UNIVERSITAT POLITÈCNICA DE VALÈNCIA

DEPARTAMENT DE MATEMÀTICA APLICADA

The specification property in linear dynamics

Tesis Doctoral

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Dirigida por:
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Valencia, diciembre de 2015

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CERTIFICAN

que la presente memoria "The specification property for operators" ha sido realizada bajo nuestra dirección por Salud Bartoll Arnau y constituye su tesis para optar al grado de Doctor en Matemáticas.

Y para que así conste en cumplimiento de la legislación vigente presentamos y apadrinamos ante la Escuela de Doctorado de la Universitat Politècnica de València la referida tesis firmando el presente certificado.

Valencia, diciembre de 2015

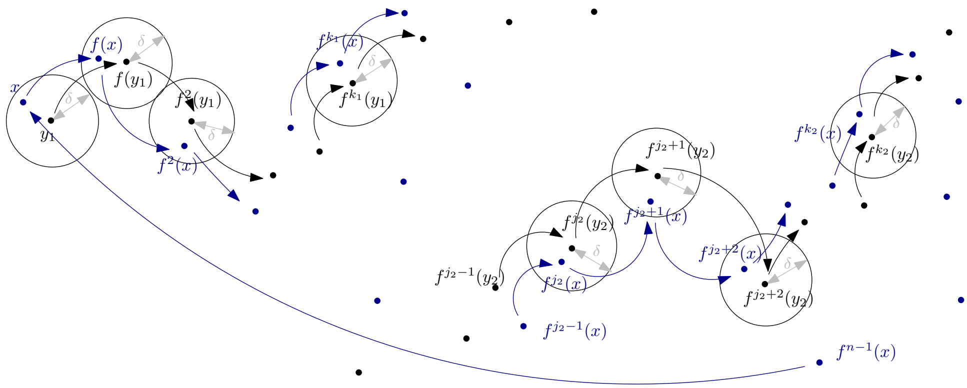
Los directores:

Félix Martínez Jiménez | Alfred Peris Manguillot

Resumen

La dinámica de operadores lineales, o simplemente dinámica lineal, estudia las órbitas generadas por las iteraciones de una transformación lineal. La hiperciclicidad es el estudio de los operadores lineales que poseen una órbita densa. Si bien G. D. Birkhoff (en 1929, [25]), G. R. MacLane (en 1952, [72]) y S. Rolewicz (en 1969, [83]) obtuvieron ejemplos de operadores lineales hipercíclicos, podemos fijar el nacimiento de la dinámica lineal en 1982 con la tesis de C. Kitai [68]. Desde entonces muchos matemáticos han contribuido al desarrollo de esta floreciente área del análisis.

La dinámica lineal conecta el análisis funcional y la dinámica. Al igual que en sistemas dinámicos clásicos, podemos estudiar la dinámica de operadores lineales desde un punto de vista topológico. En este contexto, hablamos de que un operador tiene la propiedad de especificación (SP). Precisamente, al estudio de la propiedad de especificación en sistemas dinámicos lineales está dedicada la presente tesis doctoral. Una aplicación continua en un espacio métrico satisface la propiedad de especificación si para cualquier familia de puntos podemos aproximar, con una cierta uniformidad, partes de sus órbitas por una sola órbita de un punto periódico.



La tesis es un compendio de artículos sobre la propiedad de especificación. Se estructura en cuatro partes precedidas de un capítulo dedicado a introducir la notación, definir los conceptos y enunciar los resultados de ámbito general que van a ser utilizados en el resto de la memoria.

Los operadores "shift" (desplazamiento) constituyen una de la clases más importantes, como campo de pruebas, en sistemas dinámicos lineales discretos. Debido a su estructura simple, siempre que se introduce un nuevo concepto en dinámica lineal es habitual comprobarlo sobre shifts ponderados. Por este motivo, en la primera parte de esta memoria, se estudia la propiedad de especificación para operadores desplazamiento unilaterales y bilaterales en espacios ℓ^p ponderados y la relación con otras propiedades dinámicas como, por ejemplo, el caos de Devaney. Los resultados que aparecen en el capítulo 2 han sido publicados en [8].

En el capítulo 3 se generalizan los resultados sobre la propiedad SP a operadores desplazamiento en F-espacios separables de sucesiones. Un F-espacio es un espacio vectorial, dotado de una F-norma, que es completo con la métrica inducida. La noción de F-norma tiene la ventaja de que permite trabajar como en un espacio de Banach llevando cuidado con la homogeneidad de la norma que ahora no se cumple. Los espacios ℓ^p con $0 < p < 1$ son ejemplos de F-espacios. La elección de estos espacios se debe principalmente a que los resultados básicos en dinámica lineal usan argumentos basados en el teorema de categoría de Baire y, es conocido que, todo F-espacio es un espacio de Baire (la intersección numerable de abiertos densos es densa). El contenido de este capítulo se encuentra publicado en [7].

Los sistemas dinámicos caóticos han recibido gran atención en los últimos años. De acuerdo con Kolyada y Snoha [69] el término caos fue usado por Li y Yorke en 1957 aunque ellos no dieron una definición formal. Fue la definición de Devaney (en 1989) la que llegó a hacerse más popular. Un operador lineal es caótico si algún elemento tiene una órbita densa, posee un conjunto denso de puntos periódicos y tiene una cierta dependencia sensible de las condiciones iniciales. Banks et al. [4] probaron que esta tercera condición es redundante. Así pues, un operador hipercíclico es caótico si admite un conjunto denso de puntos periódicos. La propiedad de especificación es una noción de caos (en el sentido topológico) más potente que la debida a Devaney.

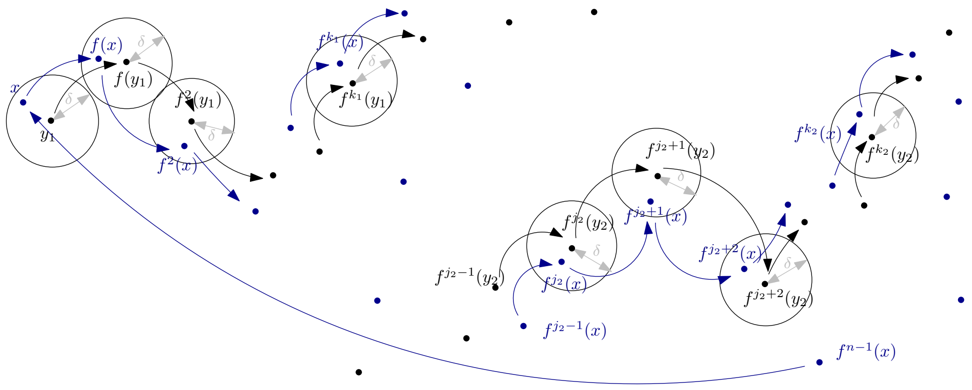
Otra variante más fuerte que la hiperciclicidad es la hiperciclicidad frequente. Este concepto fue introducido por Bayart y Grivaux [13] motivados por el teorema ergódico de Birkhoff. Un operador es frecuentemente hipercíclico si algún elemento tiene una órbita que corta muy a menudo a cada conjunto abierto no vacío. En el capítulo 4 de esta tesis se estudia con profundidad la propiedad de especificación para operadores lineales y continuos definidos en F-espacios separables. También se incide en la conexión de dicha propiedad con otras propiedades dinámicas como mezclante, caos de Devaney e hiperciclicidad frecuente. Los resultados que presentamos, recogidos en [5], han sido aceptados para su publicación en Journal of Mathematical Analysis and Applications.

Finalmente, en la cuarta parte de este trabajo, se extiende la propiedad de especificación a semigrupos de operadores fuertemente continuos en espacios de Banach, esto es, C_0-semigrupos. Estos operadores pueden verse como la versión continua del caso discreto correspondiente a las iteraciones de un único operador; en otras palabras, el papel de las iteraciones en el caso discreto lo asume el parámetro en el caso continuo. Ahora, la labor de los operadores desplazamiento en espacios de sucesiones como clases de prueba la desempeñan los semigrupos de traslación. Al igual que en capítulos anteriores, se estudia la relación de la propiedad SP para C_0-semigrupos con otras propiedades dinámicas: mezclante, caos de Devaney e hiperciclicidad frecuente. En [6] se encuentran los resultados expuestos en el capítulo 5.

Resum

La dinàmica d'operadors lineals, o simplement dinàmica lineal, estudie les òrbites generades per les iteracions d'una transformació lineal. La hiperciclicitat es el estudi dels operadors lineal que posseeixen una òrbita densa. Si bé G. D. Birkhoff (en 1929, [25]), G. R. MacLane (en 1952, [72]) y S. Rolewicz (en 1969, [83]) van obtenir exemples d'operadors lineals hipercíclics, podem fixar el naixement de la dinàmica lineal en 1982 amb la tesi de C. Kitai [68]. Des de llavors molts matemàtics han contribuït al desenvolupament d'esta florent area de l'anàlisi.

La dinàmica lineal connecta el anàlisi funcional y la dinàmica. Igual que en sistemes dinàmics clàssics, podem estudiar la dinàmica d'operadors lineals des d'un punt de vista topològic. En eixe context, parlem que un operador té la propietat d'especificació (SP). Precisament, al estudi de la propietat d'especificació en sistemes dinàmics lineals està dedicada la present tesi doctoral. Una aplicació continua en un espai mètric compleix la propietat d'especificació si per a qualsevol família de punts podem aproximar, amb certa uniformitat, parts de les seues òrbites per una sola òrbita d'un punt periòdic.



La tesi es un compendi de articles sobre la propietat d'especificació. S'estructura en quatre parts precedides d'un capítol dedicat a introduir la notació, definir els conceptes i enunciar els resultats d'àmbit general que seran utilitzats en la resta de la memòria.

Els operadors "shifts" (desplaçaments) constitueixen una de les classes més importants, com a camp de proves, en sistemes dinàmics lineals discrets. Degut a la seua estructura simple, sempre que es introdueix un nou concepte en dinàmica lineal es habitual comprovar-ho sobre shifts ponderats. Per esta raó, en la primera part d'esta memòria, s'estudia la propietat d'especificació per a operadors desplaçament unilaterals i bilaterals en espais ℓ^p ponderats i la relació amb altres propietats dinàmiques com, per exemple, el caos de Devaney. Els resultats que apareixen en el capítol 2 han segut publicats en [8].

En el capítol 3 es generalitzen els resultats sobre la propietat SP a operadors desplaçament en F-espais separables de successions. Un F-espai es un espai vectorial, dotat d'una F-norma, que és complet amb la mètrica induida. La noció de F-norma té l'avantatge que permet treballar com en un espai de Banach anant en compte amb l'homogeneitat de la norma que ara no es compleix. Els espais ℓ^p amb $0 < p < 1$ són exemples de F-espais. L'elecció d'aquests espais es deu principalment al fet que els resultats bàsics en dinàmica lineal fan servir arguments basats en el teorema de categoria de Baire i, és conegut que, tot F-espai és un espai de Baire (la intersecció numerable d'oberts densos és densa). El contingut d'aquest capítol es trobe publicat en [7].

Els sistemes dinàmics caòtics han rebut gran atenció en els últims anys. D'acord amb Kolyada i Snoha [69] el terme caos va ser utilitzat per Li i Yorke en 1957 encara que ells no van donar una definició formal. Va ser la definició de Devaney (en 1989) la que va arribar a fer-se més popular. Un operador lineal és caòtic si algun element té una òrbita densa, posseeix un conjunt dens de punts periòdics i té una dependència sensible de les condicions inicials. Banks et al. [4] van provar que esta tercera condició és redundant. Així doncs, un operador hipercíclic és caòtic si admet un conjunt dens de punts periòdics. La propietat d'especificació és una noció de caos (en el sentit topològic) més potent que la deguda a Devaney.

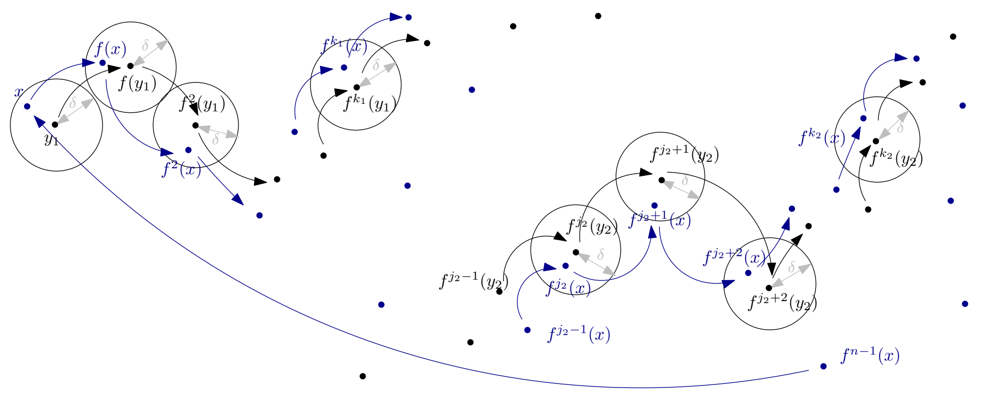
Una altra variant més forta que la hiperciclicitat és la hiperciclicitat freqüent. Aquest concepte va ser introduït per Bayart i Grivaux [13] motivats per el teorema ergòdic de Birkhoff. Un operador és freqüentment hipercíclic si algun element té una òrbita que talle molt sovint a cada conjunt obert no vuit. En el capítol 4 d'esta tesi se estudie amb profunditat la propietat d'especificació per a operadors lineals i continus definits en F-espais separables. També s'incideix en la connexió de dita propietat amb altres propietats dinàmiques com mesclant, caos de Devaney i hiperciclicitat freqüent. Els resultats que presentem, arreplegats en [5], han estat acceptats per a la seva publicació en Journal of Mathematical Analysis and Applications.

Finalment, en la quarta part d'aquest treball, s'estén la propietat d'especificació a semigrups d'operadors fortament continus en espais de Banach, això és, C_0-semigrups. Aquests operadors poden veure's com la versió continua del cas discret corresponen a les iteracions d'un únic operador; en altres paraules, el paper de les iteracions en el cas discret ho assumeix el paràmetre en el cas continu. Ara, la labor del operadors desplaçament en espais de successions com classes de prova l'exerceixen els semigrups de translació. Igual que en capítols anteriors, s'estudia la relació de la propietat SP per a C_0-semigrups amb altres propietats dinàmiques: mesclant, caos Devaney i hiperciclicitat freqüent. En [6] es troben els resultats exposats en el capítol 5.

Summary

The dynamics of linear operators, namely linear dynamics, is mainly concerned with the behaviour of iterates of linear transformations. Hypercyclicity is the study of linear operators that possess a dense orbit. Although the first examples of hypercyclic operators are due to G. D. Birkhoff (in 1929, [25]), G. R. MacLane (in 1952, [72]) and S. Rolewicz (in 1969, [83]), we can date the birth of the linear dynamics in 1982 with the unpublished PhD thesis of C. Kitai [68]. Since then, many mathematicians have contributed to the development of this flourishing new area of the analysis.

Linear dynamics connects functional analysis and dynamics. As for the classical dynamical systems, one can study the dynamics of linear operators from a topological point of view. In this context, we state that an operator has the specification property (SP). Precisely, the aim of this PhD thesis is to study the specification property on linear dynamical systems. A continuous map on a compact metric space satisfies the specification property if one can approximate pieces of orbits by a single periodic orbits with a certain uniformity.



This Doctoral dissertation is a compendium of articles on the specification property. It is structured in four parts preceded by a chapter which introduces the notation, definitions and the basic results that will be needed throughout the thesis.

The shift operators on sequence spaces constitute one of the most important test ground for discrete linear dynamical systems. Due to its simple structure, every time you introduce a new property in linear dynamics it is common to check it on weighted shifts operators. It is for this reason that the first part of this research work is devoted to study the specification property for unilateral and bilateral backward shift operators on weighted ℓ^p-spaces and the relationship with other dynamical properties such as Devaney chaos. The results that appear in Chapter 2 have been published in [8].

In Chapter 3 we extend the results on the SP to shift operators on separable sequence F-spaces. An F-space is a vector space that is endowed with an F-norm and that is complete under the induced metric. The notion of an F-norm has the advantage that one can largely argue as if one was working in a Banach space. One need to be aware of the fact that the positive homogeneity of a norm is no longer available. The spaces ℓ^p with $0 < p < 1$ are F-spaces. We have chosen to go no further that the F-space setting because the Baire Category Theorem is a basic tool in linear dynamics and it is well known that F-spaces are Baire spaces (every intersection of countably many dense open sets is dense). The contents of this chapter are gathered in [7].

Chaotic dynamical systems have received a great deal of attention in recent years. According to Kolyada and Snoha, the term chaos was first used by Li and Yorke in 1975 although they did not give a formal definition. Devaney's definition of chaos (in 1989) has become rather popular. An operator is chaotic if some element has a dense orbit, it has a dense set of periodic points, and it exhibits a certain sensitivity to initial conditions. Banks et al. [4] proved that sensitivity to initial conditions is redundant in Devaney's definition of chaos. Therefore, a hypercyclic operator is chaotic if and only if it has a dense set of periodic points. The specification property is an interesting and rather strong notion of chaos (in the topological sense).

We also consider a qualitative strengthening of hypercyclicity namely frequent hypercyclicity. It was introduced by Bayart and Grivaux [13], motivated by Birkhoff's ergodic theorem. An operator is frequently hypercyclic if there is some element whose orbit meets every non-empty open set very often. In Chapter 4 the specification property is deeply studied for linear and continuous operators on separable F-spaces. In addition, we are interested in finding out its relation with other dynamical properties such as mixing, Devaney chaos and frequent hypercyclicity. The results that we have achieved are collected in [5] and have been accepted to be publish in Journal of Mathematical Analysis and Applications.

Finally, in the last chapter of this dissertation, we examine the specification property for strongly continuous semigroups on Banach spaces, that is, for C_0-semigroups. They can viewed as the continuous-time analogue of the discrete-time case of iterates of a single operator; in other words, the parameter in the continuous case plays the role of the iterations in the discrete case. Now the translation semigroups substitute the shift operators as test classes. Once again, we study the relationship between the specification property and mixing, chaos and frequent hypercyclicity properties of a C_0-semigroup. The results of Chapter 5 will appear in [6].

Agradecimientos

La realización de esta memoria ha sido posible gracias a la inestimable ayuda y generosidad del profesor Dr. Alfred Peris Manguillot y el profesor Dr. Félix Martínez Jiménez. Ha sido un auténtico privilegio poder trabajar con ellos y aprender de ellos. Por su magisterio y su estímulo, quiero expresarles mi admiración, gratitud y respeto.

Durante la elaboración de este trabajo he tenido la fortuna de pertenecer a un magnífico grupo de investigación. Ha resultado ser muy gratificante compartir esta experiencia con los profesores Francisco Rodenas, coautor de los resultados del Capítulo 5, y J. Alberto Conejero y con mis compañeros Marina Murillo, Xavier Barrachina y Javier Aroza. A todos ellos les doy las gracias por su paciencia, comprensión y por su valioso apoyo.

También quiero dar las gracias a mis compañeros, y a la vez amigos, de la Unidad Docente de Matemáticas por poder contar siempre con ellos y por hacer mucho más fácil mi trabajo.

De manera muy especial me gustaría trasmitir a mi hermano mi más sincero reconocimiento por el cariño y el respaldo que siempre me ha brindado.

A la memoria de mi queridísima madre,
Salud A. Noviembre

Contents

Chapter 1

Introduction and basic concepts

This chapter is devoted to introduce the notation, definitions and the basic results that we will use throughout the thesis. Most of the results related to linear dynamics can be found in [15] and [62].

1.1 Topological dynamics

Dynamical systems appear naturally in the study of the behavior of evolving systems. Let X be a set of elements that describes the different acceptable states of a system. If $x_n \in X$ is the state of the system at time $n \geq 0$, then its evolution will be given by a linear map $T : X \to X$ such that $x_{n+1} = T(x_n)$.

Definition 1.1.1 (Discrete dynamical system)**.** Let X be a metric space and let T be a continuous map $T : X \to X$. A *discrete dynamical system* is a pair (X, T). We define the *orbit* of a point $x \in X$ as the set $\mathrm{Orb}(x, T) = \{T^n x : n \in \mathbb{N}_0\}$, where T^n denotes the n-th iterate of a map T. We will often simply say that T or $T : X \to X$ is a dynamical system.

Definition 1.1.2. Let $S : Y \to Y$ and $T : X \to X$ be dynamical systems.

1. Then T is called *quasi-conjugate* to S if there exists a continuous map $\phi : Y \to X$ with dense range such that $T \circ \phi = \phi \circ S$; that is, the following diagram commutes.

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ X & \xrightarrow[T]{} & X \end{array}$$

2. If ϕ can be chosen to be a homeomorphism, then S and T are called *conjugated.*

Definition 1.1.3. We say that a property $\mathcal{P}$ for dynamical systems is preserved under (quasi-)conjugacy if the following holds: if a dynamical system $S : Y \to Y$ has property $\mathcal{P}$ then every dynamical system $T : X \to X$ that is (quasi-) conjugate to S also has property $\mathcal{P}$.

Definition 1.1.4. Let $T : X \to X$ be a dynamical system. Then $Y \subset X$ is called T*-invariant* or invariant under T if $T(Y) \subset Y$.

Definition 1.1.5. We say that $x \in X$ is a *fixed point for the dynamical system* $T : X \to X$ if $Tx = x$, and we say that $x \in X$ is a *periodic point for the dynamical system* T if $T^n x = x$ for some $n \in \mathbb{N}_0$. The set of all periodic points is denoted by $\operatorname{Per}(T)$. If $x \in \operatorname{Per}(T)$ then the smallest positive integer n such that $T^n x = x$ is called a *primary period* of x.

Definition 1.1.6. Let $T : X \to X$ be a dynamical system. For any pair of nonempty open sets U, V the return set is defined as $N(U, V) = \{n \in \mathbb{N}_0 : T^n(U) \cap V \neq \emptyset\}$. Then we have that (X, T) is:

(i) *topologically transitive* if for any pair U,$V \subset X$ of nonempty open sets, the return set $N(U, V)$ is non-empty;

(ii) *weakly mixing* if the map $T \times T$ is topologically transitive;

(iii) *mixing* if for any pair U,$V \subset X$ of nonempty open sets, $N(U, V)$ is cofinite.

(iv) *topologically ergodic* if for any pair of nonempty open sets $U, V \subset X$ $N(U,V)$ is syndetic, that is, there exists $p \in \mathbb{N}$, such that $\{n, n+1, \ldots, n+p\} \cap N(U,V) \neq \emptyset$ for any $n \in \mathbb{N}_0$.

A result due to Furstenberg [53] is the following:

Theorem 1.1.7. Let $T : X \to X$ be a weakly mixing dynamical system. Then the n-fold product $T \times \ldots \times T$ is weakly mixing for each $n \geq 2$.

Remark 1.1.8. For any linear dynamical system,

$$\begin{aligned}\text{mixing} &\Longrightarrow \text{topologically ergodic}\\ &\Longrightarrow \text{weakly mixing}\\ &\Longrightarrow \text{topologically transitive.}\end{aligned}$$

In 1989 Robert L. Devaney proposed the first good definition of chaos; see [43]. This concept reflects the unpredictability of chaotic systems because the definition contains a *sensitive dependence on initial conditions*, i.e.:

Definition 1.1.9. Let X be a metric space without isolated points. Then the dynamical system $T : X \to X$ is said to have *sensitive dependence on initial conditions* if there exists some $\delta > 0$ such that, for every $x \in X$ and $\varepsilon > 0$, there exists some $y \in X$ with $d(x,y) < \varepsilon$ such that, for some $n \geq 0$, $d(T^n x, T^n y) > \delta$. The number δ is called a *sensitivity constant* for T.

Definition 1.1.10 (Devaney chaos)**.** A dynamical system $T : X \to X$ is called chaotic in the sense of *Devaney* if it satisfies the following properties:

(i) T is topologically transitive,

(ii) $\mathrm{Per}(T)$ is dense in X,

(iii) T has sensitive dependence on initial conditions.

However, Banks, Brooks, Cairns, Davis and Stacey proved in 1992 ([4]), that one can drop sensitive dependence from Devaney's definition because it is implied by the other two conditions.

Theorem 1.1.11 ([4])**.** Let X be a metric space without isolated points. If a dynamical system $T : X \to X$ is topologically transitive and has a dense set of periodic points then T has sensitive dependence on initial conditions with respect to any metric defining the topology of X.

Proposition 1.1.12. The following properties are preserved by quasi-conjugacy:

(i) Topological transitivity.

(ii) The property of having a dense orbit.

(iii) The property of having a dense set of periodic points.

(iv) Devaney Chaos.

(v) The mixing property.

(vi) The weak-mixing property.

(vii) Topological ergodicity.

1.2 Hypercyclic and chaotic operators

Dynamical systems are defined by continuous maps on metric spaces. For linear dynamical systems, the underlying space must in addition have a linear structure, as is the case for Hilbert spaces and Banach spaces. We will give definitions of linear dynamical systems on spaces of a more general type, topological vector spaces.

Definition 1.2.1. Let $||.|| : X \to \mathbb{R}_+$ be a functional on a vector space X that satisfies:

(i) $||x + y|| \leq ||x|| + ||y||$

(ii) $||\lambda x|| \leq ||x|| \quad \text{if} \quad |\lambda| \leq 1$

(iii) $\lim_{\lambda \to 0} ||\lambda x|| = 0$

(iv) $||x|| = 0$ implies that $x = 0$.

Then $||.|| : X \to \mathbb{R}_+$ is called an F-norm. If $(X, ||.||)$ is complete under the induced metric $d(x, y) = ||x - y||$, then X is an F-space.

A particular case of F-spaces are Fréchet spaces.

Definition 1.2.2. A Fréchet space is a vector space X, endowed with a separating increasing sequence $(p_n)_n$ of seminorms, which is complete under the metric given by

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \min(1, p_n(x - y)).$$

Definition 1.2.3. Let X and Y be topological vector spaces. Then a continuous linear map $T : X \to Y$ is called an *operator*. The space of all operators is denoted by $L(X, Y)$. If $Y = X$ we say that T is an operator on X, with $L(X) = L(X, X)$.

A link between chaos theory and linear operator theory was established by Birkhoff's Transitivity Theorem in 1922. In this theorem, he showed that topological transitivity was equivalent to the notion of hypercyclicity that Beauzamy established in 1987.

Definition 1.2.4 ([17])**.** An operator $T : X \to X$ is said to be *hypercyclic* if there is some $x \in X$ whose orbit under T is dense in X. In that case, x is called a *hypercyclic vector* for T. The set of hypercyclic vectors is denoted by $HC(T)$.

Theorem 1.2.5 (Birkhoff Transitivity theorem, [24])**.** An operator T is hypercyclic if and only if it is topologically transitive. If one of these conditions holds then, the set $HC(T)$ of hypercyclic vectors is a dense G_δ-set; i.e., $HC(T)$ is a countable intersection of open dense sets.

In 1991 Godefroy and Shapiro adopted Devaney's definition for linear chaos.

Definition 1.2.6 ([55])**.** An operator $T : X \to X$ is called *chaotic in the sense of Devaney* if:

(i) T is hypercyclic.

(ii) $\mathrm{Per}(T)$ is dense in X.

Example 1.2.7. The first examples of hypercyclic operators were found by G.D.Birkhoff in 1929 ([25]), G.R. Maclane in 1952 ([72]) and S.Rolewickz in 1969 ([83]).

(i) (**Birkhoff's operators**) The translation operators given by

$$T_a f(z) = f(z+a), a \neq 0.$$

on the space $H(\mathbb{C})$ of entire functions are hypercyclic for all $a \neq 0$.

(ii) (**MacLane's operator**) The differentiation operator:

$$D : f \to f'$$

on $H(\mathbb{C})$ is hypercyclic.

(iii) (**Rolewicz's operators**) On the spaces $X = \ell^p$, $1 \leq p < \infty$, or $X = c_0$ we consider the multiple

$$T = \lambda B : X \to X, \ (x_1, x_2, x_3 \ldots) \to \lambda(x_2, x_3, x_4, \ldots)$$

of the backward shift, where $\lambda \in \mathbb{K}$. T is hypercyclic whenever $|\lambda| > 1$.

Moreover, these operators are chaotic. We first need the following results.

Proposition 1.2.8. Let T be a linear map on a complex vector space X. Then the set of periodic points of T is given by

$$\mathrm{Per}(T) = \mathrm{span}\{x \in X; \quad Tx = e^{\alpha\pi i}x \quad \text{for some} \quad \alpha \in \mathbb{Q}\}$$

Let e_λ denotes the exponential function $e_\lambda(z) = e^{\lambda z}$.

Lemma 1.2.9. Let $\Lambda \subset \mathbb{C}$ be a set with an accumulation point. Then the set

$$\text{span}\{e_\lambda;\ \lambda \in \Lambda\}$$

is dense in $H(\mathbb{C})$.

The lemma allows us to show that Birkhoff's and MacLane's operators are chaotic on $H(\mathbb{C})$.

Example 1.2.10. For the differentiation operator D, any function e_λ is an eigenvector of D to the eigenvalue λ. Thus, since the subspace

$$\text{span}\{e_\lambda;\ \lambda = e^{\alpha\pi i} \quad \text{for some} \quad \alpha \in \mathbb{Q}\}$$

is dense in $H(\mathbb{C})$ by lema 1.2.9, proposition 1.2.8 tells us that $Per(T)$ is dense. Since we already know that D is hypercyclic, it is also chaotic.

For the translation operators T_a, $a \in \mathbb{C}\backslash\{0\}$, any function e_λ is an eigenvector of T_a to the eigenvalue $e^{a\lambda}$. Thus, since the subspace

$$\text{span}\{e_\lambda;\ e^{a\lambda} = e^{\alpha\pi i} \quad \text{for some } \alpha \in \mathbb{Q}\} = \text{span}\{e_\lambda;\ \lambda = \frac{\alpha}{a}\pi i,\ \alpha \in \mathbb{Q}\}$$

is also dense in $H(\mathbb{C})$, we conclude as before that each T_a is chaotic.

1.3 Hypercyclic criteria

The main purpose of this section is to show several criteria under which an operator is chaotic, mixing or weakly mixing. In this section we show these criteria. This first criterion is due to Godefroy Shapiro and it is contained implicitly in their paper [55] and was isolated by Bernal [19].

Theorem 1.3.1 (Godefroy-Shapiro criterion, [55])**.** Let T be an operator. Suppose that the subspaces

$$X_0 := \text{span}\{x \in X;\ Tx = \lambda x \ \text{ for some } \lambda \in \mathbb{K} \text{ with } |\lambda| < 1\}$$

$$Y_0 := \text{span}\{x \in X;\ Tx = \lambda x \ \text{ for some } \lambda \in \mathbb{K} \text{ with } |\lambda| > 1\}$$

are dense in X.
Then T is mixing, and in particular hypercyclic.

If, moreover, X is a complex space and the subspace

$$Z_0 := \text{span}\{x \in X;\ Tx = \lambda x \ \text{ for } \lambda \in \mathbb{C},\ |\lambda|^n = 1 \ \text{ for some } n \in \mathbb{N}\}$$

is dense in X, then T is chaotic.

Example 1.3.2. **Rolewicz's operators**
Let $T = \mu B$, with $|\mu| > 1$, be the multiple of the backward shift on any space $X = \ell^p$, $1 \leq p < \infty$ or $X = c_0$. Let us consider the complex case. One easily determines the eigenvectors of B as the nonzero multiples of the sequences

$$e_\lambda := (\lambda, \lambda^2, \lambda^3, \ldots), \quad |\lambda| < 1$$

with corresponding eigenvalue λ. Therefore, e_λ is an eigenvector of $T = \mu B$ corresponding to the eigenvalue $\mu\lambda$. For any subset Λ of the unit disk that has an accumulation point inside the disk, the set $\text{span}\{e_\lambda;\ \lambda \in \Lambda\}$ is dense in X. By the Hahn-Banach theorem it suffices to show that any continuous linear functional x^* on X that vanishes on each $e_\lambda,\ \lambda \in \Lambda$ vanishes on X. Since $x^* \in X^*$, via the canonical representation it is given by a sequence $(y_n)_n \in \ell^q$ for a certain q, with $1 \leq q \leq \infty$, we have that

$$x^*(e_\lambda) =< e_\lambda, x^* >= \sum_{n=1}^{\infty} y_n \lambda^n \quad \text{if} \quad |\lambda| < 1.$$

The identity theorem for holomorphic functions implies that each y_n is zero and therefore $x^* = 0$. In particular, the subspace

$$X_0 = \text{span}\{x \in X;\ Tx = \eta x \ \text{ for } \eta \in \mathbb{K},\ |\eta| < 1\} = \text{span}\{e_\lambda;\ |\lambda| < \frac{1}{|\mu|}\}$$

is dense in X, as the suspaces Y_0 and Z_0 of the Godefroy- Shapiro criterion; note that $\frac{1}{|\mu|} < 1$. This implies that Rolewicz's operators are mixing and chaotic.

The earliest forms of the Hypercyclicity Criterion were found independently by Kitai [68] and by Gethner and Shapiro [54]. In its general form it is due to Bés and Peris [22].

Theorem 1.3.3 (Kitai's criterion, [68])**.** Let T be an operator. If there are dense subsets X_0, $Y_0 \subset X$ and a map $S : Y_0 \to Y_0$ such that, for any $x \in X_0$, $y \in Y_0$:

(i) $T^n x \to 0$,

(ii) $S^n y \to 0$,

(iii) $TSy = y$,

then T is mixing.

Example 1.3.4. (i) (**Rolewicz's operators**) Taking $X_0 = Y_0$ the set of finite sequences, which is dense in ℓ^p, and for $S : Y_0 \to Y_0$ the map $S = \frac{1}{\lambda}F$ where F is the forward shift operator $F : (x_1 x_2, \ldots) \to (0, x_1, x_2, \ldots)$ the conditions of Kitai's criterion are clearly satisfied.

(ii) (**MacLane's operators**) In this case we take for $X_0 = Y_0$ the set of polynomials, which is dense in $H(\mathbb{C})$, and for S we consider the integral operator $Sf(z) = \int_0^z f(\zeta)d\zeta$. While conditions (i) and (iii) are obvious, we note that condition (ii) is sufficient to be verified by monomials, and $S^n(z^k) = \frac{k!}{(k+n)!}z^{k+n} \to 0$ as $n \to \infty$, uniformly on compact sets, as required.

(iii) (**Birkhoff's operators**) It is sufficient to prove that $T_1 f(z) = f(z+1)$ on $H(\mathbb{C})$ is mixing. For $X_0 = Y_0$ we choose the set of functions $f_{p,\alpha,\nu} = p(z)e^{-\alpha(z-\nu)^2}$, where p is a polynomial and $\alpha > 0$, $\nu \in \mathbb{N}_0$. Since $f_{p,\alpha,\nu} \to p$ in $H(\mathbb{C})$ as $\alpha \to 0$, this set is dense in $H(\mathbb{C})$. Moreover, for S we consider the translation operator $Sf(z) = f(z-1)$. Now if $z = x + iy$ with $|y| \leq \frac{1}{2}|x|$ then we have that $|e^{-\alpha z^2}| = e^{-\alpha(x^2-y^2)} \leq e^{-\frac{3}{4}\alpha x^2}$. This implies, that for any p, α and ν, $f_{p,\alpha,\nu}(z \pm n) \to 0$ uniformly on compact sets as $n \to \infty$, which shows that conditions (i) and (ii) of Kitai's criterion hold, while condition (iii) is trivial.

Theorem 1.3.5 (Gethner-Shapiro criterion, [54]). Let T be an operator. If there are dense subsets X_0, $Y_0 \subset X$, an increasing sequence $(n_k)_k$ of positive integers, and a map $S : Y_0 \to Y_0$ such that, for any $x \in X_0$, $y \in Y_0$:

(i) $T^{n_k} x \to 0$,

(ii) $S^{n_k} y \to 0$,

(iii) $TSy = y$,

then T is weakly mixing.

Theorem 1.3.6 (Hypercyclicity criterion, [22]). Let T be an operator. If there are dense subsets X_0, $Y_0 \subset X$, an increasing sequence $(n_k)_k$ of positive integers, and maps $S_{n_k} : Y_0 \to X$, $k \geq 1$ such that, for any $x \in X_0$, $y \in Y_0$:

(i) $T^{n_k} x \to 0$,

(ii) $S_{n_k} y \to 0$,

(iii) $T^{n_k} S_{n_k} y \to y$,

then T is weakly mixing, and in particular hypercyclic.

1.4 Weighted shifts

In this section we include some basic results about weighted shifts, which make up an important class of hypercyclic and chaotic operators. Due to its simple structure, the class of weighted shifts is a favorite testing ground for operator-theorists. Salas([87]) characterized hypercyclic and weakly mixing unilateral and bilateral weighted shifts on ℓ^2 and $\ell^2(\mathbb{Z})$, respectively. The characterizations for more general sequence spaces and chaos characterizations are due to Grosse-Erdmann [60].

Definition 1.4.1. The basic model of all shifts is the *backward shift*

$$B(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Another shift is the *weighted backward shift* which is defined as:

$$B_w(x_1, x_2, x_3, \ldots) = (w_2x_2, w_3x_3, w_4x_4, \ldots),$$

where $w = (w_n)_n$ is called a *weight sequence.* The weights w_n will be assumed to be non-zero.

These operators can be defined on an arbitrary *sequence space* X, that is, a linear space of sequences or, in other words, a subspace of $w = \mathbb{K}^{\mathbb{N}}$. Moreover, X should carry a topology that is compatible with the sequence space structure of X. We interpret this as demanding that convergence in X should imply coordinatewise convergence. A Banach (Fréchet, F-) space of this kind is called a Banach (Fréchet, F-) sequence space.

Theorem 1.4.2. Let X be a Fréchet sequence space in which $(e_n)_n$ (where $e_n = (0, \ldots, 0, \underbrace{1}_{n}, 0, \ldots))$ is a basis. Suppose that the backward shift B is an operator on X. Then the following assertions are equivalent:

(i) B is hypercyclic;

(ii) B is weakly mixing;

(iii) there is an increasing sequence $(n_k)_k$ of positive integers such that $e_{n_k} \to 0$ in X as $k \to \infty$.

Example 1.4.3. Let

$$\ell_p^v = \{(x_n)_n; \sum_{n=1}^{\infty} |x_n|^p v_n < \infty\},$$

with $1 \leq p < \infty$, be a weighted ℓ_p-space, where $v = (v_n)_n$ is a positive weight sequence. Then B is an operator on ℓ_p^v if and only if there is an

$M > 0$ such that, for all $x \in \ell_p^v$

$$\left(\sum_{n=1}^{\infty} |x_{n+1}|^p v_n\right)^{\frac{1}{p}} \leq M \left(\sum_{n=1}^{\infty} |x_n|^p v_n\right)^{\frac{1}{p}}$$

which is equivalent to $\sup_{n\in\mathbb{N}} \frac{v_n}{v_{n+1}} < \infty$. Theorem 1.4.2, tells us that hypercyclicity of B is characterized by $\inf_{n\in\mathbb{N}} v_n = 0$.

The same conditions also characterize the continuity and hypercyclicity of the backward shift B on the weighted c_0-space

$$c_0(v) = \{(x_n)_n; \lim_{n\to\infty} |x_n| v_n = 0\}.$$

The following Theorem provides a charcaterization of mixing backward shifts.

Theorem 1.4.4. Let X be a Fréchet sequence space in which $(e_n)_n$ is a basis. Suppose that the backward shift B is an operator on X. Then the following assertions are equivalent:

- B is mixing;
- $e_n \to 0$ in X as $n \to \infty$.

In order to show the following results we first need the definition of unconditional convergence.

Definition 1.4.5. Let X be a Fréchet space. Then the following assertions are equivalent:

(i) $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent;

(ii) for any 0-1-sequence $(\epsilon_n)_n$, $\sum_{n=1}^{\infty} \epsilon_n x_n$ converges;

(iii) for any bounded sequence $(\alpha_n)_n$ of scalars, $\sum_{n=1}^{\infty} \alpha_n x_n$ converges;

(iv) for any $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that for any finite set $F \subset \{N, N+1, N+2, \ldots\}$ we have that

$$\left\| \sum_{n\in F} x_n \right\| < \epsilon;$$

(v) for any $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that for any 0-1-sequence $(\epsilon_n)_n$, $\sum_{n=1}^{\infty} \epsilon_n x_n$ converges and

$$\left\| \sum_{n\geq N} \epsilon_n x_n \right\| < \epsilon;$$

(vi) for any $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that whenever $\sup_{n\geq 1} |\alpha_n| \leq 1$ then $\sum_{n=1}^{\infty} \alpha_n x_n$ converges and

$$\left\| \sum_{n\geq N} \alpha_n x_n \right\| < \epsilon;$$

Definition 1.4.6. A sequence $(e_n)_n$ in a Fréchet space X is called an *unconditional basis* if it is a basis such that, for every $x \in X$, the representation

$$x = \sum_{n=1}^{\infty} a_n e_n$$

converges unconditionally.

Theorem 1.4.7. Let X be a Fréchet sequence space in which $(e_n)_n$ is an unconditional basis. Suppose that the backward shift B is an operator on X. Then the following assertions are equivalent:

(i) B is chaotic;

(ii) $\sum_{n=1}^{\infty} e_n$ converges in X;

(iii) the constant sequences belong to X;

(iv) B has a non-trivial periodic point.

It is easy to transfer results to weighted shifts by a conjugacy. Let B_w be a weighted shift on some sequence space X. We define v_n by

$$v_n = \left(\prod_{\nu=1}^{n} w_\nu \right)^{-1}, \quad n \geq 1$$

and consider the sequence space

$$X_v = \{(x_n)_n ; (x_n v_n)_n \in X\}.$$

The map $\Phi_v : X_v \to X$, $(x_n)_n \to (x_n v_n)_n$ is a vector space isomorphism and $B_w \circ \Phi_v = \Phi_v \circ B$, that is the following diagram commutes:

$$\begin{array}{ccc} X_v & \xrightarrow{\ B\ } & X_v \\ {\scriptstyle \phi_v}\big\downarrow & & \big\downarrow{\scriptstyle \phi_v} \\ X & \xrightarrow[\ B_w\ ]{} & X \end{array}$$

Thus $B_w : X \to X$ and $B : X_v \to X_v$ are conjugate operators.

Theorem 1.4.8. Let X be a Fréchet sequence space in which $(e_n)_n$ is a basis. Suppose that the weighted shift B_w is an operator on X.

1. The following assertions are equivalent:

   (i) B_w is hypercyclic;

   (ii) B_w is weakly mixing;

   (iii) there is an increasing sequence $(n_k)_k$ of positive integers such that
   $$\left(\prod_{\nu=1}^{n_k} w_\nu\right)^{-1} e_{n_k} \to 0$$
   in X as $k \to \infty$.

2. The following assertions are equivalent:

   (i) B_w is mixing;

   (ii) we have that
   $$\left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} e_n \to 0$$
   in X as $n \to \infty$;

3. Suppose that the basis $(e_n)_n$ is unconditional. Then the following assertions are equivalent:
   (i) B_w is chaotic;
   (ii) the series
   $$\sum_{n=1}^{\infty}\left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} e_n$$
   converges in X;
   (iii) the sequence
   $$\left(\left(\prod_{\nu=1}^{n} w_\nu\right)^{-1}\right)_n$$
   belongs to X;
   (iv) B_w has a non-trivial periodic point.

Example 1.4.9. A weighted backward shift B_w is an operator on a sequence space ℓ^p, $1 \leq p < \infty$, or c_0 if and only if the weights w_n are bounded. The respective characterizing conditions for B_w to be hypercyclic, mixing or chaotic on ℓ^p are

$$\sup_{n\geq 1}\prod_{\nu=1}^{n}|w_\nu| = \infty, \quad lim_{n\to\infty}\prod_{\nu=1}^{n}|w_\nu| = \infty, \quad \sum_{n=1}^{\infty}\frac{1}{\prod_{\nu=1}^{n}|w_\nu|^p} < \infty.$$

The first condition also characterizes when B_w is hypercyclic on c_0 and the second when it is mixing or equivalently chaotic on c_0. In particular for Rolewicz's operator $T = \lambda B$, $|\lambda| > 1$, we have that $\prod_{\nu=1}^{n}|w_\nu| = \lambda^n$, which implies that this operator is chaotic.

We can also study shifts on sequence spaces indexed over $\mathbb{Z}$. The bilateral backward shift is given by

$$B(x_n)_{n\in\mathbb{Z}} = (x_{n+1})_{n\in\mathbb{Z}}$$

and the bilateral weighted backward shifts are given by

$$B_w(x_n)_{n\in\mathbb{Z}} = (w_{n+1}x_{n+1})_{n\in\mathbb{Z}}$$

where $w = (w_n)_n$ is called a *weight sequence*.

Theorem 1.4.10. Let X be a Fréchet sequence space in which $(e_n)_{n\in\mathbb{Z}}$ is a basis. Suppose that the bilateral shift B is an operator on X.

1. The following assertions are equivalent:
   (i) B is hypercyclic;
   (ii) B is weakly mixing;
   (iii) there is an increasing sequence $(n_k)_k$ of positive integers such that for any $j \in \mathbb{Z}$, $e_{j-n_k} \to 0$ and $e_{j+n_k} \to 0$ in X as $k \to \infty$.
2. The following assertions are equivalent:
   - B is mixing;
   - $e_{-n} \to 0$ and $e_n \to 0$ in X as $n \to \infty$.
3. The following assertions are equivalent:
   (i) B is chaotic;
   (ii) $\sum_{n=-\infty}^{\infty} e_n$ converges in X;
   (iii) The constant sequences belong to X;
   (iv) B has a nontrivial periodic point.

Using a suitable conjugacy this result can be generalized immediately to weighted shifts. The conjugacy is given by:

$$\begin{array}{ccc} X_v & \xrightarrow{\;B\;} & X_v \\ {\scriptstyle \phi_v}\big\downarrow & & \big\downarrow{\scriptstyle \phi_v} \\ X & \xrightarrow[B_w]{} & X \end{array}$$

where

$$X_v = \{(x_n)_{n\in\mathbb{Z}}; (x_n v_n)_n \in X\}$$

and $\Phi_v : X_v \to X, (x_n)_{n\in\mathbb{Z}} \to (x_n v_n)_{n\in\mathbb{Z}}$ with

$$v_n = \left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} \quad \text{for} \quad n \geq 1, v_n = \prod_{\nu=n+1}^{0} w_\nu \quad \text{for} \quad n \leq -1, v_0 = 1.$$

Theorem 1.4.11. Let X be a Fréchet sequence space over $\mathbb{Z}$ in which $(e_n)_{n\in\mathbb{Z}}$ is a basis. Suppose that the weighted shift B_w is an operator on X.

1. The following assertions are equivalent:

   (i) B_w is hypercyclic;

   (ii) B_w is weakly mixing;

   (iii) there is an increasing sequence $(n_k)_k$ of positive integers such that, for any $j \in \mathbb{Z}$

   $$\left(\prod_{\nu=j-n_k+1}^{j} w_\nu\right) e_{j-n_k} \to 0 \quad \text{and} \quad \left(\prod_{\nu=j+1}^{j+n_k} w_\nu\right)^{-1} e_{j+n_k} \to 0$$

   in X as $k \to \infty$.

2. The following assertions are equivalent:

   (i) B_w is mixing;

   (ii) we have

   $$\left(\prod_{\nu=-n+1}^{0} w_\nu\right) e_{-n} \to 0 \quad \text{and} \quad \left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} e_n \to 0$$

   in X as $n \to \infty$.

3. Suppose that the basis $(e_n)_n$ is unconditional. Then the following assertions are equivalent:

   (i) B_w is chaotic;

   (ii) the series

   $$\sum_{n=-\infty}^{0} \left(\prod_{\nu=n+1}^{0} w_\nu\right) e_n + \sum_{n=1}^{\infty} \left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} e_n$$

   converges in X;

(iii) the sequence $(x_n)_{n\in\mathbb{Z}}$ with

$$x_n = \prod_{\nu=n+1}^{0} w_\nu (n \leq 0), x_n = \left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} \quad (n \geq 1)$$

belongs to X;

(iv) B_w has a nontrivial periodic point.

Remark 1.4.12. A weighted backward shift B_w is an operator on a sequence space $\ell^p(\mathbb{Z})$, $1 \leq p < \infty$, if and only if the weights $w_n, n \in \mathbb{Z}$ are bounded. Such an operator is then hypercyclic, mixing or chaotic if and only if the following conditions, respectively, are satisfied. There exists $(n_k)_k$ such that for all $j \in \mathbb{Z}$:

$$\lim_{k\to\infty} \prod_{\nu=j-n_k+1}^{j} w_\nu = 0 \quad \text{and} \quad \lim_{k\to\infty} \prod_{\nu=j+1}^{j+n_k} |w_\nu| = \infty;$$

$$\lim_{n\to\infty} \prod_{\nu=-n+1}^{0} w_\nu = 0 \quad \text{and} \quad \lim_{n\to\infty} \prod_{\nu=1}^{n} |w_\nu| = \infty;$$

$$\sum_{n=0}^{\infty} \prod_{\nu=-n+1}^{0} |w_\nu|^p < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{\prod_{\nu=1}^{n} |w_\nu|^p} < \infty.$$

In particular, a symmetric weight (that is, one with $w_{-n} = w_n$ for all $n \geq 0$) never defines a hypercyclic weighted shift B_w on these spaces.

1.5 C_0-semigroups

In this section we study dynamical properties of strongly continuous semigroups of operators on Banach spaces, that is, for C_0-semigroups. They can be viewed as the continuous-time analogue of the discrete-time case of iterates of a single operator. All these results about C_0-semigroups can be found in the books of Engel and Nagel ([50] and [49]) and in [62].

Definition 1.5.1. A one-parameter family $(T_t)_{t\geq 0}$ of operators on a Banach space X is called a *strongly continuous semigroup of operators* if the following three conditions are satisfied:

(i) $T_0 = I$

(ii) $T_t T_s = T_{t+s}$ for all $t, s \geq 0$

(iii) $\lim_{s\to t} T_s x = T_t x$ for all $x \in X$ and $t \geq 0$

One also refers to it as a C_0-semigroup.

The analytical theory of semigroups of bounded linear operators in a Banach space deals with the exponential functions in infinite dimensional function spaces. The easiest general construction of C_0-semigroups is via an operator A on X. Since $\sum_{n=0}^{\infty} \frac{t^n}{n!}\|A\|^n < \infty$ for any $t \geq 0$,

$$T_t = e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n, \quad t \geq 0$$

defines operators on X, and it is easily seen that $(T_t)_{t\geq 0}$ is a C_0-semigroup; we even have that, for any $t \geq 0$, $\lim_{s\to t} T_s = T_t$ in the operator norm topology. The semigroup is then called *uniformly continuous.* Moreover, for any $x \in X$, $Ax = \lim_{t\to 0} \frac{1}{t}(T_t x - x)$.

Now let $(T_t)_{t\geq 0}$ be an arbitrary C_0-semigroup on X. It can be shown that

$$Ax := \lim_{t\to 0} \frac{1}{t}(T_t x - x)$$

exists on a dense subspace of X; the set of these x, the domain of A is denoted by $D(A)$. Then A, or rather $(A, D(A))$, is called the *infinitesimal generator of the semigroup.* Moreover $T_t(D(A)) \subset D(A)$ with $AT_t x = T_t A x$, for every $t \geq 0$ and $x \in D(A)$, see for instance [91]. Another important property is provided by the point spectral mapping theorem for semigroups. If X is a complex Banach space then, for every $x \in D(A)$ and $\lambda \in \mathbb{C}$,

$$Ax = \lambda x \quad \Longrightarrow \quad T_t x = e^{\lambda t} x$$

for every $t \geq 0$.

A systematic study of the dynamical properties of semigroups, was started by Desch, Schappacher and Webb [42]. In particular they introduced the notions of hypercyclicity and chaos for semigroups.

Definition 1.5.2. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X.

(i) The semigroup is *hypercyclic* if there is some $x \in X$ whose orbit $\text{Orb}(x, T_t) = \{T_t x;\ t \geq 0\}$ is dense in X. In such a case, x is called a *hypercyclic vector* for $(T_t)_{t\geq 0}$.

(ii) The semigroup is called *topologically transitive* if for any pair U, V of nonempty open sets of X, there exists some $t_0 \geq 0$ such that $T_{t_0}(U) \cap V \neq \emptyset$.

(iii) The semigroup is *mixing* if, for any pair U, V of nonempty open sets of X, there exists some $t_0 \geq 0$ such that $T_t(U) \cap V \neq \emptyset$ for all $t \geq t_0$.

(iv) The semigroup is *weakly mixing* if $(T_t \oplus T_t)_{t\geq 0}$ is topologically transitive on $X \oplus X$.

(v) A point $x \in X$ is called a *periodic point* of $(T_t)_{t\geq 0}$ if there is some $t_0 > 0$ such that $T_{t_0} x = x$.

(vi) The semigroup is said to be *chaotic* if it is hypercyclic and its set of periodic points is dense in X.

(vii) Let $(S_t)_{t\geq 0}$ be a C_0-semigroup on a Banach space Y. Then $(T_t)_{t\geq 0}$ is called *quasiconjugate* to $(S_t)_{t\geq 0}$ if there exists a continuous map $\Phi : Y \to X$ with dense range such that $T_t \circ \Phi = \Phi \circ S_t$ for all $t \geq 0$. If Φ can be chosen to be a homeomorphism then $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are called *conjugate*.

Proposition 1.5.3. Hypercyclicity, mixing, weak mixing and chaos for a C_0-semigroup are preserved under quasiconjugacy.

The first criteria for hypercyclicicty of C_0-semigroups were found by Desch, Schappacher and Webb [42]. In the form that we give, the Hypercyclicity criterion is due to Conejero and Peris [37] and El Mourchid [46], while the criterion for mixing, is due to Bermúdez, Bonilla, Conejero and Peris [18].

Theorem 1.5.4 (Hypercyclicity Criterion for Semigroups, [37], [46]). Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. If there are dense subsets $X_0, Y_0 \subset X$, a sequence $(t_n)_n \in \mathbb{R}_+$ with $t_n \to \infty$, and maps $S_{t_n} : Y_0 \to X, n \in \mathbb{N}$, such that, for any $x \in X_0, y \in Y_0$,

(i) $T_{t_n}x \to 0$,

(ii) $S_{t_n}y \to 0$,

(iii) $T_{t_n}S_{t_n}y \to y$,

then $(T_t)_{t\geq 0}$ is weakly mixing, and in particular hypercyclic.

If in the hypercyclicity criterion one has convergence along the whole real line then we obtain a criterion for mixing.

Theorem 1.5.5 ([18]). Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. If there are dense subsets $X_0, Y_0 \subset X$, and maps $S_t : Y_0 \to X, t \geq 0$, such that, for any $x \in X_0, y \in Y_0$,

(i) $T_t x \to 0$,

(ii) $S_t y \to 0$,

(iii) $T_t S_t y \to y$,

then $(T_t)_{t\geq 0}$ is mixing.

Sometimes the Hypercyclicity Criterion is hard to be applied. In many situations we can obtain the infinitesimal generator of a semigroup although we do not have the explicit representation of its operators. Desch, Schappacher, and Webb gave a criterion which permits us to state Devaney chaos (and hypercyclicity) of a C_0-semigroup in terms of the abundance of eigenvectors of the infinitesimal generator [42].

By a *weakly holomorphic function* $f : U \to X$ on an open set $U \subset \mathbb{C}$ we understand an X-valued function such that, for every $x^* \in X^*$, the complex valued function $z \longrightarrow \langle f(z), x^* \rangle$ is holomorphic on U. In the sequel, J is a nonempty index set.

Theorem 1.5.6 ([42]). Let X be a complex separable Banach space, and $(T_t)_{t \geq 0}$ a C_0-semigroup on X with generator $(A, D(A))$. Assume that there exists an open connected subset U and weakly holomorphic functions $f_j : U \to X, j \in J$, such that

(i) $U \bigcap i\mathbb{R} \neq \emptyset$,

(ii) $f_j(\lambda) \in Ker(\lambda I - A)$ for every $\lambda \in U; j \in J$,

(iii) for any $x^* \in X^*$, if $\langle f_j(\lambda), x^* \rangle = 0$ for all $\lambda \in U$ and $j \in J$ then $x^* = 0$,

then the semigroup $(T_t)_{t \geq 0}$ is mixing and chaotic.

A more general version of this criterion can be found in [47].

Theorem 1.5.7. Let X be a complex separable Banach space, and $(T_t)_{t \geq 0}$ a C_0-semigroup on X with generator $(A, D(A))$. Assume that there are $a < b$ and continuous functions $f_j : [a, b] \to X$, $j \in J$, such that

(i) $f_j(s) \in Ker(isI - A)$ for every $s \in [a, b]$, $j \in J$,

(ii) $\text{span}\{f_j(s); s \in [a, b], j \in J\}$ is dense in X,

then the semigroup $(T_t)_{t \geq 0}$ is mixing and chaotic.

1.6 Frequent Hypercyclicity

The concept of frequent hypercyclicity was introduced by Bayart and Grivaux [13] inspired by Birkhoff's Ergodic Theorem.

Theorem 1.6.1 (Birkhoff's Ergodic Theorem, [23])**.** Let T be an operator on a Fréchet space X ergodic respect to μ then, for any μ-integrable function f on X, its time average with respect to T coincides with its space average; more precisely

$$\frac{1}{N+1}\sum_{n=0}^{N} f(T^n x) \to \int_X f d\mu$$

for μ-almost all $x \in X$ as $N \to \infty$.

First of all we recall the following definition:

Definition 1.6.2. The *lower density* of a subset $A \subset \mathbb{N}_0$ is defined as

$$\underline{\text{dens}}(A) = \liminf_{N\to\infty} \frac{\text{card}\{0 \leq n \leq N; n \in A\}}{N+1}.$$

Definition 1.6.3. An operator T on a Fréchet space X is called *frequently hypercyclic* if there is some $x \in X$ such that, for any nonempty open subset U of X,

$$\underline{\text{dens}}\{n \in \mathbb{N}_0; T^n x \in U\} > 0.$$

In this case, x is called a *frequently hypercyclic vector* for T. The set of frequently hypercyclic vectors of T is denoted by $FHC(T)$.

Proposition 1.6.4. A vector x is frequently hypercyclic for T if and only if, for any nonempty open subset U of X, there is a strictly increasing sequence $(n_k)_k$ of positive integers such that

$$T^{n_k} x \in U \quad \text{for all} \quad k \in \mathbb{N} \quad \text{and} \quad n_k = O(k).$$

By contrast, T is hypercyclic if and only if the same is true for some $(n_k)_k$, not necessarily of order $O(k)$.

Definition 1.6.5. We recall that a sequence $(T_n)_n$ of continuous mappings between topological spaces X and Y is called *frequently universal* if there exists $x \in X$ such that for every non-empty open set $U \subseteq Y$,

$$\underline{\text{dens}}\{n \in \mathbb{N}_0 \,:\, T_n x \in U\} > 0.$$

In this case, x is called a *frequently universal vector* for $(T_n)_{n\in\mathbb{N}_0}$.

The first ones that used ergodic theory for the dynamics of linear operators were Rudnicki [85] and Flytzanis [51]. The notion of frequent hypercyclicity was extended to C_0-semigroups in [3]. We recall the corresponding notion of lower density for a subset of $\mathbb{R}_+$.

Definition 1.6.6. The lower density of a measurable set $M \subset \mathbb{R}_+$ is defined by

$$\underline{\text{Dens}}(M) := \liminf_{N\to\infty} \frac{\lambda(M \bigcap [0,N])}{N},$$

where λ is the Lebesgue measure on $\mathbb{R}_+$.

Definition 1.6.7. A C_0-semigroup $(T_t)_{t\geq 0}$ is said to be *frequently hypercyclic* if there exists $x \in X$ such that $\underline{\text{Dens}}(\{t \in \mathbb{R}_+;\ T_t x \in U\}) > 0$ for any non-empty open set $U \subset X$.

Proposition 1.6.8. Frequent hypercyclicity is preserved by quasi-conjugacy.

1.7 Pettis Integral

Int his section we recall the main definitions and results about Pettis integrability. The proof of all these results can be found in [44] for the case of a finite measure space, but they easily extend to σ-finite measures spaces. Let X be a Banach space and (Ω, μ) a σ-finite measure space.

Definition 1.7.1. (i) A function $f : \Omega \to X$ is said to be weakly μ-measurable if the scalar function $\varphi \circ f$ is μ-measurable for every $\varphi \in X^*$, where X^* denotes the topological dual of X.

(ii) f is said to be μ-measurable if there exists a sequence $(f_n)_n$ of simple functions such that $\lim_{n\to+\infty} \|f_n - f\| = 0$ μ-a.e.

Lema 1.7.1 (Dunford's lemma)**.** Let f be a weakly μ-measurable and $\varphi \circ f \in L_1(\Omega, \mu)$ for every $\varphi \in X^*$, then for every measurable $E \subseteq \Omega$ there exists $x_E \in X^{**}$ such that

$$x_E(\varphi) = \int_E \varphi \circ f \, d\mu$$

for every $\varphi \in X^*$.

Definition 1.7.2. (i) If $f : \Omega \to X$ is weakly μ-measurable and $\varphi \circ f \in L_1(\Omega, \mu)$ for every $\varphi \in X^*$, then f is called Dunford integrable. The Dunford integral of f over a measurable set $E \subseteq \Omega$ is defined by the element $x_E \in X^{**}$ such that $x_E(\varphi) = \int_E \varphi \circ f \; d\mu$ for every $\varphi \in X^*$.

(ii) In the case that $x_E \in X$ for every measurable E, then f is said to be Pettis integrable and x_E is called the Pettis integral of f over E which is denoted by (P) $- \int_E f \; d\mu$.

(iii) If $\|f\|$ is integrable on Ω, then f is said to be Bochner integrable on Ω.

Clearly the Dunford and the Pettis integrals coincide if X is a reflexive space, and if f is Bochner integrable, then it is Pettis integrable. Some basic and useful results to characterize Pettis integral are the following:

Theorem 1.7.3. If f is Pettis integrable, then for every sequence $(E_n)_n$ of disjoint measurable sets in Ω

$$\int_{\bigcup_{n\in\mathbb{N}} E_n} f \; d\mu = \sum_{n\in\mathbb{N}} \int_{E_n} f \; d\mu,$$

where the series converges unconditionally.

As a consequence,

Corollary 1.7.4. If $f : [0, +\infty[\to X$ is Pettis integrable on $[0, +\infty[$, then for every $\varepsilon > 0$ there exists $N > 0$ such that for every compact set $K \subset [N, +\infty[$

$$\left\| \int_K f(t) \; dt \right\| < \varepsilon.$$

Chapter 2

The specification property for backward shifts[1]

Abstract

We characterize when backward shift operators defined on Banach sequence spaces exhibit the strong specification property. In particular, within this framework, the specification property is equivalent to the notion of chaos introduced by Devaney.

2.1 Introduction

A continuous map on a metric space is said to be chaotic in the sense of Devaney if it is topologically transitive and the set of periodic points is dense. Although there is no common agreement about what a chaotic map is, a notion of chaos stronger than Devaney's definition is the so

[1] This chapter is the revised author version of article Salud Bartoll, Félix Martínez-Giménez & Alfredo Peris (2012): The specification property for backward shifts, Journal of Difference Equations and Applications, 18:4, 599–605.
In order to use same reference numbers through all the chapters, the list of references has been eliminated and they have been collected in a single bibliography at the end of the PhD thesis.

called specification property. It was first introduced by Bowen [31]; since then, several kinds and degrees of this property have been stated [88], we will follow the definitions and terminology used in [9]. Some recent works on the specification property are [80, 81, 71].

Definition 2.1.1. A continuous map $f : X \to X$ on a compact metric space (X, d) has the strong specification property (SSP) if for any $\delta > 0$ there is a positive integer N_δ such that for any integer $s \geq 2$, any set $\{y_1, \dots, y_s\} \subset X$ and any integers $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ satisfying $j_{r+1} - k_r \geq N_\delta$ for $r = 1, \dots, s-1$, there is a point $x \in X$ such that, for each positive integer $r \leq s$ and all integers i with $j_r \leq i \leq k_r$, the following conditions hold:

$$d(f^i(x), f^i(y_r)) < \delta,$$
$$f^n(x) = x, \quad \text{where } n = N_\delta + k_s.$$

When the above property is satisfied for $s = 2$, then the dynamical system is said to satisfy the weak specification property (WSP).

Obviously, the SSP implies the WSP, and compact dynamical systems with the specification property are mixing and Devaney chaotic, among other basic properties (see, e.g., [41]).

Devaney chaos and mixing properties have been widely studied for linear operators on Banach and more general spaces [18, 27, 38, 55, 57, 61, 82]. The recent books [15] and [62] contain the basic theory, examples, and many results on chaotic linear dynamics.

We plan to study this strong specification property for bounded linear operators defined on separable Banach spaces. In this situation, the first crucial problem is that these spaces are never compact. The following definition can be considered the natural extension in this setting.

Definition 2.1.2. A bounded linear operator $T : X \to X$ on a separable Banach space X has the SSP if there exists an increasing sequence $(K_m)_m$ of T-invariant compact sets with $0 \in K_1$ and $\overline{\cup_{m\in\mathbb{N}} K_m} = X$ such that for each $m \in \mathbb{N}$ the map $T|_{K_m}$ has the SSP, that is, for any $\delta > 0$ there is a positive integer $N_{\delta,m}$ such that for every $s \geq 2$, any set $\{y_1, \dots, y_s\} \subset$

K_m and any integers $0 = j_1 \le k_1 < j_2 \le k_2 < \cdots < j_s \le k_s$ with $j_{r+1} - k_r \ge N_{\delta,m}$ for $1 \le r \le s-1$, there is a point $x \in K_m$ such that, for each positive integer $r \le s$ and integers i with $j_r \le i \le k_r$, the following conditions hold:

$$\|T^i(x) - T^i(y_r)\| < \delta,$$
$$T^n(x) = x, \quad \text{where } n = N_{\delta,m} + k_s.$$

2.2 Strong specification property for backward shift operators

For a strictly positive sequence $(v_i)_i$ (weight sequence from now on), consider the Banach sequence spaces

$$\ell^p(v) := \left\{ (x_i)_i \in \mathbb{K}^{\mathbb{N}} : \|x\| := \left(\sum_{i=1}^{\infty} |x_i|^p \, v_i \right)^{1/p} < \infty \right\}, \; 1 \le p < \infty,$$
$$c_0(v) := \left\{ (x_i)_i \in \mathbb{K}^{\mathbb{N}} : \lim_{i \to \infty} |x_i| \, v_i = 0, \; \|x\| := \sup_i |x_i| \, v_i \right\}.$$

On sequence spaces, the backward shift B is defined as $B((x_i)_i) := (x_{i+1})_i$, that is, $B(x_1, x_2, x_3, \dots) := (x_2, x_3, x_4, \dots)$. In order to have a bounded operator it is required that the weights satisfy

$$\sup_{i \in \mathbb{N}} \frac{v_i}{v_{i+1}} < \infty,$$

condition that will always be assumed to hold.

Theorem 2.2.1. For a bounded backward shift operator B defined on $\ell^p(v)$, $1 \le p < \infty$ (respectively, on $c_0(v)$) the following conditions are equivalent:

(i) $\sum_{i=1}^{\infty} v_i < \infty$ (respectively, $\lim_{i \to \infty} v_i = 0$).

(ii) B has SSP.

(iii) B is Devaney chaotic.

Proof. To see (i) implies (ii) take the compact set $K = \{(x_i)_i \in \ell^p(v) : |x_i| \leq 1, \forall i\}$ and let $\delta > 0$ be fixed. There is N such that

$$\sum_{i \geq N} v_i < \frac{\delta}{2^p}.$$

Set $N_\delta = N + 1$. Take any $\{y_1, \ldots, y_s\} \subset K$ and any sequence $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ with $j_{r+1} - k_r \geq N_\delta$ for $r = 1, \ldots, s-1$. Consider $x = (x_i)_i$ defined as follows: for $1 \leq i \leq N_\delta + k_s$

$$x_i := \begin{cases} y_{1,i} \text{ if } i \in [1, j_2[\\ y_{2,i} \text{ if } i \in [j_2, j_3[\\ \vdots \\ y_{s,i} \text{ if } i \in [j_s, N_\delta + k_s] \end{cases},$$

that is,

$$(x_1, \ldots, x_{N_\delta + k_s}) = (y_{1,1}, \ldots, y_{1,j_2 - 1}, y_{2,j_2}, \ldots, y_{2,j_3 - 1}, \ldots, y_{s,j_s}, \ldots, y_{s,N_\delta + k_s})$$

and for any other index set $x_j := x_i$ if $j \equiv i \pmod{N_\delta + k_s}$. Clearly x is a periodic point belonging to K. For $r = 1, \ldots, s-1$ and $j_r \leq i \leq k_r$ we have

$$\|B^i x - B^i y_r\|^p = \sum_{l \geq j_{r+1}} |x_l - y_{r,l}|^p v_{l-i} \leq 2^p \sum_{l \geq j_{r+1}} v_{l-i} < \delta$$

since $l - i \geq j_{r+1} - i \geq j_{r+1} - k_r \geq N_\delta$. For $j_s \leq i \leq k_s$ we have

$$\|B^i x - B^i y_s\|^p = \sum_{l \geq m} |x_l - y_{s,l}|^p v_{l-i} \leq 2^p \sum_{l \geq m} v_{l-i} < \delta$$

since $l - i \geq m - i \geq N_\delta + k_s - i \geq N_\delta$. The sequence of compact sets $(K_m := mK)_m$ satisfies the required properties so that B has SSP.

That condition (i) implies (iii) come from the characterizations of chaos for backward shift operators in weighted ℓ^p-spaces (see [75, Theorem 3.2] or [60, Theorem 8]).

The proof for the case $c_0(v)$ is similar using the supremum norm. □

Our next step is to take sequences indexed over the set of integers. For a strictly positive sequence $(v_i)_{i\in\mathbb{Z}}$, consider the Banach sequence spaces

$$\ell^p(v,\mathbb{Z}) := \left\{ (x_i)_i \in \mathbb{K}^{\mathbb{Z}} : \|x\| := \left(\sum_{i=-\infty}^{\infty} |x_i|^p \, v_i \right)^{1/p} < \infty \right\}, \ 1 \le p < \infty,$$

$$c_0(v,\mathbb{Z}) := \left\{ (x_i)_i \in \mathbb{K}^{\mathbb{Z}} : \lim_{|i|\to\infty} |x_i| \, v_i = 0, \ \|x\| := \sup_i |x_i| \, v_i \right\}.$$

The bilateral backward shift B is defined as $B((x_i)_i) := (x_{i+1})_i$, that is,

$$B(\dots, x_{-2}, x_{-1}, \overset{\triangledown}{x_0}, x_1, x_2, \dots) := (\dots, x_{-1}, x_0, \overset{\triangledown}{x_1}, x_2, x_3, \dots),$$

where the small triangle marks the coordinate corresponding to the index 0. In order to have a bounded operator it is required that the weights satisfy

$$\sup_{i\in\mathbb{Z}} \frac{v_i}{v_{i+1}} < \infty,$$

condition that will always be assumed to hold.

Theorem 2.2.2. For a bounded bilateral backward shift operator B defined on $\ell^p(v,\mathbb{Z})$, $1 \le p < \infty$ (respectively, on $c_0(v,\mathbb{Z})$) the following conditions are equivalent:

(i) $\sum_{i=-\infty}^{\infty} v_i < \infty$ (respectively, $\lim_{|i|\to\infty} v_i = 0$).

(ii) B has SSP.

(iii) B is Devaney chaotic.

Proof. Take the compact set $K = \{(x_i)_i \in \ell^p(v,\mathbb{Z}) : |x_i| \le 1, \forall i\}$ and let $\delta > 0$ be fixed. There is a positive integer N such that

$$\sum_{|i|\ge N} v_i < \frac{\delta}{2^p}.$$

Set $N_\delta = 2N+1$. Take any $\{y_1,\dots,y_s\} \subset K$ and any sequence $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ with $j_{r+1} - k_r \geq N_\delta$ for $r = 1,\dots,s-1$. Consider $x = (x_i)_i$ defined as follows: for $-N \leq i \leq k_s + N$

$$x_i := \begin{cases} y_{1,i} \text{ if } i \in [-N, j_2 - N[\\ y_{2,i} \text{ if } i \in [j_2 - N, j_3 - N[\\ \quad \vdots \\ y_{s,i} \text{ if } i \in [j_s - N, k_s + N] \end{cases},$$

that is,

$$(x_{-N},\dots,x_{k_s+N}) = (y_{1,-N},\dots,y_{1,j_2-N-1},y_{2,j_2-N},\dots,y_{2,j_3-N-1},\dots,y_{s,j_s-N},\dots,y_{s,k_s+N}),$$

and for any other index set $x_j := x_i$ if $j \equiv i \pmod{N_\delta + k_s}$. The rest of the proof goes verbatim to the one of Theorem 2.2.1. □

Some operators can be represented as a weighted backward shift operator $B_w(x_1,x_2,\dots) := (w_2x_2, w_3x_3,\dots)$ defined on a weighted $\ell^p(v)$ space. This case may be reduced to the non-weighted backward shift via topological conjugacy. Set

$$a_1 := 1,\ a_i := w_2 \dots w_i,\ i > 1,$$

and consider $\ell^p(\bar{v})$ where

$$\bar{v}_i = \frac{v_i}{\prod_{j=2}^{i} \left|w_j^p\right|}, \text{ for all } i.$$

Take $\phi_a : \ell^p(v) \to \ell^p(\bar{v})$ defined as $\phi_a(x_1,x_2,\dots) := (a_1x_1, a_2x_2,\dots)$ to construct a commutative diagram $\phi_a \circ B_w = B \circ \phi_a$. Since ϕ_a is an isometry, by topological conjugacy, we have that B has SSP (is chaotic) on $\ell^p(\bar{v})$ if and only if B_w has SSP (is chaotic) on $\ell^p(v)$.

In this situation, the required condition to have $B_w : \ell^p(v) \to \ell^p(v)$ bounded is

$$\sup_{i\in\mathbb{N}} \left|w_{i+1}^p\right| \frac{v_i}{v_{i+1}} < \infty,$$

and the corresponding characterization is as follows.

Theorem 2.2.3. For a bounded weighted backward shift operator B_w defined on $\ell^p(v)$, $1 \leq p < \infty$, (respectively, on $c_0(v)$) the following conditions are equivalent:

(i) $\displaystyle\sum_{i=1}^{\infty} \frac{v_i}{\prod_{j=2}^{i} |w_j^p|} < \infty$ (respectively, $\lim_{i\to\infty} \frac{v_i}{\prod_{j=2}^{i}|w_j|} = 0$),

(ii) B_w has SSP.

(iii) B_w is Devaney chaotic.

Similarly, for the bilateral case $B_w : \ell^p(v, \mathbb{Z}) \to \ell^p(v, \mathbb{Z})$ one should take

$$a_0 := 1,\ a_i := w_1 \dots w_i,\ a_{-i} := \frac{1}{w_0 w_{-1} \dots w_{-i+1}},\ i > 0,$$

and consider $\ell^p(\bar{v}, \mathbb{Z})$ where

$$\bar{v}_0 := v_0,\ \bar{v}_i = \frac{v_i}{\prod_{j=1}^{i} |w_j^p|},\ \bar{v}_{-i} = \prod_{j=0}^{-i+1} |w_j^p|\, v_{-i},\ i > 0.$$

The required condition to have $B_w : \ell^p(v, \mathbb{Z}) \to \ell^p(v, \mathbb{Z})$ bounded is

$$\sup_{i\in\mathbb{Z}} |w_{i+1}^p| \frac{v_i}{v_{i+1}} < \infty,$$

and the characterization for SSP in this bilateral case follows.

Theorem 2.2.4. For a bounded bilateral weighted backward shift operator B_w defined on $\ell^p(v, \mathbb{Z})$, $1 \leq p < \infty$, (respectively, on $c_0(v, \mathbb{Z})$) the following conditions are equivalent:

(i) $\displaystyle\sum_{i=1}^{\infty} \frac{v_i}{\prod_{j=1}^{i} |w_j^p|} < \infty$ and $\sum_{i=1}^{\infty} \prod_{j=0}^{-i+1} |w_j^p|\, v_{-i} < \infty$

(respectively, $\displaystyle\lim_{i\to\infty} \frac{v_i}{\prod_{j=1}^{i} |w_j|} = \lim_{i\to\infty} \prod_{j=0}^{-i+1} |w_j|\, v_{-i} = 0$).

(ii) B_w has SSP.

(iii) B_w is Devaney chaotic.

2.3 Examples

2.3.1 Weighted backward shift operators on ℓ^p

For any bounded sequence $(w_i)_i$ with $w_i \neq 0$, its associated weighted backward shift $B_w : \ell^p \to \ell^p$ is a bounded operator. Since ℓ^p corresponds to $\ell^p(v)$ with $(v_i)_i = (1)_i$, we have that B_w defined on ℓ^p has the SSP property if and only if

$$\sum_{i=1}^{\infty} \prod_{j=2}^{i} |w_j^{-p}| < \infty.$$

The particular case $(w_i)_i = (\lambda)_i$ with $\lambda \in \mathbb{K}$ reads as $\lambda B : \ell^p \to \ell^p$ has the SSP if and only if $|\lambda| > 1$.

2.3.2 The operator of differentiation on Hilbert spaces of entire functions

Let $\gamma(z)$ be an admissible comparison entire function, that is, the Taylor coefficients $\gamma_i > 0$ for all $i \in \mathbb{N}_0$ and the sequence $(i\gamma_i/\gamma_{i-1})_i$ is monotonically decreasing. We consider the Hilbert space $E^2(\gamma)$ of power series

$$g(z) = \sum_{i=0}^{\infty} \hat{g}(i) z^i$$

for which

$$\|g\|_{2,\gamma}^2 := \sum_{i=0}^{\infty} \gamma_i^{-2} \left|\hat{g}(i)\right|^2 < \infty.$$

Chan and Shapiro studied some dynamical properties of the operator of differentiation and the translation operator on $E^2(\gamma)$ (see [33]).

It is clear that $E^2(\gamma)$ is isometric to $\ell^2(v)$ with $v = (v_i)_{i\in\mathbb{N}_0} = (\gamma_i^{-2})_{i\in\mathbb{N}_0}$ and with the identification $f \mapsto (f^{(i)}(0)/i!)_{i\in\mathbb{N}_0}$. Moreover, the operator of differentiation D turns out to be a weighted backward shift with weights $w = (w_i)_i = (i)_i$ or, equivalently, as a backward shift defined on $\ell^2(\bar{v})$, where

$$\bar{v}_i = \frac{1}{(\gamma_i\ i!)^2},\ i \geq 0.$$

Since $\gamma(z)$ is an admissible comparison entire function, it is easy to check that $\sup_{i\geq 0} \bar{v}_i/\bar{v}_{i+1} < \infty$ and B is a bounded operator on $\ell^2(\bar{v})$ (this is equivalent to saying that D is a bounded operator on $E^2(\gamma)$).

Applying Theorem 2.2.3 we have that $D : E^2(\gamma) \to E^2(\gamma)$ has SSP is and only if $\sum_{i=0}^{\infty}(\gamma_i i!)^{-2} < \infty$, in particular, if $\lim_i i\gamma_i/\gamma_{i-1} > 1$ then D has SSP.

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Chapter 3

Cantor Sets, Bernoulli Shifts and Linear Dynamics[1]

Abstract

Our purpose is to review some recent results on the interplay between the symbolic dynamics on Cantor sets and linear dynamics. More precisely, we will give some methods that allow the existence of strong mixing measures invariant for certain operators on Fréchet spaces, which are based on Bernoulli shifts on Cantor spaces. Also, concerning topological dynamics, we will show some consequences for the specification properties.

[1] This chapter is the revised author version of article Salud Bartoll, Félix Martínez-Giménez, Marina Murillo Arcila & Alfredo Peris (2014): Cantor Sets, Bernoulli Shifts and Linear Dynamics, Descriptive topology and functional analysis, Springer Proc. Math. Stat. 80, 195–207.
In order to use same reference numbers through all the chapters, the list of references has been eliminated and they have been collected in a single bibliography at the end of the PhD thesis.
It should be noted that Section 4 does not belong to the main topic of this PhD. We decided to keep this section since we choose to make this dissertation as *compendium* of articles.

3.1 Introduction

Our framework is the study of the dynamics of a linear operator $T : X \to X$ on a metrizable and complete topological vector space (in short, F-space) X. Moreover, we will assume that X is separable.

We recall that T is said to be *hypercyclic* if there is a vector x in X such that its *orbit* $\mathrm{Orb}(x, T) = \{x, Tx, T^2x, \dots\}$ is dense in X. The recent books [15, 62] contain the theory and most of the recent advances on hypercyclicity and linear dynamics.

Here we want to focus on some measure-theoretic properties and notions from topological dynamics. In recent years the study of the (chaotic) dynamics of linear operators has experienced a great development. This review article pretends to focus on the interplay between the dynamics of the Bernoulli shift on the Cantor set and the dynamics of certain operators. More precisely, we will focus on the strong specification property, which is a concept from topological dynamics, and the existence of strong mixing measures, a concern in measure-theoretic dynamics.

We also recall that a continuous map on a separable metric space is said to be chaotic in the sense of Devaney if it is topologically transitive (i.e., within our framework, it admits points with dense orbit) and the set of periodic points is dense. A notion of chaos (in the topological sense) stronger than Devaney's definition is the so called specification property. It was introduced by Bowen [31] and several versions of this property (see [88]) have been studied for the dynamics on compact metric spaces. We follow here the approach given in [9]. Some recent works on the specification properties are [80, 81, 71]. A continuous map $f : X \to X$ on a compact metric space (X, d) has the strong specification property if, for any $\delta > 0$, there is a positive integer N_δ such that for each integer $s \geq 2$, for any set $\{y_1, \dots, y_s\} \subset X$, and for every tuple of integers $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ satisfying $j_{r+1} - k_r \geq N_\delta$ for $r = 1, \dots, s-1$, there is a point $x \in X$ such that, for each $r \leq s$ and for all $i \in \mathbb{N} \cup \{0\}$ with $j_r \leq i \leq k_r$, the following conditions hold:

$$d(f^i(x), f^i(y_r)) < \delta,$$
$$f^n(x) = x, \quad \text{where } n = N_\delta + k_s.$$

Compact dynamical systems with the specification property are mixing and Devaney chaotic, among other basic properties (see, e.g., [41]). We will consider a notion of the strong specification property for operators as it was introduced in [8].

With respect to the measure-theoretic properties, we recall that ergodic theory was introduced in the dynamics of linear operators by Flytzanis [51] and Rudnicki [85]. It was only in recent years that it deserved special attention thanks to the work of Bayart and Grivaux [12, 13]. The papers [3, 14, 10, 39, 59, 86] contain recent advances on the subject.

The notion of frequently hypercyclicity was introduced by Bayart and Grivaux [13] as a way to measure the frequency of hitting times in an arbitrary non-empty open set for a dense orbit. In [13] a first version of a Frequent Hypercyclicity Criterion was given. We will work with the corresponding formulation of Bonilla and Grosse-Erdmann [29] for operators on separable F-spaces. Another (probabilistic) version of it was given by Grivaux [58].

After a section containing some preliminaries and basic notions, we will present the results on topological dynamics concerning the strong specification property. The last section deals with the measure-theoretic dynamics of operators with the existence of strong mixing probability measures with full support. In both cases the Frequent Hypercyclicity Criterion will play a key role. At the end it will allow us to reduce the problem to the Bernoulli shift on countable unions of Cantor sets. The particular case of weighted shifts on sequence F-spaces is also presented, since we can go a bit further in this context and they serve as a typical test ground for linear dynamics.

3.2 Notation and preliminaries

From now on, $T : X \to X$ will be an operator defined on a separable F-space X. We introduce first the necessary notions from topological dynamics for operators.

Definition 3.2.1. An operator $T : X \to X$ on a separable F-space X has the strong specification property (SSP) if there exists an increasing sequence $(K_m)_m$ of T-invariant compact sets with $0 \in K_1$ and $\overline{\cup_{m\in\mathbb{N}} K_m} = X$ such that for each $m \in \mathbb{N}$ the map $T|_{K_m}$ has the SSP.

Shifts on sequence spaces will be a matter of study in this paper. By a *sequence space* we mean a topological vector space X which is continuously included in ω, the countable product of the scalar field $\mathbb{K}$. A *sequence F-space* is a sequence space that is also an F-space. Given a sequence $w = (w_n)_n$ of positive weights, the associated *unilateral* (respectively, *bilateral*) *weighted backward shift* $B_w : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ is defined by $B_w(x_1, x_2, \dots) = (w_2x_2, w_3x_3, \dots)$ (respectively, $B_w : \mathbb{K}^{\mathbb{Z}} \to \mathbb{K}^{\mathbb{Z}}$ is defined by $B_w(\dots, x_{-1}, x_0, x_1, \dots) = (\dots, w_0x_0, w_1x_1, w_2x_2, \dots)$). If a sequence F-space X is invariant under certain weighted backward shift T, then T is also continuous on X by the closed graph theorem. In [62, Chap. 4] contains more details about dynamical properties of weighted shifts on Fréchet sequence spaces.

We recall that a series $\sum_n x_n$ in X *converges unconditionally* if it converges and, for any 0-neighbourhood U in X, there exists some $N \in \mathbb{N}$ such that $\sum_{n\in F} x_n \in U$ for every finite set $F \subset \{N, N+1, N+2, \dots\}$.

The results on Devaney chaos for shift operators given in [62] (we refer the reader to [60] for the original results) remain valid for a unilateral (respectively, bilateral) weighted backward shift $T = B_w : X \to X$ on a sequence F-space X in which the canonical unit vectors $(e_n)_{n\in\mathbb{N}}$ (respectively, $(e_n)_{n\in\mathbb{Z}}$) form an unconditional basis. In particular, B_w is chaotic if, and only if,

$$\sum_{n=1}^{\infty}\Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1} e_n \text{ (respectively, } \sum_{n=-\infty}^{0}\Big(\prod_{\nu=n+1}^{0} w_\nu\Big) e_n + \sum_{n=1}^{\infty}\Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1} e_n)$$

converges unconditionally.

For a weight sequence $(v_i)_i$ the following Banach sequence spaces are considered

$$\ell^p(v) := \left\{ (x_i)_i \in \mathbb{K}^{\mathbb{N}} : \|x\| := \left(\sum_{i=1}^{\infty} |x_i|^p \, v_i \right)^{1/p} < \infty \right\}, \; 1 \le p < \infty,$$

$$c_0(v) := \left\{ (x_i)_i \in \mathbb{K}^{\mathbb{N}} : \lim_{i \to \infty} |x_i| \, v_i = 0, \; \|x\| := \sup_i |x_i| \, v_i \right\}.$$

In this situation, the required condition to have $B_w : \ell^p(v) \to \ell^p(v)$ bounded is

$$\sup_{i \in \mathbb{N}} \left| w_{i+1}^p \right| \frac{v_i}{v_{i+1}} < \infty,$$

condition that will always be assumed to hold.

For a weight sequence $(v_i)_{i \in \mathbb{Z}}$ indexed on the set of integers, we consider the Banach sequence spaces

$$\ell^p(v, \mathbb{Z}) := \left\{ (x_i)_i \in \mathbb{K}^{\mathbb{Z}} : \|x\| := \left(\sum_{i=-\infty}^{\infty} |x_i|^p \, v_i \right)^{1/p} < \infty \right\}, \; 1 \le p < \infty,$$

$$c_0(v, \mathbb{Z}) := \left\{ (x_i)_i \in \mathbb{K}^{\mathbb{Z}} : \lim_{|i| \to \infty} |x_i| \, v_i = 0, \; \|x\| := \sup_i |x_i| \, v_i \right\}.$$

The condition that characterizes that $B_w : \ell^p(v, \mathbb{Z}) \to \ell^p(v, \mathbb{Z})$ is bounded is

$$\sup_{i \in \mathbb{Z}} \left| w_{i+1}^p \right| \frac{v_i}{v_{i+1}} < \infty.$$

Finally, for measure-theoretic dynamics, let $(X, \mathfrak{B}, \mu)$ be a probability space, where X is a topological space and $\mathfrak{B}$ denotes the σ-algebra of Borel subsets of X. We say that a Borel probability measure μ has *full support* if for all non-empty open set $U \subset X$ we have $\mu(U) > 0$. A measurable map $T : (X, \mathfrak{B}, \mu) \to (X, \mathfrak{B}, \mu)$ is called a *measure-preserving* transformation if $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathfrak{B}$). T is *ergodic* if $T^{-1}(A) = A$ for certain $A \in \mathfrak{B}$ necessarily implies that $\mu(A)(1 - \mu(A)) = 0$. T is said to be *strongly mixing* with respect to μ if

$$\lim_{n \to \infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B) \qquad (A, B \in \mathfrak{B}),$$

and it is *exact* if given $A \in \bigcap_{n=0}^{\infty} T^{-n}\mathfrak{B}$ then either $\mu(A)(1-\mu(A)) = 0$. We refer to [45, 90] for a detailed account on these properties.

Given $A \subset \mathbb{N}$, its *lower density* is defined by

$$\underline{\text{dens}}(A) = \liminf_{n\to\infty} \frac{\text{card}(A \cap [1,n])}{n},$$

An operator T is *frequently hypercyclic* (see [13]) if there exists $x \in X$ such that for every non-empty open subset U of X, the set $\{n \in \mathbb{N}\ ;\ T^n x \in U\}$ has positive lower density.

The following version of the so called Frequently Hypercyclicity Criterion given by Bonilla and Grosse-Erdmann [29] (see also [62, Theorem 9.9 and Remark 9.10]) gives a sufficient condition for frequent hypercyclicity.

Theorem 3.2.2 ([29])**.** Let T be an operator on a separable F-space X. If there is a dense subset X_0 of X and a sequence of maps $S_n : X_0 \to X$ such that, for each $x \in X_0$,

(i) $\sum_{n=0}^{\infty} T^n x$ converges unconditionally,

(ii) $\sum_{n=0}^{\infty} S_n x$ converges unconditionally, and

(iii) $T^n S_n x = x$ and $T^m S_n x = S_{n-m} x$ if $n > m$,

then the operator T is frequently hypercyclic.

3.3 Cantor sets and the Strong Specification Property

This section is devoted to a dynamical property in the topological sense, namely, the SSP. We first study the dynamics on certain Cantor sets of the backward shift on a sequence F-space X, by following an approach slightly different from the one in [8]. Later in this section we present a more general argument that yields the SSP for operators on F-spaces satisfying the Frequent Hypercyclicity Criterion.

Theorem 3.3.1. Let $B_w : X \to X$ be a unilateral weighted backward shift on a sequence F-space X in which $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis. Then the following conditions are equivalent:

(i) B_w is chaotic;

(ii) the series
$$\sum_{n=1}^{\infty}\Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1} e_n$$
converges in X;

(iii) B_w has a nontrivial periodic point;

(iv) B_w has the SSP.

Proof. The equivalence of (i)-(ii)-(iii) was given in [60] (See also [62, Chap. 4]). Obviously, (iv) implies (iii). For the converse, we fix a countable set $M = \{z_n \ ; \ n \in \mathbb{N}\}$ of pairwise different scalars which form a dense set in $\mathbb{K}$ with $z_1 = 0$. Let $(U_n)_n$ be a basis of balanced open 0-neighbourhoods in X such that $U_{n+1} + U_{n+1} \subset U_n$, $n \in \mathbb{N}$.

Since $\sum_{n=1}^{\infty}\Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1} e_n$ converges, there exists an increasing sequence of positive integers $(N_n)_n$ with $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ for all $n \in \mathbb{N}$ such that

$$\begin{aligned}&\sum_{k>N_n} \alpha_k\Big(\prod_{\nu=1}^{k} w_\nu\Big)^{-1} e_k \in U_{n+1},\\ &\text{if } \alpha_k \in \{z_1, \ldots, z_n\}, \text{ for each } n \in \mathbb{N}.\end{aligned} \tag{3.1}$$

We define $A_m = \{z_1, \ldots z_m\}$ for $m \in \mathbb{N}$. $A_m^{\mathbb{N}}$ is a compact space when endowed with the product topology inherited from $M^{\mathbb{N}}$, $m \in \mathbb{N}$. Now we define the map $\Phi : \bigcup_{m=1}^{\infty} A_m^{\mathbb{N}} \to X$ given by

$$\Phi((\alpha_k)_{k\in\mathbb{N}}) = \sum_{k=1}^{\infty} \alpha_k\Big(\prod_{\nu=1}^{k} w_\nu\Big)^{-1} e_k.$$

Φ is well-defined and $\Phi|_{A_m^{\mathbb{N}}}$ is continuous for each $m \in \mathbb{N}$ by (3.1). We have that $K_m := \Phi(A_m^{\mathbb{N}})$ is a compact Cantor subset of X, invariant under B_w, and such that $B_w|_{K_m}$ is conjugated to $\sigma|_{A_m^{\mathbb{N}}}$ via Φ, for each $m \in \mathbb{N}$. It is well-known that $\sigma|_{A_m^{\mathbb{N}}}$ satisfies the SSP (see, e.g., [88]), and by conjugacy we obtain that $B_w|_{K_m}$ satisfies the SSP too, for every $m \in \mathbb{N}$. Finally, by density of M in the scalar field $\mathbb{K}$ we conclude that $\bigcup_{m\in\mathbb{N}} K_m$ is dense in X, therefore B_w satisfies the SSP. □

A similar argument, by considering the bilateral shift on the sets $A_m^{\mathbb{Z}}$, $m \in \mathbb{N}$, and the results of [60] (See also [62, Chap. 4]), yields the analogous characterizations for bilateral weighted shifts on sequence F-spaces.

Theorem 3.3.2. Let $B_w : X \to X$ be a bilateral weighted backward shift on a sequence F-space X in which $(e_n)_{n\in\mathbb{Z}}$ is an unconditional basis. Then the following conditions are equivalent:

(i) B_w is chaotic;

(ii) the series

$$\sum_{n=-\infty}^{0}\Big(\prod_{\nu=n+1}^{0} w_\nu\Big)e_n + \sum_{n=1}^{\infty}\Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1} e_n$$

converges in X;

(iii) B_w has a nontrivial periodic point;

(iv) B_w has the SSP.

Particular cases on which the above results are interesting are the (unilateral and bilateral) weighted shifts on (weighted or not) ℓ^p-spaces and c_0 (see [8]). Moreover, F. Bayart and I. Z. Ruzsa [16] recently proved that weighted shift operators on ℓ^p, $1 \leq p < \infty$, are frequently hypercyclic if, and only if, they are Devaney chaotic. This fact adds a new equivalence for ℓ^p-spaces.

Corollary 3.3.3. For a bounded weighted backward shift operator B_w defined on $X = \ell^p(v)$, $1 \leq p < \infty$, (respectively, on $X = c_0(v)$) the following conditions are equivalent:

(i) $\displaystyle\sum_{i=1}^{\infty} \frac{v_i}{\prod_{j=1}^{i} |w_j^p|} < \infty$ (respectively, $\lim_{i\to\infty} \frac{v_i}{\prod_{j=1}^{i}|w_j|} = 0$),

(ii) B_w has SSP.

(iii) B_w is Devaney chaotic.

Moreover, for $X = \ell^p(v)$, $1 \leq p < \infty$, the above items are equivalent to

(iv) B_w is frequently hypercyclic.

Corollary 3.3.4. For a bounded bilateral weighted backward shift operator B_w defined on $\ell^p(v, \mathbb{Z})$, $1 \leq p < \infty$, (respectively, on $c_0(v, \mathbb{Z})$) the following conditions are equivalent:

(i) $\displaystyle\sum_{i=1}^{\infty} \frac{v_i}{\prod_{j=1}^{i} |w_j^p|} < \infty$ and $\sum_{i=1}^{\infty} \prod_{j=0}^{-i+1} |w_j^p| \, v_{-i} < \infty$

(respectively, $\displaystyle\lim_{i\to\infty} \frac{v_i}{\prod_{j=1}^{i} |w_j|} = \lim_{i\to\infty} \prod_{j=0}^{-i+1} |w_j| \, v_{-i} = 0$).

(ii) B_w has SSP.

(iii) B_w is Devaney chaotic.

Moreover, for $X = \ell^p(v, \mathbb{Z})$, $1 \leq p < \infty$, the above items are equivalent to

(iv) B_w is frequently hypercyclic.

In the previous results we deduced the SSP from the dynamics on certain invariant Cantor sets. Something similar can be done under the more general assumptions that the operator satisfies the Frequent Hypercyclicity Criterion (see [5]. The idea now is to work with certain invariant factors of Cantor sets. We will take the general version of the Frequent Hypercyclicity Criterion given in [62, Remark 9.10].

Theorem 3.3.5. Let T be an operator on a separable F-space X. If there is a dense subset X_0 of X and a sequence of maps $S_n : X_0 \to X$ such that, for each $x \in X_0$,

(i) $\sum_{n=0}^{\infty} T^n x$ converges unconditionally,

(ii) $\sum_{n=0}^{\infty} S_n x$ converges unconditionally, and

(iii) $T^n S_n x = x$ and $T^m S_n x = S_{n-m} x$ if $n > m$,

then the operator T satisfies the SSP.

Proof. We suppose that $X_0 = \{x_n \ ; \ n \in \mathbb{N}\}$ with $x_1 = 0$ and $S_n 0 = 0$ for all $n \in \mathbb{N}$. Let $(U_n)_n$ be a basis of balanced open 0-neighbourhoods in X such that $U_{n+1} + U_{n+1} \subset U_n$, $n \in \mathbb{N}$. By (i) and (ii), there exists an increasing sequence of positive integers $(N_n)_n$ with $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ for all $n \in \mathbb{N}$ such that

$$\sum_{k>N_n} T^k x_{m_k} \in U_{n+1} \text{ and } \sum_{k>N_n} S_k x_{m_k} \in U_{n+1}, \qquad \text{if } m_k \in \{1, \dots, n\}, \text{ for each } n \in \mathbb{N}. \quad (3.2)$$

We let $B_m = \{1, \dots, m\}$ and define the map $\Phi : \bigcup_{m=1}^{\infty} B_m^{\mathbb{Z}} \to X$ given by

$$\Phi((n_k)_{k\in\mathbb{Z}}) = \sum_{k<0} S_{-k} x_{n_k} + x_{n_0} + \sum_{k>0} T^k x_{n_k}.$$

Φ is well-defined and $\Phi|_{B_m^{\mathbb{Z}}}$ is continuous for each $m \in \mathbb{N}$ by (3.2). As in Theorem 3.3.1 we have that $K_m := \Phi(B_m^{\mathbb{Z}})$ is a compact subset of X, invariant under the operator T, and such that $T|_{K_m}$ is conjugated to $\sigma^{-1}|_{B_m^{\mathbb{Z}}}$ via Φ, for each $m \in \mathbb{N}$. Again, by conjugacy, we obtain that $T|_{K_m}$ satisfies the SSP too, for every $m \in \mathbb{N}$ and, since $\bigcup_{m\in\mathbb{N}} K_m$ is dense in X because it contains X_0, we conclude that T satisfies the SSP. □

3.4 Mixing measures and Bernoulli shifts

For the existence of strong mixing measures with full support, certain Cantor subsets of $\mathbb{N}^N$, with either $N = \mathbb{N}$ or $N = \mathbb{Z}$, will be needed. This time we precise some finer adjustments that involve Cantor sets larger than A^N with A finite. Actually they will be of the form $C = \prod_{n\in N} A_n$, where the cardinalities of the finite sets A_n tend to infinity as $n \to \infty$. The problem now is that these type of sets are not invariant under the shift, so actually we will use invariant sets that are countable unions of Cantor sets. The contents of this section are based on [78]. We also recall that, recently, Bayart and Matheron gave very general conditions expressed on eigenvector fields associated to unimodular eigenvalues under which an operator T admits a T-invariant mixing measure [10].

In all the cases that we treat in this section the idea is to construct a model probability space $(Z, \overline{\mu})$ and a (Borel) measurable map $\Phi : Z \to X$, where

(a) $Z \subset \mathbb{N}^N$ is such that $\sigma(Z) = Z$ for the Bernoulli shift (either unilateral or bilateral),

(b) $\overline{\mu}$ is a σ-invariant strongly mixing measure,

(c) $Y := \Phi(Z)$ is a T-invariant dense subset of X, and

(d) the operator T is either (quasi)conjugated to σ or to σ^{-1} through Φ.

This way, the measure $\overline{\mu}$ induces by (quasi)conjugacy a Borel probability measure μ on X that is T-invariant and strongly mixing.

The model probability space $(Z, \overline{\mu})$.

We will construct $Z \subset \mathbb{N}^N$, invariant under the shift, where $N = \mathbb{N}$ or $N = \mathbb{Z}$. In all the cases we are given certain increasing sequence of positive integers $(N_n)_n$ with $N_{n+2} - N_{n+1} > N_{n+1} - N_n$. We define the compact space $L = \prod_{k\in N} A_k$ where

$$A_k = \{1, \dots m\} \text{ if } N_m < |k| \le N_{m+1},\ m \in \mathbb{N},\ \text{ and } A_k = \{1\},\ \text{ if } |k| \le N_1.$$

Let $L(s) := \sigma^s(L)$, $s \in N \cup \{0\}$. $L(s)$ is a subspace of $\mathbb{N}^N$, $s \in N \cup \{0\}$.

In $\mathbb{N}^N$ we define the product probability measure $\overline{\mu} = \bigotimes_{k\in N} \overline{\mu_k}$, where $\overline{\mu_k}(\{n\}) = p_n$ for all $n \in \mathbb{N}$ and $\overline{\mu_k}(\mathbb{N}) = \sum_{n=1}^{\infty} p_n = 1$, $k \in N$. The numbers $p_n \in]0,1[$, $n \in \mathbb{N}$, are so that, if

$$\beta_j := \left(\sum_{i=1}^{j} p_i\right)^{N_{j+1}-N_j}, \ j \in \mathbb{N}, \text{ then } \prod_{j=1}^{\infty} \beta_j > 0.$$

We define $Z = \bigcup_{s\in N} L(s)$, which is a countable union of Cantor sets, invariant under the shift, and satisfies

$$\overline{\mu}(Z) \geq \overline{\mu}(L) = \prod_{|k|\leq N_1} \overline{\mu}_k(\{1\}) \prod_{l=1}^{\infty} \left(\prod_{N_l<|k|\leq N_{l+1}} \overline{\mu}_k(\{1,\ldots,l\}) \right)$$
$$\geq p_1^{2N_1+1} (\prod_{l=1}^{\infty} \beta_l)^2 > 0.$$

By [90] we know that $\overline{\mu}$ is a σ-invariant strongly mixing Borel probability measure on $\mathbb{N}^N$. Since $\sigma(Z) = Z$, it has positive measure, and every strong mixing measure is ergodic, we necessarily have that $\overline{\mu}(Z) = 1$. Even more, in the case $N = \mathbb{N}$ it is known (see [90, Sect. 4.12]) that $\overline{\mu}$ is a σ-invariant exact Borel probability measure.

Once we have defined the model, as in the previous section we first analyze the situation for unilateral shifts.

Theorem 3.4.1. Let $B_w : X \to X$ be a unilateral weighted backward shift on a sequence F-space X in which $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis. If B_w is chaotic then there exists a T-invariant exact Borel probability measure on X with full support.

Proof. As in the proof of Theorem 3.3.1, we fix a countable set $M = \{z_n \ ; \ n \in \mathbb{N}\}$ of pairwise different scalars which form a dense set in $\mathbb{K}$ with $z_1 = 0$, and a basis $(U_n)_n$ of balanced open 0-neighbourhoods in X such that $U_{n+1} + U_{n+1} \subset U_n$, $n \in \mathbb{N}$.

Again, although not so immediate, the fact that $\sum_{n=1}^{\infty}\left(\prod_{\nu=1}^{n} w_\nu\right)^{-1} e_n$ converges, implies the existence of an increasing sequence of positive integers $(N_n)_n$ with $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ for all $n \in \mathbb{N}$ such that

$$\begin{aligned}&\sum_{k>N_n} \alpha_k \Big(\prod_{\nu=1}^{k} w_\nu\Big)^{-1} e_k \in U_{n+1},\\ &\text{if } \alpha_k \in \{z_1, \dots, z_{2m}\}, \text{ for } N_m < k \le N_{m+1},\ m \ge n.\end{aligned} \tag{3.3}$$

We set $Z = \bigcup_{s\ge 0} L(s) \subset \mathbb{N}^{\mathbb{N}}$ the σ-invariant set and the exact probability measure $\overline{\mu}$ considered in the model above, and we define the map

$$\Phi : Z \to X$$

given by

$$\Phi((n(k))_{k\in\mathbb{N}}) = \sum_{k=1}^{\infty} \alpha_{n(k)} \Big(\prod_{\nu=1}^{k} w_\nu\Big)^{-1} e_k.$$

Φ is well-defined and $\Phi|_{L(s)}$ is continuous for each $s \ge 0$ by (3.3). We also have that $Y := \Phi(Z)$ is a countable union of Cantor subsets of X, invariant under B_w, and such that $B_w|_Y$ is conjugated to $\sigma|_Z$ via Φ. Since we know that $\overline{\mu}$ is exact on Z, the measure $\mu(A) = \overline{\mu}(\Phi^{-1}(A))$, $A \in \mathfrak{B}(X)$, is well-defined on X, it is B_w-invariant and exact, so it only remains to show that it has full support. Indeed, given a non-empty open set U in X, we pick $y = \Phi((n(k))_k) \in Y$ satisfying $y + U_n \subset U$. Thus

$$\begin{aligned}\mu(U) &\ge \mu\left(\left\{x = y + \sum_{k>N_n} \frac{\alpha_k}{\prod_{\nu=1}^{k} w_\nu} e_k \ ;\ \alpha_k \in \{z_1, \dots, z_{2m}\},\right.\right.\\ &\qquad\qquad\qquad\qquad \left.\left.\text{for } N_m < k \le N_{m+1},\ m \ge n\right\}\right)\\ &\ge \prod_{k=1}^{N_n} \overline{\mu}_k(\{n(k)\}) \prod_{l=n}^{\infty} \left(\prod_{N_l<k\le N_{l+1}} \overline{\mu}_k(\{1, \dots, 2l\})\right)\\ &> \prod_{k=1}^{N_n} p_{n(k)} \left(\prod_{l=n}^{\infty} \beta_l\right)^2 > 0,\end{aligned}$$

by (3.3) and by the selection of the sequence $(p_n)_n$ in the model. Therefore we conclude the result. $\square$

As in the previous section, a general result can be obtained for operators satisfying the Frequent Hypercyclicity Criterion.

Theorem 3.4.2. Let T be an operator on a separable F-space X. If there is a dense subset X_0 of X and a sequence of maps $S_n : X_0 \to X$ such that, for each $x \in X_0$,

(i) $\sum_{n=0}^{\infty} T^n x$ converges unconditionally,

(ii) $\sum_{n=0}^{\infty} S_n x$ converges unconditionally, and

(iii) $T^n S_n x = x$ and $T^m S_n x = S_{n-m} x$ if $n > m$,

then there is a T-invariant strongly mixing Borel probability measure μ on X with full support.

Proof. As in Theorem 3.3.5 we suppose that $X_0 = \{x_n \ ; \ n \in \mathbb{N}\}$ with $x_1 = 0$ and $S_n 0 = 0$ for all $n \in \mathbb{N}$, and $(U_n)_n$ is a basis of balanced open 0-neighbourhoods in X such that $U_{n+1} + U_{n+1} \subset U_n$, $n \in \mathbb{N}$. Again, with a more subtle argument, by (i) and (ii) we have the existence of an increasing sequence of positive integers $(N_n)_n$ with $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ for all $n \in \mathbb{N}$ such that

$$\sum_{k>N_n} T^k x_{m_k} \in U_{n+1} \text{ and } \sum_{k>N_n} S_k x_{m_k} \in U_{n+1},$$
$$\text{if } m_k \leq 2l, \text{ for } N_l < k \leq N_{l+1}, \ l \geq n. \quad (3.4)$$

Actually, this is a consequence of the completeness of X and the fact that, for each 0-neighbourhood U and for all $l \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that $\sum_{k \in F} T^k x \in U$ and $\sum_{k \in F} S_k x \in U$ for any finite subset $F \subset]N, +\infty[$ and for each $x \in \{x_1, \dots, x_{2l}\}$.

We fix now our model space $(Z, \overline{\mu})$ with $Z = \bigcup_{s \in \mathbb{Z}} L(s) \subset \mathbb{N}^{\mathbb{Z}}$ and the measure $\overline{\mu}$ associated with the increasing sequence $(N_n)_n$.

1. **The map Φ**

   We define the map $\Phi : Z \to X$ by

   $$\Phi((n(k))_{k\in\mathbb{Z}}) = \sum_{k<0} S_{-k}x_{n(k)} + x_{n(0)} + \sum_{k>0} T^k x_{n(k)}. \tag{3.5}$$

   Φ is well-defined since, given $(n(k))_{k\in\mathbb{Z}} \in L(s)$, and for $l \geq |s|$, we have $n(k) \leq 2l$ if $N_l < |k| \leq N_{l+1}$, which shows the convergence of the series in (3.5) by (3.4).

   $\Phi|_{L(s)}$ is also continuous for each $s \in \mathbb{Z}$. Indeed, let $(\gamma_j)_j$ be a sequence of elements of $L(s)$ that converges to $\gamma \in L(s)$ and fix any $n \in \mathbb{N}$ with $n > |s|$. We will find $n_0 \in \mathbb{N}$ such that $\Phi(\gamma_j) - \Phi(\gamma) \in U_n$ for $n \geq n_0$. To do this, by definition of the topology of $L(s)$ there exists $n_0 \in \mathbb{N}$ such that

   $$\gamma(k)_j = \gamma(k) \text{ if } |k| \leq N_{n+1} \text{ and } j \geq n_0.$$

   By (3.4) we have

   $$\Phi(\gamma_j) - \Phi(\gamma) = \sum_{k<-N_{n+1}} S_{-k}(x_{\gamma(k)_j} - x_{\gamma(k)}) + \sum_{k>N_{n+1}} T^k(x_{\gamma(k)_j} - x_{\gamma(k)}) \in U_n$$

   for all $j \geq n_0$. This shows the continuity of $\Phi : L(s) \to X$ for every $s \in \mathbb{Z}$.

   The map $\Phi : Z \to X$ is then measurable (i.e., $\Phi^{-1}(A) \in \mathfrak{B}(Z)$ for every $A \in \mathfrak{B}(X)$).

2. **The measure μ on X**

   $Y := \Phi(Z)$ is a T-invariant Borel subset of X because it is a countable union of compact sets and $\Phi\sigma^{-1} = T\Phi$. As before, the measure μ on X is defined by $\mu(A) = \overline{\mu}(\Phi^{-1}(A))$ for all $A \in \mathfrak{B}(X)$, which is well-defined, T-invariant, and a strongly mixing Borel probability measure. As in Theorem 3.4.1, it only remains to show that μ has full support. Indeed, given a non-empty open set U in X, we fix $n \in \mathbb{N}$ such that $x_n + U_n \subset U$. Therefore,

$$\mu(U) \geq \mu\left(\left\{x = x_n + \sum_{k>N_n} T^k x_{m(k)} + \sum_{k>N_n} S_k x_{m(-k)} \ ; \right.\right.$$
$$\left.\left. m(k) \leq 2l, \text{ for } N_l < |k| \leq N_{l+1}, \ l \geq n \right\}\right)$$
$$\geq \overline{\mu}_0(\{n\}) \prod_{0<|k|\leq N_n} \overline{\mu}_k(\{1\}) \prod_{l=n}^{\infty} \left(\prod_{N_l<|k|\leq N_{l+1}} \overline{\mu}_k(\{1,\dots,2l\}) \right)$$
$$> p_n p_1^{2N_n} \left(\prod_{l=n}^{\infty} \beta_l \right)^2 > 0,$$

and we obtain that μ has full support. □

As a consequence, since every chaotic bilateral shift on a sequence F-space in which the natural basis $(e_n)_{n\in\mathbb{Z}}$ is an unconditional basis satisfies the Frequent Hypercyclicity Criterion (see, e.g., [62, Chap. 9]), we obtain strong mixing measures for these shifts.

Corollary 3.4.3. Let $T : X \to X$ be a chaotic bilateral weighted shift on a sequence F-space X in which $(e_n)_{n\in\mathbb{Z}}$ is an unconditional basis. Then there exists a T-invariant strongly mixing Borel probability measure on X with full support.

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Chapter 4

Operators with the specification property[1]

Abstract

We study a version of the specification property for linear dynamics. Operators having the specification property are investigated, and relationships with other well known dynamical notions such as mixing, Devaney chaos, and frequent hypercyclicity are obtained.

4.1 Introduction

A continuous map on a metric space is said to be chaotic in the sense of Devaney if it is topologically transitive and the set of periodic points is dense. Although there is no common agreement about what a chaotic map is, a notion of chaos stronger than Devaney's definition is the so

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In order to use same reference numbers through all the chapters, the list of references has been eliminated and they have been collected in a single bibliography at the end of the PhD thesis.

called specification property. It was first introduced by Bowen [31] and since then, several kinds and degrees of this property have been stated [88]. We follow the definitions and terminology used in [9]. Some recent works on the specification property are [80, 81, 71, 52, 63, 70, 8, 7].

Definition 4.1.1 ([31])**.** A continuous map $f : X \to X$ on a compact metric space (X, d) has the *specification property* (SP) if for any $\delta > 0$ there is a positive integer N_δ such that for any integer $s \geq 2$, any set $\{y_1, \dots, y_s\} \subset X$ and any integers $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ satisfying $j_{r+1} - k_r \geq N_\delta$ for $r = 1, \dots, s-1$, there is a point $x \in X$ such that, for each positive integer $r \leq s$ and any integer i with $j_r \leq i \leq k_r$, the following conditions hold:

$$d(f^i(x), f^i(y_r)) < \delta,$$
$$f^n(x) = x, \text{ where } n = N_\delta + k_s.$$

Although there are weaker versions of this property, we will be using the above version, which is in fact, the strongest one. Compact dynamical systems with the SP are mixing and Devaney chaotic, among other basic dynamical properties (see, e.g., [41]).

Devaney chaos and mixing properties have been widely studied for linear operators on Banach and more general spaces [18, 27, 38, 55, 57, 61, 82]. The recent books [15] and [62] contain the basic theory, examples, and many results on chaotic linear dynamics.

Our aim is to study the SP in the context of continuous linear operators defined on separable F-spaces. In this situation, the first crucial problem is that these spaces are never compact, therefore, our first task should be the adjusting of this property to the new context. We recall that an F-space is a topological vector space whose topology is induced by a complete translation-invariant metric. In fact, if X is an F-space, there exists a complete translation-invariant metric d such that $\|x\| = d(x, 0)$ is an F-norm.

Definition 4.1.2. Let X be a vector space. A map $\|\cdot\|$ from X to $\mathbb{R}^+$ is an *F-norm* provided for each $x, y \in X$ and $\lambda \in \mathbb{K}$ we have

1. $\|x + y\| \leq \|x\| + \|y\|$;

2. $\|\lambda x\| \leq \|x\|$ if $|\lambda| < 1$;

3. $\lim_{\lambda\to 0}\|\lambda x\| = 0$;

4. $\|x\| = 0$ implies $x = 0$.

A consequence from 1 and 2 above is that for any $x \in X$, and $\lambda \in \mathbb{K}$, we have

$$\|\lambda x\| \leq (|\lambda| + 1)\|x\|.$$

The class of F-spaces includes complete metrizable locally convex spaces (i.e., Fréchet spaces) and hence it also includes Banach spaces. For introductory texts on functional analysis that cover Fréchet spaces we refer to Rudin [84] and Meise and Vogt [76]. The notion of an F-norm can be found in Kalton, Peck and Roberts [67].

From now on the space X will a separable (infinite dimensional) F-space with F-norm $\|.\|$, and $T : X \to X$ will be a continuous linear operator (operator for short).

The following definition can be considered the natural extension of the SP in this setting.

Definition 4.1.3. An operator $T : X \to X$ on a separable F-space X has the *operator specification property* (OSP) if there exists an increasing sequence $(K_m)_m$ of T-invariant sets with $0 \in K_1$ and $\overline{\cup_{m\in\mathbb{N}} K_m} = X$ such that for each $m \in \mathbb{N}$ the map $T|_{K_m}$ has the SP, that is, for any $\delta > 0$ there is a positive integer $N_{\delta,m}$ such that for every $s \geq 2$, any set $\{y_1, \dots, y_s\} \subset K_m$ and any integers $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ with $j_{r+1} - k_r \geq N_{\delta,m}$ for $1 \leq r \leq s-1$, there is a point $x \in K_m$ such that, for each positive integer $r \leq s$ and any integer i with $j_r \leq i \leq k_r$, the following conditions hold:

$$\|T^i(x) - T^i(y_r)\| < \delta,$$
$$T^n(x) = x, \text{ where } n = N_{\delta,m} + k_s.$$

Observation 4.1.4. We would like to point out that although we removed compactness of each K_m from our definition, it is hard to think of a map having the specification property outside of the compact setting; in other words, for all cases we know of operators having the OSP, the required sets K_m are always compact.

Its is natural to study any property in linear dynamics for the most typical operators in this context, namely the weighted shifts on sequence spaces. By a *sequence space* we mean a topological vector space X which is continuously included in ω, the countable product of the scalar field $\mathbb{K}$. A *sequence F-space* is a sequence space that is also an F-space. Given a sequence $w = (w_n)_n$ of positive weights, the associated *unilateral weighted backward shift* $B_w : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ is defined by $B_w(x_1, x_2, \dots) = (w_2 x_2, w_3 x_3, \dots)$. If a sequence F-space X is invariant under certain weighted backward shift T, then T is also continuous on X by the closed graph theorem. In [8] we characterize when backward shift operators defined on certain Banach sequence spaces exhibit the OSP. We were able to extend these characterizations to the more general setting of sequence F-spaces in [7] where the next result is proved.

Theorem 4.1.5 ([7])**.** Let $B_w : X \to X$ be a unilateral weighted backward shift on a sequence F-space X in which $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis. Then the following conditions are equivalent:

(i) B_w is chaotic;

(ii) the series
$$\sum_{n=1}^{\infty} \Big(\prod_{\nu=1}^{n} w_\nu \Big)^{-1} e_n$$
converges in X;

(iii) B_w has a nontrivial periodic point;

(iv) B_w has the OSP.

The paper is organised as follows: in Section 4.2 we study the basic properties for operators with the OSP. In Section 4.3 we show the connections of the OSP with other dynamical properties for linear operators like mixing, chaos in the sense of Devaney and frequent hypercyclicity. Section 4.4 provides several examples of operators with the OSP. In the final Section 4.5 we present the conclusions and a diagram containing the implications between the different dynamical properties discussed here.

4.2 Basic properties

We first show that the OSP behaves well by quasi-conjugation.

Proposition 4.2.1. Suppose $T_i : X_i \to X_i$ is an operator on a separable F-space X_i, $i = 1, 2$, and $\phi : X_1 \to X_2$ is a uniformly continuous map with dense range such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{T_1} & X_1 \\ {\scriptstyle \phi}\big\downarrow & & {\scriptstyle \phi}\big\downarrow \\ X_2 & \xrightarrow{T_2} & X_2 \end{array}$$

conmutes. If T_1 has the OSP then so does T_2.

Proof. Without loss of generality, we may assume that $\phi(0) = 0$, otherwise take $\tilde{\phi} := \phi - \phi(0)$. Since T_1 has the OSP, let $(K^1_m)_m$ the required sequence of T_1-invariant sets satisfying all the conditions given in Definition 4.1.3. Set $(K^2_m)_m := (\phi(K^1_m))_m$. Clearly $0 \in K^2_1$ and, since ϕ has dense range we have that $\overline{\cup_{m\in\mathbb{N}} K^2_m} = X_2$. The map ϕ is uniformly continuous on each K^1_m, therefore, fixed $\delta > 0$, there exists $\delta' > 0$ such that for each $x, y \in K^1_m$ with $\|x - y\| < \delta'$, we have $\|\phi(x) - \phi(y)\| < \delta$. Since $T_1|_{K^1_m}$ has the SP, there exits $N_{\delta',m}$. Taking now $N_{\delta,m} := N_{\delta',m}$, using the commutativity of the diagram, and the uniform continuity of ϕ, it is routine to see that T_2 has the OSP. □

Remark 4.2.2. Since uniform continuity and continuity are equivalent for linear transformations on F-spaces, we have that Proposition 4.2.1 is true when $\phi : X_1 \to X_2$ is a linear continuous transformation with dense range. Even more, if ϕ is a linear homeomorphism, then T_1 has the OSP if and only if T_2 does it.

Our next result shows that each iterate of an operator having the OSP inherits that property. This is a natural question in discrete dynamics, and the most important result in this direction in the linear setting was due to Ansari [2] who proved that T^n is hypercyclic whether T is. We recall that for continuous maps on separable complete metric spaces, topological transitivity is equivalent to the existence of a dense orbit, and this concept is known as hypercyclicity in our context (see [62]).

Proposition 4.2.3. If $T : X \to X$ has the OSP, then so does T^k for every $k \in \mathbb{N}$.

Proof. To show that T^k has the OSP, take the same sequence $(K_m)_m$ of T-invariant sets, which are obviously T^k-invariant. Since $T|_{K_m}$ has the SP, given $\delta > 0$, there exists $N_{\delta,m}$ satisfying all the requirements of Definition 4.1.1 and, taking a greater index if necessary, we may assume that $N_{\delta,m}$ is a multiple of k. Now, the positive integer $N_{\delta,m}/k$ would do the job to show that $T^k|_{K_m}$ has the SP. □

Next, we study how the OSP behaves by direct sums of operators. The motivation for this question in our linear setting comes from an old problem of Herrero [64]: He asked whether $T \oplus T$ is hypercyclic whenever T is. This problem turned out to be equivalent to the question whether every hypercyclic operator satisfies the so called Hypercyclicity Criterion. The negative answer was found by de la Rosa and Read [40]. In contrast, the OSP is inherited by taking direct sums.

Proposition 4.2.4. Suppose $T_i : X_i \to X_i$ is an operator on a separable F-space X_i, $1 \leq i \leq n$. If T_i has the OSP for $1 \leq i \leq n$, then $\oplus_{i=1}^n T_i : \oplus_{i=1}^n X_i \to \oplus_{i=1}^n X_i$ has the OSP.

Proof. It is enough to do the proof for the case of two operators. We recall that there are several equivalent F-norms on $X_1 \oplus X_2$, we use here the F-norm

$$\|(x_1, x_2)\|_{X_1 \oplus X_2} := \|x_1\|_{X_1} + \|x_2\|_{X_2}, \ (x_1, x_2) \in X_1 \oplus X_2,$$

where $\|.\|_{X_i}$ is the corresponding F-norm on X_i. To make notation simpler we will avoid to specify the underlying space.

Take $(K_m)_m := (K_m^1 \times K_m^2)_m$, where $(K_m^i)_m$ is the corresponding sequence of T_i- invariant sets required by Definition 4.1.3, $i = 1, 2$. To see that $(T_1 \oplus T_2)|_{K_m}$ has the SP, given $\delta > 0$, since $T_i|_{K_m}$ has the SP, $i = 1, 2$, there exists $N^i_{\delta/2,m}$ and we take $N_{\delta,m} := \max\{N^1_{\delta/2,m}, N^2_{\delta/2,m}\}$. Now, for every $s \geq 2$, any set $\{(y_1^1, y_1^2), \ldots, (y_s^1, y_s^2)\} \subset K_m$ and any integers $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ with $j_{r+1} - k_r \geq N_{\delta,m}$ for $1 \leq r \leq s-1$, there are $x^1 \in K_m^1$ and $x^2 \in K_m^2$ such that, for each positive integer $r \leq s$ and any integer i with $j_r \leq i \leq k_r$, the following conditions hold:

$$\|T_1^i(x^1) - T_1^i(y_r^1)\| < \delta/2,$$
$$\|T_2^i(x^2) - T_2^i(y_r^2)\| < \delta/2,$$
$$\left.\begin{array}{l} T_1^n(x_1) = x_1 \\ T_2^n(x_2) = x_2 \end{array}\right\}, \ n = N_{\delta,m} + k_s.$$

Now, it is easy to check that

$$\|(T_1 \oplus T_2)^i(x^1, x^2) - (T_1 \oplus T_2)^i(y_r^1, y_r^2)\| < \delta, \ 1 \leq r \leq s, \ j_r \leq i \leq k_r,$$
$$(T_1 \oplus T_2)^n(x^1, x^2) = (x^1, x^2), \ n = N_{\delta,m} + k_s,$$

which completes the proof. □

We finish this section with two additional properties. They may appear somehow artificial and technical but they do play a crucial role in the proof of one of the main results of the next section.

Proposition 4.2.5. Let $T : X \to X$ be an operator on a separable F-space X.

i) If $\lambda_i \in \mathbb{K}$ and $K_i \subset X$ is a T-invariant set such that $T|_{K_i}$ has the SP, $1 \leq i \leq k$, then $T|_{\sum_{i=1}^k \lambda_i K_i}$ has the SP.

ii) If X is locally convex and $K \subset X$ is a T-invariant set such that $T|_K$ has the SP and $\overline{\text{co}(K)}$ is the closed convex envelope of K, then $T|_{\overline{\text{co}(K)}}$ has the SP.

Proof. i) It suffices to prove the case of two sets. Given $\delta > 0$, take $\delta' := \delta/(|\lambda_1| + |\lambda_2| + 2)$. There exists $N^i_{\delta'}$, $i = 1, 2$, and we take $N_\delta := \max\{N^1_{\delta'}, N^2_{\delta'}\}$. Now, for every $s \geq 2$, any set $\{(\lambda_1 y^1_1 + \lambda_2 y^2_1), \ldots, (\lambda_1 y^1_s + \lambda_2 y^2_s)\} \subset \lambda_1 K_1 + \lambda_2 K_2$ and any integers $0 = j_1 \leq k_1 < j_2 \leq k_2 < \cdots < j_s \leq k_s$ with $j_{r+1} - k_r \geq N_\delta$ for $1 \leq r \leq s - 1$, there are $x^1 \in K_1$ and $x^2 \in K_2$ such that for each integer $r \in [1, s]$ and for any integer i with $j_r \leq i \leq k_r$, the following conditions hold:

$$\begin{aligned} &\|T^i(x^1) - T^i(y^1_r)\| < \delta', \\ &\|T^i(x^2) - T^i(y^2_r)\| < \delta', \\ &\left.\begin{aligned} T^n(x^1) = x^1 \\ T^n(x^2) = x^2 \end{aligned}\right\}, \ n = N_\delta + k_s. \end{aligned}$$

Now, it is easy to check that

$$\|T^i(\lambda_1 x^1 + \lambda_2 x^2) - T^i(\lambda_1 y^1_r + \lambda_2 y^2_r)\| < \delta, \ 1 \leq r \leq s, \ j_r \leq i \leq k_r,$$
$$T^n(\lambda_1 x^1 + \lambda_2 x^2) = \lambda_1 x^1 + \lambda_2 x^2, \ n = N_\delta + k_s.$$

ii) By the continuity of T and all its iterates, it is clear that if $T|_K$ has the SP, then $T|_{\overline{K}}$ has the SP. Therefore, it remains to show that $T|_{\mathrm{co}(K)}$ has the SP. Since X is a Fréchet space, we can fix an increasing sequence of seminorms $(\|\cdot\|_n)_n$ that generate the topology of X. The key point to prove this is to observe that if you have two (or more points) belonging to the convex hull of a set, you can always rewrite the convex combinations in such a way that their length and coefficients are the same. This fact may appear strange at first because one usually thinks about 'minimal' convex combinations but the fact is clear if we decompose terms of the convex combination in several terms 'as needed'. For example

$$\begin{aligned} x &= 0.9x_1 + 0.1x_2 &&= 0.5x_1 + 0.3x_1 + 0.1x_1 + 0.1x_2 \\ y &= 0.5y_1 + 0.3y_2 + 0.2y_3 &&= 0.5y_1 + 0.3y_2 + 0.1y_3 + 0.1y_3 \end{aligned}$$

Using the above fact to express the points $\{y_1, \ldots, y_s\} \subset \mathrm{co}(K)$ as convex combinations with the same length and coefficients, the fact that $T|_{\mathrm{co}(K)}$ has the SP can be obtained as in one i), by computing the corresponding inequalities for an arbitrary seminorm $\|\cdot\|_n$, $n \in \mathbb{N}$. $\square$

4.3 Connections with other dynamical properties

In this section we focus in the connections of the OSP with other well know dynamical properties. To be precise, we prove that operators with the OSP are mixing, chaotic in the sense of Devaney, and they have a strong version of hypercyclicity, introduced by Bayart and Grivaux [11, 13], called frequent hypercyclicity. For completeness we recall that an operator $T : X \to X$ is topologically transitive if for any non-empty open sets U and V, there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. Moreover, if the set $\{n \in \mathbb{N} : T^n(U) \cap V \neq \emptyset\}$ is cofinite, then T is mixing.

Proposition 4.3.1. If $T : X \to X$ has the OSP, then T is mixing.

Proof. Fix a non-empty open set U and a 0-neighbourhood W. We recall that the return set from U to W is defined as $N(U, W) := \{n \in \mathbb{N} : T^n(U) \cap W \neq \emptyset\}$. We claim that $N(U, W)$ and $N(W, U)$ are cofinite and this implies T is mixing (see [62, Proposition 2.37]).

Take $u \in U$ and $\delta > 0$ such that $B(u, 2\delta) \subset U$ and $B(0, 2\delta) \subset W$. Since T has the OSP, we may find a set K such that $T|_K$ has the SP and $K \cap B(u, \delta) \neq \emptyset$. There exists N_δ (depending on K and δ).

Take $y_1 \in K \cap B(u, \delta)$, $y_2 = 0$, $m \in \mathbb{N}$, $0 = j_1 = k_1 < j_2 = N_\delta < k_2 = N_\delta + m$. Since $j_2 - k_1 \geq N_\delta$, there exists $x \in K$ such that

$$\|T^i x - T^i y_1\| < \delta, \ i = j_1, \dots, k_1$$
$$\|T^i x - T^i y_2\| < \delta, \ i = j_2, \dots, k_2.$$

This implies $\|x - y_1\| < \delta$, so $\|x - u\| < 2\delta$ and hence $x \in U$. As $T^i y_2 = 0$, we have that $T^i x \in B(0, \delta)$ for $i = N_\delta, \dots, N_\delta + m$, therefore $T^{N_\delta + m} x \in B(0, \delta) \subset W$. We have proved that $N_\delta + m \in N(U, W)$ for any $m \in \mathbb{N}$.

Take now $i = N_\delta$. Clearly $\|T^{N_\delta} x\| < \delta$, hence $T^{N_\delta} x \in B(0, \delta) \subset W$. Observing that x is periodic with period $N_\delta + k_2$, we have

$$T^{N_\delta + m}(T^{N_\delta} x) = T^{N_\delta + k_2} x = x \in U,$$

which means that $N_\delta + m \in N(W, U)$ for any $m \in \mathbb{N}$. This finishes the proof □

Proposition 4.3.2. If $T : X \to X$ has the OSP, then T is chaotic in the sense of Devaney, that is, T is topologically transitive and it admits a dense set of periodic points.

Proof. Clearly, T is transitive by the above proposition. By the mere definition of the specification property, it is also clear that any point in the space may be approximated by periodic points. □

Given $A \subset \mathbb{N}$, its lower density is defined by

$$\underline{\mathrm{dens}}(A) = \liminf_{n\to\infty} \frac{\mathrm{card}(A \cap [1,n])}{n},$$

An operator T is *frequently hypercyclic* if there exists $x \in X$ such that for every non-empty open subset U of X, the set $\{n \in \mathbb{N} : T^n x \in U\}$ has positive lower density (see [13]). Frequent hypercyclicity is a way to measure the frequency of hitting times in an arbitrary non-empty open set for a dense orbit.

Theorem 4.3.3. If $T : X \to X$ has the OSP, then T is frequently hypercyclic.

Proof. Let $(K_m)_m$ be an increasing sequence of T-invariant sets associated with the OSP for T. Fix $v_m \in K_m$, $m \in \mathbb{N}$, such that $\overline{\{v_m; m \in \mathbb{N}\}} = X$. We set inductively $\widetilde{K}_1 = K_1$, $\widetilde{K}_m = K_m - \sum_{i=1}^{m-1} \widetilde{K}_i$, $m > 1$. We know that T satisfies the OSP with respect to $(\widetilde{K}_m)_m$. Let $(r_m)_m$ be an increasing sequence in $]0,1[$ with

$$\lim_m \prod_{i=1}^m r_i > 0,$$

and fix $(p_m)_m \subset \mathbb{N}$ such that $(p_m - 1)/(p_m + 1) > r_m$, $m \in \mathbb{N}$. We apply the OSP with respect to $(\widetilde{K}_m)_m$. For $\delta_m := 2^{-m}$ we denote $N_m = N_{\delta_m,m}$, $m \in \mathbb{N}$. W.l.o.g., $(p_m + 1)N_m$ divides N_{m+1}, $m \in \mathbb{N}$. Given $m \in \mathbb{N}$ we set $j_1 = k_1 = 0$, $j_2 = N_m$, $k_2 = p_m N_m$. For $m = 1$ let $y_1 = v_1$ and, inductively, given $m > 1$ suppose we have $x_i \in \widetilde{K_i}$, $i = 1, \ldots, m-1$. Let

$$y_1 = u_m := v_m - \sum_{i=1}^{m-1} T^{N_i} x_i \in \widetilde{K}_m,$$
$$y_2 = 0.$$

By assumption, there is $x_m \in \widetilde{K}_m$ such that

$$\|x_m - u_m\| < 2^{-m}, \quad \|T^i x_m\| < 2^{-m}, \quad i = N_m, \ldots, p_m N_m, \quad \text{and}$$
$$T^n x_m = x_m \text{ for } n = (p_m + 1)N_m.$$

We will show that the vector $x := \sum_k T^{N_k} x_k$ is frequently hypercyclic for T.

Let $q_k := (p_k + 1)N_k$, $k \in \mathbb{N}$. Given $m > 1$, we have

$$\|T^{p_m N_m + j q_m}\left(\sum_{k=1}^{m} T^{N_k} x_k\right) - v_m\| = \|\left(\sum_{k=1}^{m-1} T^{N_k} x_k + u_m\right) - v_m\| < \frac{1}{2^m}, \tag{4.1}$$

for all $j \in \mathbb{N}_0$.

Fix $n > q_{m+1}$. There exists $m' > m$ such that $(p_{m'} - 1)N_{m'} < n \leq (p_{m'+1} - 1)N_{m'+1}$. Since

$$\|T^j(T^{N_k} x_k)\| < \frac{1}{2^k}, \quad \forall k \geq m' + 1, \quad j = 0, \ldots, (p_{m'+1} - 1)N_{m'+1},$$

we get,

$$\|T^j\left(\sum_{k \geq m'+1} T^{N_k} x_k\right)\| < \frac{1}{2^{m'}}, \quad j = 0, \ldots, n. \tag{4.2}$$

It remains to show the inequalities $\|T^j(T^{N_k} x_k)\| < 2^{-k}$ for $m < k \leq m'$, and for certain $j \leq n$ of the form $j = p_m N_m + j' q_m$, $j' \in \mathbb{N}_0$. To do this, we have to count the number of elements of this form contained in suitable blocks of consecutive integers. Indeed, for each $i \in \mathbb{N}_0$, the block of integers $\{iN_{m+1} + j \; ; \; j = 0, \ldots, N_{m+1} - 1\}$ contains N_{m+1}/q_m elements of the form $p_m N_m + j q_m$, $j \in \mathbb{N}_0$. Hence the block $\{iq_{m+1} + j \; ; \; j =$

$0, \ldots, (p_{m+1}-1)N_{m+1}\}$ contains $(p_{m+1}-1)N_{m+1}/q_m$ elements of the form $p_m N_m + j q_m$, $j \in \mathbb{N}_0$.

Analogously, for each $i \in \mathbb{N}_0$, the block $\{iN_{k+1}+j \; ; \; j = 0, \ldots, N_{k+1}-1\}$ contains N_{k+1}/q_k blocks of the form $\{i'q_k+j \; ; \; j = 0, \ldots, (p_k-1)N_k\}$, $i' \in \mathbb{N}_0$. Hence the block $\{iq_{k+1}+j \; ; \; j = 0, \ldots, (p_{k+1}-1)N_{k+1}\}$ contains $(p_{k+1}-1)N_{k+1}/q_k$ blocks of the form $\{i'q_k + j \; ; \; j = 0, \ldots, (p_k-1)N_k\}$, $i' \in \mathbb{N}_0$.

We assumed $n > (p_{m'}-1)N_{m'}$, then there is $k \geq 1$ such that $(p_{m'}-1)N_{m'} + (k-1)q_{m'} \leq n < (p_{m'}-1)N_{m'} + kq_{m'}$. This implies that the set of integers $A_n := \{i \in \mathbb{N}_0 \; ; \; i \leq n\}$ contains k blocks of the form $\{i'q_{m'} + j \; ; \; j = 0, \ldots, (p_{m'}-1)N_{m'}\}$, $i' \in \mathbb{N}_0$. Thus, the above considerations yield that A_n contains

$$k\left(\frac{(p_{m'}-1)N_{m'}}{q_{m'-1}}\right)\cdots\left(\frac{(p_{m+1}-1)N_{m+1}}{q_m}\right) > k(p_{m'}-1)N_{m'}\frac{\alpha_m}{q_m}$$

elements of the form $j = p_m N_m + j'q_m$, $j' \in \mathbb{N}_0$, such that $\|T^j(T^{N_l}x_l)\| < 2^{-l}$, for all $l \in \{m+1, \ldots, m'\}$, where

$$\alpha_m := \prod_{l>m} \frac{p_l-1}{p_l+1} > \prod_{l>m} r_l > 0.$$

Therefore,

$$\left|\{j \leq n \; ; \; \|T^j(\sum_{k>m} T^{N_k}x_k)\| < \frac{1}{2^m}\}\right| > \left(\frac{k}{k+1}\right)\left(\frac{p_{m'}-1}{p_{m'}+1}\right) n\beta_m \geq n\frac{\beta_m}{4}, \tag{4.3}$$

where $\beta_m := \alpha_m/q_m$.

From (4.1) and (4.3), we conclude

$$\underline{\text{dens}}\{j \in \mathbb{N} \; ; \; \|T^j x - v_m\| < \frac{1}{2^{m-1}}\} \geq \frac{\beta_m}{4} > 0,$$

which finishes the proof. □

The most usual way to prove that an operator is (frequently) hypercyclic is to use the so called (Frequent) Hypercyclicity Criterion (see [15, 62]). It should be noted that the proof of Theorem 4.3.3 does not use the Frequent Hypercyclicity Criterion, instead a frequent hypercyclicity vector is constructed. Next result shows that the Frequent Hypercyclicity Criterion is far stronger than any of the dynamical properties we have been working with in this paper. In particular, it implies the OSP. We will take the general version of the Frequent Hypercyclicity Criterion given in [29].

Theorem 4.3.4. Let $T : X \to X$ be an operator on a separable F-space X. If there is a dense subset X_0 of X and a sequence of maps $S_n : X_0 \to X$ such that, for each $x \in X_0$,

(i) $\sum_{n=0}^{\infty} T^n x$ converges unconditionally,

(ii) $\sum_{n=0}^{\infty} S_n x$ converges unconditionally, and

(iii) $T^n S_n x = x$ and $T^m S_n x = S_{n-m} x$ if $n > m$,

then the operator T has the OSP.

Proof. We suppose that $X_0 = \{x_n \ ; \ n \in \mathbb{N}\}$ with $x_1 = 0$ and $S_n 0 = 0$ for all $n \in \mathbb{N}$. Let $(U_n)_n$ be a basis of balanced open 0-neighbourhoods in X such that $U_{n+1} + U_{n+1} \subset U_n$, $n \in \mathbb{N}$. By (i) and (ii), there exists an increasing sequence of positive integers $(N_n)_n$ with $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ for all $n \in \mathbb{N}$ such that

$$\sum_{k>N_n} T^k x_{m_k} \in U_{n+1} \text{ and } \sum_{k>N_n} S_k x_{m_k} \in U_{n+1}, \qquad \text{if } m_k \in \{1, \ldots, n\}, \text{ for each } n \in \mathbb{N}. \tag{4.4}$$

We let $B_m = \{1, \ldots, m\}$ and define the map $\Phi : \bigcup_{m=1}^{\infty} B_m^{\mathbb{Z}} \to X$ given by

$$\Phi((n_k)_{k\in\mathbb{Z}}) = \sum_{k<0} S_{-k} x_{n_k} + x_{n_0} + \sum_{k>0} T^k x_{n_k}.$$

The map Φ is well-defined and $\Phi|_{B_m^{\mathbb{Z}}}$ is continuous for each $m \in \mathbb{N}$ by (4.4). We have that $K_m := \Phi(B_m^{\mathbb{Z}})$ is a compact subset of X, invariant under the operator T, and such that $T|_{K_m}$ is conjugated to $\sigma^{-1}|_{B_m^{\mathbb{Z}}}$

via Φ, for each $m \in \mathbb{N}$; where σ is the usual Bernoulli shift defined as $\sigma(\dots, n_{-1}, n_0, n_1, \dots) = (\dots, n_0, n_1, n_2, \dots)$. Since $\sigma^{-1}|_{B_m^{\mathbb{Z}}}$ has the SP (see for instance [88]), by conjugacy, we obtain that $T|_{K_m}$ satisfies the SP too, for every $m \in \mathbb{N}$ and, since $\bigcup_{m \in \mathbb{N}} K_m$ is dense in X because it contains X_0, we conclude that T has the OSP. □

4.4 Families of operators with the OSP

We already noticed in Theorem 4.1.5 that weighted backward shifts on sequence F-spaces having the OSP can be characterized in terms of the weight sequence, therefore examples of backward shifts with the OSP defined on the Banach spaces ℓ^p and c_0 are easy to find (see [8]). Also, any weighted shift so that every weight is non-zero on ω has the OSP. Theorem 4.3.4 is very useful to find more examples of operators having the OSP.

Example 4.4.1. We consider the Fréchet space $H(\mathbb{C})$ of entire functions endowed with the topology of uniform convergence on compact sets. Suppose that $T : H(\mathbb{C}) \to H(\mathbb{C})$, $T \neq \lambda I$, is an operator that commutes with the operator of differentiation D, that is, $TD = DT$. It is known that T satisfies the Frequent Hypercyclicity Criterion [28], so T has the OSP.

Example 4.4.2. Let φ be a nonconstant bounded holomorphic function on the unit disc $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, and let M_φ^* be the corresponding adjoint multiplication operator on the Hardy space H^2. Godefroy and Shapiro proved that M_φ^* is hypercyclic if and only if $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, where $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ (see [55]). They even proved that condition $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$ is equivalent to M_φ^* being chaotic and also equivalent to M_φ^* being mixing. Later, Bayart and Grivaux improved this result showing that if $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, then M_φ^* satisfies the Frequent Hypercyclicity Criterion [13]. Taking into account Theorem 4.3.4 we have the following characterization.

Theorem 4.4.3. Let φ be a nonconstant bounded holomorphic function on $\mathbb{D}$ and let M_φ^* be the corresponding adjoint multiplier on H^2. Then the following assertions are equivalent: (i) M_φ^* is hypercyclic; (ii) M_φ^* is mixing; (iii) M_φ^* is chaotic; (iv) M_φ^* is frequently hypercyclic; (v) M_φ^* has the OSP; (vi) $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$.

Example 4.4.4. Let φ be an automorphism of the unit disk $\mathbb{D}$ and let $C_\varphi f = f \circ \varphi$ be the corresponding composition operator on the Hardy space H^2. Bourdon and Shapiro proved that C_φ is hypercyclic if and only if C_φ is mixing if and only if φ has no fixed point in $\mathbb{D}$ (see [30]). Hosokawa [66] proved that C_φ is chaotic whenever it is hypercyclic and his proof shows that in fact C_φ satisfies the Frequent Hypercyclicity Criterion whenever φ has no fixed point (see also [89]). Therefore we have the following characterization.

Theorem 4.4.5. Let $\varphi \in \mathrm{Aut}(\mathbb{D})$ and C_φ be the corresponding composition operator on H^2. Then the following assertions are equivalent: (i) C_φ is hypercyclic; (ii) C_φ is mixing; (iii) C_φ is chaotic; (iv) C_φ is frequently hypercyclic; (v) C_φ has the OSP; (vi) φ has no fixed point in $\mathbb{D}$.

Example 4.4.6. Let $\Omega \subset \mathbb{C}$ be a simply connected domain, and let $\varphi : \Omega \to \Omega$ be a holomorphic function. Bès (see [21, Theorem 1]) characterized several dynamical properties for the composition operator C_φ on $H(\Omega)$, which included when C_φ satisfies the Frequent Hypercyclicity Criterion. As a consequence we obtain the following result.

Theorem 4.4.7. The following assertions are equivalent: (i) C_φ is hypercyclic; (ii) $P(C_\varphi)$ has the OSP for every non-constant polynomial P; (iii) φ is univalent and has no fixed point in Ω.

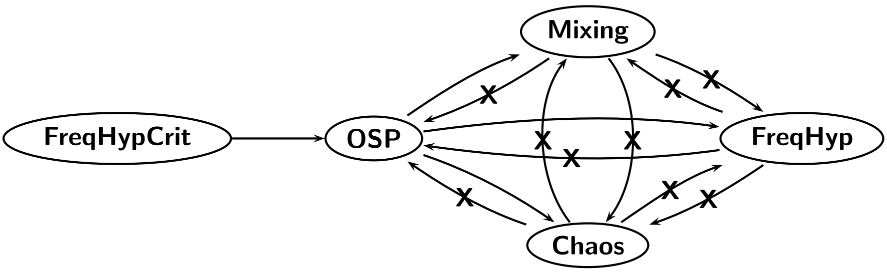
4.5 Concluding remarks

Theorems 4.3.1, 4.3.2, and 4.3.3 show that the OSP is in fact a strong dynamical property. Next we prove that neither converse of those theorems are true. Even more, we will show that there are operators defined on the Hilbert space ℓ^2 which are mixing, chaotic and frequently hypercyclic altogether but not having the OSP. To this aim we need a result from [34] concerning sets of periods of maps. We recall that $n \in \mathbb{N}$ is a period of T if there is $x \in X$ such that $T^n x = x$ but $T^i x \neq x$ for $0 < i < n$. The set of periods of T is defined as $\{n : n \text{ is a period of } T\}$.

Theorem 4.5.1 ([34]). A nonvoid subset $A \subset \mathbb{N}$ is the set of periods for certain bounded operator T on a separable complex Hilbert space H if and only if A contains $\operatorname{lcm}(a, b)$ for every $a, b \in A$. Moreover, if A is infinite, then it is possible to find a mixing, chaotic and frequently hypercyclic bounded operator T on H whose set of periods is exactly A.

Now, as set of periods take the powers of 2, that is, $A = \{2^i,\ i \in \mathbb{N}\}$. To fix ideas set the complex Hilbert space ℓ^2. Obviously A contains the least common multiple of any pair of elements of A so, by Theorem 4.5.1, there exists a mixing, chaotic and frequently hypercyclic operator $T : \ell^2 \to \ell^2$ whose periods are only the powers of 2. The operator T cannot have OSP because for any operator having this property there is a positive integer N such that any integer greater than N is a period.

At this point, relationships between OSP, mixing, chaos, frequent hypercyclicity and the Frequent Hypercyclicity Criterion are shown in the next figure.



To complete the figure, we would like to mention here the sources for the counter examples: Mixing operators which are not chaotic are easy to find; Bayart and Grivaux [14] constructed a weighted shift on c_0 that is frequently hypercyclic, but neither chaotic nor mixing; Badea and Grivaux [3] found operators on a Hilbert space that are frequently hypercyclic and chaotic but not mixing. Also, Bayart and Grivaux [13] provided easy examples of topologically mixing operators that are not frequently hypercyclic. Very recently, Menet constructed examples of chaotic operators which are not frequently hypercyclic in [77], which solved an important problem in linear dynamics.

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Chapter 5

The specification property for semigroups[1]

Abstract

We study one of the strongest versions of chaos for continuous dynamical systems, namely the specification property. We extend the definition of specification property for operators on a Banach space to strongly continuous one-parameter semigroups of operators, that is, C_0-semigroups. In addition, we study the relationships of the specification property for C_0-semigroups (SgSP) with other dynamical properties: mixing, Devaney's chaos, distributional chaos and frequent hypercyclicity. Concerning the applications, we provide several examples of semigroups which exhibit the SgSP with particular interest on solution semigroups to certain linear PDEs, which range from the hyperbolic heat equation to the Black-Scholes equation.

[1]This chapter is the revised author version of preprint Salud Bartoll, Félix Martínez-Giménez, Alfredo Peris & Francisco Rodenas (2015): The specification property for semigroups, submitted for publication.
In order to use same reference numbers through all the chapters, the list of references has been eliminated and they have been collected in a single bibliography at the end of the PhD thesis.

5.1 Introduction

A (continuous) map on a metric space satisfies the specification property (SP) if for any choice of points, one can approximate distinct pieces of orbits by a single periodic orbit with a certain uniformity. It was first introduced by Bowen [31]; since then, this property has attracted the interest of many researchers (see, for instance, the early work [88]). In a few words, the specification property requires that, for a given distance $\delta > 0$, and for any finite family of points, there is always a periodic orbit that traces arbitrary long pieces of the orbits of the family, up to a distance δ, allowing a minimum "jump time" N_δ from one piece of orbit to another one, which only depends on δ.

Definition 5.1.1. A continuous map $f : X \to X$ on a compact metric space (X, d) has the specification property if for any $\delta > 0$ there is a positive integer N_δ such that for any integer $s \geq 2$, any set $\{y_1, \dots, y_s\} \subset X$ and any integers $0 = i_1 \leq j_1 < i_2 \leq j_2 < \cdots < i_s \leq j_s$ satisfying $i_{r+1} - j_r \geq N_\delta$ for $r = 1, \dots, s-1$, there is a point $x \in X$ such that the following conditions hold:

$$d(f^i(x), f^i(y_r)) < \delta, \text{ with } i_r \leq i \leq j_r, \text{ for every } r \leq s,$$
$$f^{N_\delta + j_s}(x) = x.$$

This definition must be modified when one treats with bounded linear operators defined on separable Banach spaces which are never compact [8, 5]. Here, we denote the specification property for operators by OSP (see [8, 5] for definitions and properties). A continuous map on a metric space is said to be chaotic in the sense of Devaney if it is topologically transitive and the set of periodic points is dense. Although there is no common agreement about what a chaotic map is, the specification property is stronger than Devaney's definition of chaos. Recently, several properties of linear operators with the OSP and the connections of this OSP with other well-known dynamical properties, like mixing, chaos in the sense of Devaney and frequent hypercyclicity have been studied in [5], we will use these results throughout the paper. Other recent works on the specification property are [80, 81, 71].

A family $(T_t)_{t\geq 0}$ of linear and continuous operators on a Banach space X is said to be a C_0-*semigroup* if $T_0 = Id$, $T_tT_s = T_{t+s}$ for all $t, s \geq 0$, and $\lim_{t\to s} T_tx = T_sx$ for all $x \in X$ and $s \geq 0$.

Let $(T_t)_{t\geq 0}$ be an arbitrary C_0-semigroup on X. It can be shown that an operator defined by $Ax := \lim_{t\to 0} \frac{1}{t}(T_tx - x)$ exists on a dense subspace of X; denoted by $D(A)$. Then A, or rather $(A, D(A))$, is called the *(infinitesimal) generator* of the semigroup. It can also be shown that the infinitesimal generator determines the semigroup uniquely. If the generator A is defined on X ($D(A) = X$), the semigroup is expressed as $\{T_t\}_{t\geq 0} = \{e^{tA}\}_{t\geq 0}$.

Given a family of operators $(T_t)_{t\geq 0}$, we say that this family of operators is *transitive* if for every pair of non-empty open sets $U, V \subset X$ there exists some $t > 0$ such that $T_t(U) \cap V \neq \emptyset$. Furthermore, if there is some t_0 such that the condition $T_t(U) \cap V \neq \emptyset$ holds for every $t \geq t_0$ we say that it is *topologically mixing* or mixing.

A family of operators $(T_t)_{t\geq 0}$ is said to be *universal* if there exists some $x \in X$ such that $\{T_tx : t \geq 0\}$ is dense in X. When $(T_t)_{t\geq 0}$ is a C_0-semigroup we refer to it as *hypercyclic* instead of universal. In this setting, transitivity coincides with universality, but it is strictly weaker than mixing [18].

In addition, two notions of chaos are introduced: *Devaney chaos* and *distributional chaos*. First, we recall that an element $x \in X$ is said to be a *periodic point* of $(T_t)_{t\geq 0}$ if there exists some $t_0 > 0$ such that $T_{t_0}x = x$. A family of operators $(T_t)_{t\geq 0}$ is said to be *chaotic in the sense of Devaney* if it is hypercyclic (universal) and there exists a dense set of periodic points in X. On the other hand, it is *distributionally chaotic* if there are an uncountable set $S \subset X$ and $\delta > 0$, so that for each $\varepsilon > 0$ and each pair $x, y \in S$ of distinct points we have

$$\overline{\text{Dens}}\{s \geq 0 : ||T_sx - T_sy|| \geq \delta\} = 1 \text{ and}$$
$$\overline{\text{Dens}}\{s \geq 0 : ||T_sx - T_sy|| < \varepsilon\} = 1,$$

where $\overline{\text{Dens}}(B)$ is the upper density of a Lebesgue measurable subset $B \subset \mathbb{R}_0^+$ defined as

$$\limsup_{t\to\infty} \frac{\mu(B \cap [0, t])}{t},$$

with μ standing for the Lebesgue measure on $\mathbb{R}_0^+$. A vector $x \in X$ is said to be *distributionally irregular* for the C_0-semigroup $(T_t)_{t\geq 0}$ if for every $\varepsilon > 0$ we have

$$\overline{\text{Dens}}\{s \geq 0 : ||T_s x|| \geq \varepsilon^{-1}\} = 1 \text{ and}$$
$$\overline{\text{Dens}}\{s \geq 0 : ||T_s x|| < \varepsilon\} = 1.$$

Such vectors were considered in [20] so as to get a further insight into the phenomenon of distributional chaos, showing the equivalence between a distributionally chaotic operator and an operator having a distributionally irregular vector. This equivalence has been shown for C_0-semigroups in [1].

Devaney chaos, hypercyclicity and mixing properties have been widely studied for linear operators on Banach and more general spaces [18, 27, 38, 55, 57, 61, 82]. The recent books [15] and [62] contain the basic theory, examples, and many results on chaotic linear dynamics.

A stronger concept than hypercyclic operators is the notion of frequent hypercyclic operators introduced by Bayart and Grivaux [13] (see [62] and the references therein) trying to quantify the frequency with which an orbit meets an open set. This concept was extended to C_0-semigroups in [3]

A C_0-semigroups is $(T_t)_{t\geq 0}$ is said to be *frequently hypercyclic* if there exists $x \in X$ (called frequently hypercyclic vector) such that $\underline{\text{Dens}}(\{t \geq 0 : T_t x \in U\}) > 0$ for every non-empty open subset $U \subset X$, where $\underline{\text{Dens}}(B)$ is the lower density of a Lebesgue measurable subset $B \subset \mathbb{R}_0^+$ defined as

$$\liminf_{t\to\infty} \frac{\mu(B \cap [0,t])}{t},$$

with μ standing for the Lebesgue measure on $\mathbb{R}_0^+$. In [36] it was proved that if $x \in X$ is a frequently hypercyclic vector for $(T_t)_{t\geq 0}$, then x is a frequently hypercyclic vector for every the operator T_t, $t \geq 0$.

In [29] Bonilla and Grosse-Erdmann, based on a result of Bayart and Grivaux, provided a Frequent Hypercyclicity Criterion for operators. Later, Mangino and Peris [74] obtained a continuous version of the criterion based on Pettis integrals, which is called the Frequent Hypercyclicity Criterion for semigroups.

The aim of this work is to study the specification property for strongly continuous semigroups of operators on Banach spaces, that is, for C_0-semigroups and its relationship with other dynamical properties, like hypercyclicity, mixing, chaos and frequent hypercyclicity. The paper is structured as follows: in Section 5.2 we introduce the notion of the specification property for C_0-semigroups, from now on denoted by SgSP. Section 5.3 is devoted to study the SgSP in connection with other dynamical properties. Finally, in Section 5.4, we provide several applications of the results in previous sections to solution semigroups of certain linear PDEs, and a characterization of translation semigroups which exhibit the SgSP.

5.2 Specification property for C_0-semigroups

A first notion of the specification property for a one-parameter family of continuous maps acting on a compact metric space was given in [32]. When trying to study the specification property in the context of semigroups of linear operators defined on separable Banach spaces, the first crucial problem is that these spaces are never compact, therefore, our first task should be to adjust the SP in this context, in the vain of the discrete case, and the following definition can be considered the natural extension in this setting.

Definition 5.2.1 (Specification property for semigroups, SgSP). A C_0-semigroup $(T_t)_{t\geq 0}$ on a separable Banach space X has the SgSP if there exists an increasing sequence $(K_n)_n$ of T-invariant sets with $0 \in K_1$ and $\overline{\cup_{n\in\mathbb{N}} K_n} = X$ and there exists a $t_0 > 0$, such that for each $n \in \mathbb{N}$ and for any $\delta > 0$ there is a positive real number $M_{\delta,n} \in \mathbb{R}_+$ such that for any integer $s \geq 2$, any set $\{y_1, \dots, y_s\} \subset K_n$ and any real numbers: $0 = a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_s \leq b_s$ satisfying $b_s + M_{\delta,n} \in \mathbb{N} \cdot t_0$ and $a_{r+1} - b_r \geq M_{\delta,n}$ for $r = 1, \dots, s-1$, there is a point $x \in K_n$ such that, for each $t_r \in [a_r, b_r]$, $r = 1, 2, ..., s$, the following conditions hold:

$$\|T_{t_r}(x) - T_{t_r}(y_r)\| < \delta,$$
$$T_t(x) = x, \quad \text{where } t = M_{\delta,n} + b_s.$$

Analogously to the discrete case, the meaning of this property is that if the semigroup has the SgSP then it is possible to approximate simultaneously several finite pieces of orbits by one periodic orbit. Obviously, parameter intervals for the approximations must be disjoint. The following result is an immediate consequence of the corresponding definitions.

Proposition 5.2.2. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space$X$. Then the following assertions are equivalent:

1. $(T_t)_{t\geq 0}$ has the SgSP.

2. Some operator T_{t_0} has the OSP.

5.3 SgSP and other dynamical properties of C_0-semigroups

In this section, we study the relation between the specification property and topological mixing, Devaney chaos, distributional chaos and frequent hypercyclicity. The following observations are useful to characterize mixing semigroups (see [62]).

Remark 5.3.1. From the definition, the semigroup $(T_t)_{t\geq 0}$ is mixing if and only if for every pair of non-empty open sets $U, V \subset X$, such that the complementary of the return set $R(U,V) := \{t \geq 0 \, : \, T_t(U) \cap V \neq \emptyset\}$ is (upper) bounded.

Remark 5.3.2. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. The semigroup $(T_t)_{t\geq 0}$ is mixing if and only if for every non-empty open set $U \subset X$ and every open 0-neighbourhood W, the complementary of the return sets $R(U,W)$ and $R(W,U)$ are (upper) bounded.

Proposition 5.3.3. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. If $(T_t)_{t\geq 0}$ has the SgSP, then $(T_t)_{t\geq 0}$ is mixing.

Proof. Let us consider a non-empty open set U and a 0-neighbourhood W. We claim that there exists some $t_1 > 0$ such that $t \in R(U,W) \cap R(W,U)$, $\forall t > t_1$ and this implies $(T_t)_{t\geq 0}$ is mixing.

Fix $u \in U$ and $\delta > 0$ such that $B(u, 2\delta) \subset U$ and $B(0, 2\delta) \subset W$. By hypothesis, $(T_t)_{t\geq 0}$ has the SgSP, then there are $t_0 > 0$ and a T_{t_0}-invariant set K such that the restriction of T_{t_0} to K has the SP and $K \cap B(u, \delta) \neq \emptyset$. From Definition 5.2.1, there exists M (depending on K and δ, which we suppose $M \in \mathbb{N} \cdot t_0$) such that if we choose $y_1 \in K \cap B(u, \delta)$, $y_2 = 0$, $s > 0$ with $s \in \mathbb{N} \cdot t_0$, and $0 = a_1 = b_1 < a_2 = M < b_2 = M + s$ then there exists a periodic point $x \in K$ with period $2M + s$ such that

$$\|T_t(x) - T_t(y_1)\| < \delta, \; a_1 \leq t \leq b_1,$$
$$\|T_t(x) - T_t(y_2)\| < \delta, \; a_2 \leq t \leq b_2.$$

This implies $\|x - y_1\| < \delta$, so $\|x - u\| < 2\delta$ and hence $x \in U$. From the second line of the previous equation, we have $T_t(x) \in B(0, \delta) \subset W$ for $M \leq t \leq M + m$. Therefore $t \in R(U, W)$ for any $t \geq M$.

Taking now $t > M$, we select $t' \in [M, M + s]$ such that $t + t' \in \mathbb{N} \cdot (2M + s)$. We have $\|T_{t'}(x)\| < \delta$, hence $T_{t'}(x) \in B(0, \delta) \subset W$. Since x is periodic with period $2M + s$, then $T^t(T^{t'}(x)) = x \in U$. Therefore $t \in R(W, U)$ for any $t > M$.

We have proved that the complementary of $R(U, W) \cap R(W, U)$ is (upper) bounded and this finishes the proof. □

Proposition 5.3.4. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. If $(T_t)_{t\geq 0}$ has the SgSP then $(T_t)_{t\geq 0}$ is Devaney chaotic.

Proof. By Proposition 5.3.3, $(T_t)_{t\geq 0}$ is topologically transitive and, by the definition of SgSP, it is clear that any vector in the space may be approximated by a periodic point. □

Proposition 5.3.5. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. If $(T_t)_{t\geq 0}$ has the SgSP with respect to an increasing sequence $(K_n)_n$ of invariant compact sets, then $(T_t)_{t\geq 0}$ is distributionally chaotic.

Proof. We first recall that for single maps on compact metric spaces, Oprocha [80] showed that the SP implies distributional chaos in our sense. Since there is $t_0 > 0$ such that T_{t_0} has the OSP, and by hypothesis the associated increasing sequence $(K_n)_n$ of invariant sets consists of compact sets, then $T_{t_0}|_{K_n}$ is distributionally chaotic for every $n \in \mathbb{N}$, thus the operator T_{t_0} is distributionally chaotic. Applying Theorem 3.1 in [1] we obtain that the semigroup is distributionally chaotic. □

It is well-known [62, 36, 74] that a C_0-semigroup is hypercyclic (respectively mixing, respectively Devaney chaotic, respectively frequently hypercyclic) if and only if it admits a hypercyclic (resp. mixing, resp. Devaney chaotic, resp. frequently hypercyclic) discretization $(T_{t_n})_n$. In particular, it is useful for our purposes the following characterization of frequent hypercyclicity for semigroups in terms of the frequent hypercyclicity of some of its operators [36, 74].

Proposition 5.3.6 ([36, 74])**.** Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. Then the following conditions are equivalent:

(i) $(T_t)_{t\geq 0}$ is frequently hypercyclic.

(ii) For every $t > 0$ the operator T_t is frequently hypercyclic.

(iii) There exists $t_0 > 0$ such that T_{t_0} is frequently hypercyclic.

The implication (i)→(ii) was proved in [36] and the other was pointed out in [74].

We point out the connection between the frequent hypercyclicity for semigroups and the specification property SgSP.

Proposition 5.3.7. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. If $(T_t)_{t\geq 0}$ has the SgSP, then $(T_t)_{t\geq 0}$ is frequently hypercyclic.

Proof. By proposition 5.2.2, if $(T_t)_{t\geq 0}$ has the SgSP, then there exists $t_0 > 0$ such that the operator T_{t_0} has the OSP, then the operator T_{t_0} is frequently hypercyclic and, therefore, the C_0-semigroup $(T_t)_{t\geq 0}$ is frequently hypercyclic (see Proposition 5.3.6 [36, 74]).

Where we have used a result from [5] about the OSP which says that if an operator T on a Banach space satisfies the OSP then it is frequently hypercyclic. □

It is obvious that if the semigroup $(T_t)_{t\geq 0}$ has the SgSP, then $(T_t)_{t\geq 0}$ is frequently hypercyclic.

Proposition 5.3.8. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable Banach space X. If $(T_t)_{t\geq 0}$ satisfies the Frequently Hypercyclic Criterion for semigroups of [74], then every operator T_t, $t \geq 0$, has the OSP and, therefore, the semigroup $(T_t)_{t\geq 0}$ has the SgSP.

Proof. If $(T_t)_{t\geq 0}$ satisfies the Frequently Hypercyclic Criterion for semigroups of [74] then every operator T_t $(t \geq 0)$ satisfies the Frequently Hypercyclic Criterion for operators of [29]. Using a result from [5] about the OSP on operators which says that if an operator T on a Banach space satisfies the Frequently Hypercyclic Criterion then it has the OSP, hence the operator T_t has the OSP for every $t \geq 0$, and finally, by using Proposition 5.2.2, we conclude that the semigroup $(T_t)_{t\geq 0}$ has the SgSP. □

In Corollary 2.3 in [74] it was showed that under some conditions, expressed in terms of eigenvector fields for the infinitesimal generator A of the C_0-semigroup $(T_t)_{t\geq 0}$, the semigroup is frequent hypercyclic, in fact, it was proved in [74] that it satisfies the Frequently Hypercyclic Criterion. As a result, we also obtain the result of the existence of the SgSP under the same conditions.

Proposition 5.3.9. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on a separable complex Banach space X and let A be its infinitesimal generator. Assume that there exists a family $(f_j)_{j\in\Gamma}$ of locally bounded measurable maps $f_j : I_j \to X$ such that I_j is an interval in $\mathbb{R}$, $f_j(I_j) \subset D(A)$, where $D(A)$ denotes the domain of the generator, $Af_j(t) = itf_j(t)$ for every $t \in I_j$, $j \in \Gamma$ and $\text{span}\{f_j(t) \ : \ j \in \Gamma, \ t \in I_j\}$ is dense in X. If either

a) $f_j \in C^2(I_j, X)$, $j \in \Gamma$, or

b) X does not contain c_0 and $\langle \varphi, f_j \rangle \in C^1(I_j)$, $\varphi \in X'$, $j \in \Gamma$,

then the semigroup $(T_t)_{t\geq 0}$ has the SgSP.

Proof. The result directly follows from the Corollary 2.3 in [74] and Proposition 5.3.8. □

Remark 5.3.10. It was pointed out in Remarks 2.4 in [74] that the spectral criterion for chaos in [42] of C_0- semigroups implies frequent hypercyclicity. As a consecuence, if a semigroup satisfies the spectral criterion, then it has the SgSP.

5.4 Applications and examples

In this section we will present several examples of C_0-semigroups exhibiting the specification property, with particular interest in solution semigroups to certain PDEs. A characterization of translation semigroups with the SgSP is also provided.

In the following examples, in order to ensure whether the solution semigroup has the SgSP, we will check the conditions of Proposition 5.3.9 (*i.e.*, the conditions of Corollary 2.3 in [74]) or the spectral criterion in [42] for chaos.

Example 5.4.1 (The solution semigroup of the hyperbolic heat transfer equation). Let us consider the hyperbolic heat transfer equation (HHTE):

$$\begin{cases} \tau \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \\ u(0,x) = \varphi_1(x), x \in \mathbb{R}, \\ \frac{\partial u}{\partial t}(0,x) = \varphi_2(x), x \in \mathbb{R} \end{cases}$$

where φ_1 and φ_2 represent the initial temperature and the initial variation of temperature, respectively, $\alpha > 0$ is the thermal diffusivity, and $\tau > 0$ is the thermal relaxation time.

The dynamical behaviour presented by the solutions of the classical heat equation was studied by Herzog [65] on certain spaces of analytic functions with certain growth control. Later, the dynamical properties of the solution semigroup for the hyperbolic heat transfer equation were also established in [35, 62].

The HHTE can be expressed as a first-order equation on the product of a certain function space with itself $X \oplus X$. We set $u_1 = u$ and $u_2 = \frac{\partial u}{\partial t}$.

Then the associated first-order equation is:

$$\begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ \frac{\alpha}{\tau}\frac{\partial^2}{\partial x^2} & \frac{-1}{\tau}I \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \\ \\ \begin{pmatrix} u_1(0,x) \\ u_2(0,x) \end{pmatrix} = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}, x \in \mathbb{R} \end{cases}$$

We fix $\rho > 0$ and consider the space [65]

$$X_\rho = \{f : \mathbb{R} \to \mathbb{C}; f(x) = \sum_{n=0}^{\infty} \frac{a_n \rho^n}{n!} x^n, (a_n)_{n\geq 0} \in c_0\}$$

endowed with the norm $||f|| = \sup_{n\geq 0} |a_n|$, where c_0 is the Banach space of complex sequences tending to 0.

Since

$$A := \begin{pmatrix} 0 & I \\ \frac{\alpha}{\tau}\frac{\partial^2}{\partial x^2} & \frac{-1}{\tau}I \end{pmatrix}.$$

is an operator on $X := X_\rho \oplus X_\rho$, we have that $(T_t)_{t\geq 0} = (e^{tA})_{t\geq 0}$ is the C_0-semigroup solution of the HHTE. We know from [35] and [62] that, given α, τ and ρ such that $\alpha\tau\rho > 2$, the solution semigroup $(e^{tA})_{t\geq 0}$ defined on $X_\rho \oplus X_\rho$ is mixing and chaotic since it satisfies the hypothesis of the spectral criterion [42]. Therefore, it satisfies the hypothesis of Corollary 2.3 in [74] which implies that the solution semigroup fulfilles the Frequent Hypercyclicity Criterion and, by the Proposition 5.3.8, it follows that the solution semigroup of the HHTE has the SgSP.

Remark 5.4.2. With minor changes, we can apply the previous argument to the wave equation

$$\left\{ \begin{array}{rll} u_{tt}t & = \alpha u_{xx} & \\ u(0,x) & = \varphi_1(x), & x \in \mathbb{R} \\ u_t(0,x) & = \varphi_2(x), & x \in \mathbb{R} \end{array} \right\}$$

which can be expressed as a first order equation in $X_\rho \oplus X_\rho$ (see [62]), in order to state that its semigroup solution has the SgSP.

Remark 5.4.3. This result can be extended to the solution semigroup of an abstract Cauchy problem of the form:

$$\left\{ \begin{array}{rl} u_t & = Au \\ u(0,x) & = \varphi(x) \end{array} \right\},$$

where A is a linear operator on a Banach space X and the generator of the solution semigroup. If A satisfy the conditions of Corollary 2.3 in [74], then the semigroup$(T_t)_{t\geq 0}$ with generator A has the SgSP

Example 5.4.4 (C_0-semigroup solution of the Black-Scholes equation). Black and Scholes proposed in [26] a mathematical model which gives a theoretical estimate of the price of stock options. The model is based on a partial differential equation, called the Black-Scholes equation, which estimates the price of the option over time. They proved that under some assumptions about the market, the value of a stock option $u(x,t)$, as a function of the current value of the underlying asset $x \in \mathbb{R}^+ = [0,\infty)$ and time, satisfies the final value problem:

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} = -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} - rx\frac{\partial u}{\partial x} + ru & \text{in } \mathbb{R}^+ \times [0,T] \\ u(0,T) = 0 & \text{for } t \in [0,T] \\ u(x,T) = (x-p)^+ & \text{for } x \in \mathbb{R}^+ \end{array} \right.$$

where $p > 0$ represents a given strike price , $\sigma > 0$ is the volatility, $r > 0$ is the interest rate and

$$(x-p)^+ = \left\{ \begin{array}{ll} x-p & \text{if } x > p \\ 0 & \text{if } x \leq p. \end{array} \right.$$

Let $v(x,t) = u(x,T-t)$, then it satisfies the forward Black-Scholes equation, defined for all time $t \in \mathbb{R}^+$ by

$$\left\{ \begin{array}{ll} \frac{\partial v}{\partial t} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} - rv & \text{in } \mathbb{R}^+ \times \mathbb{R}^+ \\ v(0,T) = 0 & \text{for } t \in \mathbb{R}^+ \\ v(x,0) = (x-p)^+ & \text{for } x \in \mathbb{R}^+ \end{array} \right.$$

This problem can be expressed in an abstract form:

$$\left\{ \begin{array}{l} \frac{\partial v}{\partial t} = \mathcal{B}v, \\ v(0,T) = 0, \\ v(x,0) = (x-p)^+ \quad \text{for } x \in \mathbb{R}^+. \end{array} \right.$$

where $\mathcal{B} = (D_\nu)^2 + \gamma(D_\nu) - rI$, being $D_\nu = \nu x \frac{\partial}{\partial x}$ with $\nu = \frac{\sigma}{\sqrt{2}}$ and $\gamma = \frac{r}{\nu} - \nu$.

It was shown that the Black-Scholes equation admits a C_0-semigroup solution which can be represented by $T_t := f(tD_\nu)$, where

$$f(z) = e^{g(z)} \text{ with } g(z) = z^2 + \gamma z - r\,.$$

In [56], a new explicit formula for the solution of the Black-Scholes equation was given in certain spaces of functions $Y^{s,\tau}$ defined by

$$Y^{s,\tau} = \left\{ u \in C((0,\infty))\ ;\ \lim_{x\to\infty} \frac{u(x)}{1+x^s} = 0, \quad \lim_{x\to 0} \frac{u(x)}{1+x^{-\tau}} = 0 \right\}$$

endowed with the norm

$$||u||_{Y^{s,\tau}} = \sup_{x>0} \left| \frac{u(x)}{(1+x^s)(1+x^{-\tau})} \right|.$$

Later, it was proved in [48] that the Black-Scholes semigroup is strongly continuous and chaotic for $s > 1, \tau \geq 0$ with $s\nu > 1$ and it was showed in [79] that it satisfies the spectral criterion in [42] under the same restrictions on the parameters and, therefore, the hypothesis of Corollary 2.3 of [74] and, consequently, the Black-Scholes semigroup has the SgSP.

There exist other C_0-semigroups related with PDEs which present the SgSP. In fact, the examples given in [79] in the context of strong mixing measures, satisfy either the conditions of Corollary 2.3 of [74] or the spectral criterion in [42] and, therefore, they have the SgSP. The examples provided in [79] include the semigroup generated by a linear perturbation of the one-dimensional Ornstein-Uhlenbeck operator, the solution C_0-semigroup of a partial differential equation of population dynamics, the solution C_0-semigroup associated to Banasiak and Moszyński models of *birth-and-death* processes.

Let $1 \leq p < \infty$ and let $v : \mathbb{R}_+ \to \mathbb{R}$ be a strictly positive locally integrable function, that is, v is measurable with $\int_0^b v(x)\,dx < \infty$ for all $b > 0$. We consider the space of weighted p-integrable functions defined as

$$X = L^p_v(\mathbb{R}_+) = \{f : \mathbb{R}_+ \to \mathbb{K}\ ;\ f \text{ is measurable and } \|f\| < \infty\},$$

where

$$\|f\| = \Big(\int_0^\infty |f(x)|^p v(x)\,dx\Big)^{1/p}.$$

The *translation semigroup* is then given by

$$T_t f(x) = f(x+t), \quad t, x \geq 0.$$

This defines a C_0-semigroup on $L^p_v(\mathbb{R}_+)$ if and only if there exist $M \geq 1$ and $w \in \mathbb{R}$ such that, for all $t \geq 0$, the following condition

$$v(x) \leq Me^{wt} v(x+t) \text{ for almost all } x \geq 0.$$

is satisfied. In that case, v is called an *admissible weight function* and we will assume in the sequel that v belongs to this class of weight functions.

For the translation semigroup defined on $L^p_v(\mathbb{R}_+)$, there was proved in [74] that $(T_t)_{t\geq 0}$ is chaotic if and only if it satisfies the Frequent Hypercyclicity Criterion for semigroups and that $(T_t)_{t\geq 0}$ is chaotic if and only if every operator T_t satisfies the Frequent Hypercyclicity Criterion for operators. A more complete characterization of the frequently hypercyclic criterion for the translation semigroup on $L^p_v(\mathbb{R}_+)$ was given in [73]:

Theorem 5.4.5 (Theorem 3.10, [73])**.** Let v be an admissible weight function on R. The following assertions are equivalent:

(1) The translation semigroup $(T_t)_{t\geq 0}$ is frequently hypercyclic on $L^p_v(\mathbb{R}_+)$.

(2) $\sum_{k\in\mathbb{Z}} v(k) < \infty$.

(3) $\int_{-\infty}^{\infty} v(t)dt < \infty$.

(4) $(T_t)_{t\geq 0}$ is chaotic on $L^p_v(\mathbb{R}_+)$.

(5) $(T_t)_{t\geq 0}$ satisfies the Frequently Hypercyclicity Criterion.

This result allows us to give a characterization of the SgSP for the translation semigroup on the space $X = L^p_v(\mathbb{R}_+)$.

Theorem 5.4.6. Let us consider the translation semigroup on the space $X = L^p_v(\mathbb{R}_+)$, where $1 \leq p < \infty$ and $v : \mathbb{R}_+ \to \mathbb{R}$ is an admissible weight function. We claim that the following assertions are equivalent:

(i) $\int_0^\infty v(x)\,dx < \infty$.

(ii) $(T_t)_{t\geq 0}$ has SgSP.

(iii) $(T_t)_{t\geq 0}$ is Devaney chaotic.

(iv) $(T_t)_{t\geq 0}$ satisfies the Frequently Hypercyclicity Criterion.

(v) The translation semigroup $(T_t)_{t\geq 0}$ is frequently hypercyclic.

Proof. By Theorem 5.4.5 [73] and Propositions 5.3.7 and 5.3.8, it is obvious that for the translation semigroup the SgSP is equivalent to satisfy the Frequently Hypercyclicity Criterion and the SgSP is equivalent to frequently hypercyclic. □

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