

Proceedings of the Workshop on Applied Topological Structures

editors

Jesús Rodríguez-López and Pedro Tirado



WATS'15

Valencia, Spain, September 3-4, 2015

Universitat Politècnica de València

EDITORS

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Editorial Universitat Politècnica de València

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Editors: Jesús Rodríguez-López
Pedro Tirado

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Workshop on Applied Topological Structures WATS'15

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Preface

General Topology has become one of the fundamental parts of mathematics. Nowadays, as a consequence of an intensive research activity, this mathematical branch has been shown to be very useful in modeling several problems which arise in some branches of applied sciences as Economics, Artificial Intelligence and Computer Science. Due to this increasing interaction between applied and topological problems, we have promoted the creation of an annual or biennial workshop to encourage the collaboration between different national and international research groups in the area of General Topology and its Applications. This year it has been given the name of Workshop on Applied Topological Structures (WATS).

This book contains a collection of papers presented by the participants in this workshop which took place in Valencia (Spain) from September 3 to 4, 2015.

All the papers of the book have been strictly refereed.

We would like to thank all participants, the plenary speakers and the regular ones, for their excellent contributions.

We express our gratitude to the Ministerio de Economía y Competitividad, grant MTM2012-37894-C02-01, and Instituto de Matemática Pura y Aplicada for their financial support without which this workshop would not have been possible.

We are certain of all participants have established fruitful scientific relations during the Workshop.

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WATS'15

LECTURES

On fixed point theorems for generalized set valued maps with mw -distances

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ABSTRACT

In this paper the notion of mw -distance on a quasi-metric space is discussed. Some fixed point theorems in the context of quasi-metric spaces using that notion are included.

Key words: *quasi-metric space, complete quasi-metric space, fixed point, set-valued map, generalized contraction, mw -distance.*

1. INTRODUCTION AND PRELIMINARIES

Kada et al. [11] introduced the notion of w -distance on a metric space and improved some classical fixed point theorems by replacing the metric with a w -distance in the contraction conditions. Later, Park [19] extended this notion of

¹The authors acknowledge the support of the Ministry of Economy and Competitiveness of Spain, Grant MTM2012-37894-C02-01.

w -distance to quasi-metric spaces. Using these concepts, Alegre, Marín and Romaguera ([4], [5]) obtained fixed point theorems for generalized contractions on complete metric and quasi-metric spaces which involve w -distances and Meir-Keeler or Jachymski functions.

A metric d on X is a w -distance on the metric space (X, d) . Nevertheless, if d is a quasi-metric on X , d is not necessarily a w -distance on the quasi-metric space (X, d) . Motivated from this fact, we introduced in [2] the notion of mw -distance on a quasi-metric space, slightly modifying the definition of w -distance given by Park. This new notion generalizes the concept of quasi-metric. We also showed that mw -distance and w -distance are two different notions, both in the metric case and quasi-metric case.

By using mw -distances, it has been possible to obtain new fixed point theorems for generalized contractions on quasi-metric spaces [2] and generalizations of well-known fixed points theorems in metric spaces (see [1], [16]). Currently, our purpose is to obtain fixed point theorems for multivalued maps on quasi-metric spaces with mw -distances.

Throughout this paper the letters \mathbb{R} , \mathbb{R}^+ and \mathbb{N} will denote the set of real numbers, the set of non-negative real numbers and the set of positive integer numbers, respectively. Our basic references for quasi-metric spaces and asymmetric normed spaces are [9], [13] and [7].

A quasi-metric on a set X is a function $d : X \times X \rightarrow \mathbb{R}^+$ such that for all $x, y, z \in X$: (i) $d(x, y) = d(y, x) = 0$ if and only if $x = y$; (ii) $d(x, y) \leq d(x, z) + d(z, y)$. If in addition it is fulfilled (iii) $d(x, y) = d(y, x)$ for all $x, y \in X$, d is a metric on X .

If the quasi-metric d satisfies the stronger condition (i) $d(x, y) = 0$ if and only if $x = y$, we say that d is a T_1 quasi-metric on X .

A (T_1) quasi-metric space is a pair (X, d) such that X is a non-empty set and d is a (T_1) quasi-metric on X .

Each quasi-metric d on a set X induces a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x, r) : x \in X, r > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Note that if d is quasi-metric then τ_d is a T_0 topology, and if d is a T_1 quasi-metric then τ_d is a T_1 topology on X .

Given a quasi-metric d on X , the function d^{-1} defined by $d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$, is also a quasi-metric on X , and the function d^s defined by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ for all $x, y \in X$, is a metric on X .

2. mw -DISTANCES IN A QUASI-METRIC SPACE AND EXAMPLES

Definition 1 ([2]). An mw -distance on a quasi-metric space (X, d) is a function $q : X \times X \rightarrow \mathbb{R}^+$ satisfying the following conditions:

(W1) $q(x, y) \leq q(x, z) + q(z, y)$ for all $x, y, z \in X$;

(W2) $q(x, \cdot) : X \rightarrow \mathbb{R}^+$ is lower semicontinuous on $(X, \tau_{d^{-1}})$ for all $x \in X$;

(mW3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $q(x, z) \leq \delta$ and $q(y, x) \leq \delta$ then $d(y, z) \leq \varepsilon$.

Note that every quasi-metric d on X is an mw -distance on (X, d) .

Definition 2 ([2]). A **strong- mw -distance** on a **quasi-metric** space (X, d) is an mw -distance $q : X \times X \rightarrow \mathbb{R}^+$ satisfying the following condition:

(mW2) $q(\cdot, x) : X \rightarrow \mathbb{R}^+$ is lower semicontinuous on $(X, \tau_{d^{-1}})$ for all $x \in X$.

It is easy to prove that every strong- mw -distance on a quasi-metric space (X, d) generates a w -distance on this quasi-metric space.

Proposition 3. *Let (X, d) be a quasi-metric space and let q be a strong- mw -distance on (X, d) . Then, for all $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta > 0$, the function $q_1 : X \times X \rightarrow \mathbb{R}^+$ defined by $q_1(x, y) = \alpha q(x, y) + \beta q(y, x)$ is a w -distance on the quasi-metric space (X, d) .*

We will show now some examples of mw -distances and strong- mw -distances defined on quasi-metric spaces. These examples are included in [2].

Example 1. Let $X = \mathbb{R}$ and let d_S be the quasi-metric on X given by $d_S(x, y) = y - x$ if $x \leq y$, and $d_S(x, y) = 1$ if $x > y$. The quasi-metric d_S induces the Sorgenfrey topology on \mathbb{R} . Since d_S is a quasi-metric, d_S is an mw -distance on

the T_1 quasi-metric space (X, d_S) . Furthermore, d_S is a strong- mw -distance on (X, d_S) .

Example 2. Let $(X, \preceq, \|\cdot\|)$ be a normed lattice. Denote by X^+ the positive cone of X , i.e., $X^+ := \{x \in X : \mathbf{0} \preceq x\}$, and let $\|\cdot\|^+$ be the asymmetric norm on X given by $\|x\|^+ = \|x \vee \mathbf{0}\|$ for all $x \in X$ (see e.g. [8]). The function d defined by $d(x, y) = \|y - x\|^+$ for all $x, y \in X$, is a quasi-metric on X , then (X^+, d_+) is a quasi-metric space, where d_+ denotes the restriction of d to X^+ . The function q defined by $q(x, y) = \|y\|$ for all $x, y \in X^+$, is a strong- mw -distance on (X^+, d_+) .

Example 3. Consider the quasi-metric space (\mathbb{R}, d) where $d(x, y) = (y - x) \vee 0$. Then $q = d$ is an mw -distance but q is not a strong- mw -distance, because the condition (mW2) does not hold.

3. PARTIAL METRICS AND mw -DISTANCES

In [16] it is studied the relation between mw -distances and partial metrics (quasi-metrics). A partial metric (quasi-metric) is a generalization of the notion of metric (quasi-metric) such that the distance of a point from itself is not necessarily zero. The notion of partial metric was introduced by Matthews [18] as a part of the study of programming language semantics. Later on, Künzi [14] extended this notion to nonsymmetric case.

Definition 4 ([18]). A **partial metric** on a set X is a function $p : X \times X \rightarrow [0, \infty)$ satisfying:

- (1.a) $p(x, x) \leq p(x, y)$;
- (1.b) $p(x, x) \leq p(y, x)$;
- (2) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$;
- (3) $x = y \leftrightarrow p(x, x) = p(x, y)$ and $p(y, y) = p(y, x)$;
- (4) $p(x, y) = p(y, x)$.

A **partial metric space** is a pair (X, p) such that X is a set and p is a partial metric on X . The partial metric p induces a T_0 topology τ_p on X which has a base the family of open p -balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < \varepsilon\}$.

Definition 5 ([14]). A **partial quasi-metric** on a set X is a function $p : X \times X \rightarrow [0, \infty)$ that verifies the conditions (1.a), (1.b), (2) and (3) of Definition 4. A **partial quasi-metric space** is a pair (X, p) such that X is a set and p is a partial quasi-metric on X . If p satisfies all these conditions except (1b), the function p is called **lopsided partial quasi-metric**.

In a partial (quasi-)metric space (X, p) the partial (quasi-)metric p induces a quasi-metric d_p on X given by $d_p(x, y) = p(x, y) - p(x, x)$ for all $x, y \in X$, and the topology generated by p is the same that the topology generated by d_p . In addition, the function $p^s : X \times X \rightarrow [0, \infty)$ given by $p^s(x, y) = d_p(x, y) + d_p(y, x) = p(x, y) + p(y, x) - p(x, x) - p(y, y)$ is a metric on X .

The relationship between mw -distances and partial metrics (quasi-metrics) appears in a natural way as shown in the following result.

Proposition 6. (a) *If (X, p) is a partial metric space then p is both an mw -distance and a w -distance on the quasi-metric space (X, d_p) , where $d_p(x, y) = p(x, y) - p(x, x)$.*

(b) *If (X, p) is a partial quasi-metric space then p is a mw -distance on the quasi-metric space (X, d_p) , where $d_p(x, y) = p(x, y) - p(x, x)$. But p is not necessarily a w -distance on (X, d_p) .*

4. RESULTS

There are several notions of Cauchy sequence and of complete quasi-metric space in the literature (see e.g. [13]). In this paper we shall use the following general notions.

A sequence $(x_n)_{n \in \mathbb{N}}$ in a quasi-metric space (X, d) is said to be Cauchy if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ whenever $n_0 \leq n \leq m$. A quasi-metric space (X, d) is called complete if every Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ in the metric space (X, d) converges with respect to the topology $\tau_{d^{-1}}$ (i.e., there exists $z \in X$ such that $d(x_n, z) \rightarrow 0$).

Recently, we have obtained a fixed point theorem for generalized contractions with respect to mw -distances on complete quasi-metric spaces. Our approach uses a kind of functions considered by Jachymski in [10].

Theorem 7 (Theorem 2 of [2]). *Let f be a self-map of a complete quasi-metric space (X, d) . If there exist a strong- mw -distance q on (X, d) and a Jachymski function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\phi(t) < t$ for all $t > 0$, and*

$$q(fx, fy) \leq \phi(q(x, y)),$$

for all $x, y \in X$, then f has a unique fixed point $z \in X$. Moreover $q(z, z) = 0$.

On the other hand, we have proved in [3] a quasi-metric version of Caristi's fixed point theorem [6] by using mw -distances. Our result generalizes a recent result obtained by Karapinar and Romaguera in [12].

Theorem 8 (Theorem 1 of [3]). *Let T be a self mapping of a complete quasi-metric space (X, d) and let q be an mw -distance on (X, d) . If there exists a proper bounded below and nearly lower semicontinuous function for $\tau_{d^{-1}}$, $\varphi : X \rightarrow \mathbb{R} \cup \{\infty\}$ such that for all $x \in X$:*

$$q(x, Tx) + \varphi(Tx) \leq \varphi(x)$$

then there exists $z \in X$ such that $\varphi(Tz) = \varphi(z)$ and $q(z, Tz) = 0$.

As we mentioned before, our next aim is to obtain fixed point theorems for multivalued maps on quasi-metric spaces with mw -distances. Latif and Al-Mezel [15] extended Mizoguchi-Takahashi's theorem to complete T_1 quasi-metric spaces by using w -distances. Later on, Marín, Romaguera and Tirado [17] generalized this result for multivalued maps.

Theorem 9 (Theorem 1 of [17]). *Let (X, \preceq, d) be a complete preordered quasi-metric space and $T : X \rightarrow C_d(X)$ be a generalized w_{\preceq} -contractive set-valued map. Then T has a fixed point.*

At present, we are trying to prove a result similar to this one by replacing the w -distance with an mw -distance.

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An application of Grabiec's theorem to the Baire fuzzy quasi-metric space

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ABSTRACT

We present a fuzzy quasi-metric space of type Baire which improves the useful advantages of the Baire quasi-metric space and where we can also apply a known quasi-metric version of Grabiec's fixed point theorem for fuzzy metric spaces. This construction will be used to analyze the complexity of the average case of the Quicksort algorithm.

Let Σ be a nonempty alphabet. Let Σ^∞ be the set of all finite and infinite sequences (“words”) over Σ , where we adopt the convention that the empty sequence ϕ is an element of Σ^∞ . Denote by \sqsubseteq the prefix order on Σ^∞ , i.e. $x \sqsubseteq y \Leftrightarrow x$ is a prefix of y .

Now, for each $x \in \Sigma^\infty$ denote by $\ell(x)$ the length of x . Then $\ell(x) \in [1, \infty]$ whenever $x \neq \phi$ and $\ell(\phi) = 0$. For each $x, y \in \Sigma^\infty$ let $x \sqcap y$ be the common prefix of x and y . For all $x, y, z \in \Sigma^\infty$.

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In the theory of computation the fact $x \sqsubseteq y$ means that the element y contains all the information provided by x . Thus the partially defined objects (finite words) customarily represent stages of a computational process for which the totally defined objects (infinite words) contain exactly the amount of information provided by Σ^∞ .

Recall that the function d_{\sqsubseteq} defined on $\Sigma^\infty \times \Sigma^\infty$ by

$$\begin{aligned} d_{\sqsubseteq}(x, y) &= 0 \quad \text{if } x \sqsubseteq y, \\ d_{\sqsubseteq}(x, y) &= 2^{-\ell(x \sqcap y)} \quad \text{otherwise,} \end{aligned}$$

is a quasi-metric on Σ^∞ . (We adopt the convention that $2^{-\infty} = 0$)

Actually d_{\sqsubseteq} is a non-Archimedean quasi-metric on Σ^∞ (see, for instance, [3], Example 8 (b)).

We also observe that the non-Archimedean metric $(d_{\sqsubseteq})^s$ is the Baire metric on Σ^∞ , i.e.

$$(d_{\sqsubseteq})^s(x, x) = 0$$

and

$$(d_{\sqsubseteq})^s(x, y) = 2^{-\ell(x \sqcap y)}$$

for all $x, y \in \Sigma^\infty$ such that $x \neq y$.

It is well known that $(d_{\sqsubseteq})^s$ is complete. From this fact it clearly follows that d_{\sqsubseteq} is bicomplete.

The quasi-metric d_{\sqsubseteq} , which was introduced by Smyth [6], will be called the Baire quasi-metric. Observe that condition $d_{\sqsubseteq}(x, y) = 0$ can be used to distinguish between the case where x is a prefix of y and the remaining cases.

However, the quasi-metric d_{\sqsubseteq} does not provide us with any information about the degree of approximation to a word z from two different prefixes x, y of z . For instance, if we consider the totally defined object π and the partially defined ones $x = 3.14$ and $y = 3.141$, then it is clear that y contains more information of π than x , nevertheless $d_{\sqsubseteq}(x, \pi) = d_{\sqsubseteq}(y, \pi) = 0$, so d_{\sqsubseteq} is not sensitive to this amount of information.

Motivated by this fact J. Rodríguez-López, S. Romaguera and J.M. Sánchez-Álvarez constructed in [4] a fuzzy quasi-metric on Σ^∞ that preserves the advantages of d_{\sqsubseteq} and that, in addition, allows us to measure, with the help of the parameter t , the degree of approximation to a given word of each of its prefixes. This new fuzzy quasi-metric space $(\Sigma^\infty, M_1, \wedge)$ is given by:

$$M_1(x, y, 0) = 0 \quad \text{for all } x, y \in \Sigma^\infty,$$

$$M_1(x, x, t) = 1 \quad \text{for all } x \in \Sigma^\infty \text{ and } t > 0,$$

$$M_1(x, y, t) = 1 \quad \text{if } x \sqsubseteq y, \text{ and } t > 2^{-\ell(x)},$$

$$M_1(x, y, t) = 1 - 2^{-\ell(x \sqcap y)} \quad \text{otherwise.}$$

Therefore it is satisfied that for all $x, y, z \in \Sigma^\infty$ $M_1(x, y, t + s) \geq M_1(x, z, t) \wedge M_1(z, y, s)$ for all $t, s \geq 0$. In addition it is easy to see that for all $x, y, z \in \Sigma^\infty$, $M_1(x, y, t) \geq M_1(x, z, t) \wedge M_1(z, y, t)$, so $(\Sigma^\infty, M_1, \wedge)$ is a non-Archimedean fuzzy quasi-metric space.

Observe that if x, y are prefixes of z , with $x \neq y$, and one obtains that for some $t_0 > 0$, $M(x, z, t_0) < 1$ and $M(y, z, t_0) = 1$, then $2^{-\ell(y)} < t_0 \leq 2^{-\ell(x)}$, so that $\ell(x) \leq \ell(y)$, i.e., $x \sqsubseteq y$; which shows that y is better approximation to z than x .

Then (see [4]), for each $z \in \Sigma^\infty \setminus \{\phi\}$, and each $x \sqsubseteq y$ the degree of approximation of x to z , associated to (M_1, \wedge) , is defined as the number $DA(x, z) = 1/t_x$ where $t_x = \inf\{t > 0 : M_1(x, z, t) = 1\}$. It is clear that $DA(x, z) = 2^{\ell(x)}$.

It easy to see that $(\Sigma^\infty, M_1, \wedge)$ is a bicomplete non-Archimedean fuzzy quasi-metric space, so by [5, Theorem 3] $(\Sigma^\infty, M_1, \wedge)$ is a G-bicomplete fuzzy quasi-metric space. Nevertheless $\lim_{t \rightarrow \infty} M_1(x, y, t) \neq 1$, so we can not apply [5, Theorem 2] which is the extended version of Grabiec's fixed point theorem to the fuzzy quasi-metric case. To this end we modify the previous fuzzy quasi-metric in the following manner:

$$M_2(x, y, 0) = 0 \quad \text{for all } x, y \in \Sigma^\infty,$$

$$M_2(x, x, t) = 1 \quad \text{for all } x \in \Sigma^\infty \text{ and } t > 0,$$

$$M_2(x, y, t) = 1 \quad \text{if } x \sqsubseteq y, \text{ and } t > 2^{-\ell(x)},$$

$$M_2(x, y, t) = 1 - 2^{-\ell(x \sqcap y)} \quad \text{if } x \sqsubseteq y, \text{ and } t \leq 2^{-\ell(x)},$$

$$M_2(x, y, t) = 1 - 2^{-\ell(x \sqcap y)} \quad \text{if } x \text{ is not a prefix of } y \text{ and } t \leq 1,$$

$$M_2(x, y, t) = 1 - \frac{2^{-\ell(x \sqcap y)}}{t} \quad \text{if } x \text{ is not a prefix of } y \text{ and } t > 1.$$

Then we have the following

Proposition. $(\Sigma^\infty, M_2, \wedge)$ is a non-Archimedean fuzzy quasi-metric space.

Proof. The different case between (M_1, \wedge) and (M_2, \wedge) is the last condition on (M_2, \wedge) , i.e., $M_2(x, y, t) = 1 - \frac{2^{-\ell(x \sqcap y)}}{t}$ if x is not a prefix of y and $t > 1$. So we will prove that $M_2(x, y, t) \geq M_2(x, z, t) \wedge M_2(z, y, t)$ if x is not a prefix of y and $t > 1$. To this end recall that $l(x \sqcap y) \geq \min\{l(x \sqcap z), l(z \sqcap y)\}$ for all $x, y, z \in \Sigma^\infty$, so $1 - \frac{2^{-\ell(x \sqcap y)}}{t} \geq \min\{1 - \frac{2^{-\ell(x \sqcap z)}}{t}, 1 - \frac{2^{-\ell(z \sqcap y)}}{t}\}$. Moreover if x is not a prefix of y we have:

$$\text{if } x \sqsubseteq z \text{ then } \min\{l(x \sqcap z), l(z \sqcap y)\} = l(z \sqcap y),$$

$$\text{if } z \sqsubseteq y \text{ then } \min\{l(x \sqcap z), l(z \sqcap y)\} = l(x \sqcap z).$$

A). Suppose that $x \sqsubseteq z$ then $\min\{l(x \sqcap z), l(z \sqcap y)\} = l(z \sqcap y)$, and z is not a prefix of y (otherwise $x \sqsubseteq y$ and the inequality is satisfied), so $M_2(x, y, t) = 1 - \frac{2^{-\ell(x \sqcap y)}}{t} \geq 1 - \frac{2^{-\ell(z \sqcap y)}}{t} = M_2(z, y, t)$, then $M_2(x, y, t) \geq M_2(x, z, t) \wedge M_2(z, y, t)$.

B). Suppose that $z \sqsubseteq y$ then $\min\{l(x \sqcap z), l(z \sqcap y)\} = l(x \sqcap z)$, and x is not a prefix of z (otherwise $x \sqsubseteq y$ and the inequality is satisfied), so $M_2(x, y, t) = 1 - \frac{2^{-\ell(x \sqcap y)}}{t} \geq 1 - \frac{2^{-\ell(x \sqcap z)}}{t} = M_2(x, z, t)$, then $M_2(x, y, t) \geq M_2(x, z, t) \wedge M_2(z, y, t)$. \square

Corollary. For each continuous t-norm $*$, $(\Sigma^\infty, M_2, *)$ is a fuzzy quasi-metric space.

Definition. The fuzzy quasi-metric (M_2, \wedge) is said to be the Baire fuzzy quasi-metric and the space $(\Sigma^\infty, M_2, \wedge)$ is said to be the Baire fuzzy quasi-metric space.

It is easy to see that $(\Sigma^\infty, M_2, \wedge)$ is a bicomplete fuzzy quasi-metric space. So by [5, Theorem 3] we have that $(\Sigma^\infty, M_2, \wedge)$ is a G-bicomplete fuzzy quasi-metric space, moreover $\lim_{t \rightarrow \infty} M_2(x, y, t) = 1$.

Example. The average case analysis of Quicksort is discussed in [2] (see also [1]), where the following recurrence equation is obtained:

$$\begin{aligned} T(1) &= 0, \quad \text{and} \\ T(n) &= \frac{2(n-1)}{n} + \frac{n+1}{n}T(n-1), \quad n \geq 2. \end{aligned}$$

Consider as an alphabet Σ the set of nonnegative real numbers, i.e. $\Sigma = [0, \infty)$. We associate to T the functional $\Phi : \Sigma^\infty \rightarrow \Sigma^\infty$ given by $(\Phi(x))_1 = T(1)$ and

$$(\Phi(x))_n = \frac{2(n-1)}{n} + \frac{n+1}{n}x_{n-1}$$

for all $n \geq 2$ (if $x \in \Sigma^\infty$ has length $n < \infty$, we write $x := x_1x_2\dots x_n$, and if x is an infinite word we write $x := x_1x_2\dots$).

Next we show that Φ satisfies the contraction in the sense of [5, Theorem 2] on the G-bicomplete non-Archimedean Baire fuzzy quasi-metric space $(\Sigma^\infty, M_2, \wedge)$, with contraction constant $1/2$.

To this end, we first note that, by construction, we have $\ell(\Phi(x)) = \ell(x) + 1$ for all $x \in \Sigma^\infty$ (in particular, $\ell(\Phi(x)) = \infty$ whenever $\ell(x) = \infty$).

Furthermore, it is clear that $x \sqsubseteq y \iff \Phi(x) \sqsubseteq \Phi(y)$,

and consequently

$$\Phi(x \sqcap y) \sqsubseteq \Phi(x) \sqcap \Phi(y)$$

for all $x, y \in \Sigma^\infty$. Hence

$$\ell(\Phi(x \sqcap y)) \leq \ell(\Phi(x) \sqcap \Phi(y))$$

Then we will prove that

$$M_2(\Phi(x), \Phi(y), t) \geq M_2(x, y, 2t) \text{ for all } x, y \in \Sigma^\infty, t > 0.$$

A). $x \sqsubseteq y$ and $t \leq 2^{-\ell(\Phi(x))}$. So $\Phi(x) \sqsubseteq \Phi(y)$ and $2t \leq 2^{-\ell(x)}$. Then we have:

$$\begin{aligned} M_2(\Phi(x), \Phi(y), t) &= 1 - 2^{-\ell(\Phi(x) \sqcap \Phi(y))} \\ &\geq 1 - 2^{-\ell(\Phi(x \sqcap y))} = 1 - 2^{-(\ell(x \sqcap y)+1)} \\ &= 1 - \frac{1}{2} 2^{-\ell(x \sqcap y)} \geq 1 - 2^{-\ell(x \sqcap y)} \\ &= M_2(x, y, 2t) \end{aligned}$$

B). x is not a prefix of y and $2t < 1$. Then we have:

$$\begin{aligned} M_2(\Phi(x), \Phi(y), t) &= 1 - 2^{-\ell(\Phi(x) \sqcap \Phi(y))} \\ &\geq 1 - 2^{-\ell(\Phi(x \sqcap y))} = 1 - 2^{-(\ell(x \sqcap y)+1)} \\ &= 1 - \frac{1}{2} 2^{-\ell(x \sqcap y)} \geq 1 - 2^{-\ell(x \sqcap y)} \\ &= M_2(x, y, 2t) \end{aligned}$$

C). x is not a prefix of y and $t > 1$. Then we have:

$$\begin{aligned} M_2(\Phi(x), \Phi(y), t) &= 1 - \frac{2^{-\ell(\Phi(x) \sqcap \Phi(y))}}{t} \\ &\geq 1 - \frac{2^{-\ell(\Phi(x \sqcap y))}}{t} = 1 - \frac{2^{-(\ell(x \sqcap y)+1)}}{t} \\ &= 1 - \frac{2^{-\ell(x \sqcap y)}}{2t} = M_2(x, y, 2t) \end{aligned}$$

D). x is not a prefix of y and $t \in (1/2, 1]$, i.e., $t \leq 1$ and $2t > 1$. Then we have:

$$\begin{aligned} M_2(\Phi(x), \Phi(y), t) &= 1 - 2^{-\ell(\Phi(x) \sqcap \Phi(y))} \\ &\geq 1 - 2^{-\ell(\Phi(x \sqcap y))} = 1 - 2^{-(\ell(x \sqcap y)+1)} \\ &= 1 - \frac{1}{2} 2^{-\ell(x \sqcap y)} \geq 1 - \frac{2^{-\ell(x \sqcap y)}}{2t} \\ &= M_2(x, y, 2t) \end{aligned}$$

Therefore Φ is a contraction in the sense of [5, Theorem 2] on the G-bicomplete non-Archimedean Baire fuzzy quasi-metric space $(\Sigma^\infty, M_2, \wedge)$ with contraction constant $1/2$. So, by [5, Theorem 2], Φ has a unique fixed point $z = z_1 z_2 \dots$, which is obviously the unique solution to the recurrence equation T , i.e. $z_1 = 0$ and

$$z_n = \frac{2(n-1)}{n} + \frac{n+1}{n} z_{n-1}$$

for all $n \geq 2$.

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The Banach contraction principle in quasi-metric spaces revisited

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ABSTRACT

We update the Banach Contraction Principle in the realm of quasi-metric spaces and give some illustrative examples.

1. INTRODUCTION AND PRELIMINARIES

In 1982, Reilly, Subrahmanyam and Vamanamurthy [14] obtained a quasi-metric version of the celebrated Banach Contraction Principle. Since then, and specially in the last seven years, several authors have contributed to the development of the fixed point theory in the framework of quasi-metric spaces (see e.g. [1, 2, 3, 5, 10, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17]). In particular, other versions of the Banach

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Contraction Principle, different to the one obtained in [14] have been established. In this note we update such versions and present two examples that illustrate the results.

Our basic reference for quasi-metric spaces is [4].

By a quasi-metric on set X we mean a function $d : X \times X \rightarrow [0, \infty)$ such that for all $x, y, z \in X$:

$$(i) \quad x = y \Leftrightarrow d(x, y) = d(y, x) = 0;$$

$$(ii) \quad d(x, z) \leq d(x, y) + d(y, z).$$

A quasi-metric space is a pair (X, d) such that X is a set and d is a quasi-metric on X .

Given a quasi-metric d on X , then the function d^{-1} defined by $d^{-1}(x, y) = d(y, x)$, is also a quasi-metric on X , called the conjugate of d , and the function d^s defined by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is a metric on X .

Each quasi-metric d on X induces a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x, r) : x \in X, r > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If τ_d is a T_1 (resp. a T_2) topology on X , we say that (X, d) is a T_1 (resp. a Hausdorff) quasi-metric space.

Note that a quasi-metric space (X, d) is T_1 if and only if for each $x, y \in X$, condition $d(x, y) = 0$ implies $x = y$.

A quasi-metric space (X, d) is called bicomplete if the metric space (X, d^s) is complete.

A sequence $(x_n)_{n \in \mathbb{N}}$ in a quasi-metric space (X, d) is called left K-Cauchy if for each $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ whenever $m \geq n \geq n_0$.

A sequence $(x_n)_{n \in \mathbb{N}}$ in a quasi-metric space (X, d) is called right K-Cauchy if it is a left K-Cauchy sequence in the quasi-metric space (X, d^{-1}) .

The quasi-metric space (X, d) is called left (right) K-sequentially complete if every left (right) K-Cauchy sequence converges with respect to the topology τ_d .

(X, d) is called d -sequentially complete if every Cauchy sequence in the metric space (X, d^s) converges with respect to the topology τ_d .

Remark 1. It is obvious that both bicompleteness and left (right) K-sequential completeness imply d -sequential completeness. However, the rest of implications does not hold in general.

2. CONTRACTION MAPPINGS AND FIXED POINTS IN QUASI-METRIC SPACES

The celebrated Banach Contraction Principle states that every contraction on a complete metric space has a unique fixed point.

Let us recall that a contraction on a metric space (X, d) is a self mapping T on X such that there exists a constant $c \in [0, 1)$ satisfying

$$d(Tx, Ty) \leq cd(x, y),$$

for all $x, y \in X$.

This suggests, in a natural way, the following notion.

Definition 2. Let (X, d) be a quasi-metric space.

A d -contraction on (X, d) is a mapping $T : X \rightarrow X$ such that there is a constant $c \in [0, 1)$ satisfying $d(Tx, Ty) \leq cd(x, y)$, for all $x, y \in X$.

A d^{-1} -contraction on (X, d) is a mapping $T : X \rightarrow X$ such that there is a constant $c \in [0, 1)$ satisfying $d(Tx, Ty) \leq cd(y, x)$, for all $x, y \in X$.

Then, we have the following easy but useful consequence of the Banach Contraction Principle for metric spaces.

Proposition 3. *Let (X, d) be a quasi-metric space. If T is a d -contraction or a d^{-1} -contraction on (X, d) , then the following hold:*

- (1) T is a contraction on the metric space (X, d^s) .
- (2) For any $x_0 \in X$, the sequence $(T^n x_0)_{n \in \mathbb{N}}$ is a Cauchy sequence in the metric space (X, d^s) .

Proof. (1) Suppose that T is a d -contraction of (X, d) . Then, there exists a constant $c \in [0, 1)$ such that

$$d(Tx, Ty) \leq cd(x, y),$$

for all $x, y \in X$. Thus, given $x, y \in X$, we have

$$d(Tx, Ty) \leq cd(x, y) \quad \text{and} \quad d(Ty, Tx) \leq cd(y, x).$$

So $d^s(Tx, Ty) \leq cd^s(x, y)$.

Similarly, we show that if T is a d^{-1} -contraction on (X, d) , then it is a contraction on (X, d^s) .

(2) Since, by (1), T is a contraction on the metric space (X, d^s) , it follows from the proof of classical Banach Contraction Principle, that $(T^n x_0)_{n \in \mathbb{N}}$ is a Cauchy sequence in the metric space (X, d^s) . \square

By using the preceding proposition, three well-known quasi-metric versions of the Banach Contraction Principle are easily deduced.

Theorem 4 ([17]). *Every d - (resp. every d^{-1} -) contraction on a bicomplete quasi-metric space (X, d) has a unique fixed point.*

Proof. Let T be a d -contraction or a d^{-1} -contraction on the bicomplete quasi-metric space (X, d) . Since (X, d^s) is a complete metric space and, by Proposition 3 (1), T is a contraction on (X, d^s) , we deduce, from the classical Banach Contraction Principle, that T has a unique fixed point. \square

Theorem 5 ([14]). *Every d -contraction on a Hausdorff d -sequentially complete quasi-metric space (X, d) has a unique fixed point.*

Proof. Let T be a d -contraction on the Hausdorff d -sequentially complete quasi-metric space (X, d) . Fix an $x_0 \in X$. By Proposition 3 (2), the sequence $(T^n x_0)_{n \in \mathbb{N}}$ is a Cauchy sequence in the metric space (X, d^s) . Hence, there is $y \in X$ such that $(T^n x_0)_{n \in \mathbb{N}}$ converges to y with respect to τ_d , i.e., $d(y, T^n x_0) \rightarrow 0$ as $n \rightarrow \infty$. Since T is a d -contraction, there exists $c \in [0, 1)$ such that

$$d(Ty, T^{n+1}x_0) \leq cd(y, T^n x_0),$$

for all $n \in \mathbb{N}$. Consequently $d(Ty, T^{n+1}x_0) \rightarrow 0$ as $n \rightarrow \infty$. From Hausdorffness of (X, d) we deduce that $y = Ty$. Finally, suppose that $z \in X$ is a fixed point of T . Then

$$d(y, z) = d(Ty, Tz) \leq cd(y, z),$$

and thus $y = z$. This concludes the proof. \square

Corollary 6. *Every d -contraction on a quasi-metric space (X, d) such that (X, d^{-1}) is Hausdorff and d^{-1} -sequentially complete has a unique fixed point.*

Proof. Let T be a d -contraction on (X, d) . Put $q = d^{-1}$. Then T is a q -contraction on the Hausdorff q -sequentially complete quasi-metric space (X, q) . From Theorem 5 we deduce that T has a unique fixed point. \square

Theorem 7 ([10]). *Every d^{-1} -contraction on a T_1 d -sequentially complete quasi-metric space (X, d) has a unique fixed point.*

Proof. Let T be a d^{-1} -contraction on the T_1 d -sequentially complete quasi-metric space (X, d) . Fix an $x_0 \in X$. As in the proof of Theorem 5 (see Proposition 3), $(T^n x_0)_{n \in \mathbb{N}}$ is a Cauchy sequence in the metric space (X, d^s) . Hence, there is $y \in X$ such that $(T^n x_0)_{n \in \mathbb{N}}$ converges to y with respect to τ_d , i.e., $d(y, T^n x_0) \rightarrow 0$ as $n \rightarrow \infty$. Since T is a d^{-1} -contraction, there exists $c \in [0, 1)$ such that

$$d(T^{n+1}x_0, Ty) \leq cd(y, T^n x_0),$$

for all $n \in \mathbb{N}$. Consequently $d(Ty, T^{n+1}x_0) \rightarrow 0$ as $n \rightarrow \infty$. From the triangle inequality we deduce $d(y, Ty) = 0$. Therefore $y = Ty$ because (X, d) is a T_1 quasi-metric space. Finally, suppose that $z \in X$ is a fixed point of T . Then

$$d(y, z) = d(Ty, Tz) \leq cd(z, y) = cd(Tz, Ty) \leq c^2 d(y, z),$$

and thus $y = z$. This concludes the proof. \square

Corollary 8. *Every d^{-1} -contraction on a T_1 quasi-metric space (X, d) such that (X, d^{-1}) is d^{-1} -sequentially complete has a unique fixed point.*

Proof. Let T be a d^{-1} -contraction on (X, d) . Put $q = d^{-1}$. Then T is a q^{-1} -contraction on the T_1 q -sequentially complete quasi-metric space (X, q) . From Theorem 7 we deduce that T has a unique fixed point. \square

Remark 9. Note that, by Remark 1, Theorems 5 and 7 remain valid if “ d -sequentially complete” is replaced with “left K -sequentially complete” or “right K -sequentially complete”.

The following example shows that Theorem 5 cannot be generalized to T_1 d -sequentially complete quasi-metric spaces.

Example 10. Let $X = \mathbb{N}$ and let d be the T_1 quasi-metric on X given by $d(n, n) = 0$ for all $n \in X$, and $d(n, m) = 1/m$ otherwise. Clearly (X, d) is both left and right K -sequentially complete, so it is d -sequentially complete.

Now define $T : X \rightarrow X$ as $Tn = 2n$ for all $n \in X$. Of course T has no fixed point. However it is a d -contraction since for each $n, m \in X$ with $n \neq m$ one has

$$d(Tn, Tm) = d(2n, 2m) = \frac{1}{2m} = \frac{1}{2}d(n, m).$$

We conclude the paper with an example which shows that Theorem 7 cannot be generalized to d -sequentially complete quasi-metric spaces.

Example 11. Let $X = \mathbb{N} \cup \{0\}$ and let d be the quasi-metric on X given by $d(n, n) = 0$ for all $n \in X$, $d(0, n) = d(n, 1) = 0$ for all $n \in \mathbb{N}$, $d(n, 0) = 1$ for all $n \in \mathbb{N}$, and $d(n, m) = 2^{-(n+1)} + 2^{-(m+1)}$ otherwise. Observe that (X, d) is both left and right K -sequentially complete, and hence, d -sequentially complete, because every sequence in X converges to 0 with respect to τ_d .

Now define $T : X \rightarrow X$ as $Tn = n + 1$ for all $n \in X$. Of course T has no fixed point. However, it is a d^{-1} -contraction since for each $n \in \mathbb{N}$ one has

$$d(T0, Tn) = d(1, n + 1) = \frac{1}{4} + \frac{1}{2^{n+2}} < \frac{1}{2} = \frac{1}{2}d(n, 0),$$

and

$$d(Tn, T0) = d(n + 1, 1) = 0,$$

whereas for $n, m \in \mathbb{N}$ with $n \neq m$, one has

$$d(Tn, Tm) = d(n + 1, m + 1) = \frac{1}{2^{n+2}} + \frac{1}{2^{m+2}} = \frac{1}{2} \left(\frac{1}{2^{n+1}} + \frac{1}{2^{m+1}} \right) = \frac{1}{2}d(m, n).$$

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On fuzzy number valued functions

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ABSTRACT

Two aspects of the set of fuzzy number valued continuous functions are treated: the compactness and the density of its subspaces.

1. INTRODUCTION

Let $F(\mathbb{R}) := \{u : \mathbb{R} \rightarrow [0, 1]\}$; that is, the family of all fuzzy subsets on the real numbers \mathbb{R} . For $u \in F(\mathbb{R})$ and $\lambda \in [0, 1]$, the λ -level sets of u are defined by

$$[u]^\lambda := \{x \in \mathbb{R} / u(x) \geq \lambda\}, \quad \lambda \in]0, 1]$$

$$[u]^0 := cl_{\mathbb{R}}\{x \in \mathbb{R} / u(x) > 0\}.$$

Definition 1. The fuzzy number space \mathbb{E}^1 is the set of elements u of $F(\mathbb{R})$ satisfying the following properties:

- (1) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ with $u(x_0) = 1$;

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- (2) u is convex, i.e., $u(\lambda x + (1 - \lambda)y) \geq \min \{u(x), u(y)\}$ for all $x, y \in \mathbb{R}, \lambda \in [0, 1]$;
- (3) u is upper-semicontinuous;
- (4) $[u]^0$ is a compact set in \mathbb{R} .

If $u \in \mathbb{E}^1$, then every λ -level set $[u]^\lambda$ of u is a compact interval for each $\lambda \in [0, 1]$.

We denote $[u]^\lambda = [u^-(\lambda), v^+(\lambda)]$.

Every real number r can be considered a fuzzy number since r can be identified with the fuzzy number \tilde{r} defined by

$$\tilde{r}(x) = \begin{cases} 1 & \text{if } x = r, \\ 0 & \text{if } x \neq r. \end{cases}$$

Theorem 2 ([6]). *Let $u \in \mathbb{E}^1$ and $[u]^\lambda = [0, 1]$, $\lambda \in [0, 1]$. Then the pair of functions $u^-(\lambda)$ and $u^+(\lambda)$ has the following properties:*

- (1) $u^-(\lambda)$ is a bounded left continuous nondecreasing function on $]0, 1]$;
- (2) $u^+(\lambda)$ is a bounded left continuous nonincreasing function on $]0, 1]$;
- (3) $u^-(\lambda)$ and $u^+(\lambda)$ are right continuous at $\lambda = 0$;
- (4) $u^-(1) \leq u^+(1)$.

Conversely, if a pair of functions $\alpha(\lambda)$ and $\beta(\lambda)$ satisfies the above conditions (i)-(iv), then there exists a unique $u \in \mathbb{E}^1$ such that $[u]^\lambda = [\alpha(\lambda), \beta(\lambda)]$ for each $\lambda \in [0, 1]$.

2. TWO TOPOLOGIES ON \mathbb{E}^1 AND THEIR RESPECTIVE SPACES OF CONTINUOUS \mathbb{E}^1 -VALUED FUNCTIONS.

Let A and B be two compact subsets of \mathbb{R}^n . The Hausdorff distance between A and B is defined as:

$$d_H(A, B) := \max\{d(A, B), d(B, A)\}$$

$$d_H([a, b], [c, d]) = \max\{|a - c|, |b - d|\}$$

We can then endow \mathbb{E}^1 with the following metric:

$$\begin{aligned} d_\infty(u, v) &:= \sup_{\lambda \in [0,1]} d_H([u]^\lambda, [v]^\lambda) = \\ &= \sup_{\lambda \in [0,1]} \max \{ |u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)| \} \end{aligned}$$

It is known (see, e.g., [1]) that (\mathbb{E}^1, d_∞) is a non-separable complete metric space.

Notice also that, by the definition of d_∞ , \mathbb{R} endowed with the euclidean topology can be topologically identified with the closed subspace $\tilde{R} = \{\tilde{x}/x \in \mathbb{R}\}$ of (\mathbb{E}^1, d_∞) where $\tilde{x}^+(\lambda) = \tilde{x}^-(\lambda) = x$ for all $\lambda \in [0, 1]$.

Definition 3. Let $C(K, (\mathbb{E}^1, d_\infty))$ denote the set of all fuzzy-valued continuous functions on K , a compact subset of a metric space. Given $f, g \in C(K, \mathbb{E}^1)$, we can define

$$D(f, g) := \sup_{t \in K} d_\infty(f(t), g(t))$$

Theorem 4 ([3]). $(C(K, (\mathbb{E}^1, d_\infty)), D)$ is a complete metric space.

On the other hand (see, e.g., [8]), let $(u_k)_{k \in D}$ be a net in \mathbb{E}^1 , where D is a directed set. It is said that $(u_k)_{k \in D}$ *levelly converges* to $u \in \mathbb{E}^1$ if, for each $\lambda \in [0, 1]$,

$$\lim_{k \in D} d_H([u_k]^\lambda, [u]^\lambda) = 0$$

That is,

$$\begin{aligned} \lim_{k \in D} u_k^-(\lambda) &= u^-(\lambda) \\ \lim_{k \in D} u_k^+(\lambda) &= u^+(\lambda). \end{aligned}$$

Let $\tau(l)$ stand for the topology associated to the level convergence.

Proposition 5 ([2]). For each $u \in \mathbb{E}^1$, $\epsilon > 0$ and $\lambda \in [0, 1]$, we can define

$$U_u(\lambda, \epsilon) := \{v \in \mathbb{E}^1 / d_H([u]^\lambda, [v]^\lambda) < \epsilon\}$$

Then $B_u := \{U_u(\lambda, \epsilon) / \epsilon > 0, \lambda \in [0, 1]\}$ is a local subbase of u in $(\mathbb{E}^1, \tau(l))$.

Theorem 6 ([2]). $(\mathbb{E}^1, \tau(l))$ satisfies the first countability axiom but is not complete.

Theorem 7 ([4]). $(\mathbb{E}^1, \tau(l))$ is separable and a Baire space.

Definition 8. Let $f : X \longrightarrow (\mathbb{E}^1, \tau(l))$ be a continuous function, where X is topological Hausdorff space. We say that f is *level-continuous*. Indeed

$$\lim_{x \rightarrow x_0} [f(x)]^-(\lambda) = [f(x_0)]^-(\lambda)$$

$$\lim_{x \rightarrow x_0} [f(x)]^+(\lambda) = [f(x_0)]^+(\lambda)$$

for each $\lambda \in [0, 1]$ and all $x_0 \in X$

3. ARZELA-ASCOLI TYPE RESULTS

In this section we survey some results concerning the compactness of a subset of the space of continuous fuzzy number valued functions.

Let $f_L : [0, 1] \times [a, b] \longrightarrow \mathbb{R}$ be defined as $f_L(\lambda, t) := [f(t)]^-(\lambda)$.

Let $f_R : [0, 1] \times [a, b] \longrightarrow \mathbb{R}$ be defined as $f_R(\lambda, t) := [f(t)]^+(\lambda)$.

Theorem 9 ([10]). *A subset $F \subset (C([a, b], (\mathbb{E}^1, d_\infty)), D)$ is relatively compact if, and only if, $F_L := \{f_L/f \in F\}$ and $F_R := \{f_R/f \in F\}$ are relatively compact subsets of $(C([0, 1] \times [a, b], \mathbb{R}), \|\cdot\|_\infty)$.*

Theorem 10 ([3]). *A closed subset $F \subset C(K, (\mathbb{E}^1, d_\infty))$ is compact if, and only if, the following conditions are satisfied:*

- (i) F is uniformly d_∞ -bounded on K ;
- (ii) F is equi-continuous on K ; i.e., for each $\epsilon > 0$, there exists $\delta > 0$ such that $D(f(t), f(t')) < \epsilon$ for all $f \in F$ and $d(t, t') < \delta$;
- (iii) For each $t \in K$, $\{[f(t)]^+(\cdot)/f \in F\}$ and $\{[f(t)]^-(\cdot)/f \in F\}$ are equi-left-continuous on $(0, 1]$.

This result is based on the following:

Theorem 11 ([3]). *A closed subset M of (\mathbb{E}^1, d_∞) is compact if, and only if, the following two conditions are satisfied:*

- (1) M is uniformly support-bounded, i.e., there is a constant $L > 0$ such that $|u^+(0)| \leq L$ and $|u^-(0)| \leq L$ for all $u \in M$;

- (2) $\{u^+/u \in M\}$ and $\{u^-/u \in M\}$ are *equi-left-continuous* on $(0, 1]$; i.e., for each $\epsilon > 0$ there exists $\delta > 0$ such that $|u^+(\lambda') - u^+(\lambda)| < \epsilon$ (resp. $|u^-(\lambda') - u^-(\lambda)| < \epsilon$) for all $u \in M$ whenever $\lambda, \lambda' \in]0, 1]$ with $\lambda' \in (\lambda - \delta, \lambda]$.

Unfortunately, this result (and, consequently, Theorem 10) is not true as the following counterexample shows:

Let

$$u(x) = \begin{cases} 0 & \text{if } x \notin [0, 1], \\ 1 & \text{if } x \in [0, \frac{1}{2}], \\ \frac{1}{2} & \text{if } x \in]\frac{1}{2}, 1]. \end{cases}$$

Then

$$u^-(\lambda) = 0; \quad u^+(\lambda) = \begin{cases} 1 & \text{if } \lambda \in [0, \frac{1}{2}], \\ \frac{1}{2} & \text{if } \lambda \in]\frac{1}{2}, 1]. \end{cases}$$

It is apparent that $M := \{u(x)\}$ is a compact subset of (\mathbb{E}^1, d_∞) , but u^+ is not equi-left-continuous on $]0, 1]$.

In [5], the authors provide a right characterization of the compact subsets of (\mathbb{E}^1, d_∞) . We first need two definitions:

Definition 12. Let $\{f_i\}_{i \in I}$ be a family of functions defined from the unit interval $[0, 1]$ into the reals. Given $\lambda_0 \in [0, 1[$ such that $f_i(\lambda_0+)$ exists for all $i \in I$, the family $\{f_i\}_{i \in I}$ is said to be *right equicontinuous at λ_0* if for every $\epsilon > 0$, there is $\delta > 0$ such that $|f_i(\lambda) - f_i(\lambda_0+)| < \epsilon$ for all $i \in I$ whenever $\lambda \in]\lambda_0, \lambda_0 + \delta[$.

Definition 13. Let $\{f_i\}_{i \in I}$ be a family of functions defined from the unit interval $[0, 1]$ into the reals. Given $\lambda_0 \in [0, 1[$ such that $f_i(\lambda_0-)$ exists for all $i \in I$, the family $\{f_i\}_{i \in I}$ is said to be *left equicontinuous at λ_0* if for every $\epsilon > 0$, there is $\delta > 0$ such that $|f_i(\lambda) - f_i(\lambda_0-)| < \epsilon$ for all $i \in I$ whenever $\lambda \in]\lambda_0 - \delta, \lambda_0[$.

Theorem 14 ([5]). *A closed subset M of (\mathbb{E}^1, d_∞) is compact if, and only if, it satisfies the following properties:*

- (1) M is *uniformly support-bounded*, i.e., there is $\epsilon > 0$ such that $d_\infty(u, \tilde{0}) \leq \epsilon$ for all $u \in M$
- (2) $\{u^+/u \in M\}$ and $\{u^-/u \in M\}$ are *two-sided equicontinuous* on $[0, 1]$.

Hence, a corrected version of Theorem 10 would be the following:

Theorem 15. *A closed subset $F \subset C(X, (\mathbb{E}^1, d_\infty))$ is compact if, and only if, the following conditions are satisfied:*

- (i) *F is uniformly d_∞ -bounded on X ;*
- (ii) *F is equi-continuous on X .*
- (iii) *For each $t \in X$, $\{[f(t)]^+(\cdot)/f \in F\}$ and $\{[f(t)]^-(\cdot)/f \in F\}$ are two-sided equicontinuous on $(0, 1]$.*

When we deal with level continuous functions, similar results can be obtained:

Theorem 16 ([4]). *A closed subset K in $(\mathbb{E}^1, \tau(l))$ is compact if and only if*

- (1) *K is uniformly support-bounded,*
- (2) *K is pointwise closed.*

Theorem 17 ([4]). *A closed subset $F \subset C(X, (\mathbb{E}^1, \tau(l)))$ is compact in the compact-open topology if, and only if, the following conditions are satisfied:*

- (i) *$F[x]$ is pointwise closed and uniformly support-bounded for any $x \in X$;*
- (ii) *F is evenly-equicontinuous.*

4. WEIERSTRASS-STONE TYPE RESULTS

In this section we survey some results concerning the density of certain subsets of the space of continuous fuzzy number valued functions:

Theorem 18 ([9]). *Let $f \in C([a, b], (\mathbb{E}^1, d_\infty))$ and $\epsilon > 0$. Then the polynomial*

$$P(x) = \sum_{i=0}^n \binom{n}{i} x^i (1-x)^{n-i} f\left(\frac{i}{n}\right)$$

satisfies, for all $x \in [a, b]$,

$$d_\infty(P(x), f(x)) \leq \epsilon.$$

That is, $D(P, f) \leq \epsilon$.

Theorem 19 ([9]). *Let $f \in C([a, b], (\mathbb{E}^1, d_\infty))$ and $\epsilon > 0$. Then there exists a four-layer regular fuzzy neural network, say,*

$$N(x) = \sum_{i=1}^n W_i \left(\sum_{j=1}^m c_{ij} \sigma(y_j \cdot x + \theta_j) \right),$$

where $W \in \mathbb{E}^1$, $c_{ij}, y_j, \theta_j \in \mathbb{R}$ and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$, satisfies, for all $x \in [a, b]$,

$$d_\infty(N(x), f(x)) \leq \epsilon.$$

That is, $D(N, f) \leq \epsilon$.

Theorem 20 ([7]). *Let $f \in C([a, b], (\mathbb{E}^1, \tau(l)))$ and $\epsilon > 0$. Then there exists a sequence of four-layer regular fuzzy neural networks $(N_n(x))$ such that*

$$d_H([N_n(x)]^\lambda, [f(x)]^\lambda) \longrightarrow 0$$

uniformly in $[a, b]$ for each $\lambda \in [0, 1]$.

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Generating a probability measure from a fractal structure

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ABSTRACT

The main goal of this work is to generate a probability measure from a fractal structure. Since we want to define a measure, we take into account the theorems on construction of measures (Method I and Method II).

First, we define a first measure on the bicompletion of X and then we explore conditions to ensure that the restriction of the measure to the original space is a probability measure.

1. PRELIMINARIES.

First we introduce from [1] (see also [3] for a survey on fractal structures) the concept of fractal structure.

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Let X be a set and Γ_1 and Γ_2 be coverings of X . Γ_2 is said to be a strong refinement of Γ_1 if it is a refinement (that is, each element of Γ_2 is contained in some element of Γ_1) and for each $A \in \Gamma_1$ it is satisfied that $A = \cup\{B \in \Gamma_2 : B \subseteq A\}$.

Definition 1. A fractal structure Γ on a set X is a countable family of coverings $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$ such that each cover Γ_{n+1} is a strong refinement of Γ_n for each $n \in \mathbb{N}$. Cover Γ_n is called level n of the fractal structure.

For some background on fractal structures and the construction of the bicompletion \tilde{X} we refer the reader to [4].

In the rest of the section we recall from [2] two of the most important theorems on construction of outer measures, as well as other theorems on outer measures. First, we recall Method I.

Theorem 2 (Method I). *Let \mathcal{A} be a family of subsets of X that covers X . Let $c : \mathcal{A} \rightarrow [0, \infty]$ be any function, there is a unique outer measure $\overline{\mathcal{M}}$ on X such that*

- (1) $\overline{\mathcal{M}}(A) \leq c(A)$ for all $A \in \mathcal{A}$.
- (2) If $\overline{\mathcal{N}}$ is any outer measure on X with $\overline{\mathcal{N}}(A) \leq c(A)$ for all $A \in \mathcal{A}$ then $\overline{\mathcal{N}}(B) \leq \overline{\mathcal{M}}(B)$ for all $B \subseteq X$.

Furthermore, for any subset B of X , the definition of $\overline{\mathcal{M}}$ is given by $\overline{\mathcal{M}}(B) = \inf \sum_{A \in \mathcal{D}} c(A)$, where the infimum is over all countable covers \mathcal{D} of B by sets of \mathcal{A} .

Second, we recall Method II.

Theorem 3 (Method II). *Let \mathcal{A} be a family of subsets of a metric space S , and suppose that, for every $x \in S$ and $\varepsilon > 0$, there exists $A \in \mathcal{A}$ with $x \in A$ and $\text{diam } A \leq \varepsilon$. Suppose $c : \mathcal{A} \rightarrow [0, \infty]$ is a given function. An outer measure will be constructed based on this data. For each $\varepsilon > 0$, let $\mathcal{A}_\varepsilon = \{A \in \mathcal{A} : \text{diam } A \leq \varepsilon\}$. Let $\overline{\mathcal{M}}_\varepsilon$ be the previous theorem outer measure determined by c using the family \mathcal{A}_ε . Then, for a given set E , when ε decreases, $\overline{\mathcal{M}}_\varepsilon(E)$ increases. Define $\overline{\mathcal{M}}(E) = \lim_{\varepsilon \rightarrow 0} \overline{\mathcal{M}}_\varepsilon(E) = \sup_{\varepsilon > 0} \overline{\mathcal{M}}_\varepsilon(E)$.*

The next results are important in order to construct a measure:

Theorem 4. *The outer measure $\overline{\mathcal{M}}$ defined by Method II is a metric outer measure.*

Theorem 5. *A metric outer measure $\overline{\mathcal{M}}$ is a measure on the Borel σ -algebra.*

2. RESULTS

Defining a measure on X directly is not easy, so we will proceed in three steps.

2.1. Defining a measure on \tilde{X} .

Let $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a fractal structure on X , and let $(\tilde{X}, \tilde{\mathbf{\Gamma}})$ the completion of $(X, \mathbf{\Gamma})$ given in [4].

We will use the notation $U_{xn} = X \setminus \bigcup_{x \notin A, A \in \Gamma_n} A$, $U_{xn}^{-1} = \bigcap_{x \in A, A \in \Gamma_n} A$ and $U_{xn}^* = U_{xn} \cap U_{xn}^{-1}$ for each $x \in X$ and $n \in \mathbb{N}$. For $(\tilde{X}, \tilde{\mathbf{\Gamma}})$ we will use the notation \tilde{U}_{xn} , \tilde{U}_{xn}^{-1} and \tilde{U}_{xn}^* . These latter subsets are like the former ones, but with respecto to the fractal structure $\tilde{\mathbf{\Gamma}}$, instead of $\mathbf{\Gamma}$.

Let $G_n = \{U_{xn}^* : x \in X\}$, then G_n is a partition of X (see [4]).

Let us denote by $\mathcal{G} = \bigcup_{n \in \mathbb{N}} G_n = \{U_{xn}^* : x \in X, n \in \mathbb{N}\}$. Let ω be a function $\omega : \mathcal{G} \rightarrow [0, 1]$ (a pre-measure) such that:

- (1) $\sum \{\omega(U_{x_1}^*) : U_{x_1}^* \in G_1\} = 1$.
- (2) $\omega(U_n^*) = \sum \{\omega(U_{y,n+1}^*) : U_{y,n+1}^* \in G_{n+1}; y \in U_{xn}^*\}$ for each $U_{xn}^* \in G_n$ and each $n \in \mathbb{N}$.

Now, let $\tilde{\mathcal{G}} = \{\tilde{U}_{xn}^* : x \in \tilde{X}; n \in \mathbb{N}\}$. We define $\tilde{\omega} : \tilde{\mathcal{G}} \rightarrow [0, 1]$ by $\tilde{\omega}(\tilde{U}_{xn}^*) = \omega(\tilde{U}_{xn}^* \cap X)$ for each $x \in X$ and $n \in \mathbb{N}$ (note that $\tilde{U}_{xn}^* \cap X \in G_n$, see [4]).

Let μ be the method I outer measure determined by $\tilde{\mathcal{G}}$ and $\tilde{\omega}$. Then $\mu(A) = \inf \{\sum_{i=1}^{\infty} \tilde{\omega}(\tilde{U}_{x_i n_i}^*) ; A \subseteq \bigcup \tilde{U}_{x_i n_i}^*\}$, for each $A \subseteq \tilde{X}$.

A fractal structure induces a non archimedean quasi pseudo metric. Let d be the non archimedean quasi pseudo metric induced by $\mathbf{\Gamma}$ on X and \tilde{d} be the non archimedean quasi pseudo metric induced by $\tilde{\mathbf{\Gamma}}$ on \tilde{X} .

The next results show that μ is a probability measure on \tilde{X} and that it is an extension of ω and $\tilde{\omega}$.

Proposition 6. μ is a measure on the Borel sigma-algebras of (\tilde{X}, \tilde{d}^*) and (\tilde{X}, \tilde{d}) .

Proposition 7. μ is an extension of ω and $\tilde{\omega}$. In fact, $\mu(\tilde{U}_{xn}^*) = \omega(U_{xn}^*) = \tilde{\omega}(\tilde{U}_{xn}^*)$ for all $x \in X$ and $\mu(\tilde{U}_{xn}^*) = \tilde{\omega}(\tilde{U}_{xn}^*)$ for all $x \in \tilde{X}$ and $n \in \mathbb{N}$.

Proposition 8. $\mu(\tilde{X}) = 1$ and hence μ is a probability measure on \tilde{X} .

2.2. Defining a measure on \tilde{X} from the fractal structure.

In this section, we will define the pre-measure ω for the elements of Γ_n , instead of G_n . We will suppose that Γ is a *tiling* fractal structure, that is, all levels Γ_n are tilings (the elements of Γ_n are regularly closed with disjoint interiors).

Given $A \in \Gamma_n$, let $i_n(A) = A \setminus \bigcup_{B \in \Gamma_n; B \neq A} B$ and $\omega : \bigcup_{n \in \mathbb{N}} \Gamma_n \rightarrow [0, 1]$ be a function such that:

- (1) $\sum_{A \in \Gamma_1} \omega(A) = 1$.
- (2) $\omega(A) = \sum_{B \in \Gamma_{n+1}, B \subseteq A} \omega(B)$, for each $A \in \Gamma_n$ and each $n \in \mathbb{N}$.

From ω , we can define a function (which we will call ω too) on \mathcal{G} as follows:

$$\omega(U_{xn}^*) = \begin{cases} \omega(A) & \text{if } x \in i_n(A) \\ 0 & \text{otherwise} \end{cases}$$

This function ω satisfies the conditions of the previous section:

Proposition 9. $\omega : \mathcal{G} \rightarrow [0, 1]$ verifies the following conditions:

- (1) $\sum \{\omega(U_{x1}^*) : U_{x1}^* \in G_1\} = 1$.
- (2) $\omega(U_n^*) = \sum \{\omega(U_{y,n+1}^*) : U_{y,n+1}^* \in G_{n+1}; y \in U_{xn}^*\}$ for each $U_{xn}^* \in G_n$ and each $n \in \mathbb{N}$.

So by the previous section $\omega : \mathcal{G} \rightarrow [0, 1]$ can be extended to a metric outer measure μ , which is a measure on the Boreal σ -algebra. Moreover, μ is somehow an extension of $\omega : \bigcup_{n \in \mathbb{N}} \Gamma_n \rightarrow [0, 1]$, as shown in the next

Proposition 10. $\mu(\tilde{A}) = \omega(A)$ for all $A \in \Gamma_n$ and $n \in \mathbb{N}$.

2.3. Defining a measure on X .

Finally, we explore some conditions in order to get a probability measure on X , instead of \tilde{X} .

Definition 11. For each $n \in \mathbb{N}$ we define $\mathcal{C}_n = \bigcup\{A \cap B : A, B \in \Gamma_n; A \neq B\}$.

Given $C \subseteq X$, we define $St(C, \Gamma_n) = \bigcup\{A \in \Gamma_n : A \cap C \neq \emptyset\}$.

A fractal structure $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$ is said to be Cantor complete if for each sequence $(A_n)_{n \in \mathbb{N}}$ with $A_n \in \Gamma_n$ and $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$, it holds that $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$.

The next result gives conditions in order to get a probability measure on X from the pre-measure defined on the fractal structure, which is our main goal.

In the next result, $\omega(St(\mathcal{C}_n, \Gamma_m))$ will denote $\sum\{\omega(U_{xm}^*) : U_{xm}^* \in G_m; U_{xm}^* \subseteq St(\mathcal{C}_n, \Gamma_m)\}$ if ω is defined on \mathcal{G} . Note that if ω is defined on $\bigcup_{n \in \mathbb{N}} \Gamma_n$, then $\sum\{\omega(U_{xm}^*) : U_{xm}^* \in G_m; U_{xm}^* \subseteq St(\mathcal{C}_n, \Gamma_m)\} = \sum\{\omega(A) : A \in \Gamma_m; A \cap \mathcal{C}_n \neq \emptyset\}$.

Theorem 12. *Let Γ be a Cantor complete tiling fractal structure on X and suppose that for each $n \in \mathbb{N}$, $\omega(St(\mathcal{C}_n, \Gamma_m)) \rightarrow 0$. Then X is μ -measurable and $\mu(X) = 1$.*

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Convergence and continuity in fuzzy metric spaces

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ABSTRACT

In this paper we recompile some aspects of principal fuzzy metrics and add some new results. Then we propose to do a similar study about convergence of sequences and continuity of mappings in the class of s -fuzzy metric and coprincipal fuzzy metric spaces, replacing p -convergence by other conditions of convergence.

Keywords: *fuzzy metric space; principal (fuzzy metric space); coprincipal (fuzzy metric space); p -convergence.*

MSC: *54A40; 54D35; 54E50.*

1. INTRODUCTION AND PRELIMINARIES

In this paper we deal with the concept of fuzzy metric due to George and Veeramani. Nevertheless all definitions given here (and some of the results exposed)

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could be stated for fuzzy metric spaces in the sense of Kramosil and Michalek in the modern version found in [1, 3].

Definition 1 (George and Veeramani [1]). A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t -norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X, s, t > 0$:

- (GV1) $M(x, y, t) > 0$
- (GV2) $M(x, y, t) = 1$ if and only if $x = y$
- (GV3) $M(x, y, t) = M(y, x, t)$
- (GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
- (GV5) $M(x, y, -) :]0, \infty[\rightarrow]0, 1[$ is continuous.

As usual, if $(X, M, *)$ is a fuzzy metric space we will say that $(M, *)$, or, simply, M is a fuzzy metric on X .

Lemma 2 ([3]). *The real function $M(x, y, -)$ of Axiom (GV5) is increasing for all $x, y \in X$.*

Due to the appearance of the parameter t in the definition of a fuzzy metric M , then in the concepts given in this fuzzy setting, in general, it appears the parameter t . For instance

$$B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$$

represents an open ball centred at $x \in X$ with radius $\epsilon \in]0, 1[$ and parameter $t > 0$. When distinction is not necessary we write B instead of B_M .

The family $\{B(x, \epsilon, t) : x \in X, \epsilon \in]0, 1[, t > 0\}$ is a base for a topology τ_M on X , that we say generated by M . This topology is characterized by the next proposition. $(X, M, *)$ is called compact if (X, τ_M) is compact.

Proposition 3. *A sequence $\{x_n\}$ in the topological space (X, τ_M) converges to x_0 if and only if $\lim_n M(x_n, x_0, t) = 1$ for all $t > 0$.*

Let (X, d) be a metric space and let $\varphi :]0, \infty[\rightarrow]0, \infty[$ be a non-decreasing continuous function (for the usual topology of \mathbb{R}). Let $M_{d, \varphi}$ be a fuzzy set on $X^2 \times]0, \infty[$

defined by

$$M_{d,\varphi}(x, y, t) = \frac{\varphi(t)}{\varphi(t) + d(x, y)}.$$

Then $(X, M_{d,\varphi}, \cdot)$ is a fuzzy metric space [9] and $\tau_{M_{d,\varphi}}$ coincides with the topology $\tau(d)$ on X deduced by d . If φ is the identity function then we obtain the so called standard fuzzy metric M_d on X given by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

If (X, M_*) is a fuzzy metric space then the family $\{B((x, \frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N})\}$ is a local base at x , and the family $\{U_n : n \in \mathbb{N}\}$ is a base for an uniformity \mathcal{U}_M compatible with τ_M , where $U_n = \{(x, y) \in X \times X : M(x, y, \frac{1}{n}) > 1 - \frac{1}{n}\}$ for all $n \in \mathbb{N}$. Since $\{U_n : n \in \mathbb{N}\}$ is countable then τ_M is metrizable and consequently a topological space is fuzzy metrizable if and only if it is metrizable [2, 10].

In both cases, balls and bands are defined using only a letter “n”, but in both concepts exist two parameters, r and t , which take simultaneously the same value.

A fuzzy metric space $(X, M, *)$, or M , is called stationary [12] if M does not depend on t . So, we can omit the parameter t in the definition of M . In this case the concepts stated in fuzzy setting are, in general, very similar to their corresponding ones in metric spaces. For instance $B(x, \epsilon) = \{y \in X : M(x, y) > 1 - \epsilon\}$ represents the ball centered at x with radius $\epsilon \in]0, 1[$.

Here we are not interested in stationary fuzzy metrics, but in three classes of fuzzy metrics which are characterized because they have special local bases. These classes are: principal fuzzy metrics, coprincipal fuzzy metrics and s -fuzzy metrics.

2. PRINCIPAL FUZZY METRIC SPACES

In this section $(X, M, *)$ is a fuzzy metric space and (X, τ_M) is the topological space induced by M .

The following is a weaker concept than convergence, which was defined in order to obtain a fixed point theorem in this context.

Definition 4 (D. Mihet [13]). A sequence $\{x_n\}$ is said to be p -convergent to $x_0 \in X$, for $t_0 > 0$, if $\lim M(x_n, x_0, t) = 1$, or equivalently, there exists $t_0 > 0$ such that for $\epsilon \in]0, 1[$ we can find n_0 , which depends on ϵ and t_0 , such that $x_n \in B(x_0, \epsilon, t_0)$ for each $n \geq n_0$ (i.e. $M(x_n, x_0, t_0) > 1 - \epsilon$ for all $n \geq n_0$).

Clearly, if $\{x_n\}$ is p -convergent for t_0 then it is so for all $t > t_0$.

Definition 5. We will say that $(X, M, *)$, or simply M , is *principal* at $x_0 \in X$ if $\{B(x_0, \epsilon, t) : \epsilon \in]0, 1[\}$ is a local base at $x_0 \in X$, for all $t > 0$. If $(X, M, *)$ is principal at each point of X then it is called principal.

In [4] the following results was obtained.

Proposition 6. *They are equivalent:*

- (i) M is principal at $x_0 \in X$.
- (ii) every p -convergent sequence to x_0 is convergent (to x_0).

Corollary 7. *They are equivalent:*

- (i) M is principal.
- (ii) every p -convergent sequence in X is convergent.

Next we will show some examples of principal fuzzy metric spaces.

Example 8. (a) Clearly, stationary fuzzy metrics are principal.

- (b) Consider the fuzzy metric space $(\mathbb{R}^+, M^\varphi, \cdot)$, see [9], where M^φ is given by

$$M^\varphi(x, y, t) = \frac{\min\{x, y\} + \varphi(t)}{\max\{x, y\} + \varphi(t)}$$

and where $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a non-decreasing continuous function.

For $x_0 \in X$, $\epsilon \in]0, 1[$ and $t > 0$ we have that

$$B(x_0, \epsilon, t) = \left] x_0 - \epsilon(x_0 + \varphi(t)), x_0 + \frac{\epsilon}{1 - \epsilon}(x_0 + \varphi(t)) \right[\cap]0, \infty[.$$

τ_{M^φ} is the usual topology of \mathbb{R} restricted to \mathbb{R}^+ since the diameter of this open interval, in the real line, tends to 0 as ϵ and t tend to 0. Now, this last assertion is true for a fixed $t > 0$ and for every $x \in X$ and hence

$\{B(x, \epsilon, t) : \epsilon \in]0, 1[\}$ is a local base at x for every $x \in X$, and so M^φ is principal.

(Alternatively, let $\{x_n\}$ be a p -convergent sequence to x_0 , for some $t_0 > 0$. We have that

$$\lim_n M^\varphi(x_n, x_0, t_0) = \lim_n \frac{\min\{x_n, x_0\} + \varphi(t_0)}{\max\{x_n, x_0\} + \varphi(t_0)} = 1.$$

Then $\lim_n \min\{x_n, x_0\} = \lim_n \max\{x_n, x_0\}$ and thus it is easy to conclude that $\lim_n x_n = x_0$, in the usual topology of \mathbb{R} , and by the above arguments $\{x_n\}$ converges to x_0 in τ_{M^φ}).

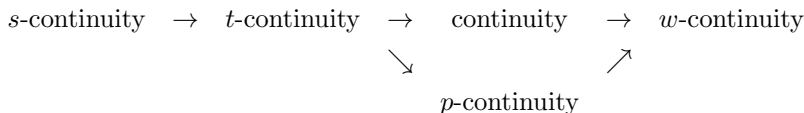
In a similar way we can prove:

- (c) $M_{d,\varphi}$ (and, in particular, the standard fuzzy metric) is principal.
- (d) compact fuzzy metric spaces are principal [8].

In [6] it was proposed the next question which, as up we know, is unsolved (For details about completion the reader is referred to [11, 12]).

Problem 9. *If the principal fuzzy metric space $(X, M, *)$ admits completion $(\tilde{X}, \tilde{M}, *)$, is it also principal?*

Clearly, proving that $\{x_n\}$ converges to x_0 in a fuzzy metric space $(X, M, *)$ is easier if M is principal. Indeed, it is only necessary to prove that $\lim_n M(x_n, x_0, t) = 1$ for some t . Consequently, to characterize a continuous mapping $f : X \rightarrow Y$, where $(X, M, *)$ and (Y, N, \diamond) are two principal fuzzy metric spaces is simpler than in the general case. Several of these characterizations suggested to the authors in [5] to introduce for a mapping $f : X \rightarrow Y$, where $(X, M, *)$ and (Y, N, \diamond) are fuzzy metric spaces, the concept of s -continuity, t -continuity, p -continuity and w -continuity. These concepts satisfy the next chain of implications



and all these concepts coincide when M and N are principal.

The most natural concept is the t -continuity ($f : X \rightarrow Y$ is t -continuous at x_0 if $\lim_n M(x_n, x_0, t) = 1$ implies $\lim_n N(f(x_n), f(x_0), 1) = 1$). This definition states

that if $\{x_n\}$ is p -convergent to x_0 for $t > 0$ then $\{f(x_n)\}$ is p -convergent to $f(x_0)$ for the same t .

Remark 10. The principal fuzzy metric's framework is ideal for working with topics involving convergence, because verifying that a sequence is convergent is easier than in the general case. Further, if also the fuzzy metric M is strong then the behaviour of M is very similar to stationary fuzzy metrics and then it is similar to classical metrics.

3. s -FUZZY METRIC AND COPRINCIPAL FUZZY METRIC SPACES

We have just seen that principal fuzzy metric spaces are characterized because each point has a particular type of local base. There are in the literature other two classes of fuzzy metrics which have a particular local base at each point, that we revise in this section.

3.1. s -fuzzy metric spaces. A sequence $\{x_n\}$ is called s -convergent to x_0 if $\lim_n M(x_n, x_0, \frac{1}{n}) = 1$.

A s -convergent sequence is convergent and the converse is false [7].

A fuzzy metric space $(X, M, *)$, or simply M , is called s -fuzzy metric if every convergent sequence is s -convergent. The fuzzy metric space (\mathbb{R}^+, M, \cdot) , where $M(x, y, t) = \frac{\min\{x, y\} + t}{\max\{x, y\} + t}$, is s -fuzzy metric, and the standard fuzzy metric M_d , except trivial cases, is not s -fuzzy metric.

The fuzzy metric space $(X, M, *)$ is s -fuzzy metric if and only if $\bigcap_{t>0} B(x, r, t)$ is a neighbourhood of x , for each $x \in X$ and each $t > 0$, and then the family $\{\bigcap_{t>0} B(x, r, t); r \in]0, 1[\}$ is a local base at x , for each $x \in X$.

3.2. Coprincipal fuzzy metric spaces. If we have in mind the theory of metric spaces it has no sense to study when a family of the form $\{B(x_0, \epsilon, t) : t > 0\}$ could be a local base at x_0 . In fact, this is not possible for stationary fuzzy metrics. We will see in this section that this study has sense in our general fuzzy setting.

Definition 11. A fuzzy metric space $(X, M, *)$, or simply M , is called coprincipal at $x_0 \in X$ for a fixed $\epsilon \in]0, 1[$ if the family $\{B(x_0, \epsilon, t) : t > 0\}$ is a local base at x_0 .

Definition 12. A fuzzy metric space $(X, M, *)$, or simply M , is called coprincipal if the family $\{B(x, \epsilon, t) : t > 0\}$ is a local base at x , for each $x \in X$ and each $\epsilon \in]0, 1[$.

The standard fuzzy metric M_d is coprincipal. Notice that it is satisfied that $\lim_{t \rightarrow 0} M_d(x, y, t) = 0$ for all $x, y \in X$, with $x \neq y$. Now, the authors have not found any relationship between the condition of being M coprincipal and the condition $\lim_{t \rightarrow 0} M(x, y, t) = 0$, for all $x, y \in X$ with $x \neq y$.

The authors think that it could be interesting to continue the study of s -fuzzy metrics and coprincipal fuzzy metrics imitating the study done with principal fuzzy metrics.

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On fixed point theory in fuzzy metric spaces

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ABSTRACT

In this paper we recall some fixed point theorems in fuzzy metric spaces, in the sense of George and Veeramani, and two unsolved questions related to them.

Keywords: fixed point; fuzzy metric space.

MSC: 54A40; 54D35; 54E50.

1. INTRODUCTION AND PRELIMINARIES

In 1994 A. George and P. Veeramani give the next concept of fuzzy metric space.

Definition 1 (George and Veeramani [1]). A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t -norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X, s, t > 0$:

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- (GV1) $M(x, y, t) > 0$;
- (GV2) $M(x, y, t) = 1$ if and only if $x = y$;
- (GV3) $M(x, y, t) = M(y, x, t)$;
- (GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
- (GV5) $M(x, y, -) :]0, \infty[\rightarrow]0, 1[$ is continuous.

Further, the authors showed that M generates a topology τ_M on X which has a base the family of open sets of the form $\{B_M(x, \epsilon, t) : x \in X, 0 < \epsilon < 1, t > 0\}$, where $B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $x \in X, \epsilon \in]0, 1[$ and $t > 0$ [1]. A characterization of convergent sequences for this topology is the next one.

Proposition 2 (George and Veeramani [1]). *A sequence $\{x_n\}$ in X converges to x if and only if $\lim_n M(x_n, x, t) = 1$, for all $t > 0$.*

Also, George and Veeramani introduced the next concept of Cauchy sequence and complete fuzzy metric space.

Definition 3 (George and Veeramani [1]). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be *M-Cauchy*, or simply *Cauchy*, if for each $\epsilon \in]0, 1[$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$ or, equivalently, $\lim_{n,m} M(x_n, x_m, t) = 1$ for all $t > 0$. X is said to be *complete* if every Cauchy sequence in X is convergent with respect to τ_M . In such a case M is also said to be complete.

Later, Gregori and Romaguera [3] proved that the class of topological spaces metrizable coincides with the class of topological spaces fuzzy metrizable, i.e. fuzzy metric spaces and classical metrics are topologically identical. But from the metrical point of view we can find some differences between them. For instance, there exist fuzzy metric spaces which do not admit completion. The fixed point theory also constitutes a difference between classical metrics and fuzzy metrics. In fact, there is not in the literature an analogous version of the Banach Fixed Point Theorem in fuzzy setting for complete fuzzy metric spaces, in the sense of George and Veeramani. Nevertheless, several authors have approached this topic giving fixed point theorems for a stronger completeness or demanding additional conditions to the completeness.

In the next we recall some approaches to extend the Banach's fixed point theorem to the theory of fuzzy metric spaces, in the sense of George and Veeramani.

2. ON THE GREGORI AND SAPENA'S FIXED POINT THEOREM

The first attempt was given by Gregori and Sapena [5]. They introduced the next concepts of fuzzy contractive mapping and fuzzy contractive sequence, respectively.

Let $(X, M, *)$ be a fuzzy metric space. A mapping $f : X \rightarrow X$ is said to be fuzzy *GS*-contractive if there exists $k \in]0, 1[$ such that for all $x, y \in X$ and $t > 0$ it is satisfied

$$(1) \quad 1 - \frac{1}{M(f(x), f(y), t)} \leq k \left(1 - \frac{1}{M(x, y, t)} \right).$$

A sequence $\{x_n\}$ is said to be fuzzy *GS*-contractive, if for each $n \in \mathbb{N}$ and each $t > 0$ we have that

$$(2) \quad 1 - \frac{1}{M(x_{n+2}, x_{n+1}, t)} \leq k \left(1 - \frac{1}{M(x_{n+1}, x_n, t)} \right).$$

Using these two concepts they proved the next theorem.

Theorem 4. *Let $(X, M, *)$ be a complete fuzzy metric space in which fuzzy *GS*-contractive sequences are Cauchy. Let $f : X \rightarrow X$ be a fuzzy *GS*-contractive mapping. Then, f has a unique fixed point.*

As well as, they proposed the next question.

Question 5. *Is a *GS*-fuzzy contractive sequence a Cauchy sequence?*

3. ON THE MIHET'S FIXED POINT THEOREM

Later, Mihet [6] generalized the concept of fuzzy contractive mapping given by Gregori and Sapena and others appeared in the literature. He introduced the next concepts and gave his theorem for *KM*-fuzzy metric spaces. In this paper we do a slight modification in order to adapt them to fuzzy metric spaces (see [6]).

Let Ψ be the class of continuous increasing functions $\psi : [0, 1] \rightarrow [0, 1]$ such that $\psi(z) > z$ for all $z \in]0, 1[$.

Example 6. Given $k \in]0, 1[$. The following are examples of functions belonging to the class Ψ .

- (1) $\psi_1(z) = \frac{z}{z+k(1-z)}$;
- (2) $\psi_2(z) = z^k$;
- (3) $\psi_3(z) = 1 - k(1 - z)$.

A mapping $f : X \rightarrow X$ is said to be fuzzy ψ -contractive for ψ if there exists $\psi \in \Psi$ such that for all $x, y \in X$ and $t > 0$ it is satisfied

$$(3) \quad M(f(x), f(y), t) \geq \psi(M(x, y, t))$$

Remark 7. A fuzzy ψ -contractive mapping for ψ_1 is a fuzzy *GS*-contractive mapping.

Mihet restricted his result to a class of fuzzy metric spaces that we recall here.

Definition 8. A fuzzy metric $(X, M, *)$ is said to be *strong* if for all $x, y, z \in X$ and all $t > 0$ satisfies

$$M(x, z, t) \geq M(x, y, t) * M(y, z, t).$$

Theorem 9. *Let $(X, M, *)$ be a complete strong fuzzy metric space and let $f : X \rightarrow X$ be a fuzzy ψ -contractive mapping. Then, f has a unique fixed point.*

D. Mihet generalized the concept of fuzzy contractive sequence as follows:

A sequence $\{x_n\}$ is said to be fuzzy ψ -contractive [6], if for each $n \in \mathbb{N}$ and each $t > 0$ we have that

$$(4) \quad M(x_{n+2}, x_{n+1}, t) \geq \psi(M(x_{n+1}, x_n, t)).$$

Attending to Question 5 and taking into account that the last result is given for a subclass of complete fuzzy metric spaces, the authors in [2] propose the next more general question.

Question 10. *Is a fuzzy ψ -contractive sequence a Cauchy sequence?*

Imitating the proof of [5, Theorem 4.4] it is easy to verify the next theorem.

Theorem 11. *Let $(X, M, *)$ be a complete fuzzy metric space in which fuzzy ψ -contractive sequences are Cauchy. Let $f : X \rightarrow X$ be a fuzzy ψ -contractive mapping. Then, f has a unique fixed point.*

Remark 12. From the proof of Theorem 9 one can deduce that in a strong fuzzy metric space the answer to Question 10 is affirmative.

4. ON THE GREGORI AND MIÑANA'S FIXED POINT THEOREM

Recently, Gregori and Miñana have characterized in [2] those complete fuzzy metric spaces in which a fuzzy ψ -contractive mapping has a unique fixed point by means of the next result.

Theorem 13. *Let $(X, M, *)$ be a complete fuzzy metric space and let $f : X \rightarrow X$ be a fuzzy ψ -contractive mapping. Then,*

f has a unique fixed point iff there exists $x \in X$ such that $\bigwedge_{t>0} M(x, f(x), t) > 0$.

An immediate corollary of the last theorem, in which the last condition is demanded on the whole space is the next one.

Corollary 14. *Let $(X, M, *)$ be a complete fuzzy metric space in which for each $x, y \in X$ we have that $\bigwedge_{t>0} M(x, y, t) > 0$ and let $f : X \rightarrow X$ be a fuzzy ψ -contractive mapping. Then, f has a unique fixed point.*

Remark 15. From the proof of Theorem 13 one can deduce that Question 10 has affirmative answer for the class of fuzzy metrics that satisfy the next condition:

$$\bigwedge_{t>0} M(x, y, t) > 0 \text{ for each } x, y \in X.$$

Remark 16. It is the purpose of these authors to continue the study of those fuzzy metric spaces in which fuzzy ψ -contractive sequences are Cauchy. We will try to provide partial answers to Question 10 for some subclass of Ψ or demanding another conditions on the whole fuzzy metric space.

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On compatible pairs of convergence and Cauchyness in fuzzy metric spaces

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ABSTRACT

The notion of compatible pair of convergence and Cauchyness in fuzzy metric spaces has been recently introduced by Gregori and Miñana in [7]. In this paper we study some well-motivated notions of convergence and Cauchyness in fuzzy metric spaces which have appeared in the literature and we survey the corresponding compatible pair defined for each one of them.

1. INTRODUCTION

Several well-motivated notions of convergence and Cauchyness in fuzzy metric spaces have been recently introduced in the literature. In particular, M. Grabiec introduced in [2] the notion of Cauchy sequence, which we will call G -Cauchy sequence, in order to prove his celebrated fixed point theorem. After, Mihet introduced in [9] the notion of p -convergent sequence in order to find new results on fixed point theory. In [11], Ricarte and Romaguera introduced the notion of

standard Cauchy sequence in order to extend the domain theory to fuzzy metric spaces. As Mihet proposed in his paper, the authors in [4] found a notion of p -Cauchyness related to p -convergence. Similarly and following a natural way, Morillas and Sapena introduced the concept of standard convergent sequence (which we call std^* -convergence). Nevertheless, in this case, the expected diagram of implications is not fulfilled. This fact has motivated the appearance of the notion of compatible pair of convergence and Cauchyness in fuzzy metric spaces (Definition 12) to define a notion of Cauchyness (resp. convergence) for each notion of convergence (resp. Cauchyness) to satisfy some demanded conditions. In this paper we survey the corresponding compatible pairs for several notions which have appeared recently in the context of fuzzy metric spaces.

2. PRELIMINARIES

Definition 1 (A. George and P. Veeramani, 1994). A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (nonempty) set, $*$ is a continuous t -norm and M is a fuzzy set on $X \times X \times \mathbb{R}^+$ satisfying the following conditions, for all $x, y, z \in X, s, t > 0$:

- (GV1) $M(x, y, t) > 0$
- (GV2) $M(x, y, t) = 1$ if and only if $x = y$
- (GV3) $M(x, y, t) = M(y, x, t)$
- (GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
- (GV5) $M(x, y, -) : \mathbb{R}^+ \rightarrow]0, 1]$ is continuous

If $(X, M, *)$ is a fuzzy metric space we say that $(M, *)$, or simply M , is a fuzzy metric on X . Also, we say that (X, M) or, simply, X is a fuzzy metric space.

Definition 2 (A. George and P. Veeramani, 1994). Let (X, d) be a metric space. Denote by $a \cdot b$ the usual product for all $a, b \in [0, 1]$, and let M_d be the fuzzy set defined on $X \times X \times \mathbb{R}^+$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then (M_d, \cdot) is a fuzzy metric on X called *standard fuzzy metric* induced by d .

George and Veeramani proved that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \varepsilon, t) : x \in X, 0 < \varepsilon < 1, t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ for all $x \in X, \varepsilon \in]0, 1[$ and $t > 0$.

In the case of the standard fuzzy metric M_d it is well-known that the topology $\tau(d)$ on X deduced from d satisfies $\tau(d) = \tau_{M_d}$. From now on we will suppose X endowed with the topology τ_M .

Definition 3 (V. Gregori and S. Romaguera, 2002). $(X, M, *)$ is called compact if (X, τ_M) is compact.

Definition 4 (V. Gregori and S. Romaguera, 2004). A fuzzy metric M on X is said to be *stationary* if M does not depend on t , i.e. if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant. In this case we write $M(x, y)$ instead of $M(x, y, t)$,

Proposition 5 (A. George and P. Veeramani, 1994). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ converges to x_0 if and only if $\lim_n M(x_0, x_n, t) = 1$, for all $t > 0$.

Definition 6 (A. George and P. Veeramani, 1994). A sequence $\{x_n\}_{n \in \mathbb{N}}$ in a fuzzy metric space $(X, M, *)$ is called *Cauchy* if for each $\varepsilon \in]0, 1[$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$ or equivalently $\lim_{m,n} M(x_n, x_m, t) = 1$ for all $t > 0$.

3. COMPATIBLE PAIRS OF CONVERGENCE AND CAUCHYNESS

The classical idea which associates a Cauchy notion to a corresponding notion of convergence have been extended in a natural way to the fuzzy setting. Nevertheless, the natural way to extend each classical notion to define the corresponding one in the fuzzy setting is not always satisfactory and several problems have arisen.

First we will see three well-motivated definitions which have been recently introduced in the literature.

In order to establish a Banach Contraction Principle in the context of fuzzy metric spaces in the sense of Kramosil and Michalek, M. Grabiec gave the following weaker concept than Cauchy sequence that we denote G -Cauchy.

Definition 7 (M. Grabiec, 1988). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called G -Cauchy if $\lim_n M(x_n, x_{n+p}, t) = 1$ for each $t > 0$ and each $p \in \mathbb{N}$.

After, in 2007, D. Mihet [9] introduced the following definition in order to extend the fixed point theory in fuzzy metric spaces.

Definition 8 (D. Mihet, 2007). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called p -convergent to x_0 if $\lim_n M(x_n, x_0, t_0) = 1$ for some $t_0 > 0$.

Later, in 2013, S. Romaguera and L.A. Ricarte introduced the following definition in order to extend the domain theory to the fuzzy metric context.

Definition 9 (S. Romaguera and L.A. Ricarte, 2013). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called standard Cauchy if for each $\varepsilon \in]0, 1[$ there exists $n_0 \in \mathbb{N}$, depending on ε , such that

$$M(x_n, x_m, t) > \frac{t}{t + \varepsilon},$$

for all $n, m \geq n_0$ and $t > 0$.

Now, it arises to give appropriate definitions of G -convergence, p -Cauchyness and standard convergence associated to their corresponding pair in such a manner which we call a natural way. That is, as in the classical case, by replacing the simple limit by a double limit or vice-versa. So, chronologically, the notion of p -convergence was the first to be associated to a Cauchy notion. In fact, the authors in [4] gave the following definition.

Definition 10 (V. Gregori et al., 2007). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called p -Cauchy to x_0 if $\lim_{n,m} M(x_n, x_m, t_0) = 1$ for some $t_0 > 0$.

The authors proved that the following diagram of implications was accomplished.

$$\begin{array}{ccc}
 \text{convergence} & \rightarrow & p\text{-convergence} \\
 \downarrow & & \downarrow \\
 \text{Cauchy} & \rightarrow & p\text{-Cauchy}
 \end{array}$$

Now, when considering the concept of standard Cauchy sequence, S. Morillas and A. Sapena introduced in [10] the following definition.

Definition 11 (S. Morillas and A. Sapena, 2013). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called *std** convergent if for each $\varepsilon \in]0, 1[$ there exist $x_0 \in X$ and $n_0 \in \mathbb{N}$ such that

$$M(x_n, x_0, t) > \frac{t}{t + \varepsilon},$$

for all $n \geq n_0$ and $t > 0$.

A question that arises when considering the previous definition is if it satisfies the corresponding diagram of implications. Nevertheless, as Gregori and Miñana pointed out in [5], there exist standard convergent sequences which are not standard Cauchy and so, the diagram of implications is not fulfilled in this case.

In the *G*-Cauchy case, if we try to define in a natural way a corresponding notion of *G*-convergence we have the notion of convergence.

To overcome this inconveniences, Gregori and Miñana have introduced in [5] the following definition to extend the classical idea that convergence implies Cauchy-ness and not reciprocally to the fuzzy setting.

Definition 12 (V. Gregori and J.J. Miñana, 2015). Suppose that it is given a sequential stronger (weaker, respectively) concept than Cauchy sequence, say *s*-Cauchy sequence (*w*-Cauchy, respectively). A concept of convergence, say *s*-convergence (*w*-convergence, respectively), is said to be compatible with *s*-Cauchy (*w*-Cauchy, respectively), and vice-versa, if the diagram of implications below on the left (on the right, respectively) is fulfilled.

$$\begin{array}{ccccc}
 s\text{-convergence} & \rightarrow & \text{convergence} & & \text{convergence} & \rightarrow & w\text{-convergence} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 s\text{-Cauchy} & \rightarrow & \text{Cauchy} & & \text{Cauchy} & \rightarrow & w\text{-Cauchy}
 \end{array}$$

and there is not any other implication, in general, among these concepts. In such a case we also say that w -convergence and w -Cauchy (respectively, s -convergence and s -Cauchy) is a compatible pair.

3.1. On p -convergence. In order to define a compatible pair for p -convergent sequences, Gregori et al. introduced the notion of principal fuzzy metric space as follows.

Definition 13 (V. Gregori et al., 2009). A fuzzy metric space $(X, M, *)$ is said to be *principal* (or simply, M is principal) if $\{B(x, r, t) : r \in]0, 1[\}$ is a local base at $x \in X$, for each $x \in X$ and each $t > 0$.

This definition was used to characterize those fuzzy metric spaces where p -convergent sequences are convergent since a fuzzy metric space $(X, M, *)$ is principal if and only if every p -convergent sequence in X is convergent in (X, τ_M) .

Now, the following notion was introduced.

Definition 14 (V. Gregori et al., 2009). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called p -Cauchy if there exists $t_0 > 0$ such that for each $\epsilon \in]0, 1[$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t_0) > 1 - \epsilon$ for all $n, m \geq n_0$, or equivalently, $\lim_{n,m} M(x_n, x_m, t_0) = 1$ for some $t_0 > 0$.

It can be proved the the pair p -convergence and p -Cauchyness is a compatible pair.

3.2. On G -Cauchyness. In order to define a compatible pair of G -convergence and G -Cauchyness, the authors in [8] have introduced the following definition.

Definition 15. We will say that a sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is G -convergent to x_0 if $\{x_n\}$ has a subsequence converging to x_0 (i.e., x_0 is a cluster point of $\{x_n\}$) and $\lim_n M(x_n, x_{n+1}, t) = 1$ for all $t > 0$.

With the previous definition it can be proved that G -convergence and G -Cauchyness is a compatible pair.

3.3. On standard Cauchyess. As commented before, Gregori and Miñana proved the existence of std^* -convergent sequences which are not standard Cauchy and, in consequence, std^* -convergence is not compatible with standard Cauchy. To overcome this inconvenience, they gave the following definition.

Definition 16 (V. Gregori and J-J. Miñana, 2014). A sequence $\{x_n\}$ is called standard convergent if it is convergent and standard Cauchy.

It is easy to verify that standard convergence and standard Cauchyess is a compatible pair.

3.4. On s -convergence and s -Cauchyess. The following stronger concept than convergence, called s -convergence, tries to extend the classical metric formulation of convergence using a simple limit and it has been recently introduced by Gregori and Miñana in [7].

Definition 17. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is s -convergent to $x_0 \in X$ if $\lim_n M(x_n, x_0, \frac{1}{n}) = 1$.

A fuzzy metric space in which every convergent sequence is s -convergent is said to be an s -fuzzy metric space (or M is an s -fuzzy metric on X).

The previous definition allows to characterize s -fuzzy metric spaces as follows.

Proposition 18. Let $(X, M, *)$ be a fuzzy metric space and consider $N(x, y) = \bigwedge_{t>0} M(x, y, t)$ for all $x, y \in X$. Then:

- (i) $(N, *)$ is a stationary fuzzy metric on X .
- (ii) $(X, M, *)$ is an s -fuzzy metric space if and only if $\tau_N = \tau_M$.

The corresponding concept of Cauchyess deduced in a *natural way* from the s -convergence is the following.

Definition 19. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is s^* -Cauchy if $\lim_{n,m} M(x_n, x_m, \frac{1}{n}) = 1$.

Unfortunately, in [7] the authors have proved that there exist s -convergent sequences which are not x^* -Cauchy. Consequently, s^* -Cauchy is not compatible with s -convergence. To overcome this inconvenience, the following definition has been introduced.

Definition 20. A sequence in a fuzzy metric space $(X, M, *)$ is s -Cauchy if $\lim_{m,n} M(x_n, x_m, \frac{m+n}{mn}) = 1$.

It is easy to verify that an s -Cauchy sequence is Cauchy.

Proposition 21. *Every s -convergent sequence in a fuzzy metric space $(X, M, *)$ is s -Cauchy.*

So, the pair s -convergence and s -Cauchyness is a compatible pair.

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Completion of a fractal structure

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ABSTRACT

Fractal structures are somehow equivalent to non archimedean quasi pseudo metrics, though its recursiveness allows its use in many different topics where non archimedean quasi pseudo metrics are not used in a natural way. In this talk, we show a constructive way to construct the (bi)completion of a fractal structure.

1. FRACTAL STRUCTURES AND NON ARCHIMEDEAN QUASI METRICS

Fractal structures were introduced in [1] to study non archimedean quasi metrization, but they have also been used to study other topological or uniform concepts like metrization ([2, 3]), compactification and completion ([4, 5, 7]), topological dimension ([6]), topological properties like normal, paracompact, compactness, etc. ([17]) or continua ([8, 18]). On the other hand, its recursive character allows the use of fractal structures to study some other fields like fractals,

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in particular, self homeomorphic or self similar sets ([9]) or fractal dimension ([10, 11, 12, 13, 14, 15, 16, 19]).

Let X be a set and Γ_1 and Γ_2 be coverings of X . Γ_2 is said to be a strong refinement of Γ_1 if it is a refinement (that is, each element of Γ_2 is contained in some element of Γ_1) and for each $A \in \Gamma_1$ it is satisfied that $A = \cup\{B \in \Gamma_2 : B \subseteq A\}$.

Definition 1. A fractal structure $\mathbf{\Gamma}$ on a set X is a countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ such that each cover Γ_{n+1} is a strong refinement of Γ_n for each $n \in \mathbb{N}$. Cover Γ_n is called level n of the fractal structure.

A fractal structure induces a transitive base of quasi-uniformity, given by $\{U_{\Gamma_n} : n \in \mathbb{N}\}$, where entourages U_{Γ} are defined as $U_{\Gamma} = \{(x, y) \in X \times X : y \notin \bigcup\{A \in \Gamma : x \notin A\}\}$.

We will use the notation $U_{xn} = \{y \in X : (x, y) \in U_{\Gamma_n}\}$, $U_{xn}^{-1} = \{y \in X : (y, x) \in U_{\Gamma_n}\}$ and $U_{xn}^* = U_{xn} \cap U_{xn}^{-1}$.

A quasi pseudo metric on a set X is a function $d : X \times X \rightarrow [0, \infty[$ such that:

- (1) $d(x, x) = 0$, for each $x \in X$.
- (2) $d(x, z) \leq d(x, y) + d(y, z)$ for each $x, y, z \in X$.

d is called a pseudo metric if it also satisfies that $d(x, y) = d(y, x)$ for each $x, y \in X$. A quasi pseudo metric (resp. a pseudo metric) is said to be a quasi metric (resp. a metric) if $d(x, y) = d(y, x) = 0$ implies that $x = y$, for each $x, y \in X$.

If d is a quasi pseudo metric, the function defined by $d^{-1}(x, y) = d(y, x)$ is also a quasi (pseudo) metric, called conjugate quasi (pseudo) metric of d . Furthermore, the function $d^*(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is a (pseudo) metric.

A quasi pseudo metric is said to be non archimedean if $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for each $x, y, z \in X$.

If d is a non archimedean quasi (pseudo) metric, then d^{-1} is also a non archimedean quasi (pseudo) metric and d^* is a non archimedean (pseudo) metric.

Definition 2. A quasi (pseudo) metric d is said to be bicomplete if the (pseudo) metric d^* is complete.

A fractal structure Γ induces a non archimedean quasi pseudo metric d_Γ given by:

$$d_\Gamma(x, y) = \begin{cases} \frac{1}{2^n} & \text{if } y \in U_{xn} \setminus U_{x, n+1} \\ 1 & \text{if } y \notin U_{x1} \end{cases}$$

Conversely, a non archimedean quasi pseudo metric d induces a fractal structure by defining level n as $\Gamma_n = \{B_{d^{-1}}(x, \frac{1}{2^n}) : x \in X\}$, where $B_{d^{-1}}(x, \frac{1}{2^n})$ is the ball with respect to the conjugate quasi pseudo metric d^{-1} , given as usual by $B_{d^{-1}}(x, \frac{1}{2^n}) = \{y \in X : d^{-1}(x, y) < \frac{1}{2^n}\}$.

Proposition 3. *Let $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$ be a fractal structure on a set X . Some properties of the previous defined sets are the following ones:*

- (1) $U_{xn} = X \setminus \bigcup_{x \notin A, A \in \Gamma_n} A$, for each $x \in X$ and $n \in \mathbb{N}$.
- (2) $U_{xn}^{-1} = \bigcap_{x \in A, A \in \Gamma_n} A$, for each $x \in X$ and $n \in \mathbb{N}$.
- (3) $U_{xn}^* = U_{xn} \cap U_{xn}^{-1} = \bigcap_{x \in A, A \in \Gamma_n} A \setminus \bigcup_{x \notin A, A \in \Gamma_n} A$, for each $x \in X$ and $n \in \mathbb{N}$.
- (4) $y \in U_{xn}^*$ if and only if x and y belong exactly to the same elements of level n of the fractal structure.
- (5) $U_{xn}^* = U_{yn}^*$ or $U_{xn}^* \cap U_{yn}^* = \emptyset$ for each $x, y \in X$ and $n \in \mathbb{N}$.
- (6) $G_n = \{U_{xn}^* : x \in X\}$ is a partition of X for each $n \in \mathbb{N}$.

Now, we can construct an extension of X from the sets G_n , defined in the previous proposition, as follow (see [1]):

First, we can define the projection of X onto G_n by $\rho_n : X \rightarrow G_n$ given by $\rho_n(x) = U_{xn}^*$.

Note that G_n is a partially ordered set with the order $\rho_n(x) \leq \rho_n(y)$ iff $y \in U_{xn}$.

Then we define the bonding maps $\phi_n : G_{n+1} \rightarrow G_n$ given by $\phi_n(\rho_{n+1}(x)) = \rho_n(x)$. So we can consider the inverse limit $\varprojlim G_n = \{(g_1, g_2, \dots) \in \prod_{n=1}^{\infty} G_n : \phi(g_{n+1}) = g_n, \forall n \in \mathbb{N}\}$, which will be our extension space, so we will use the notation $\tilde{X} = \varprojlim G_n$. Finally, the embedding of X into the inverse limit is given by $\rho : X \rightarrow \tilde{X}$ defined as $\rho(x) = (\rho_n(x))_{n \in \mathbb{N}}$.

2. COMPLETION OF A FRACTAL STRUCTURE

Next, we define the fractal structure on the extension \tilde{X} .

Definition 4. Let $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$ be a fractal structure on a set X , and \tilde{X} the extension defined in the previous section. We define $\tilde{\Gamma} = \{\tilde{\Gamma}_n : n \in \mathbb{N}\}$, where $\tilde{\Gamma}_n = \{\tilde{A} : A \in \Gamma_n\}$ and $\tilde{A} = \{(\rho_k(x_k))_{k \in \mathbb{N}} \in \tilde{X} : x_n \in A\}$ for each $A \in \Gamma_n$.

If we identify $x \equiv \rho(x)$, then we can see that the previous definition is an extension, not only of the fractal structure, but also of all the structures associated with a fractal structure.

Proposition 5. *Some properties of the fractal structure $\tilde{\Gamma}$ defined on the extension \tilde{X} :*

- (1) $\tilde{A} \cap X = A$, for each $A \in \Gamma_n$ and $n \in \mathbb{N}$.
- (2) $\tilde{A} = \text{Cl}_{d_{\tilde{\Gamma}}}^*(A)$, for each $A \in \Gamma_n$ and $n \in \mathbb{N}$.
- (3) $\tilde{U}_{xn} \cap X = U_{xn}$ for each $x \in X$ and $n \in \mathbb{N}$.
- (4) $\tilde{U}_{xn}^{-1} \cap X = U_{xn}^{-1}$ for each $x \in X$ and $n \in \mathbb{N}$.
- (5) $\tilde{U}_{xn}^* \cap X = U_{xn}^*$ for each $x \in X$ and $n \in \mathbb{N}$.
- (6) $d_{\tilde{\Gamma}}(x, y) = d_{\Gamma}(x, y)$ for each $x, y \in X$.

This extension is a completion in the sense of bicompleteness of the induced quasi metrics.

Theorem 6. *Let Γ be a fractal structure on X . Then $(\tilde{X}, d_{\tilde{\Gamma}})$ is a bicompletion of (X, d_{Γ}) .*

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C-compactness and 2-pseudocompactness in paratopological groups

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ABSTRACT

*A paratopological group (G, τ) can be regarded as a bitopological space (G, τ, τ^{-1}) where τ^{-1} is the conjugate topology of τ . In this paper we summarize several known results related to two bitopological properties of a paratopological groups: *C*-compactness and 2-pseudocompactness.*

1. INTRODUCTION

Let G be a semigroup which is also a topological space. G is said to be a topological semigroup if the operation $\cdot : S \times S \rightarrow S$ is continuous. Following Bourbaki [4], a topological semigroup which is algebraically a group is called a paratopological group. Paratopological groups have received considerable attention in the last decades. The interesting reader can consult the surveys [17, 21]. Topological groups are paratopological groups with the inverse operation continuous.

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A ordered triple (X, σ, τ) is said to be a *bitopological space* (in short, a *bispace*) if both σ and τ are topologies on X . If $\xi(e)$ is the filter of neighborhoods of the identity e of a paratopological group (G, τ) , then the family $\{U^{-1} \mid U \in \xi(e)\}$ defines a topology τ^{-1} , named the conjugate of τ , in such a way that (G, τ^{-1}) is a paratopological group homeomorphic to (G, τ) (but, in general, (G, τ) and (G, τ^{-1}) fail to be isomorphic, see [3, Example 4]). So a paratopological group (G, τ) can be regarded in a natural way as the bispace (G, τ, τ^{-1}) . It is a well-known fact that $(G, \tau \vee \tau^{-1})$ is a topological group which is a Hausdorff space whenever (G, τ) is a T_0 space. So far as the author knows, the first to use this approach in the framework of paratopological groups were Raghavan and Reilly ([12]).

The paper is organized as follows. In the second section we present two useful tools in the study of paratopological groups: the semiregularization and the T_0 reflection of a paratopological group. Section 3 is devoted to C -compact paratopological groups and the last section to 2-pseudocompact paratopological groups. Our terminology and notation is standard. Our basic reference for bispaces is [11] and for paratopological groups is [2].

2. TWO HELPFUL TOOLS

We explain in this section two notions that became useful devices in the theory of paratopological groups: the semiregularization of a paratopological group and the T_0 -reflection of a paratopological group.

Given a topological space (X, τ) Stone [19] and Katětov [10] consider the topology τ_{sr} on X generated by the base consisting of all canonically open sets of the space (X, τ) . This topology is called the semiregularization of the topology τ .

One of the main results regarding the semiregularization of paratopological groups is the following theorem:

Theorem 1 ([13, 5]). *The semiregularization (G, τ_{sr}) of an arbitrary paratopological group (G, τ) is again a paratopological group satisfying the T_3 separation axiom. Therefore, if (G, τ) is Hausdorff, then (G, τ_{sr}) is a regular paratopological*

group. Furthermore, if H is an arbitrary subgroup of G , then the quotient space G/H satisfies the T_3 separation axiom.

Here for T_3 separation axiom it is understood the following: if $x \notin F$ with F a closed subset, then there exist pairwise open subsets U, V such that $x \in U$ and $F \subset V$. Thus, a *regular* space is a space that satisfies $T_1 + T_3$.

We turn now to the notion of T_i -*reflection* of a paratopological group (G, τ) , $T_i(G)$ ($i \in \{0, 1, 2, 3\}$), which is defined as a pair $(H, \varphi_{G,i})$ where H is a paratopological group satisfying the T_i separation axiom and $\varphi_{G,i}$ is a continuous homomorphism of G onto H with the following property: *for every continuous mapping $f: G \rightarrow X$ to a T_i -space X , there exists a continuous mapping $h: H \rightarrow X$ such that $f = h \circ \varphi_{G,i}$, that is, the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\varphi_{G,i}} & H \\ f \downarrow & \swarrow h & \\ X & & \end{array}$$

commutes. Similarly, a *regular reflection* of a paratopological group (G, τ) is a pair $(H, \varphi_{G,r})$, where H is a regular paratopological group and $\varphi_{G,r}$ is a continuous homomorphism of G onto H such that every continuous mapping of G to a regular space admits a continuous factorization through $\varphi_{G,r}$. The corresponding group H is denoted by $Reg(G)$.

Abusing of terminology, we will usually refer to $T_0(G)$, $T_1(G)$, $T_2(G)$, and $Reg(G)$ as the T_0 -, T_1 -, Hausdorff, and regular reflection, respectively, of the paratopological group (G, τ) . It is shown in [20] that for every paratopological group (G, τ) , the T_0 -, T_1 -, Hausdorff, and regular reflections of (G, τ) exist and are unique up to a topological isomorphism. The T_0 -refelction can be easily described:

Theorem 2 ([20, Theorem 3.1]). *Let (G, τ) be a paratopological group, $P = \bigcap \{N \mid N \in \xi(e)\}$, and $K = P \cap P^{-1}$. Then K is an invariant subgroup of G and $(G/K, \pi)$ is the T_0 -refelction of G , where $\pi: G \rightarrow G/K$ is the quotient homomorphism. Further, the homomorphism π satisfies $U = \pi^{-1}(\pi(U))$, for each open set $U \subseteq G$. Moreover (G, τ) is a topological group if and only if its T_0 -refelction is a topological group.*

3. C -COMPACT PARATOPOLOGICAL GROUPS

A function f from a bispace (X, σ, τ) into a bispace (Y, σ^*, τ^*) is said to be bi-continuous if both $f: (X, \sigma) \rightarrow (Y, \sigma^*)$ and $f: (X, \tau) \rightarrow (Y, \tau^*)$ are continuous functions. The letters u and l stand for the upper topology and the lower topology, respectively, on the reals \mathbb{R} . The family of all continuous functions from a bispace (X, σ, τ) into (\mathbb{R}, u, l) will denote by $BC(X, \sigma, \tau)$.

A bispace (X, σ, τ) is said to be C -compact if $(f \vee g)(X)$ is compact in (\mathbb{R}, u, l) for every $f \in BC(X, \sigma, \tau)$ and every $g \in BC(X, \tau, \sigma)$. This concept was introduced in [9] and characterized in [9, Theorem 1.7]. A paratopological group (G, τ) is named C -compact if the bispace (G, τ, τ^{-1}) is C -compact and it is said to be locally C -compact if the identity has a C -compact neighborhood. The most important result about (locally) C -compact paratopological groups is the following theorem. Recall that a Tychonoff space (X, τ) is pseudocompact if every continuous real-valued function on X is bounded.

Theorem 3 ([16, Proposition 3.4], [18, Theorem 4]). *A T_0 (locally) C -compact paratopological group is a (locally) pseudocompact topological group.*

In this type of situations it highlights the importance of the T_0 -reflexion. In fact, it is easy to show that (locally) C -compactness is preserved in the passage from a paratopological group (G, τ) to its T_0 -reflexion. Thus, we can remove the T_0 separation axiom of the previous theorem.

Theorem 4. *Every (locally) C -compact paratopological group is a (locally) pseudocompact topological group.*

4. 2-PSEUDOCOMPACT PARATOPOLOGICAL GROUPS

A bispace (X, σ, τ) is said to be 2-pseudocompact if every bicontinuous function $f: (X, \sigma, \tau) \rightarrow (\mathbb{R}, u, l)$ is bounded in \mathbb{R} . The concept of 2-pseudocompactness was considered implicitly by Brümmer in [6] and by Brümmer and Salbany in [7]. It was explicitly introduced and systematically studied in [9].

In this section we present some results on paratopological groups (G, τ) such that (G, τ, τ^{-1}) is a 2-pseudocompact bispaces (in short, 2-pseudocompact paratopological groups). In this setting, the following characterization of 2-pseudocompactness will be useful.

Theorem 5 ([9, Corrolary 1.5]). *A bispaces (X, σ, τ) is 2-pseudocompact if and only if for every decreasing sequence $(U_n)_{n < \omega}$ of non-empty σ -open subsets of X , and for every decreasing sequence $(V_n)_{n < \omega}$ of non-empty τ -open subsets of X , we have $\bigcap_{n < \omega} \text{cl}_{(X, \tau)} U_n \neq \emptyset$ and $\bigcap_{n < \omega} \text{cl}_{(X, \sigma)} V_n \neq \emptyset$.*

In particular, the previous theorem implies that a bispaces (X, σ, τ) is 2-pseudocompact whenever the topologies σ and τ are countably compact. Thus, a countably compact paratopological group (G, τ) is 2-pseudocompact.

The following proposition is straightforward

Proposition 6. *A paratopological group (G, τ) is 2-pseudocompact if, and only if, for every decreasing sequence $(U_n)_{n < \omega}$ of non-empty open sets, we have*

$$\bigcap_{n < \omega} \text{cl}_{(G, \tau)} U_n^{-1} \neq \emptyset.$$

Some authors take the above property as the definition of a 2-pseudocompact paratopological group: the reason is that this property involves the topology τ but not its conjugate.

It is worth noting that a 2-pseudocompact paratopological can fail to be a topological group. A topological space (X, τ) is said to be *feebly compact* if every locally finite family of pairwise disjoint open sets is finite (no separation axioms are assumed). Notice that for Tychonoff spaces feebly compactness is equivalent to pseudocompactness.

Example 7 ([18, Theorem 1]). There exists a feebly compact, 2-pseudocompact Hausdorff paratopological group G with the Fréchet-Urysohn property which is not a topological group.

A useful property of 2-pseudocompact paratopological groups is given in the following theorem. Recall that a topological space (X, τ) is said to be a Baire space if the intersection of every decreasing sequence of dense open sets is a dense set.

Theorem 8 ([1, Theorem 2.2]). *Every 2-pseudocompact paratopological group is a Baire space.*

The following proposition characterizes 2-pseudocompact group which are topological groups. Recall that a topological group (G, τ) is called ω -bounded if for every neighborhood U of the identity, there exists a countable set A such that $G = AU$.

Theorem 9 ([1, Theorem 2.4]). *For a paratopological group (G, τ) , the following conditions are equivalent:*

- (i) (G, τ) is a pseudocompact topological group.
- (ii) (G, τ) is a 2-pseudocompact paratopological group and $(G, \tau \vee \tau^{-1})$ is an ω -bounded topological group.

The minimum cardinality of a family \mathcal{U} of open sets in a paratopological group (G, τ) such that $\bigcap_{U \in \mathcal{U}} U = \{e\}$ is called the pseudocharacter of (G, τ) . Two interesting particular cases of Theorem 9 are:

Theorem 10 ([14, Proposition 6]). *Each 2-pseudocompact paratopological group of countable pseudocharacter is a topological group.*

Theorem 11 ([14, Lemma 10]). *Every T_3 2-pseudocompact paratopological group is a pseudocompact topological group.*

The previous theorem allows us to point out how the semiregularization works in the field of paratopological groups. Throughout what follows, we shall freely use without explicit mention the elementary fact that a topological space is feebly compact if, and only if, its semiregularization is feebly compact.

Theorem 12 ([14, Proposition 2]). *Each 2-pseudocompact paratopological group is feebly compact.*

Proof. Let (G, τ) be such a group. It is an easy matter to show that the semiregularization (G, τ_{sr}) of the paratopological group (G, τ) is 2-pseudocompact. Since (G, τ_{sr}) is a T_3 space, Theorem 11 tells us that (G, τ_{sr}) is a pseudocompact topological group. Thus, (G, τ) is feebly compact. \square

The converse of Theorem 12 fails to be true: there exists a Hausdorff feebly compact paratopological group which is a Baire space and that it is not 2-pseudocompact ([18, Theorem 2]).

An outstanding theorem by Comfort and Ross states that an arbitrary product of Hausdorff pseudocompact topological groups is a pseudocompact topological group ([8]). By means of the notion of semiregularization and the concept of T_0 -reflection of a paratopological group is not difficult to show that an arbitrary product of feebly compact paratopological groups is a feebly compact paratopological group (see [15, Proposition 20] for details). Thus, a natural question to ask is whether this result continues to hold for 2-pseudocompact paratopological groups.

Question ([15, Problem 5]) Is an arbitrary product of 2-pseudocompact paratopological groups 2-pseudocompact? What about the product of two 2-pseudocompact paratopological groups?

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