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A spatio-temporal approach to brain dynamics

Joan C. Micó
Departament de Matemàtica Aplicada. Universitat Politècnica de Valéncia
jmico@mat.upv.es

Maria T. Sanz
Departament de Didàctica de les Matemàtiques. Universitat de València
m.teresa.sanz@uv.es

Abstract

A spatio-temporal response model of brain activity is presented in this paper. The model is a non-autonomous reaction-diffusion model, i.e., a non-autonomous parabolic partial differential equation. The model is deduced as a spatial generalization of the time response model presented in other works. The analytical solution of the model is obtained for an idealized brain geometry given by a parallelepiped. The boundary conditions provide the quantization of the brain frequencies and the initial conditions are needed to obtain the coefficients of a Fourier series that take part of the model solution. The ways of validation are discussed, particularly those related with the brain resting state, and those related with getting the theoretical values of the so-called brain wave frequencies and comparing them with the experimental values. Some generalization hypotheses are commented to the model becomes more realistic and useful in a future investigation.

Key words: brain activity; spatio-temporal dynamics; partial differential equation; brain geometry.

Resumen

Se presenta en este artículo un modelo de respuesta espacio-temporal de la actividad cerebral. El modelo es una ecuación de reacción-difusión no autónoma, es decir, una ecuación en derivadas parciales parabólica y no autónoma. El modelo es una generalización espacial del modelo de respuesta temporal presentado en otros artículos. La solución analítica del modelo se obtiene para una geometría idealizada del cerebro dada por un paralelepípedo. Las condiciones de contorno proporcionan la cuantificación de las frecuencias del cerebro y las condiciones iniciales son necesarias para obtener los coeficientes de una serie de Fourier que forman parte de la solución del modelo. Se describen formas teóricas de validación, en particular la relacionada con el estado del cerebro en reposo, y la relacionada con la obtención de los valores teóricos de las frecuencias de las conocidas ondas cerebrales. Se comentan algunas hipótesis para generalizar el modelo, con el fin de hacerlo más realista y útil en una investigación futura.

Palabras clave: actividad cerebral; dinámica espacio-temporal; ecuación en derivadas parciales; geometría cerebral.

1. Introduction

A spatio-temporal model of brain dynamics is presented. It is here called as the *spatio-temporal response model*. The prediction of the spatial brain activity from a known initial one is possible with this model. Its mathematical structure is deduced from the *time response model* that predicts the *time brain activity* as a consequence of a stimulus, particularly of a stimulant drug (Amigó et al., 2008, 2010; Amigó et al., in press; Caselles et al. 2011, Caselles et al., 2012).

Let us difference between the *time brain activity*, which no details of spatial structure, and the *spatio-temporal brain activity*, which is a spatial-density (in brain volume) distribution of the brain activity. In fact, the *time brain activity* can be deduced from the *spatio-temporal brain activity* at any time as the spatial integral over the brain spatial domain.

On a hand, the *time response model* is a non autonomous first order differential equation whose dynamics depends on the particular stimulus. The *time response model* permits to compute the dynamics of the *time brain activity*. Its dynamics has an attractor, the tonic level, particular of each individual brain, to which every dynamics tends as the stimulus vanishes. On the other hand, the *spatio-temporal response model* permits to compute the dynamics of the *spatio-temporal brain activity*. It is a non-autonomous parabolic differential equation, which has been deduced from the *response model*, considering the following hypotheses:

1. Many stimuli of different kinds can affect the brain activity.

2. It has not a tonic level, but this level is substituted by the continuous presence of the stimulus that is a consequence of the heart beatings.

3. The spatio-temporal dynamics is obtained by substituting the time dependence on the *global brain activity* by the spatio-temporal dependence on the *brain activity*.

4. The addition of a diffusion term can describe the particular spatial dynamics.

The analytical solution of the *spatio-temporal response model* is obtained for an idealized brain given by a parallelepiped of known dimensions. The boundary conditions given by the null flow on the brain walls provide the quantization of the wave numbers, and thus, of the brain frequencies. In addition, the initial conditions provide the coefficients of a tridimensional Fourier series that takes part of the analytical solution.

A theoretical experimental design is stated to validate the model in a future. It consists in studying the model predictions of the brain resting state, i.e., the state given by the only stimulus produced by the heart beatings. Another way to validate the model is comparing the quantized brain frequencies with the experimental frequencies of the so-called *brain waves*. The investigation of the actual speed signal in brain is essential to set out this comparing.

Section 2 is devoted to explain the mathematical structure and meaning of the *time response model*. In Section 3 the *spatio-temporal response model* is deduced from the *time response model*. Section 4 is devoted to obtain the analytical solution of the *spatio-temporal response model*, including the boundary and the initial conditions, presenting the theoretical experimental design to validate it. Section 5 provides some results as numerical solutions. Section 6 is devoted to a short discussion of the *brain waves* from the model perspective and its possible use to validate the model. Section 7 discusses the main results of the paper and future ways of research about the topic.

All computations have been done by using MATHEMATICA 8.0.1.0.

2. The time response model

Let $y(t)$, b and y_0 be respectively the time brain activity variable of brain, its tonic level and its initial value. The dynamics of the time brain activity as a consequence of different stimuli $s_i(t)$, i=1, 2, …, n, (which can be of different natures: blood flow as a consequence of heart beating, the amount in blood of drug non consumed by cells, etc.), is given by:

$$\left.\begin{array}{c}\frac{dy(t)}{dt} = a\big(b - y(t)\big) + \sum_i p_i \cdot s_i(t) - \\ - \sum_i q_i \cdot s_i(t - \tau_i) \cdot y(t - \tau_i) \\ y(0) = y_0\end{array}\right\} \quad (1)$$

Where t is time; $a\big(b - y(t)\big)$ is the *homeostatic control*, i.e., the cause of the fast recovering of the tonic level b, being a the "power" of this control; $p_i \cdot s_i(t)$ are the different *excitation effects*, which tend to increase the time brain activity, being p_i the *excitation effect powers*; $q_i \cdot s_i(t - \tau_i) \cdot y(t - \tau_i)$ are the different *inhibitor effects*, which tend to decrease the time brain activity and are the cause of the its slow recovering, being q_i the *inhibitor effect powers* and being τ_i the *inhibitor effect delays*, times after which the inhibitor effects take place (which means that an "all or nothing" effect occurs, similar to the electrochemical transmission in the neuron axon).

Equation (1) defines the time response model for the time brain activity. Amigó et al. (2008) demonstrate that the time response model explains the evolution of the time brain activity as a consequence of a unique stimulus given by a stimulant drug. It reproduces the dynamic patterns forecasted by Solomon and Corbit (1974), Grossberg (2000) and Amigó (2005), and it can be considered theoretically validated through the scientific literature about the subject (Amigó et al., 2008). It has been also validated experimentally when the unique stimulus is caffeine in the work of Caselles et al., (2011), and for a continuous-delayed version, when the unique stimulus is methylphenidate, by Micó et al., (2012).

Two considerations must be done to simplify (1). On a hand, brain is here considered as an open system. In fact, at least in the resting state, one stimulus is influencing on it: the stimulus of blood flow as a consequence of heart beating. Thus, the tonic level must be consequence of this stimulus, and considering b=0 is correct, because this stimulus is always present. On the other hand, the inhibitor effect delays are considered also to be zero, in order to study the most simplified model. This particular time response model without delays has been validated by Micó et al. (2008) when the stimulus is caffeine. The equation obtained under these two considerations is the following:

$$\left.\begin{array}{c}\frac{dy(t)}{dt} = -a \cdot y(t) + sp(t) - sq(t) \cdot y(t) \\ y(0) = y_0\end{array}\right\} \quad (2\text{-}1)$$

Where:

$$sp(t) = \sum_i p_i \cdot s_i(t) \quad (2\text{-}2)$$

$$sq(t) = \sum_i q_i \cdot s_i(t) \quad (2\text{-}3)$$

Equation (2-1), together (2-2) and (2-3), is the time response model from which the spatio-temporal response model is deduced. Taking into account that it is a linear first order differential equation, its analytical solution is:

$$y(t) = e^{-a \cdot t - \int_0^t sq(u)du}\left(y_0 + \int_0^t dv\, ps(v) \cdot e^{a \cdot v + \int_0^v sq(u)du}\right) \quad (3)$$

3. The spatio-temporal response model

The method followed to obtain the spatio-temporal response model of the spatio-temporal brain activity is:

1. Considering that the time brain activity variable *y(t)* must be substituted by a field that represents the spatio-temporal brain activity as a space-density depending on time t and on the three space rectangular variables $\boldsymbol{r} = (x_1, x_2, x_3)$. Then, the time derivative in (2-1) must be a time partial derivative. Let $\Omega(t, \boldsymbol{r})$ be this field, thus:

$$y(t) = \iiint_{\partial V} \Omega(t, \boldsymbol{r}) d\boldsymbol{r} \quad (4)$$

In (4), ∂V is the integration volume of $\boldsymbol{r} = (x_1, x_2, x_3)$ that depends on the brain geometry considered.

2. The space dynamics dependence in (2-1) is introduced as a diffusion phenomenon through a Laplacianne function of $\Omega(t,\boldsymbol{r})$.

From items 1 and 2, Equation (2-1) becomes:

$$\left.\begin{array}{l}\frac{\partial\Omega(t,r)}{\partial t}=-a\cdot\Omega(t,\boldsymbol{r})+sp(t)\\ \quad -sq(t)\cdot\Omega(t,\boldsymbol{r})\\ \quad +\sigma\cdot\nabla^2\Omega(t,\boldsymbol{r})\\ \quad \Omega(0,\boldsymbol{r})=\Omega_0(\boldsymbol{r})\end{array}\right\} \quad (5)$$

In (5), σ is the diffusion coefficient, here considered positive-valued, and the other parameters conserve the same meaning than in (2-1) (and they are also positive-valued). The initial conditions in *t*=0 must be provided through the spatial distribution of the brain activity, $\Omega_0(\boldsymbol{r})$, in this instant. In addition, the boundary conditions must be also provided, depending on the brain geometry considered. They are stated in Section 5 for parallelepiped geometry.

Equation (5) is the sought spatio-temporal response model. It is a parabolic partial differential equation, with a mathematical structure similar to a non-autonomous reaction-diffusion model. This structure seems to be logical due to the pulse translation on a neuron axon is done by a reaction-diffusion model in the direction of an only spatial direction given by the axon (Scott, 2002). Thus, the spatio-temporal response model can be considered as a three dimensional generalization of this model.

4. Analytical solution of the spatio-temporal response model

Consider the following formal solution for (5):

$$\Omega(t,\boldsymbol{r})=\varphi(t)+\Phi(t,\boldsymbol{r}) \quad (6)$$

Substituting (6) in (5):

$$\varphi'(t)+\frac{\partial\Phi(t,\boldsymbol{r})}{\partial \mathrm{t}}=-a\cdot\varphi(t)-a\cdot\Phi(t,\boldsymbol{r})$$
$$+sp(t)-sq(t)\cdot\varphi(t)-sq(t)\cdot\Phi(t,\boldsymbol{r})$$
$$+\sigma\cdot\nabla^2\Phi(t,\boldsymbol{r}) \quad (7)$$

In (7) we force to the following equation holds:

$$\varphi'(t)+\big(a+sq(t)\big)\varphi(t)=sp(t) \quad (8)$$

Equation (8) is a linear first order differential equation, whose solution is:

$$\varphi(t)=\mathrm{e}^{-a\cdot t-\int_0^t sq(u)du}\Big(k+\int_0^t dv\, ps(v)\cdot\mathrm{e}^{a\cdot v+\int_0^v sq(u)du}\Big) \quad (9)$$

If (8) holds, then (7) becomes:

$$\frac{\partial\Phi(t,r)}{\partial \mathrm{t}}=-(a+sq(t))\Phi(t,\boldsymbol{r})+\sigma\cdot\nabla^2\Phi(t,\boldsymbol{r}) \quad (10)$$

Equation (10) can be solved by separating variables:

$$\Phi(t,\boldsymbol{r})=\theta(t)\cdot\omega(\boldsymbol{r}) \quad (11)$$

Whose substitution in (10) and subsequent division by the product $\theta(t)\cdot\omega(\boldsymbol{r})$ provides:

$$\frac{\theta'(\mathrm{t})}{\theta(t)}+a+sq(t)=\sigma\cdot\frac{\nabla^2\omega(\boldsymbol{r})}{\omega(\boldsymbol{r})} \quad (12)$$

In order to (12) holds, both members of the equation must be a constant. Let δ be this constant. From the temporal part:

$$\theta'(\mathrm{t})+\big(a-\delta+sq(t)\big)\vartheta(\mathrm{t})=0 \quad (13)$$

Whose solution by separating variables is:

$$\theta(t) = \theta_0 e^{-a\cdot t - \int_0^t sq(u)du + \delta\cdot t} \qquad (14)$$

Being θ_0 a constant. From the spatial part of (12):

$$\nabla^2 \omega(\boldsymbol{r}) = \frac{\delta}{\sigma}\omega(\boldsymbol{r}) \qquad (15)$$

Which is a Helmoltz equation, and it can be solved by separating variables for many coordinate systems. The solution here considered is obtained choosing the rectangular coordinates, idealizing the brain geometry by a parallelepiped of dimensions L_1 (depth), L_2 (length) and L_3 (height), that is:

$$\boldsymbol{r} = (x_1, x_2, x_3) \in \left[-\frac{L_1}{2}, \frac{L_1}{2}\right] \times \left[-\frac{L_2}{2}, \frac{L_2}{2}\right]\left[-\frac{L_3}{2}, \frac{L_3}{2}\right] \qquad (16)$$

Taking into account that the Laplacianne operator takes de following form:

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \qquad (17)$$

Thus, separating variables in (15) as:

$$\omega(\boldsymbol{r}) = \omega_1(x_1)\cdot\omega_2(x_2)\cdot\omega_3(x_3) \qquad (18)$$

And substituting (18) in (15), and subsequently dividing by the product $\omega_1(x_1)\cdot\omega_2(x_2)\cdot\omega_3(x_3)$:

$$\frac{1}{\omega_1}\frac{d^2\omega_1}{dx_1^2} + \frac{1}{\omega_2}\frac{d^2\omega_2}{dx_2^2} + \frac{1}{\omega_3}\frac{d^2\omega_3}{dx_3^2} = \frac{\delta}{\sigma} \qquad (19)$$

In order to (19) holds, each member of the addition must be a constant. Let $-k_i^2$ be, i=1, 2, 3, these constants. They must be negative-valued to obtain an oscillatory dynamics, thus:

$$\frac{1}{\omega_i}\frac{d^2\omega_i}{dx_i^2} = -k_i^2 \qquad i=1, 2, 3. \qquad (20)$$

And from (19) and (20):

$$\delta = -\sigma(k_1^2 + k_2^2 + k_3^2) \qquad (21)$$

From (20):

$$\omega_i(x_i) = A_i\cos(k_i x_i) + B_i\sin(k_i x_i) \qquad i=1, 2, 3. \qquad (22)$$

being A_i and B_i constants. Considering (6), (11) and (15) and putting as common factor the term $\exp\left(-a\cdot t - \int_0^t sq(u)du\right)$, the sought analytical solution is obtained:

$$\Omega(t,\boldsymbol{r}) = \varphi(t) + \theta(t)\cdot\omega(\boldsymbol{r}) = e^{-a\cdot t - \int_0^t sq(u)du}\left(k + \int_0^t dv\, ps(v)\cdot e^{a\cdot v + \int_0^v sq(u)du} + e^{-\sigma(k_1^2+k_2^2+k_3^2)\cdot t}\prod_{i=1}^3 (A_i\cos(k_i x_i) + B_i\sin(k_i x_i))\right) \qquad (23)$$

Observe in (23) that the constant θ_0 has been integrated in the constants A_i and B_i.

Two boundary conditions can be stated for (23): (a) it cancels on the brain walls; and (b) its spatial flow through the brain walls cancels. The experimental cases show that brain activity can be measured on the brain walls, thus, (b) seems to be the most plausible hypothesis. That is:

$$\left.\frac{\partial\Omega(t,\boldsymbol{r})}{\partial x_i}\right|_{x_i = L_i/2} = 0 \qquad i=1, 2, 3. \qquad (24\text{-}1)$$

$$\left.\frac{\partial\Omega(t,\boldsymbol{r})}{\partial x_i}\right|_{x_i = -L_i/2} = 0 \qquad i=1, 2, 3. \qquad (24\text{-}2)$$

The most general case for which (24-1) and (24-2) hold is (take into account the parity properties of the sine and cosine functions):

$$-k_i \cdot A_\mathrm{i} \sin\left(k_i \frac{L_\mathrm{i}}{2}\right) + k_i \cdot B_i \cos\left(k_i \frac{L_\mathrm{i}}{2}\right) = 0 \quad (25\text{-}1)$$

$$k_i \cdot A_\mathrm{i} \sin\left(k_i \frac{L_\mathrm{i}}{2}\right) + k_i \cdot B_i \cos\left(k_i \frac{L_\mathrm{i}}{2}\right) = 0 \quad (25\text{-}2)$$

In (25-1) and (25-2), i=1, 2, 3. The system holds determined if either A_i or B_i cancels. Due to the sine and the cosine functions have similar topological properties, we choose arbitrarily canceling B_i. Thus:

$$\sin\left(k_i \frac{L_\mathrm{i}}{2}\right) = 0 \to k_i \frac{L_\mathrm{i}}{2} = n_i\pi \to k_i = \frac{2\pi}{L_i} n_i$$

$$n_i = 1,2,\ldots \quad i=1,2,3. \quad (26)$$

Equation (26) represents the quantization of spatio-temporal brain activity as function of three positive integers. Substituting these values in (23), taking into account that $B_\mathrm{i} = 0$, and applying the superposition principle (Equation (5) is a linear partial differential equation):

$$\Omega(t,\boldsymbol{r}) = \mathrm{e}^{-a\cdot t - \int_0^t sq(u)du}\Bigg(k + \int_0^t dv\, ps(v) \cdot \mathrm{e}^{a\cdot v + \int_0^v sq(u)du} + \sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty} A_{n_1 n_2 n_3} \mathrm{e}^{-4\pi^2 \sigma \cdot t \cdot \sum_{i=1}^{3}\left(\frac{n_i}{L_i}\right)^2} \cdot \prod_{i=1}^{3} \sin\left(\frac{2\pi\cdot n_i}{L_i} x_i\right)\Bigg) \quad (27)$$

In (27) the sums run from $n_i = 1$ to $+\infty$, i=1,2,3.

Stating the initial condition of Equation (5) for (27):

$$\Omega_0(\boldsymbol{r}) = k + \sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty} A_{n_1 n_2 n_3} \prod_{i=1}^{3} \sin\left(\frac{2\pi\cdot n_i}{L_i} x_i\right) \quad (28)$$

In order to find the values of the k constant, the three-dimensional integral can be applied to the two members of Equation (28) over the integration domain established for the rectangular coordinates of brain in (16):

$$\int_{-\mathrm{L}_3/2}^{\mathrm{L}_3/2}\int_{-\mathrm{L}_2/2}^{\mathrm{L}_2/2}\int_{-\mathrm{L}_1/2}^{\mathrm{L}_1/2} \mathrm{dx}_3\, \mathrm{dx}_2\, \mathrm{dx}_1\, \Omega_0(\boldsymbol{r}) = k \int_{-\mathrm{L}_3/2}^{\mathrm{L}_3/2}\int_{-\mathrm{L}_2/2}^{\mathrm{L}_2/2}\int_{-\mathrm{L}_1/2}^{\mathrm{L}_1/2} \mathrm{dx}_3\, \mathrm{dx}_2\, \mathrm{dx}_1 + \sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty} A_{n_1 n_2 n_3} \int_{-\mathrm{L}_1/2}^{\mathrm{L}_1/2} \mathrm{dx}_1 \sin\left(\frac{2\pi\cdot n_1}{L_1} x_1\right) \cdot \int_{-\mathrm{L}_2/2}^{\mathrm{L}_2/2} \mathrm{dx}_2 \sin\left(\frac{2\pi\cdot n_2}{L_2} x_2\right) \cdot \int_{-\mathrm{L}_3/2}^{\mathrm{L}_3/2} \mathrm{dx}_3 \sin\left(\frac{2\pi\cdot n_3}{L_3} x_3\right) \cdot \quad (29)$$

Taking into account that the first term of (29) can be interpreted as coming from Equation (4) for t=0, and the fact that any sine integral is zero, Equation (29) provides $K = y_0/V$, where $V = L_1 \cdot L_2 \cdot L_3$ is the brain volume, that is:

$$\Omega_0(\boldsymbol{r}) = \frac{y_0}{L_1\cdot L_2\cdot L_3} + \sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty} A_{n_1 n_2 n_3} \prod_{i=1}^{3} \sin\left(\frac{2\pi\cdot n_i}{L_i} x_i\right) \quad (30)$$

Multiplying in (30) by $\prod_{i=1}^{3} \sin\left(\frac{2\pi\cdot m_i}{L_i} x_i\right)$, for arbitrary integers $m_i = 1,2,\ldots$, i=1, 2, 3, in the two parts of the equation, taking the triple integral over the same domain, and considering on a hand the fact that any sine integral is zero, and on the other hand that:

$$\int_{-\mathrm{L_i}/2}^{\mathrm{L_i}/2} \mathrm{dx_i} \sin\left(\frac{2\pi\cdot n_i}{L_i} x_i\right) \cdot \sin\left(\frac{2\pi\cdot m_i}{L_i} x_i\right) = \begin{cases} \frac{L_i}{2} : & n_i = m_i \\ 0 : & n_i \neq m_i \end{cases} \quad n_i = 1,2,\ldots \quad m_i = 1,2,\ldots$$

i=1, 2, 3

(31)

The conclusions are:

$$y_0 = \prod_{i=1}^{3} \int_{-\mathrm{L_i}/2}^{\mathrm{L_i}/2} dx_\mathrm{i}\, \Omega_0(\boldsymbol{r}) \quad (32\text{-}1)$$

$$A_{n_1 n_2 n_3} = \frac{8}{L_1 \cdot L_2 \cdot L_3} \prod_{i=1}^{3} \int_{-L_i/2}^{L_i/2} dx_i \sin\left(\frac{2\pi \cdot n_i}{L_i} x_i\right) \cdot \cdot \Omega_0(\boldsymbol{r}) \quad (32\text{-}2)$$

Equations (32-1) and (32-2) provide respectively the values y_0 and $A_{n_1 n_2 n_3}$ from the $\Omega_0(\boldsymbol{r})$ function. Thus, either the mathematical structure of this function in *t*=0 must be known or a data base with which to obtain the three-dimensional Fourier series must be provided. For instance, suppose that the initial condition $\Omega_0(\boldsymbol{r})$ is given by:

$$\Omega_0(\boldsymbol{r}) = \frac{15}{L_1 \cdot L_2 \cdot L_3} + x_1 \cdot x_2 \cdot x_3 \quad (33)$$

Figure 1 shows the tridimensional space distribution of the initial condition (33). It represents the activation density in the different parts of brain. In addition, its integral (32-1) provides the value $y_0 = 15$, belonging to the *GFP-MAACLR* scale [0,60]. See Amigó et al., (2010), Caselles et al. (2010) or Micó et al. (2012) for the way to evaluate dynamically the global brain activity with this scale.

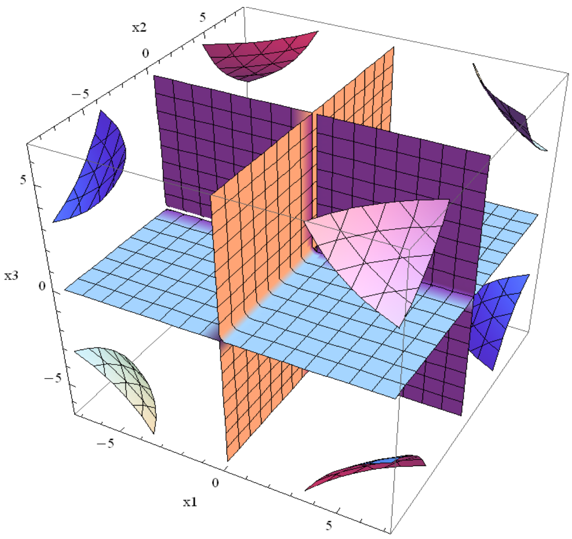

Figure 1: Spatial brain activity of the initial condition.

On the other hand, the Fourier coefficients $A_{n_1 n_2 n_3}$ must be computed through (32-2):

$$A_{n_1 n_2 n_3} = -\frac{L_1 \cdot L_2 \cdot L_3}{\pi^3 n_1 \cdot n_2 \cdot n_3} (-1)^{n_1+n_2+n_3} \quad (34)$$

Thus, taking into account this initial condition, from (33) and (34), Equation (27) can be rewritten as:

$$\Omega(t, \boldsymbol{r}) = \mathrm{e}^{-a \cdot t - \int_0^t sq(u)du} \left(\frac{15}{L_1 \cdot L_2 \cdot L_3} + \int_0^t dv\, p \cdot s(v) \cdot \mathrm{e}^{a \cdot v + \int_0^v sq(u)du} - -\frac{L_1 \cdot L_2 \cdot L_3}{\pi^3} \sum_{n_1}^{\infty} \sum_{n_2}^{\infty} \sum_{n_3}^{\infty} \frac{(-1)^{n_1+n_2+n_3}}{n_1 \cdot n_2 \cdot n_3} \mathrm{e}^{-4\pi^2 \sigma \cdot t \cdot \sum_{i=1}^{3} \left(\frac{n_i}{L_i}\right)^2} \cdot \prod_{i=1}^{3} \sin\left(\frac{2\pi \cdot n_i}{L_i} x_i\right) \right) \quad (35)$$

Observe that in order to obtain the exact solution of (35), the three arising sums are separable and they take the form:

$$\sum_{n_i=1}^{\infty} \frac{(-1)^{n_i}}{n_i} \mathrm{e}^{-\sigma \cdot t \left(\frac{2\pi \cdot n_i}{L_i}\right)^2} \cdot \sin\left(\frac{2\pi \cdot n_i}{L_i} x_i\right) \quad i=1,2,3 \quad (36)$$

The sums (36) should be computed in order to complete the analytical solution. By the moment no analytical solution has been found.

5. Numerical predictions of the model

By the moment, the ignorance of any spatial brain activity distribution in different periods, and the impossibility to perform an experimental design to obtain this quantitative information, avoids a model validation. On the other hand, some theoretical problems, such as the outcomes of the sums (36), must be solved. However, even these sums are obtained for a theoretical example rather than an experimental one. Thus, two numerical model predictions are presented below on the base of the theoretical initial condition (33).

Before obtaining the model predictions, a first question to elucidate is: which are the mathematical structures of the different stimuli that influence the brain dynamics?. It seems obvious that these structures depend on the kind of stimulus. The stimulus chosen for the theoretical prediction is the one produced by the blood flow due to the heart beatings. We suppose that this mathematical structure is the same presented by Amigó et al. (2008) for a stimulant drug:

$$s(t) = \begin{cases} \frac{\alpha \cdot M}{\beta - \alpha}(\exp(-\alpha \cdot t) - \exp(-\beta \cdot t)) : \alpha \neq \beta \\ \alpha \cdot M \cdot t \cdot \exp(-\alpha \cdot t) : \alpha = \beta \end{cases} \quad (37)$$

In (37), M is the initial amount of blood that crosses the brain walls, α is the blood assimilation rate and β is the blood distribution rate. Observe that the continuous presence of the blood flow stimulus on brain is always assumed. Thus, the fact is assumed that the only presence of this stimulus reproduces the resting state of the brain. Figure 2 shows its dynamics during a period of 10 seconds, for an individual with one heart beating per second. The parameter values have been calibrated by a trial and error process.

A second question is the brain dimensions. The normal values used for a mean brain are L_1 = 14 cm (depth), L_2 = 17 cm (length) and L_3 = 13 cm (height).

The third question is related with the values of the model parameters α, β, p, q, a and σ. They will also be calibrated by a trial and error process.

Figures 3 and 4 present respectively the numerical predictions of Equation (5) for the brain activity after 5 seconds and 10 seconds.

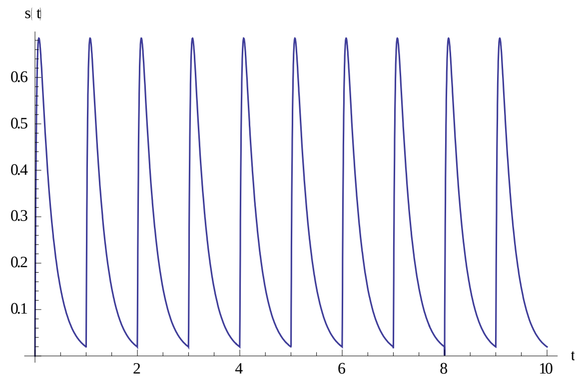

Figure 2: Dynamics of a heart beating during 10 seconds for an individual that has one heart beating per second. Parameter values: α = 4.0 s^{-1}; β = 30.0 s^{-1}; M = 7.0 ml.

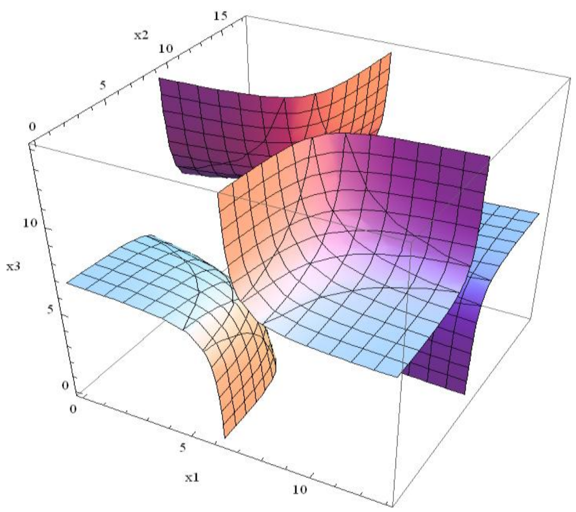

Figure 3: Numeral prediction of brain activity at 5 seconds.

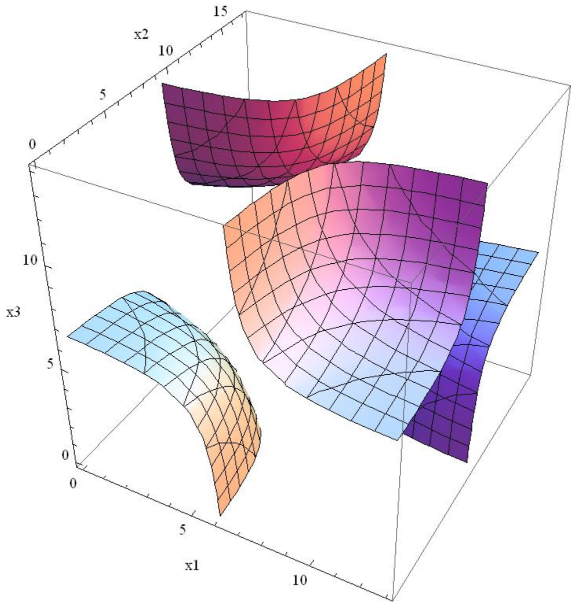

Figure 4: Numeral prediction of brain activity at 10 seconds.

6. The brain waves

Another way to validate the spatio-temporal response model is to obtain the frequencies of the so-called brain waves from the model and to compare them with the experimental values known. The key to this comparing is the set of k_i parameters from Equation (26). They can be interpreted as the three components of a vector $\boldsymbol{k} = (k_1, k_2, k_3)$, whose module k is the inverse of the wave length λ, that is:

$$\frac{1}{\lambda} = k = \sqrt{k_1^2 + k_2^2 + k_3^2} \qquad (38)$$

In addition, due to λ is the division between of the wave frequency ν and the wave speed c:

$$\lambda = \frac{\nu}{c} \qquad (39)$$

From (26), (38), (39):

$$\nu = 2\pi \cdot c \cdot \sqrt{\left(\frac{n_1}{L_1}\right)^2 + \left(\frac{n_2}{L_2}\right)^2 + \left(\frac{n_3}{L_3}\right)^2} \qquad (40)$$

In order to compute the wave frequencies from (40), the wave speed must be known. For instance, Scott (2002) provides a signal speed of a neuron axon of 21.2 $m \cdot s^{-1}$. The substitution of this value in (40) together with the values of the brain dimensions considered above and the different values of the integers do not reproduce the experimental values known of the brain waves. However, the speed value of a signal on a neuron axon may not coincide with the actual value of the wave speed in brain. Then, further investigations about the actual speed of waves in brain must be done to help to validate the model

7. Conclusions

The spatio-temporal response model has been deduced from the time response model. Observe that the first model predicts the spatial distribution of the brain activity and the second one the global brain activity. Both brain activities are related by the spatial integration. Thus, this relationship can be a convenient tool to approach the body-mind problem, if the global brain activity is contextualized in a psychological scale such as, for instance, the above mentioned *GFP-MAACLR* scale [0,60] (Amigó et al., (2010).

Besides the possibility to forecast spatio-temporal brain activities and its important relationship with a psychological scale, the problem of the model validation must be discussed. On a hand, the resting state as a consequence of the blood flow coming from the heart beatings is important in the scientific literature. For instance, Smith el al., (2009) study the brain connectivity during the brain state in the context of the studies of brain image. On the other hand, experimental designs that consider other kind of stimuli are important to validate the model. For instance, those related with stimulant drugs consumption.

The importance of the so-called brain waves must be emphasized to validate the spatio-temporal response model. The investigation of the brain signal speed as communication between the brain parts must be refaced in order to find the theoretical values of the brain wave frequencies and to compare it with the experimental values.

The future research must reinsert the inhibitor effect delays in the model. To do this, a model such as the one presented by Micó et al. (2012), where delays are continuous is hypothetically a good option. A future investigation should present as well a more realistic geometry of brain. For instance, half an ellipsoid would be a good option. Finally, the investigation of the model presented by assemblies is a long term proposal after solving the other proposed future research.

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