{\frac{1}{3}} = \|f\|_{L^{3/2}[0,1]}\|g\|_{L^{3/2}(m)},$$

hence $v \in L^1(m) = L^2[0,1]$ and $\|v\|_{L^1(m)} \le \|v\|_{(L^{3/2}[0,1],L^{3/2}(m))_{1/2}}$.

(ii) Let ν be any Rybakov control measure ν of m. Then there is $h \in B_{L^2[0,1]}$ such that $\nu = |\langle m, h\rangle|$. Notice that $\langle m, h\rangle(A) = \langle m(A), h\rangle = \int_A h\,d\lambda$ for all $A \in \Sigma$, so $\nu = |h|\,d\lambda$. Take $A \in \Sigma$ with $\lambda(A) > 0$ such that h is bounded on A, that is, for some $b > 0$ we have $|h(t)| \le b$ for all $t \in A$. The restrictions of λ and ν to the trace σ-algebra $\Sigma_A := \{A \cap E : E \in \Sigma\}$ on A are denoted by λ_A and ν_A, respectively. An easy computation shows that each $f \in L^{3/2}(\lambda_A)$ belongs to $L^{3/2}(\nu_A)$ and $\|f\|_{L^{3/2}(\nu_A)} \le b^{2/3}\|f\|_{L^{3/2}(\lambda_A)}$. Now we argue by contradiction. Suppose that $L^{3/2}(\nu) \hookrightarrow L^1(m)$. Then there is $C > 0$ such that each $f \in L^{3/2}(\lambda_A)$ belongs to $L^2(\lambda_A)$ and $\|f\|_{L^2(\lambda_A)} \le Cb^{2/3}\|f\|_{L^{3/2}(\lambda_A)}$. Hence the 'identity' mapping is an isomorphism between $L^{3/2}(\lambda_A)$ and $L^2(\lambda_A)$, a contradiction. □

Remark 2.6. *Actually the same proof of part* (ii) *gives*

(ii)' $L^{3/2}(\nu) \not\hookrightarrow L^1(m)$ *for every control measure ν of m with $L^1(m) \hookrightarrow L^1(\nu)$. Hence, the integration map $I : L^{3/2}(m) \to X$ is not $3/2$-concave. However I must be $(3/2, 1/2)$-concave (and in fact $(3/2, \theta)$-concave for all $\theta \ge 1/2$).*

The same kind of arguments can provide more examples in the setting of Lorentz spaces $L^{p,q}[0,1]$.

Definition 2.7. *Let $T : Z \to Y$ be an operator between Banach spaces.*

(i) *T is called (p,θ)-absolutely continuous (where $1 \le p < \infty$ and $0 \le \theta < 1$) if there is a constant $K > 0$ such that*

$$\sum_{i=1}^n \big\|T(z_i)\big\|_Y^{\frac{p}{1-\theta}} \le K \sup_{z' \in B_{Z'}} \sum_{i=1}^n \big|\langle z_i, z'\rangle\big|^p \big\|z_i\big\|_Z^{\frac{\theta p}{1-\theta}} \tag{2.4}$$

for every $z_1, \dots, z_n \in Z$, $n \in \mathbb{N}$.

(ii) *If Z is a Banach lattice, then T is called positive (p,θ)-absolutely continuous if there is $K > 0$ such that* (2.4) *holds for every $z_1, \dots, z_n \in Z^+$, $n \in \mathbb{N}$.*

Notice that for $\theta = 0$ the notion of (positive) (p,θ)-absolutely continuous operator coincides with that of (positive) p-summing operator.

The following result is an extension of [**2**, Theorem 2.7].

THEOREM 2.8. *Let $1 \leq p < \infty$ and $0 \leq \theta < 1$. The following statements are equivalent:*

(a) *$I : L^1(m) \to X$ is positive (p,θ)-absolutely continuous.*
(b) *$I : L^1(m) \to X$ is positive $\frac{p}{1-\theta}$-summing.*
(c) *$L^1(m)$ is order isomorphic to the L^1 space of a non-negative scalar measure.*

PROOF. (b)⇔(c) follows from [**2**, Theorem 2.7].

(a)⇒(b). Let $K > 0$ be as in Definition 2.7. Fix $f_1, \dots, f_n \in L^1(m)^+$. For each $r_1, \dots, r_n \in B_{L^\infty(\mu)}$ we have

$$\sum_{i=1}^n \left\| \int_\Omega f_i r_i \, dm \right\|_X^{\frac{p}{1-\theta}} \leq K \sup_{h \in B_{L^1(m)'}} \sum_{i=1}^n \left| \int_\Omega f_i r_i h \, d\mu \right|^p \|f_i r_i\|_{L^1(m)}^{\frac{\theta p}{1-\theta}} \leq$$
$$\leq K \sup_{h \in B_{L^1(m)'}} \sum_{i=1}^n \left(\int_\Omega f_i |h| \, d\mu \right)^p \|f_i\|_{L^1(m)}^{\frac{\theta p}{1-\theta}} \leq$$
$$\overset{(*)}{\leq} K \sup_{h \in B_{L^1(m)'}} \left(\sum_{i=1}^n \left(\int_\Omega f_i |h| \, d\mu \right)^{\frac{p}{1-\theta}} \right)^{1-\theta} \left(\sum_{i=1}^n \|f_i\|_{L^1(m)}^{\frac{p}{1-\theta}} \right)^\theta,$$

where $(*)$ follows from Hölder's inequality. Taking into account that

$$\|f_i\|_{L^1(m)} = \sup_{r \in B_{L^\infty(\mu)}} \left\| \int_\Omega f_i r \, dm \right\|_X,$$

cf. [**10**, (3.64)], we obtain

$$\sum_{i=1}^n \|f_i\|_{L^1(m)}^{\frac{p}{1-\theta}} \leq K \sup_{h \in B_{L^1(m)'}} \left(\sum_{i=1}^n \left(\int_\Omega f_i |h| \, d\mu \right)^{\frac{p}{1-\theta}} \right)^{1-\theta} \left(\sum_{i=1}^n \|f_i\|_{L^1(m)}^{\frac{p}{1-\theta}} \right)^\theta$$

and therefore

$$\sum_{i=1}^n \|f_i\|_{L^1(m)}^{\frac{p}{1-\theta}} \leq C \sup_{h \in B_{L^1(m)'}} \sum_{i=1}^n \left(\int_\Omega f_i |h| \, d\mu \right)^{\frac{p}{1-\theta}},$$

where $C = K^{1/(1-\theta)}$. It follows that

$$\sum_{i=1}^n \left\| \int_\Omega f_i \, dm \right\|^{\frac{p}{1-\theta}} \leq \sum_{i=1}^n \|f_i\|_{L^1(m)}^{\frac{p}{1-\theta}} \leq$$
$$\leq C \sup_{h \in B_{L^1(m)'}} \sum_{i=1}^n \left(\int_\Omega f_i |h| \, d\mu \right)^{\frac{p}{1-\theta}} \leq C \sup_{h \in B_{L^1(m)'}} \sum_{i=1}^n \left| \int_\Omega f_i h \, d\mu \right|^{\frac{p}{1-\theta}}.$$

Consequently, the integration operator is positive $\frac{p}{1-\theta}$-summing.

(b)⇒(a). Just bear in mind that for each $f \in L^1(m)$ and $h \in B^+_{L^1(m)'}$ we have

$$\left|\langle f, h\rangle\right|^{\frac{p}{1-\theta}} = \left|\langle f, h\rangle\right|^p \left|\langle f, h\rangle\right|^{\frac{\theta p}{1-\theta}} \leq \left|\langle f, h\rangle\right|^p \left\|f\right\|^{\frac{\theta p}{1-\theta}}_{L^1(m)}.$$

The proof is over. □

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