Relationships between different sets involving group and Drazin projectors and nonnegativity

Alicia Herrero[a], Francisco J. Ramírez[b], Néstor Thome[a]

[a]*Instituto de Matemática Multidisciplinar*
Universidad Politécnica de Valencia.
46022 Valencia, Spain,
{aherrero,njthome}@mat.upv.es
[b]*Instituto Tecnológico de Santo Domingo*
Av. Los Próceres, Galá, Santo Domingo
Dominican Republic
framiresj@hotmail.com

Abstract

This paper deals with nonnegativity of matrices and their group or Drazin inverses. Firstly, the nonnegativity of a square matrix A, its group inverse $A^{\#}$ and its group projector $AA^{\#}$ is used to define different sets for which relationships and characterizations are given. Next, an extension of the previous results for index greater than 1 is presented covering all the situations. Similar sets are introduced and studied for Drazin inverses and Drazin projectors considering the core-nilpotent decomposition. In addition, the above results are applied to study the $\{l\}$-Drazin periodic matrices for $l \geq 1$.

Keywords: Drazin inverse, group inverse, core-nilpotent, nonnegativity
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1. Introduction

Different aspects of the nonnegativity of generalized inverses have been studied in the literature during the last years. A classical result of Berman and Plemmons gives necessary and sufficient conditions for a square matrix to be group monotone. These matrices have nonnegative group inverse and their characterization involves the positive orthant and the null space of the matrix. A similar result is valid for Drazin monotone matrices [2].

In the seventies, Flor presented some special results for nonnegativity of idempotent matrices [6]. Later, using these results, Jain, Goel, Kwak, and Tynan obtained some characterizations for nonnegative matrices having nonnegative group inverse, nonnegative Drazin inverse or nonnegative group projector [9, 10, 11]. Some of these results have been applied to characterize $A^m = A$ with $m > 2$ for a nonnegative matrix A [10]. Other application of these results can be seen in [8] where the nonnegativity of a singular control system has been studied. Moreover, some more results are presented for idempotent matrices in [5].

The main aim of this paper is to cover different situations not considered up to now in the literature. For that, we consider the nonnegativity of a square matrix A, its group inverse $A^\#$ and its group projector $AA^\#$. Specifically, we combine these conditions to obtain different sets and characterize them. Moreover, in a more general way, all these results are extended for matrices having index greater than 1.

We call index of $A \in \mathbb{R}^{n\times n}$ to the smallest nonnegative integer k such that A^k and A^{k+1} have the same rank and it is denoted by $k = \mathrm{ind}(A)$. For a given matrix $A \in \mathbb{R}^{n\times n}$ of index k, a matrix $A^D \in \mathbb{R}^{n\times n}$ is called its Drazin inverse if the properties $A^D A A^D = A^D$, $AA^D = A^D A$, and $A^{k+1}A^D = A^k$ hold. When this matrix exists, it is unique [1, 4]. In the important special case $k = 1$, this matrix is called group inverse of A and denoted by $A^\#$. It is well-known that AA^D is a projector on the range of A^k along the null space of A^k. In order to distinguish this projector among others defined by using other generalized inverses, we will call Drazin projector to AA^D or group projector in the case $k = 1$.

We will stand $A \geq O$ for a matrix A with nonnegative entries, $A \ngeq O$ when there is some negative entry, A^T for the transpose of A, and I for the identity matrix of adequate size. We will also need the notion of $\{l\}$-Drazin periodic matrices. These matrices are square and satisfy that $A^D = A^{l-1}$ where $l \in \{2, 3, \dots\}$ [3].

The main goal of this paper is to give a characterization of square matrices A satisfying different conditions. In Section 2 we introduce the sets covering all the possibilities where the group or Drazin inverses are involved and some first relationships are presented. The sets are defined by the following inequalities: (i) $A^\# \geq O$, (ii) $AA^\# \geq O$, (iii) $A \geq O$ and $A^\# \geq O$, (iv) $A \geq O$ and $AA^\# \geq O$, (v) $A^\# \geq O$ and $AA^\# \geq O$; and the corresponding for the Drazin inverse. Section 3 is devoted to characterize all the aforementioned sets corresponding to index 1 matrices. Next, in Section 4 the case of index greater than 1 is analyzed extending the results obtained for index 1. In addition, further results are presented in Section 5 in order to characterize

$\{l\}$-Drazin periodic matrices.

2. Preliminaries

In this section we introduce different sets using group or Drazin inverses of a matrix. These sets will be studied in detail in the following sections. We start with the sets related to the group inverse.

The set of nonnegative idempotent matrices is defined as

$$\mathcal{PI} = \{A \in \mathbb{R}^{n\times n} :\ A \geq O,\ A^2 = A\}$$

and it considers a special case of matrices of index 1.

Group monotone matrices are considered in the following set:

$$\mathcal{PG} = \{A \in \mathbb{R}^{n\times n} :\ A^{\#} \geq O\}$$

and by adding nonnegativity on the matrix A we have nonnegative group monotone matrices:

$$\mathcal{PPG} = \{A \in \mathbb{R}^{n\times n} :\ A \geq O,\ A^{\#} \geq O\}.$$

On the other hand, when group projectors are involved, we have nonnegative matrices with nonnegative group projector:

$$\mathcal{PPGP} = \{A \in \mathbb{R}^{n\times n} :\ A \geq O,\ AA^{\#} \geq O\}$$

and matrices with nonnegative group projector

$$\mathcal{PGP} = \{A \in \mathbb{R}^{n\times n} :\ AA^{\#} \geq O\}$$

when nonnegativity of A is removed.

Note that these sets cover all the possibilities combining $A \geq O$, $A^{\#} \geq O$, and $AA^{\#} \geq O$. In fact, it is not necessary to consider the sets $\{A \in \mathbb{R}^{n\times n} : \mathrm{ind}(A) = 1,\ A \geq O\}$ and $\{A \in \mathbb{R}^{n\times n} : A^{\#} \geq O,\ AA^{\#} \geq O\}$ because there are two bijective correspondences between each one of them and $\mathcal{PG}$ and $\mathcal{PPGP}$, respectively. The functions

$$\phi : \{A \in \mathbb{R}^{n\times n} : \mathrm{ind}(A) = 1,\ A \geq O\} \rightarrow \mathcal{PG}$$

and

$$\varphi : \{A \in \mathbb{R}^{n\times n} : A^{\#} \geq O,\ AA^{\#} \geq O\} \rightarrow \mathcal{PPGP}$$

given by $\phi(A) = A^\#$ and $\varphi(A) = A^\#$ define the mentioned correspondences. Is is clear that ϕ and φ are well-defined and it can be shown that they are bijections because the group inverse property $(A^\#)^\# = A$ holds.

In a similar way, taking into account that the Drazin inverse extends the concept of the group inverse, we can define several sets by using this generalized inverse. In this case we also consider different situations depending on the nonnegativity of the own matrix, of its Drazin inverse or even of its Drazin projector.

Drazin monotone matrices are considered in the following set:

$$\mathcal{PD} = \{A \in \mathbb{R}^{n\times n} : \ A^D \geq O\}$$

and by adding nonnegativity of the matrix A we have nonnegative Drazin monotone matrices

$$\mathcal{PPD} = \{A \in \mathbb{R}^{n\times n} : \ A \geq O, \ A^D \geq O\}.$$

On the other hand, when Drazin projectors are involved, we have nonnegative matrices with nonnegative Drazin projector:

$$\mathcal{PPDP} = \{A \in \mathbb{R}^{n\times n} : \ A \geq O, \ AA^D \geq O\},$$

matrices with only nonnegative Drazin projector:

$$\mathcal{PDP} = \{A \in \mathbb{R}^{n\times n} : \ AA^D \geq O\},$$

and Drazin monotone matrices with nonnegative Drazin projector

$$\mathcal{PDPDP} = \{A \in \mathbb{R}^{n\times n} : \ A^D \geq O, \ AA^D \geq O\}$$

when $A^D \geq O$ is added.

Note that in this case the same bijections as before can not be established because, in general, $(A^D)^D \neq A$. Moreover, other relationships can be considered between these sets.

Lemma 2.1. *The sets defined before satisfy the following relations:*

(a) $\mathcal{PI} \subset \mathcal{PPG}$*.*

(b) $\mathcal{PPG} \subset \mathcal{PPGP} \subset \mathcal{PGP}$*.*

(c) $\mathcal{PPG} \subset \mathcal{PG}$*.*

(d) $\mathcal{PPG} = \mathcal{PPGP} \cap \mathcal{PG}$.

(e) $\mathcal{PPD} \subset \mathcal{PD}$.

(f) $\mathcal{PPD} \subset \mathcal{PPDP} \subset \mathcal{PDP}$.

(g) $\mathcal{PPD} = \mathcal{PPDP} \cap \mathcal{PD}$.

(h) $\mathcal{PPD} \subset \mathcal{PDPDP}$.

(i) $\mathcal{PDPDP} = \mathcal{PDP} \cap \mathcal{PD}$.

In general, all the inclusions are strict.

Proof. The condition $A^2 = A$ implies that $A^3 = A$ and then $A = A^{\#} \geq O$ for a nonnegative matrix A. So, $\mathcal{PI}$ is included in $\mathcal{PPG}$. Moreover, the inclusion is strict because the nonnegative matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

satisfies that $A = A^{\#} \geq O$ but $A^2 \neq A$. So, item (a) has been shown.

All the remaining inclusions and equalities can be easily seen. Then, we focus on showing that the inclusions are strict. For that, we give the corresponding counterexamples.

First, we start showing that $\mathcal{PPG} \subset \mathcal{PPGP}$. The nonnegative matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

satisfies that

$$A^{\#} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ngeq O \qquad \text{and} \qquad AA^{\#} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \geq O.$$

Next, we prove that $\mathcal{PPGP} \subset \mathcal{PGP}$ and $\mathcal{PPG} \subset \mathcal{PG}$ because the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is not nonnegative with

$$A^{\#} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix} \geq O \qquad \text{and} \qquad AA^{\#} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \geq O.$$

Then, items (b) and (c) have been proved.

Now, $\mathcal{PPD} \subset \mathcal{PD}$, $\mathcal{PPDP} \subset \mathcal{PDP}$, and $\mathcal{PPD} \subset \mathcal{PDPDP}$ because the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is not nonnegative and its index equals 3. Moreover,

$$A^{D} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \geq O \qquad \text{and} \qquad AA^{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \geq O.$$

Finally, we treat the case $\mathcal{PPD} \subset \mathcal{PPDP}$. The matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is nonnegative and its index equals 3. Moreover,

$$A^{D} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \ngeq O \qquad \text{and} \qquad AA^{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \geq O.$$

So, items (e), (f), and (h) have been proved. This ends the proof. ■

The relations established in Lemma 2.1 are shown in Figure 1.

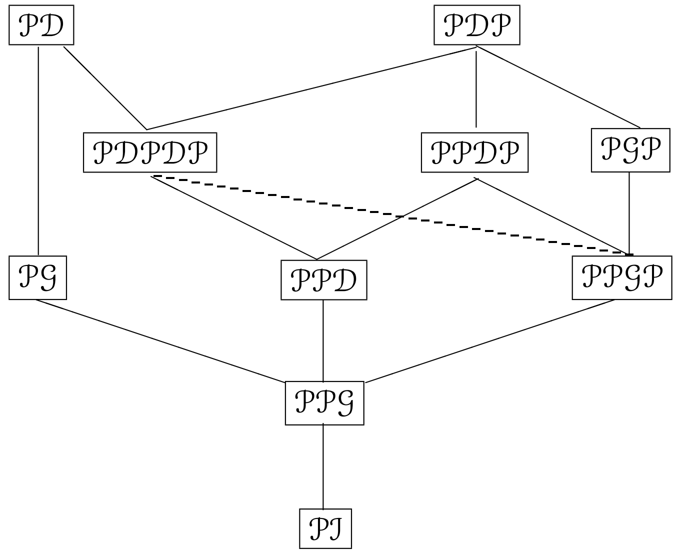

Figure 1: Inclusion relationships between different sets.

3. Characterization of the sets whose matrices have index 1

In this section we study the sets defined in the previous section involving matrices of index 1. So, we assume that all the involved matrices have group inverse. We start with a result partially given in [7] where the converse is also included here.

Lemma 3.1. *Let $A \in \mathbb{R}^{n\times n}$ a matrix with $\operatorname{rank}(A) = r > 0$. Then $A \in \mathcal{PI}$ if and only if there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$A = P \begin{bmatrix} XY & XYM & O \\ O & O & O \\ NXY & NXYM & O \end{bmatrix} P^T \tag{1}$$

where M, N are arbitrary nonnegative matrices of appropriate sizes and $X = \operatorname{diag}(x_1, \dots, x_r)$, $Y = \operatorname{diag}(y_1^T, \dots, y_r^T)$ being x_i and y_j positive column vectors with $i, j \in \{1, \dots, r\}$ such that $YX = I$.

Proof. The necessity is given in the proof of the Lemma 2.1 in [7]. Since P, M, N, X, and Y are nonnegative matrices, from (1) we get that $A \geq O$. Again from (1) and making a simple block multiplication we get $A^2 = A$. The sufficiency is then proved. ■

An important particular case is shown in the following corollary.

Corollary 3.2. *Let $A \in \mathbb{R}^{n\times n}$ with* $\mathrm{rank}(A) = r > 0$*. Then $A \in \mathcal{PGP}$ if and only if there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$AA^{\#} = P \begin{bmatrix} XY & XYM & O \\ O & O & O \\ NXY & NXYM & O \end{bmatrix} P^T \tag{2}$$

where M, N are arbitrary nonnegative matrices of appropriate size and $X = \mathrm{diag}(x_1, \ldots, x_r)$, $Y = \mathrm{diag}(y_1^T, \ldots, y_r^T)$ *being x_i and y_j positive column vectors with $i, j \in \{1, \ldots, r\}$ such that $YX = I$.*

Proof. The group inverse $A^{\#}$ of the matrix A satisfies that $(AA^{\#})^2 = AA^{\#}AA^{\#} = AA^{\#}$, that is $AA^{\#}$ is an idempotent matrix. Then, the results follow directly by applying Lemma 3.1 to the matrix $AA^{\#}$ when its nonnegativity is assumed since $\mathrm{rank}(AA^{\#}) = \mathrm{rank}(A)$. ■

Now, the importance of this last result is that the condition $A \geq O$ is suppressed and besides the form of the matrix A is slightly simplified with respect to that given in Theorem 1 in [11]. More precisely, taking into account that for a given matrix $A \in \mathbb{R}^{n\times n}$ it follows that $AA^{\#}$ is idempotent, the nonnegativity of $AA^{\#}$ allows to factorize this product. This factorization leads to the following result on the matrix A.

Theorem 3.3. *Let $A \in \mathbb{R}^{n\times n}$ with* $\mathrm{rank}(A) = r > 0$*. Then $A \in \mathcal{PGP}$ if and only if there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$A = P \begin{bmatrix} XTY & XTYM & O \\ O & O & O \\ NXTY & NXTYM & O \end{bmatrix} P^T \tag{3}$$

where M, N are arbitrary nonnegative matrices of appropriate size, $T \in \mathbb{R}^{r\times r}$ is nonsingular and $X = \mathrm{diag}(x_1, \ldots, x_r)$, $Y = \mathrm{diag}(y_1^T, \ldots, y_r^T)$ *being x_i and y_j positive column vectors with $i, j \in \{1, \ldots, r\}$ such that $YX = I$.*

In this case,

$$A^{\#} = P \begin{bmatrix} XT^{-1}Y & XT^{-1}YM & O \\ O & O & O \\ NXT^{-1}Y & NXT^{-1}YM & O \end{bmatrix} P^T. \tag{4}$$

Proof. From Corollary 3.2 we can write $AA^{\#}$ in the following form

$$AA^{\#} = P \begin{bmatrix} XY & XYM & O \\ O & O & O \\ NXY & NXYM & O \end{bmatrix} P^T \tag{5}$$

being P, M, N, X, and Y as there. We now partition adequately the matrix A in a 3×3 block matrix as in (5) as follows

$$A = P \begin{bmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{bmatrix} P^T, \tag{6}$$

and we apply the property $(AA^{\#})A = A$. The form of $AA^{\#}$ and the partition of A lead to: A_4, A_5, and A_6 are null matrices, $A_i = XYA_i$, and $A_{i+6} = NA_i$ for $i = 1, 2, 3$. Since $A^{\#}A = AA^{\#}$, the same property $A(A^{\#}A) = A$ also yields to $A_3 = O$, $A_1 = A_1XY$, and $A_2 = A_1M$. Summarizing,

$$A = P \begin{bmatrix} A_1 & A_1M & O \\ O & O & O \\ NA_1 & NA_1M & O \end{bmatrix} P^T, \tag{7}$$

where $A_1 = XYA_1 = A_1XY$. Then $A_1 = XTY$ being $T = YA_1X$. Note that $\operatorname{rank}(A_1) = \operatorname{rank}(A) = r$. Thus, $r = \operatorname{rank}(XTY) \leq \operatorname{rank}(T) \leq r$ because $T \in \mathbb{R}^{r\times r}$. The converse is evident. The proof is then completed. ■

In the following result the nonnegativity of the matrix A is added.

Theorem 3.4. *Let $A \in \mathbb{R}^{n\times n}$ with* $\operatorname{rank}(A) = r > 0$. *Then $A \in \mathcal{PPGP}$ if and only if there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$A = P \begin{bmatrix} XTY & XTYM & O \\ O & O & O \\ NXTY & NXTYM & O \end{bmatrix} P^T \tag{8}$$

where M, N are arbitrary nonnegative matrices of appropriate size, $T \in \mathbb{R}^{r\times r}$ is a nonnegative nonsingular matrix, $X = \operatorname{diag}(x_1, \ldots, x_r)$, $Y = \operatorname{diag}(y_1^T, \ldots, y_r^T)$ being x_i and y_j positive column vectors with $i, j \in \{1, \ldots, r\}$ such that $YX = I$.

Proof. Applying Theorem 3.3 we can deduce that A has the required form (8). From $A \geq O$ and $P \geq O$ we have that $P^T AP \geq O$ and thus, in particular, we get $XTY \geq O$ by (8). Since $X \geq O$, $Y \geq O$ and $YX = I$, premultiplying and postmultiplying the inequality $XTY \geq O$ by Y and X, respectively, we obtain $T \geq O$. The converse is evident. ■

A result related to the nonnegativity of A and $A^{\#}$ independently is given immediately.

Theorem 3.5. *Let $A \in \mathbb{R}^{n\times n}$ with $\operatorname{rank}(A) = r > 0$. Then $A \in \mathcal{PPG}$ if and only if there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$A = P \begin{bmatrix} XTY & XTYM & O \\ O & O & O \\ NXTY & NXTYM & O \end{bmatrix} P^T \tag{9}$$

where M, N are arbitrary nonnegative matrices of appropriate size, $T \in \mathbb{R}^{r\times r}$ is a nonnegative nonsingular matrix with $T^{-1} \geq O$, $X = \operatorname{diag}(x_1, \ldots, x_r)$, $Y = \operatorname{diag}(y_1^T, \ldots, y_r^T)$ being x_i and y_j positive column vectors with $i, j \in \{1, \ldots, r\}$ such that $YX = I$.

Proof. By Theorem 3.3, the matrix A has the required form (9) with $T \in \mathbb{R}^{r\times r}$ a nonsingular matrix. Under these assumptions, the nonnegativity of A is equivalent to the nonnegativity of T as proved in Theorem 3.4. Analogously, from (4) we get the equivalence between the nonnegativity of $A^{\#}$ and T^{-1}. This ends the proof. ■

We close this section remembering that a characterization of the set $\mathcal{PG}$ was given in [2].

4. Analysis of the sets involving matrices of index greater than 1

It is well-known that a square matrix $A \in \mathbb{R}^{n\times n}$ of index $k > 0$ can be always written as [4]

$$A = S \begin{bmatrix} C & O \\ O & N_1 \end{bmatrix} S^{-1} = B_A + N_A, \tag{10}$$

denoting by

$$B_A = S \begin{bmatrix} C & O \\ O & O \end{bmatrix} S^{-1} \qquad \text{and} \qquad N_A = S \begin{bmatrix} O & O \\ O & N_1 \end{bmatrix} S^{-1}$$

where S and C are nonsingular and N_1 is nilpotent of nilpotence index k. Consequently, B_A has index 1 and N_A is nilpotent of nilpotence index k. This expression is called the core-nilpotent decomposition of the matrix A and it will be used in the analysis of the sets presented in the second section.

Note that if the matrix $A \in \mathbb{R}^{n\times n}$ has index $k > 1$, it follows that $AA^D = O$ if and only if A is the zero matrix or A is nilpotent of index k. So, in what follows we will consider the remaining cases.

Theorem 4.1. *Let $A \in \mathbb{R}^{n\times n}$ and $B_A \in \mathbb{R}^{n\times n}$ be the matrix of the core-nilpotent decomposition (10) of A with* $\operatorname{rank}(B_A) = r > 0$. *Then $A \in \mathcal{PDP}$ if and only if there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that the matrix B_A is given by*

$$B_A = P \begin{bmatrix} XTY & XTYM & O \\ O & O & O \\ NXTY & NXTYM & O \end{bmatrix} P^T$$

where M, N are arbitrary nonnegative matrices of appropriate size, $T \in \mathbb{R}^{r\times r}$ is nonsingular and $X = \operatorname{diag}(x_1, \ldots, x_r)$, $Y = \operatorname{diag}(y_1^T, \ldots, y_r^T)$ being x_i and y_j positive column vectors with $i, j \in \{1, \ldots, r\}$ such that $YX = I$.

In this case,

$$A^D = P \begin{bmatrix} XT^{-1}Y & XT^{-1}YM & O \\ O & O & O \\ NXT^{-1}Y & NXT^{-1}YM & O \end{bmatrix} P^T. \tag{11}$$

Proof. From (10), we get $A^D = B_A^\#$ and $AA^D = B_A B_A^\# + N_A B_A^\# = B_A B_A^\#$ given that $N_A B_A^\# = O$. Since $B_A B_A^\# = AA^D \geq O$, Theorem 3.3 assures that the matrix B_A has the desired form because $\operatorname{rank}(B_A) > 0$. ■

Another set to be studied is $\mathcal{PDPDP}$.

Theorem 4.2. *Let $A \in \mathbb{R}^{n\times n}$ and $B_A \in \mathbb{R}^{n\times n}$ be the matrix of the core-nilpotent decomposition (10) of A with* $\operatorname{rank}(B_A) = r > 0$. *Then $A \in \mathcal{PDPDP}$ if and only if there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that the matrix B_A is given by*

$$B_A = P \begin{bmatrix} XTY & XTYM & O \\ O & O & O \\ NXTY & NXTYM & O \end{bmatrix} P^T$$

where M, N are arbitrary nonnegative matrices of appropriate size, $T \in \mathbb{R}^{r\times r}$ is nonsingular with $T^{-1} \geq O$ and $X = \mathrm{diag}(x_1, \ldots, x_r)$, $Y = \mathrm{diag}(y_1^T, \ldots, y_r^T)$ being x_i and y_j positive column vectors with $i, j \in \{1, \ldots, r\}$ such that $YX = I$.

Proof. Applying Theorem 4.1 we get that B_A has the required form with $T \in \mathbb{R}^{r\times r}$ a nonsingular matrix. In order to assure the nonnegativity of the matrix T^{-1} we observe that $A^D = B_A^{\#}$ has the form (11). Now, simple computations imply that the nonnegativity of T^{-1} is guaranteed by the nonnegativity of A^D. The converse is evident. This ends the proof. ■

We now study the remaining sets.

Theorem 4.3. *Let $A \in \mathbb{R}^{n\times n}$ and $B_A \in \mathbb{R}^{n\times n}$ be the matrix of the core-nilpotent decomposition (10) of A with $\mathrm{rank}(B_A) = r > 0$. If $A \in \mathcal{PPDP}$ then there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that the matrix B_A is given by*

$$B_A = P \begin{bmatrix} XTY & XTYM & O \\ O & O & O \\ NXTY & NXTYM & O \end{bmatrix} P^T \tag{12}$$

where M, N are arbitrary nonnegative matrices of appropriate size, $T \in \mathbb{R}^{r\times r}$ is nonsingular with $T \geq O$ and $X = \mathrm{diag}(x_1, \ldots, x_r)$, $Y = \mathrm{diag}(y_1^T, \ldots, y_r^T)$ being x_i and y_j positive column vectors with $i, j \in \{1, \ldots, r\}$ such that $YX = I$.

Proof. Applying Theorem 4.1 we get that B_A has the required form with $T \in \mathbb{R}^{r\times r}$ a nonsingular matrix. Using the core-nilpotent decomposition of A we arrive to

$$\begin{aligned} AA^DA &= (B_A + N_A)B_A^{\#}(B_A + N_A) \\ &= B_AB_A^{\#}B_A + N_AB_A^{\#}B_A + B_AB_A^{\#}N_A + N_AB_A^{\#}N_A = B_A \end{aligned}$$

since $N_AB_A^{\#} = B_A^{\#}N_A = O$. Then, from $A \geq O$ and $AA^D \geq O$, we have that $B_A \geq O$ and this implies that $T \geq O$ after simple computations. The proof is then completed. ■

Note that the converse of previous theorem is, in general, not valid as the following example shows.

Example 4.1. *The matrix*

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is not nonnegative and its core-nilpotent decomposition is $A = B_A + N_A$ *where*

$$B_A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq O \qquad \text{and} \qquad N_A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

being N_A *nilpotent,* B_A *of index 1 written as (12) with*

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \geq O \qquad \text{and nonsingular,}$$

and $M = N = O$, $X = Y = I$, *and* $P = I$.

However, the converse can be only partially established.

Remark 1. *Following the same notation as in Theorem 4.3, we have that if* $A = B_A + N_A$ *with* B_A *in the form (12) and* T *a nonnegative nonsingular matrix then* $AA^D \geq O$ *and* $B_A \geq O$.

Theorem 4.4. *Let* $A \in \mathbb{R}^{n\times n}$ *and* $B_A \in \mathbb{R}^{n\times n}$ *be the matrix of the core-nilpotent decomposition (10) of* A *with* $\mathrm{rank}(B_A) = r > 0$. *If* $A \in \mathcal{PPD}$ *then there exists a permutation matrix* $P \in \mathbb{R}^{n\times n}$ *such that the matrix* B_A *is given by*

$$B_A = P \begin{bmatrix} XTY & XTYM & O \\ O & O & O \\ NXTY & NXTYM & O \end{bmatrix} P^T \tag{13}$$

where M, N *are arbitrary nonnegative matrices of appropriate size,* $T \in \mathbb{R}^{r\times r}$ *is nonsingular with* $T \geq O$, $T^{-1} \geq O$ *and* $X = \mathrm{diag}(x_1, \ldots, x_r)$, $Y = \mathrm{diag}(y_1^T, \ldots, y_r^T)$ *being* x_i *and* y_j *positive column vectors with* $i, j \in \{1, \ldots, r\}$ *such that* $YX = I$.

Proof. From Theorem 4.2 we get that B_A has the required form with $T \in \mathbb{R}^{r\times r}$ a nonsingular matrix having $T^{-1} \geq O$. As $A \geq O$ and $A^D \geq O$ we have $AA^D \geq O$ and $A \geq O$ which are the hypothesis of Theorem 4.3, and so, we can assure that $T \geq O$. This completes the proof. ■

As before, the converse of Theorem 4.4 is, in general, not valid as the following example shows.

Example 4.2. *In this case, the matrix*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

of index $k = 2$ is not nonnegative and its core-nilpotent decomposition is $A = B_A + N_A$ where

$$B_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq O \qquad \text{and} \qquad N_A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

being N_A nilpotent, B_A of index 1 written as (13) with $T = I$, $M = N = O$, $X = Y = I$, and $P = I$.

Again, the converse can be only partially established.

Remark 2. *Following the same notation as in Theorem 4.4, we have that if $A = B_A + N_A$ with B_A in the form (12) and T and T^{-1} nonnegative matrices then $A^D \geq O$ and $B_A \geq O$.*

We close this section considering the nonnegativity of the matrix A.

Remark 3. *Following the same notations as before, we derive the following equivalences under the assumption $A \geq O$:*

(a) $AA^D \geq O$ if and only if B_A has the form (12) with $T \geq O$ and nonsingular.

(b) $A^D \geq O$ if and only if B_A has the form (12) with $T \geq O$ and $T^{-1} \geq O$.

5. Further results and applications

In this section, we will use the results previously established and the core-nilpotent decomposition (10) of a square matrix A of index $k > 0$ to characterize the $\{l\}$-Drazin periodic matrices, for any integer $l > 1$. We recall that this kind of matrices are those that satisfy $A^D = A^{l-1}$. By simple computations we can firstly show that $A^D = A^{l-1}$ is equivalent to $A^{l-1} = A^{2l-1}$ and $A^k = A^{k+l}$. For that, we use the properties of the Drazin inverse of a square matrix $A \in \mathbb{R}^{n\times n}$.

We now continue with a previous result to the characterization of the $\{l\}$-Drazin periodic matrices.

Lemma 5.1. *Let $A \in \mathbb{R}^{n\times n}$ and $B_A \in \mathbb{R}^{n\times n}$ be the matrix of the core-nilpotent decomposition (10) of A with* $\operatorname{rank}(B_A) = r > 0$, *and let $l > 1$ such that $B_A^l \geq O$. If $A^D = A^{l-1}$ then there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$B_A = P \begin{bmatrix} XTY & XTYM & O \\ O & O & O \\ NXTY & NXTYM & O \end{bmatrix} P^T$$

where M, N are arbitrary nonnegative matrices of appropriate size, $T \in \mathbb{R}^{r\times r}$ is a nonsingular matrix such that $T^l = I$, and $X = diag(x_1, \ldots, x_r)$, $Y = diag(y_1^T, \ldots, y_r^T)$, x_i and y_i are positive column vectors with $i, j \in \{1, \ldots, r\}$ such that $YX = I$.

Proof. The condition $A^D = A^{l-1}$ implies that $B_A^{\#} = (B_A + N_A)^{l-1} = B_A^{l-1} + N_A^{l-1}$ since $B_A N_A = N_A B_A = O$. Then, $B_A B_A^{\#} = B_A^l \geq O$, so we can apply Theorem 3.3, having that B_A and $B_A^{\#}$ are in the form (3) and (4), respectively. Moreover, by using these expressions, the equality $B_A B_A^{\#} = B_A^l$ can be written as the equivalent one

$$\begin{bmatrix} XY & XYM & O \\ O & O & O \\ NXY & NXYM & O \end{bmatrix} = \begin{bmatrix} XT^lY & XT^lYM & O \\ O & O & O \\ NXT^lY & NXT^lYM & O \end{bmatrix}.$$

Then, it follows that $XY = XT^lY$. Premultiplying by Y, postmultipliying by X, and using that $YX = I$, we obtain $T^l = I$. ■

Note that this last lemma gives only necessary conditions to get $A^D = A^{l-1}$. In order to obtain necessary and sufficient conditions we present the following result.

Theorem 5.2. *Let $A \in \mathbb{R}^{n\times n}$ and $B_A \in \mathbb{R}^{n\times n}$ be the matrix of the core-nilpotent decomposition (10) of A with* $\mathrm{rank}(B_A) = r > 0$*, and let $l > 1$ such that $B_A^l \geq O$. Then*

$$A^D = \begin{cases} A^{l-1} & l \geq k+1 \\ A^{l-1} - N_A^{l-1} & l < k+1 \end{cases}$$

if and only if there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that

$$B_A = P \begin{bmatrix} XTY & XTYM & O \\ O & O & O \\ NXTY & NXTYM & O \end{bmatrix} P^T \tag{14}$$

where M, N are arbitrary nonnegative matrices of appropriate size, $T \in \mathbb{R}^{r\times r}$ is a nonsingular matrix such that $T^l = I$, and $X = diag(x_1, \ldots, x_r)$, $Y = diag(y_1^T, \ldots, y_r^T)$, x_i and y_i are positive column vectors with $i, j \in \{1, \ldots, r\}$ such that $YX = I$.

Proof. To prove the necessity, by Lemma 5.1 we only have to show that condition $A^D = A^{l-1} - N_A^{l-1}$ implies that B_A has the form (14) when $l < k+1$. Clearly, $A^D = B_A^\# = (B_A + N_A)^{l-1} - N_A^{l-1} = B_A^{l-1}$. Then, $B_A B_A^\# = B_A^l \geq O$. Following a similar reasoning as in Lemma 5.1 we get that B_A has the form (14) with $T^l = I$.

For the converse, by making some computations, we have that $A^D = B_A^\# = B_A^{l-1}$ since $T^l = I$. Then,

$$A^{l-1} = B_A^{l-1} + N_A^{l-1} = \begin{cases} A^D & l \geq k+1 \\ A^D + N_A^{l-1} & l < k+1 \end{cases}$$

and the proof is then completed. ■

In the following corollary we can see that the above result includes the case $k = 1$.

Corollary 5.3. *Let $A \in \mathbb{R}^{n\times n}$ with* $\mathrm{rank}(A) = r > 0$, $\mathrm{ind}(A) = 1$, *and let $l \geq 1$ such that $A^l \geq O$. Then $A^{l+1} = A$ if and only if there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$A = P \begin{bmatrix} XTY & XTYM & O \\ O & O & O \\ NXTY & NXTYM & O \end{bmatrix} P^T \tag{15}$$

where M, N are arbitrary nonnegative matrices of appropriate size, $T \in \mathbb{R}^{r\times r}$ is a nonsingular matrix such that $T^l = I$, and $X = diag(x_1, \ldots, x_r)$, $Y = diag(y_1^T, \ldots, y_r^T)$, x_i and y_i are positive column vectors with $i, j \in \{1, \ldots, r\}$ such that $YX = I$.

Proof. The case $l > 1$ is a direct consequence of Theorem 5.2 because $A^{\#} = A^{l-1}$ is equivalent to $A^{l+1} = A$ and the core-nilpotent decomposition becomes $A = B_A$ for $k = 1$. The case $l = 1$ corresponds to Lemma 3.1 (where $A^2 = A$ and $T = I$). ■

Moreover, for the case $A \geq O$, the additional condition T nonnegative must be added, that is, for $l \geq 1$:

$$A \geq O \text{ and } A^{l+1} = A \Longleftrightarrow A \text{ has the form (15) with } T \geq O \text{ and } T^l = I.$$

However, for index greater than 1 the relations are the following.

Remark 4. *The cases corresponding to $A^D \geq O$ or $A \geq O$ can be studied as in Theorem 5.2. We derive the following results for $l > 1$.*

(a) $A^D \geq O$ and

$$A^D = \begin{cases} A^{l-1} & l \geq k+1 \\ A^{l-1} - N_A^{l-1} & l < k+1 \end{cases}$$

if and only if B_A has the form (14) with $T^{-1} \geq O$ and $T^l = I$.

(b) If $A \geq O$ and

$$A^D = \begin{cases} A^{l-1} & l \geq k+1 \\ A^{l-1} - N_A^{l-1} & l < k+1 \end{cases}$$

then B_A has the form (14) with $T \geq O$ and $T^l = I$ (moreover, $T^{-1} \geq O$ when $l \geq k+1$).

Example 4.2 can be used to see that the converse of (b) is not valid.

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