

Document downloaded from:

<http://hdl.handle.net/10251/62348>

This paper must be cited as:

Rodríguez López, J.; Romaguera Bonilla, S.; Sánchez Álvarez, JM. (2010). The Hausdorff fuzzy quasi-metric. *Fuzzy Sets and Systems*. 161:1078-1096. doi:10.1016/j.fss.2009.09.019.



The final publication is available at

<http://dx.doi.org/10.1016/j.fss.2009.09.019>

Copyright Elsevier

Additional Information

The Hausdorff fuzzy quasi-metric

J. Rodríguez-López, S. Romaguera, J.M. Sánchez-Álvarez*

Instituto Universitario de Matemática Pura y Aplicada,

Universidad Politécnica de Valencia, 46071 Valencia, Spain

jrlopez@mat.upv.es sromague@mat.upv.es jossnclv@mat.upv.es

Abstract

Removing the condition of symmetry in the notion of a fuzzy (pseudo)metric in Kramosil and Michalek's sense, we obtain the notion of a fuzzy quasi-(pseudo-)metric. Then for each fuzzy quasi-pseudo-metric on a set X we construct a fuzzy quasi-pseudo-metric on the collection of all nonempty subsets of X , called the Hausdorff fuzzy quasi-pseudo-metric. We investigate several properties of this structure and present several illustrative examples as well as an application to the domain of words. The notion of Hausdorff fuzzy quasi-pseudo-metric when quasi-(pseudo-)metric **fuzziness** is considered in the sense of George and Veeramani is also discussed.

Key words: The Hausdorff fuzzy quasi-pseudo-metric, complete, precompact, preorder, the domain of words.

1 Introduction

It is well known that the Hausdorff distance has **an** undoubted importance not only in general topology but also in other areas of Mathematics and Computer Science, such **as** convex analysis and optimization [6, 31, 37], dynamical systems [11, 34, 47, 61], mathematical morphology [56], fractals [3, 12], image processing [22, 30, 55, 63], programming language and semantics [4, 5],

*This research is partially supported by the Spanish Ministry of Education and Science, and FEDER, grant MTM2006-14925-C02-01.

computational biology [21, 57], etc. In [13], Egbert extended the classical construction of the Hausdorff distance of a metric space to Menger spaces. Later on, Tardiff [59] (see also [52, 54]), generalized Egbert's construction to probabilistic metric spaces, obtaining in this way a suitable notion of a Hausdorff probabilistic distance. Since fuzzy metric spaces, in the sense of Kramosil and Michalek, are closely related to Menger spaces [23], one can easily define, from Egbert-Tardiff's construction, a Hausdorff fuzzy distance for a given fuzzy metric space. In connection with these constructions, a notion of Hausdorff fuzzy metric for fuzzy metric spaces in the sense of George and Veeramani [17, 18] was discussed in [40].

On the other hand, it is well known that several structures of asymmetric topology like quasi-uniformities and (fuzzy) quasi-metrics, constitute efficient tools to formulate and solve problems in hyperspaces, function spaces, topological algebra, asymmetric functional analysis, point-free geometry, complexity of algorithms, theoretical computer science, etc. (see, for instance, Chapters 11 and 12 of [25], Section 3 of [26], and also [1, 2, 10, 16, 19, 27, 32, 41, 42, 43, 46, 49, 53, 62, etc] for recent contributions).

In this paper we introduce and study notions of Hausdorff fuzzy quasi-metric (in the senses of Kramosil and Michalek, and George and Veeramani, respectively) that generalize to the asymmetric setting the corresponding notions of Hausdorff fuzzy metric. In this way, we partially reconcile the theory of fuzzy metric hyperspaces with the theory of asymmetric topology. Furthermore, we apply our approach to the domain of words, a paradigmatic example of a space that naturally appear in the theory of computation.

The paper is organized as follows. In Section 2 we present the basic notions and results which will need later on. In Section 3 we construct and discuss a notion of Hausdorff fuzzy quasi-metric, based on the notion of fuzzy (quasi-)metric of Kramosil and Michalek. In Section 4 we shall show that this new concept has several nice properties of completeness, precompactness and compactness. In Section 5 we consider a notion of Hausdorff fuzzy quasi-metric, based on the notion of fuzzy (quasi-)metric in the sense of George and Veeramani. In Section 6 we apply the theory developed in the preceding sections to the domain of words and we point out some advantages of the use of fuzzy quasi-metrics instead of classical metrics and quasi-metrics. Finally, we present our conclusions.

2 Basic notions and preliminary results

In the sequel the letters \mathbb{R} , \mathbb{R}^+ and \mathbb{N} will denote the set of real numbers, the set of nonnegative real numbers and the set of positive integer numbers, respectively.

Our basic references for quasi-metric spaces and quasi-uniform spaces are [15] and [25], and for general topology it is [14].

Let us recall that a quasi-uniformity on a set X is a filter \mathcal{U} on $X \times X$ such that:

- (i) for each $U \in \mathcal{U}$, $\Delta \subseteq U$, where $\Delta = \{(x, x) : x \in X\}$;
- (ii) for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$, where $V^2 = \{(x, y) \in X \times X : \text{there is } z \in X \text{ with } (x, z) \in V \text{ and } (z, y) \in V\}$.

By a quasi-uniform space we mean a pair (X, \mathcal{U}) such that X is a nonempty set and \mathcal{U} is a quasi-uniformity on X .

Each quasi-uniformity \mathcal{U} on X generates a topology $\tau_{\mathcal{U}}$ on X such that a neighborhood base for each point $x \in X$ is given by $\{U(x) : U \in \mathcal{U}\}$, where $U(x) = \{y \in X : (x, y) \in U\}$.

Given a quasi-uniformity \mathcal{U} on X , then the filter \mathcal{U}^{-1} defined on $X \times X$ by $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ is also a quasi-uniformity on X , called the conjugate of \mathcal{U} , and the filter $\mathcal{U}^s = \mathcal{U} \vee \mathcal{U}^{-1}$ is a uniformity on X . (As usual, $U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$.)

An extended quasi-pseudo-metric on a set X is a function $d : X \times X \rightarrow [0, +\infty]$ such that for all $x, y, z \in X$:

- (i) $d(x, x) = 0$;
- (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

Following the modern terminology (see, for instance, [25, Chapter 11]), an extended quasi-metric on X is an extended quasi-pseudo-metric d on X which satisfies the condition:

- (i') $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$.

An extended quasi-(pseudo-)metric d on X such that $d(x, y) < +\infty$ for all $x, y \in X$, is said to be a quasi-(pseudo-)metric on X .

By a quasi-(pseudo-)metric space we mean a pair (X, d) such that X is a nonempty set and d is a quasi-(pseudo-)metric on X .

The following is an easy but paradigmatic example of a quasi-metric space.

Example 1. Let ℓ be the function defined on $\mathbb{R} \times \mathbb{R}$ by $\ell(x, y) = \max\{x - y, 0\}$. Then ℓ is a quasi-metric on \mathbb{R} such that ℓ^s is the Euclidean

metric on \mathbb{R} .

Each extended quasi-pseudo-metric d on X generates a topology τ_d on X which has as a base the family of open balls $\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Observe that if d is an extended quasi-metric on X , then τ_d is a T_0 topology on X .

A topological space (X, τ) is said to be quasi-(pseudo-)metrizable if there is a quasi-(pseudo-)metric d on X such that $\tau = \tau_d$.

Given an extended quasi-(pseudo-)metric d on X , then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$, is also an extended quasi-(pseudo-)metric on X , called the conjugate of d , and the function d^s defined on $X \times X$ by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is an extended (pseudo-)metric on X . Of course, d^{-1} and d^s are a quasi-(pseudo-)metric and a (pseudo-)metric, respectively, whenever d is a quasi-(pseudo-)metric on X .

A subset A of a quasi-(pseudo-)metric space (X, d) is called bounded if A is bounded in the (pseudo-)metric space (X, d^s) .

Each (extended) quasi-pseudo-metric d on X induces a quasi-uniformity \mathcal{U}_d on X which has as a base the countable family

$$\{\{(x, y) \in X \times X : d(x, y) < 2^{-n}\} : n \in \mathbb{N}\}.$$

In connection with this fact we have the following useful result which can be found, for instance, in [15, Theorem 1.5].

Proposition 1. *Let (X, \mathcal{U}) be a quasi-uniform space. Then, there exists a quasi-pseudo-metric d on X such that $\mathcal{U}_d = \mathcal{U}$ if and only if \mathcal{U} has a countable base.*

If d is an extended quasi-pseudo-metric on a set X , then the relation \leq_d on X given by $x \leq_d y \Leftrightarrow d(x, y) = 0$, is a preorder on X (i.e., \leq_d is reflexive and transitive).

It is clear that d is an extended quasi-metric on a set X if and only if \leq_d is a (partial) order on X (i.e., the preorder \leq_d is antisymmetric, which means that $x \leq_d y$ and $y \leq_d x$, implies $x = y$).

In this case, \leq_d is called the specialization order.

Note that in Example 1 the specialization order coincides with the usual order on \mathbb{R} .

Remark 1. The **natural** connection between asymmetric distances and order, described above, provides some advantages in certain settings, if one works with quasi-metrics instead of metrics. Thus, in modeling a computational process on a collection X of elements (for example, chains of information, words of an alphabet in a programming language, complexity functions in analysis of algorithms, etc.) we can define a preorder \leq on X given by $x \leq y$ if and only if the element y contains all the information provided by the element x , and then it is possible, in many cases, to construct a suitable (extended) quasi-pseudo-metric d on X such that the information provided by \leq is **transmitted** in this way to the quasi-pseudo-metric space (X, d) (see, for instance, [33, 50]).

Next we recall the construction of the Hausdorff (extended) quasi-metric of a given quasi-metric space.

If (X, τ) is a topological space, we denote by $\mathcal{P}_0(X)$, $\mathcal{C}_0(X)$ and $\mathcal{K}_0(X)$, the collection of all nonempty subsets of X , the collection of all nonempty closed subsets of X and the collection of all nonempty compact subsets of X , and if A is a subset of X we denote by \overline{A}^τ the closure of A with respect to τ .

If (X, d) is a quasi-pseudo-metric space, we define

$$\mathcal{C}_\cap(X) = \{\overline{A}^{\tau_d} \cap \overline{A}^{\tau_{d^{-1}}} : A \in \mathcal{P}_0(X)\}.$$

Remark 2. The following inclusions are obvious: $\mathcal{C}_0(X) \subseteq \mathcal{C}_\cap(X) \subseteq \mathcal{P}_0(X)$. Moreover, if (X, d) is a metric space, then $\mathcal{K}_0(X) \subseteq \mathcal{C}_0(X)$ and $\mathcal{C}_0(X) = \mathcal{C}_\cap(X)$. It is well known (see Example 2 below) that the situation is quite different for quasi-metric spaces).

Now, for each $A, B \in \mathcal{P}_0(X)$ let

$$H_d^-(A, B) = \sup_{a \in A} d(a, B), \quad H_d^+(A, B) = \sup_{b \in B} d(A, b),$$

and

$$H_d(A, B) = \max\{H_d^-(A, B), H_d^+(A, B)\}.$$

Then H_d^- , H_d^+ and H_d are extended quasi-pseudo-metrics on $\mathcal{P}_0(X)$ (see [7, 28, 38, 39, etc]). Moreover H_d is an extended quasi-metric on $\mathcal{C}_\cap(X)$ (compare [28, Lemma 2]), and it is a quasi-metric on the set of all bounded subsets of X that are in $\mathcal{C}_\cap(X)$. In this case we say that H_d is the Hausdorff quasi-metric of d . Note that if (X, d) is a metric space, then H_d is the extended

Hausdorff metric of d (on $\mathcal{C}_0(X)$).

Example 2. Let $X = \{a, b, c\}$ and let d be the quasi-metric on X given by $d(b, a) = d(c, b) = d(c, a) = 1$, and $d(x, y) = 0$ otherwise. Let $F = \{a, c\}$. Then $\overline{F}^{r_d} \cap \overline{F}^{r_{d^{-1}}} = X$, because $d(b, c) = d(a, b) = 0$. Therefore $F \notin \mathcal{C}_\cap(X)$, and hence $\mathcal{K}_0(X) \not\subseteq \mathcal{C}_\cap(X)$.

We conclude this section by recalling the concepts and results on fuzzy quasi-metric spaces which we will need in the rest of the paper. They are taken from [20] (see also [8]). Moreover, we shall observe that the attractive relationship between quasi-metrics and order, recalled in Remark 1, is preserved in this framework.

According to [52], a binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if $*$ satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $*$ is continuous; (iii) $a * 1 = a$ for every $a \in [0, 1]$; (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0, 1]$.

It is well known and easy to see that for each continuous t-norm $*$ one has $*$ \leq \wedge , where \wedge is the continuous t-norm given by $a \wedge b = \min\{a, b\}$ for all $a, b \in [0, 1]$.

By a KM-fuzzy quasi-pseudo-metric on a set X we mean a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0, +\infty)$ such that for all $x, y, z \in X$:

- (i) $M(x, y, 0) = 0$;
- (ii) $M(x, x, t) = 1$ for all $t > 0$;
- (iii) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s \geq 0$;
- (iv) $M(x, y, -) : [0, +\infty) \rightarrow [0, 1]$ is left continuous.

A KM-fuzzy quasi-metric on X is a KM-fuzzy quasi-pseudo-metric $(M, *)$ on X which satisfies the following condition: (ii') $x = y$ if and only if $M(x, y, t) = M(y, x, t) = 1$ for all $t > 0$.

A KM-fuzzy (pseudo-)metric on X is a KM-fuzzy quasi-(pseudo-)metric $(M, *)$ on X such that for each $x, y \in X$: (v) $M(x, y, t) = M(y, x, t)$ for all $t > 0$.

A KM-fuzzy quasi-(pseudo-)metric space is a triple $(X, M, *)$ such that X is a (nonempty) set and $(M, *)$ is a KM-fuzzy quasi-(pseudo-)metric on X . The notion of a KM-fuzzy (pseudo-)metric space is defined in the obvious manner. Note that the KM-fuzzy metric spaces are exactly the fuzzy metric spaces in the sense of Kramosil and Michalek [23].

Each KM-fuzzy quasi-pseudo-metric $(M, *)$ on X generates a topology

τ_M on X which has as a base the family of open balls $\{B_M(x, \varepsilon, t) : x \in X, \varepsilon \in (0, 1), t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ for all $x \in X, \varepsilon \in (0, 1)$ and $t > 0$. Observe that if $(M, *)$ is a **KM-fuzzy** quasi-metric on X , then τ_M is a T_0 topology on X .

It is obvious from the definition of τ_M that a sequence $(x_n)_n$ in a KM-fuzzy quasi-pseudo-metric **space** $(X, M, *)$ converges to a point $x \in X$ with respect to τ_M if and only if $\lim_n M(x, x_n, t) = 1$ for all $t > 0$.

If $(M, *)$ is a KM-fuzzy quasi-(pseudo-)metric on a set X , then $(M^{-1}, *)$ is also a KM-fuzzy quasi-(pseudo-)metric on X , where M^{-1} is the fuzzy set in $X \times X \times [0, +\infty)$ defined by $M^{-1}(x, y, t) = M(y, x, t)$. Moreover, if we denote by M^i the fuzzy set in $X \times X \times [0, +\infty)$ given by $M^i(x, y, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\}$, then $(M^i, *)$ is, clearly, a KM-fuzzy (pseudo-)metric on X .

It is well known [20, Proposition 1] that if $(X, M, *)$ is a KM-fuzzy quasi-pseudo-metric space, then, for each $x, y \in X$ the function $M(x, y, -)$ is non-decreasing.

In the rest of the paper, KM-fuzzy quasi-(pseudo-)metrics and KM-fuzzy quasi-(pseudo-)metric spaces will be simply called fuzzy quasi-(pseudo-)metrics and fuzzy quasi-(pseudo-)metric spaces, respectively.

Remark 3. Notice that if $(M, *)$ is a fuzzy quasi-pseudo-metric on a set X , then the relation \leq_M on X given by

$$x \leq_M y \iff M(x, y, t) = 1 \text{ for some } t > 0,$$

is a preorder on X .

Of course, the relation $\leq_{M,t}$ on X given by

$$x \leq_{M,t} y \iff M(x, y, t) = 1 \text{ for all } t > 0,$$

is also a preorder on X . Moreover $\leq_{M,t}$ is a (partial) order on X if and only if $(M, *)$ is a fuzzy quasi-metric on X .

Example 3 (compare [20, Example 2.16]). Let d be a (n extended) quasi-(pseudo-)metric on a set X and let M_d be the function defined on $X \times X \times [0, +\infty)$ by $M(x, y, 0) = 0$ and

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

for all $t > 0$. Then, for each continuous t-norm $*$, $(X, M_d, *)$ is a fuzzy quasi-(pseudo-)metric space called **the** standard fuzzy quasi-(pseudo-)metric space and $(M_d, *)$ is the standard fuzzy quasi-(pseudo-)metric of (X, d) . Furthermore, it is easy to check that $(M_d)^{-1} = M_{d^{-1}}$ and $(M_d)^i = M_{d^s}$, and that the topology τ_d , generated by d , coincides with the topology τ_{M_d} generated by $(M_d, *)$.

We say that a topological space (X, τ) admits a compatible fuzzy quasi-(pseudo-)metric if there is a fuzzy quasi-(pseudo-)metric $(M, *)$ on X such that $\tau = \tau_M$.

It follows from Example 3 that every quasi-(pseudo-)metrizable topological space admits a compatible fuzzy quasi-(pseudo-)metric.

Conversely, we have:

Proposition 2 [20, 44]. *Let $(X, M, *)$ be a fuzzy quasi-pseudo-metric space. Then $\{U_n : n \in \mathbb{N}\}$ is a base for a quasi-uniformity \mathcal{U}_M on X such that $\tau_{\mathcal{U}_M} = \tau_M$, where $U_n = \{(x, y) \in X \times X : M(x, y, 1/n) > 1 - 1/n\}$ for all $n \in \mathbb{N}$. Moreover the conjugate quasi-uniformity $(\mathcal{U}_M)^{-1}$ coincides with $\mathcal{U}_{M^{-1}}$ and $\tau_{(\mathcal{U}_M)^{-1}} = \tau_{M^{-1}}$.*

From Propositions 1 and 2 we deduce the following:

Corollary. *Let $(X, M, *)$ be a fuzzy quasi-(pseudo-)metric space. Then there is a quasi-(pseudo-)metric d on X such that $\mathcal{U}_d = \mathcal{U}_M$.*

3 Construction of the Hausdorff fuzzy quasi-metric

We start this section by recalling the construction of the Hausdorff fuzzy metric of a fuzzy metric space $(X, M, *)$. The construction is a simple adaptation to the fuzzy setting of the definition of the Hausdorff probabilistic metric of a probabilistic metric space [13, 52, 54, 59].

Given $x \in X$, $A \in \mathcal{P}_0(X)$ and $t > 0$, set $M(x, A, t) = \sup_{a \in A} M(x, a, t)$.

Now, for each $A, B \in \mathcal{P}_0(X)$ let

$$H_M^-(A, B, 0) = H_M^+(A, B, 0) = 0,$$

$$H_M^-(A, B, t) = \sup_{0 < s < t} \inf_{a \in A} M(a, B, s), \quad H_M^+(A, B, t) = \sup_{0 < s < t} \inf_{b \in B} M(A, b, s),$$

for all $t > 0$, and

$$H_M(A, B, t) = \min\{H_M^-(A, B, t), H_M^+(A, B, t)\},$$

for all $t \geq 0$.

Then H_M^- , H_M^+ and H_M are fuzzy pseudo-metrics on $\mathcal{P}_0(X)$. Furthermore H_M is a fuzzy metric on $\mathcal{C}_0(X)$, called the Hausdorff fuzzy metric of $(X, M, *)$.

In the light of the above notions and of the construction of Liu and Li [29, p. 67] of a ‘‘Hausdorff fuzzy metric’’ in their recent study of coincidence point theorems for multivalued maps in complete fuzzy metric spaces, one can attempt to define the Hausdorff fuzzy metric in a more simplified way, as follows:

$$H_M(A, B, t) = \min\{\inf_{a \in A} M(a, B, t), \inf_{b \in B} M(A, b, t)\},$$

for all $A, B \in \mathcal{C}_0(X)$ and $t > 0$.

The following example shows that, for this alternative definition, H_M is not a fuzzy metric, in general.

Example 4. Let $X = \mathbb{N} \cup \{0\}$, and let $d : X \times X \rightarrow \mathbb{R}^+$ given by

$$d(x, y) = 2^{-(x \wedge y)} - 2^{-(x \vee y)},$$

for all $x, y \in X$. It is clear that d is a metric on X . Then, the pair (M, \wedge) is a fuzzy metric on X , where $M(x, y, t) = 0$ if $d(x, y) \geq t$, and $M(x, y, t) = 1$ if $d(x, y) < t$.

Note also that τ_M is the discrete topology on X because for each $x \in X$ and $\varepsilon \in (0, 1)$, we have $B_M(x, \varepsilon, 2^{-(x+1)}) = \{x\}$. Indeed, fix $x \in X$ and let $y \in X$ with $y \neq x$. If $y < x$, we have

$$2^{-(x \wedge y)} - 2^{-(x \vee y)} = 2^{-y} - 2^{-x} \geq 2^{-(x-1)} - 2^{-x} = 2^{-x},$$

and if $y > x$, we have

$$2^{-(x \wedge y)} - 2^{-(x \vee y)} = 2^{-x} - 2^{-y} \geq 2^{-x} - 2^{-(x+1)} = 2^{-(x+1)}.$$

Therefore $M(x, y, 2^{-(x+1)}) = 0$. Hence $B_M(x, \varepsilon, 2^{-(x+1)}) = \{x\}$, so τ_M is the discrete topology on X .

Finally, consider the elements A, B of $\mathcal{C}_0(X)$, where $A = \mathbb{N}$ and $B = \{0\}$. Note that $M(x, 0, 1) = 1$ for all $x \in \mathbb{N}$, and that for each $t \in (0, 1)$, there is $x_t \in \mathbb{N}$ such that $1 - 2^{-x_t} \geq t$, so $M(x_t, 0, t) = 0$. Hence

$$\min\{\inf_{a \in A} M(a, B, 1), \inf_{b \in B} M(A, b, 1)\} = \min\{\inf_{x \in \mathbb{N}} M(x, \{0\}, 1), M(\mathbb{N}, 0, 1)\} = 1,$$

and, for each $t \in (0, 1)$,

$$\min\{\inf_{a \in A} M(a, B, t), \inf_{b \in B} M(A, b, t)\} = \min\{\inf_{x \in \mathbb{N}} M(x, \{0\}, t), M(\mathbb{N}, 0, t)\} = 0.$$

We conclude that $H_M(\mathbb{N}, \{0\}, -)$ is not left continuous at $t = 1$, for the alternative definition of H_M suggested above, so it is not a fuzzy metric on $\mathcal{C}_0(X)$.

Next we shall construct the Hausdorff fuzzy quasi-metric of a fuzzy quasi-metric space $(X, M, *)$.

Let $(X, M, *)$ be a fuzzy quasi-metric space. If A is a subset of X , the sets \overline{A}^M and $\overline{A}^{M^{-1}}$ will be simply denoted by \overline{A}^M and $\overline{A}^{M^{-1}}$, respectively.

The sets $\mathcal{P}_0(X)$, $\mathcal{C}_0(X)$, $\mathcal{K}_0(X)$ and $\mathcal{C}_\cap(X)$ are defined in the obvious manner, as in Section 1. In particular, we have

$$\mathcal{C}_\cap(X) = \{\overline{A}^M \cap \overline{A}^{M^{-1}} : A \in \mathcal{P}_0(X)\}.$$

Remark 4. It is straightforward to show (compare [41]) that if $A \in \mathcal{P}_0(X)$, then $A \in \mathcal{C}_\cap(X)$ if and only if $A = \overline{A}^M \cap \overline{A}^{M^{-1}}$.

As in the fuzzy metric case, given $x \in X$, $A \in \mathcal{P}_0(X)$ and $t > 0$, put $M(x, A, t) = \sup_{a \in A} M(x, a, t)$.

The following easy result will be useful later on.

Lemma 1. *Let $(X, M, *)$ be a fuzzy quasi-metric space. Then for each $x \in X$ and $A \in \mathcal{P}_0(X)$, the following hold:*

- (1) $M(x, A, s) \leq M(x, A, t)$ whenever $0 \leq s < t$.
- (2) $x \in \overline{A}^M \iff M(x, A, t) = 1$ for all $t > 0$.

Now, for a given fuzzy quasi-metric space $(X, M, *)$ and for each $A, B \in \mathcal{P}_0(X)$, define

$$H_M^-(A, B, 0) = H_M^+(A, B, 0) = 0,$$

and

$$H_M^-(A, B, t) = \sup_{0 < s < t} \inf_{a \in A} M(a, B, s), \quad H_M^+(A, B, t) = \sup_{0 < s < t} \inf_{b \in B} M(A, b, s),$$

for all $t > 0$. Then we obtain:

Lemma 2. *For each $A, B, C \in \mathcal{P}_0(X)$, the following hold:*

- (1a) $A \subseteq \overline{B}^M \iff H_M^-(A, B, t) = 1$ for all $t > 0$.
- (1b) $B \subseteq \overline{A}^{M^{-1}} \iff H_M^+(A, B, t) = 1$ for all $t > 0$.
- (2a) $H_M^-(A, C, t + s) \geq H_M^-(A, B, t) * H_M^-(B, C, s)$ for all $t, s \geq 0$.
- (2b) $H_M^+(A, C, t + s) \geq H_M^+(A, B, t) * H_M^+(B, C, s)$ for all $t, s \geq 0$.
- (3a) $H_M^-(A, B, -) : [0, +\infty) \rightarrow [0, 1]$ is *left continuous*.
- (3b) $H_M^+(A, B, -) : [0, +\infty) \rightarrow [0, 1]$ is *left continuous*.

Proof. (1a) Suppose that $A \subseteq \overline{B}^M$. Then for each $a \in A$ and $s > 0$, $M(a, B, s) = 1$ by Lemma 1 (2), so $\inf_{a \in A} M(a, B, s) = 1$. Choose any $t > 0$. Then

$$H_M^-(A, B, t) = \sup_{0 < s < t} \inf_{a \in A} M(a, B, s) = 1.$$

Conversely, choose $t > 0$. Then by hypothesis, there is a sequence $(s_k)_k$, with $s_k \in (0, t)$ for all $k \in \mathbb{N}$, such that

$$\inf_{a \in A} M(a, B, s_k) > 1 - 1/k,$$

for all $k \in \mathbb{N}$. So, by Lemma 1 (1),

$$\inf_{a \in A} M(a, B, t) \geq \inf_{a \in A} M(a, B, s_k) > 1 - 1/k,$$

for all $k \in \mathbb{N}$. Consequently $M(a, B, t) = 1$ for all $a \in A$ and $t > 0$. So, by Lemma 1 (2), $A \subseteq \overline{B}^M$.

(1b) It follows similarly to the proof of (1a), so it is omitted.

(2a) Fix $t, s \geq 0$. Since the inequality is obvious if $t = 0$ or $s = 0$, we assume that $t, s > 0$. Let $r \in (0, t)$, and $r' \in (0, s)$. Then, for each $a \in A$, with $M(a, B, r) > 0$, and each $\varepsilon \in (0, M(a, B, r))$, there exists $b_a \in B$ such that $M(a, B, r) - \varepsilon \leq M(a, b_a, r)$. Hence

$$(M(a, B, r) - \varepsilon) * \inf_{b \in B} M(b, C, r') \leq M(a, b_a, r) * M(b_a, C, r') \leq M(a, C, r + r').$$

So, by the continuity of $*$, it follows

$$M(a, B, r) * \inf_{b \in B} M(b, C, r') \leq M(a, C, r + r'),$$

for each $a \in A$ with $M(a, B, r) > 0$ (Note that the preceding inequality obviously holds if $M(a, B, r) = 0$). Therefore

$$\inf_{a \in A} M(a, B, r) * \inf_{b \in B} M(b, C, r') \leq \inf_{a \in A} M(a, C, r + r').$$

Consequently, by definition of “sup” and by the continuity of $*$, it follows from standard arguments that

$$\begin{aligned} & \sup_{0 < r < t} \inf_{a \in A} M(a, B, r) * \sup_{0 < r' < s} \inf_{b \in B} M(b, C, r') \\ & \leq \sup_{0 < r < t, 0 < r' < s} \inf_{a \in A} M(a, C, r + r'). \end{aligned}$$

Finally, since

$$\sup_{0 < r < t, 0 < r' < s} \inf_{a \in A} M(a, C, r + r') = \sup_{0 < r'' < t+s} \inf_{a \in A} M(a, C, r''),$$

we conclude that

$$H_M^-(A, B, t) * H_M^-(B, C, s) \leq H_M^-(A, C, t + s).$$

(2b) It follows similarly to the proof of (2a), so it is omitted.

(3a) Let $A, B \in \mathcal{P}_0(X)$, $t > 0$ and let $(t_k)_k$ be a strictly increasing sequence in \mathbb{R}^+ such that $t_k \rightarrow t$.

Since for each $k \in \mathbb{N}$, $t_k < t$, it immediately follows from Lemma 1 (1), that

$$H_M^-(A, B, t_k) \leq H_M^-(A, B, t),$$

for all $k \in \mathbb{N}$.

Now take an arbitrary $\varepsilon \in (0, 1)$. Then there is $s_\varepsilon \in (0, t)$ such that

$$H_M^-(A, B, t) < \varepsilon + \inf_{a \in A} M(a, B, s_\varepsilon).$$

Let $k_0 \in \mathbb{N}$ such that $s_\varepsilon < t_k - 1/k$ for all $k \geq k_0$. Then $M(a, B, s_\varepsilon) \leq M(a, B, t_k - 1/k)$ for all $a \in A$ and $k \geq k_0$, so

$$H_M^-(A, B, t) < \varepsilon + \inf_{a \in A} M(a, B, t_k - \frac{1}{k}),$$

for all $k \geq k_0$. Therefore

$$H_M^-(A, B, t) < \varepsilon + \sup_{0 < s < t_k} \inf_{a \in A} M(a, B, s),$$

for all $k \geq k_0$. We have proved that

$$H_M^-(A, B, t_k) \leq H_M^-(A, B, t) < \varepsilon + H_M^-(A, B, t_k),$$

for all $k \geq k_0$, and, consequently, $H_M^-(A, B, \cdot)$ is left continuous on $[0, +\infty)$.

(3b) It follows similarly to the proof of (3a), so it is omitted. ■

Now we define a fuzzy set H_M on $\mathcal{P}_0(X) \times \mathcal{P}_0(X) \times [0, +\infty)$, by

$$H_M(A, B, t) = \min\{H_M^-(A, B, t), H_M^+(A, B, t)\},$$

for all $A, B \in \mathcal{P}_0(X)$ and $t \geq 0$.

From the above lemma we obtain the following result.

Theorem 1. *For a fuzzy quasi-pseudo-metric space $(X, M, *)$ the following hold:*

- (1) $(H_M^-, *)$, $(H_M^+, *)$ and $(H_M, *)$ are fuzzy quasi-pseudo-metrics on $\mathcal{P}_0(X)$.
- (2) If $(X, M, *)$ is a fuzzy quasi-metric space, then $(H_M, *)$ is a fuzzy quasi-metric on $\mathcal{C}_\cap(X)$.

Proof. (1) From Lemma 2 (1a), (2a) and (3a), it follows that $(H_M^-, *)$ is a fuzzy quasi-pseudo-metric on $\mathcal{P}_0(X)$. Moreover, from Lemma 2 (1b), (2b) and (3b), it follows that $(H_M^+, *)$ is a fuzzy quasi-pseudo-metric on $\mathcal{P}_0(X)$. From these facts and the definition of H_M it immediately follows that $(H_M, *)$ is also a fuzzy quasi-pseudo-metric on $\mathcal{P}_0(X)$.

(2) Since, by (1), $(H_M, *)$ is a fuzzy quasi-pseudo-metric on $\mathcal{C}_\cap(X)$, we only need to show that for $A, B \in \mathcal{C}_\cap(X)$, we have $A = B$ whenever $H_M(A, B, t) = H_M(B, A, t) = 1$ for all $t > 0$.

Indeed, suppose that $H_M(A, B, t) = H_M(B, A, t) = 1$ for all $t > 0$. Then, by Lemma 2 (1a), $A \subseteq \overline{B}^M$ and $B \subseteq \overline{A}^M$, and by Lemma 2 (1b), $B \subseteq \overline{A}^{M^{-1}}$ and $A \subseteq \overline{B}^{M^{-1}}$. Hence $A \subseteq \overline{B}^M \cap \overline{B}^{M^{-1}} = B$, and $B \subseteq \overline{A}^M \cap \overline{A}^{M^{-1}} = A$, so $A = B$. We conclude that $(H_M, *)$ is a fuzzy quasi-metric on $\mathcal{C}_\cap(X)$. ■

The fuzzy quasi-pseudo-metrics $(H_M^-, *)$, $(H_M^+, *)$ and $(H_M, *)$ of Theorem 1 are called the lower Hausdorff fuzzy quasi-pseudo-metric, the upper **Hausdorff** quasi-pseudo-metric and the Hausdorff quasi-pseudo-metric, of $(M, *)$ on $\mathcal{P}_0(X)$, respectively. **Similarly**, $(H_M, *)$ is called the Hausdorff fuzzy quasi-metric of $(M, *)$ on $\mathcal{C}_\cap(X)$.

Example 5. Let (X, d) be a quasi-(pseudo-)metric space. Then $H_{M_d} = M_{H_d}$ on $\mathcal{P}_0(X)$, i.e., the Hausdorff fuzzy quasi-pseudo-metric of the standard fuzzy quasi-(pseudo-)metric $(M_d, *)$ coincides with the standard fuzzy quasi-pseudo-metric of the Hausdorff quasi-pseudo-metric of d on $\mathcal{P}_0(X)$.

Indeed, first note that $M_{H_d} = \min\{M_{H_d^-}, M_{H_d^+}\}$.

Now, given $A, B \in \mathcal{P}_0(X)$ and $s > 0$, an easy computation shows (compare [60, Result 2.6] or [40, Proposition 3]) that

$$M_d(a, B, s) = \frac{s}{s + d(a, B)},$$

and then

$$\inf_{a \in A} M_d(a, B, s) = \frac{s}{s + \sup_{a \in A} d(a, B)}.$$

Hence, for each $t > 0$,

$$H_{M_d}^-(A, B, t) = \sup_{0 < s < t} \frac{s}{s + \sup_{a \in A} d(a, B)} = \frac{t}{t + H_d^-(A, B)} = M_{H_d^-}(A, B, t).$$

Similarly, we obtain that

$$H_{M_d}^+(A, B, t) = M_{H_d^+}(A, B, t),$$

and consequently

$$H_{M_d}(A, B, t) = \min\{M_{H_d^-}(A, B, t), M_{H_d^+}(A, B, t)\} = M_{H_d}(A, B, t).$$

We conclude that $H_{M_d} = M_{H_d}$ on $\mathcal{P}_0(X)$.

4 Some properties of the Hausdorff fuzzy quasi-metric

In this section we study properties of completeness, precompactness and compactness of the Hausdorff fuzzy quasi-metric.

In order to help ~~to~~ the reader we first recall some pertinent concepts and results.

Let (X, \mathcal{U}) be a quasi-uniform space. For each $U \in \mathcal{U}$ put

$$H_U^- = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : A \subseteq U^{-1}(B)\}$$

and

$$H_U^+ = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : B \subseteq U(A)\}.$$

Then $\{H_U^- : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $H_{\mathcal{U}}^-$ on $\mathcal{P}_0(X)$ and $\{H_U^+ : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $H_{\mathcal{U}}^+$ on $\mathcal{P}_0(X)$ [7, 28]. The quasi-uniformity $H_{\mathcal{U}} = H_{\mathcal{U}}^- \vee H_{\mathcal{U}}^+$ is said to be the Hausdorff quasi-uniformity of \mathcal{U} on $\mathcal{P}_0(X)$.

The following result was obtained by Berthiaume [7].

Theorem 2. *Let (X, d) be a quasi-pseudo-metric space. Then $\mathcal{U}_{H_d^-} = H_{\mathcal{U}_d}^-$, $\mathcal{U}_{H_d^+} = H_{\mathcal{U}_d}^+$ and $\mathcal{U}_{H_d} = H_{\mathcal{U}_d}$ on $\mathcal{P}_0(X)$.*

In our next result we present the analogue to this theorem for fuzzy quasi-pseudo-metric spaces. Furthermore, and similarly to the fuzzy metric setting (see [40, Theorem 2]), it will be the key to deduce in a direct way several properties of fuzzy quasi-pseudo-metric spaces from the corresponding well-known properties for quasi-pseudo-metric and quasi-uniform spaces.

Theorem 3. *Let $(X, M, *)$ be a fuzzy quasi-pseudo-metric space. Then $\mathcal{U}_{H_M^-} = H_{\mathcal{U}_M}^-$, $\mathcal{U}_{H_M^+} = H_{\mathcal{U}_M}^+$ and $\mathcal{U}_{H_M} = H_{\mathcal{U}_M}$ on $\mathcal{P}_0(X)$.*

Proof. Let $n \in \mathbb{N}$. If $A, B \in \mathcal{P}_0(X)$ verify $A \subseteq U_{n+1}^{-1}(B)$, then for each $a \in A$ there is $b_a \in B$ such that $M(a, b_a, 1/(n+1)) > 1 - 1/(n+1)$. Hence, for each $s \in (1/(n+1), 1/n)$, we have

$$M(a, B, s) \geq M(a, b_a, s) \geq M(a, b_a, 1/(n+1)) > 1 - 1/(n+1),$$

so

$$\inf_{a \in A} M(a, B, s) \geq 1 - 1/(n+1) > 1 - 1/n,$$

and consequently

$$H_M^-(A, B, 1/n) > 1 - 1/n.$$

Thus, we have shown that $H_{\mathcal{U}_M}^- \subseteq \mathcal{U}_{H_M^-}$ on $\mathcal{P}_0(X)$.

On the other hand, if $H_M^-(A, B, 1/n) > 1 - 1/n$, then there is $s \in (0, 1/n)$ such that $M(a, B, s) > 1 - 1/n$ for all $a \in A$, and hence $A \subseteq U_n^{-1}(B)$.
Consequently $\mathcal{U}_{H_M^-} \subseteq H_{\mathcal{U}_M}^-$ on $\mathcal{P}_0(X)$.

Similarly we prove that $\mathcal{U}_{H_M^+} = H_{\mathcal{U}_M}^+$ on $\mathcal{P}_0(X)$. Hence $\mathcal{U}_{H_M} = H_{\mathcal{U}_M}$ on $\mathcal{P}_0(X)$. ■

In the sequel we discuss the completeness of the Hausdorff fuzzy quasi-pseudo-metric. We shall show that, in this context, right K-sequential completeness provides a satisfactory notion of (fuzzy) quasi-metric completeness. It is interesting to recall that right K-sequential completeness constitutes a suitable notion of quasi-metric completeness in the realm of spaces of functions and hyperspaces, respectively (see [25, Chapter 9]).

Following [36], a sequence $(x_n)_n$ in an extended quasi-pseudo-metric space (X, d) is said to be right K-Cauchy if for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ whenever $n_0 \leq n \leq m$.

(X, d) is called right K-sequentially complete if every right K-Cauchy sequence is convergent with respect to τ_d [36].

In the fuzzy setting we propose the following notions.

Definition 1. A sequence $(x_n)_n$ in a fuzzy quasi-pseudo-metric space $(X, M, *)$ is called right K-Cauchy if for each $t > 0$ and each $\varepsilon \in (0, 1)$ there is $n_0 \in \mathbb{N}$ such that $M(x_m, x_n, t) > 1 - \varepsilon$ whenever $n_0 \leq n \leq m$.

Definition 2. A fuzzy quasi-pseudo-metric space $(X, M, *)$ is called right K-sequentially complete if every right K-Cauchy sequence is convergent with respect to τ_M .

Probabilistic quasi-pseudo-metric versions of the above concepts may be found in [8, p. 120].

The proof of the next result follows immediately from Proposition 2 and its Corollary, so it is omitted.

Proposition 3. *Let $(X, M, *)$ be a fuzzy quasi-pseudo-metric space and let d be a (n extended) quasi-pseudo-metric on X such that $\mathcal{U}_d = \mathcal{U}_M$. Then:*

(1) *A sequence in X is right K-Cauchy in $(X, M, *)$ if and only if it is right K-Cauchy in (X, d) .*

(2) *$(X, M, *)$ is right K-sequentially complete if and only if (X, d) is right*

K-sequentially complete.

Künzi and Ryser proved in [28] the following result (see also [48]).

Theorem 4. *Let (X, d) be quasi-pseudo-metric space. Then $(\mathcal{P}_0(X), H_d)$ is right K-sequentially complete if and only if (X, d) is right K-sequentially complete.*

The next result provides the fuzzy counterpart of the preceding theorem.

Theorem 5. *Let $(X, M, *)$ be a fuzzy quasi-pseudo-metric space. Then $(\mathcal{P}_0(X), H_M, *)$ is right K-sequentially complete if and only if $(X, M, *)$ is right K-sequentially complete.*

Proof. Let d be a quasi-pseudo-metric d on X such that $\mathcal{U}_d = \mathcal{U}_M$. Then $H_{\mathcal{U}_d} = H_{\mathcal{U}_M}$; so, by Theorems 2 and 3, $\mathcal{U}_{H_d} = \mathcal{U}_{H_M}$. It then follows from Proposition 3 (2) that $(\mathcal{P}_0(X), H_M, *)$ is right K-sequentially complete if and only if $(\mathcal{P}_0(X), H_d)$ is right K-sequentially complete. Now the conclusion follows from Theorem 4 and Proposition 3 (2). ■

It is interesting to obtain a version of Theorem 5 for $\mathcal{C}_\cap(X)$, because in this case H_M is a fuzzy quasi-metric. Such a version is established in the next result.

Corollary. *Let $(X, M, *)$ be a fuzzy quasi-metric space. Then $(\mathcal{C}_\cap(X), H_M, *)$ is right K-sequentially complete if and only if $(X, M, *)$ is right K-sequentially complete.*

Proof. Suppose that $(\mathcal{C}_\cap(X), H_M, *)$ is right K-sequentially complete. Let $(x_n)_n$ be a right K-Cauchy sequence in $(X, M, *)$. Since $x_n = \overline{\{x_n\}}^M \cap \overline{\{x_n\}}^{M^{-1}}$ for all $n \in \mathbb{N}$, it follows that $(\{x_n\})_n$ is a right K-Cauchy sequence in $(\mathcal{C}_\cap(X), H_M, *)$, so it converges to some $C \in \mathcal{C}_\cap(X)$ with respect to τ_{H_M} . Then, it is immediate to check that each $c \in C$ is a cluster point of the sequence $(x_n)_n$ with respect to τ_M . Since $(x_n)_n$ is right K-Cauchy we deduce that it converges to each $c \in C$ with respect to τ_M . Therefore $(X, M, *)$ is right K-sequentially complete.

Conversely, let $(A_n)_n$ be a right K-Cauchy sequence in $(\mathcal{C}_\cap(X), H_M, *)$.

By Theorem 5, $(A_n)_n$ converges to some $C \in \mathcal{P}_0(X)$ with respect to τ_{H_M} . We shall show that $(A_n)_n$ converges to \overline{C}^M with respect to τ_{H_M} . First note that $(A_n)_n$ converges to \overline{C}^M with respect to $\tau_{H_M^+}$ because $C \subseteq \overline{C}^M$. Now choose $x \in \overline{C}^M$ and $U \in \mathcal{U}_M$. Then there exist $c \in C$ and $n_0 \in \mathbb{N}$ such that $c \in U(x)$ and $C \subseteq U^{-1}(A_n)$ for all $n \geq n_0$. Hence $x \in U^{-2}(A_n)$ for all $n \geq n_0$. Thus $\overline{C}^M \subseteq U^{-2}(A_n)$ for all $n \geq n_0$, so $(A_n)_n$ converges to \overline{C}^M with respect to $\tau_{H_M^-}$. Since $\overline{C}^M \in \mathcal{C}_\cap(X)$ we conclude that $(\mathcal{C}_\cap(X), H_M, *)$ is right K-sequentially complete. ■

We finish this section by analyzing precompactness, total boundedness and compactness of the Hausdorff fuzzy quasi-metric.

Let us recall that a quasi-uniform space (X, \mathcal{U}) is precompact [15, Chapter 3] provided that for each $U \in \mathcal{U}$ there is a finite subset A of X such that $X = \bigcup_{a \in A} U(a)$.

A quasi-uniform space (X, \mathcal{U}) is totally bounded provided that the uniform space (X, \mathcal{U}^s) is totally bounded [15, Chapter 3].

It is well known that each totally bounded quasi-uniform space is precompact and that, contrarily to the uniform case, there exist precompact quasi-uniform spaces that are not totally bounded [15, Chapter 3].

In the fuzzy case, we have the following concepts (compare [44]):

A fuzzy quasi-pseudo-metric space $(X, M, *)$ is precompact (respectively, totally bounded) provided that the quasi-uniform space (X, \mathcal{U}_M) is precompact (respectively, totally bounded).

Theorem 6 [28]. *Let (X, \mathcal{U}) be a quasi-uniform space. Then:*

- (1) $(\mathcal{P}_0(X), H_{\mathcal{U}})$ is precompact if and only if (X, \mathcal{U}) is precompact.
- (2) $(\mathcal{P}_0(X), H_{\mathcal{U}})$ is totally bounded if and only if (X, \mathcal{U}) is totally bounded.
- (3) $(\mathcal{P}_0(X), (H_{\mathcal{U}})^s)$ is compact if and only if (X, \mathcal{U}^s) is compact.

Related to the statement (3) of the above theorem, it is given in [28, Example 1] an example of a compact quasi-uniform space (X, \mathcal{U}) such that $(\mathcal{P}_0(X), H_{\mathcal{U}})$ is not compact.

In the fuzzy setting we have the following:

Theorem 7. *Let $(X, M, *)$ be a fuzzy quasi-pseudo-metric space. Then:*

- (1) $(\mathcal{P}_0(X), H_M, *)$ is precompact if and only if $(X, M, *)$ is precompact.

(2) $(\mathcal{P}_0(X), H_M, *)$ is totally bounded if and only if $(X, M, *)$ is totally bounded.

(3) $(\mathcal{P}_0(X), (H_M)^i)$ is compact if and only if $(X, M^i, *)$ is compact.

Proof. We only show the statement (1), because (2) and (3) follow similarly. Indeed, by Theorem 6 (1) we have that $(\mathcal{P}_0(X), H_{\mathcal{U}_M}, *)$ is precompact if and only if (X, \mathcal{U}_M) is precompact. Since, by Theorem 3, $\mathcal{U}_{H_M} = H_{\mathcal{U}_M}$ on $\mathcal{P}_0(X)$, we deduce that $(\mathcal{P}_0(X), H_M, *)$ is precompact if and only if $(X, M, *)$ is precompact. ■

5 The Hausdorff GV-fuzzy quasi-metric

Following [20], by a GV-fuzzy quasi-pseudo-metric on a set X we mean a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times (0, +\infty)$ such that for all $x, y, z \in X, t, s > 0$:

- (i) $M(x, y, t) > 0$;
- (ii) $M(x, x, t) = 1$;
- (iii) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
- (iv) $M(x, y, -) : (0, +\infty) \rightarrow (0, 1]$ is continuous.

A GV-fuzzy quasi-metric on X is a GV-fuzzy quasi-pseudo-metric $(M, *)$ on X which satisfies the following condition: (ii') $x = y \Leftrightarrow M(x, y, t) = M(y, x, t) = 1$ for some $t > 0$.

A GV-fuzzy (pseudo-)metric on X is a GV-fuzzy quasi-(pseudo-)metric $(M, *)$ on X such that for each $x, y \in X$: (v) $M(x, y, t) = M(y, x, t)$ for all $t > 0$.

The notion of a GV-fuzzy quasi-(pseudo-)metric space is defined in the obvious manner. Note that the GV-fuzzy metric spaces are exactly the fuzzy metric spaces in the sense of George and Veeramani [17].

If $(M, *)$ is a GV-fuzzy quasi-(pseudo-)metric on X , then the fuzzy sets in $X \times X \times (0, +\infty)$, M^{-1} and M^i given by $M^{-1}(x, y, t) = M(y, x, t)$ and $M^i(x, y, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\}$, are, as in the KM-case, a GV-fuzzy quasi-(pseudo-)metric and a GV-fuzzy (pseudo-)metric on X , respectively.

Obviously, each GV-fuzzy quasi-(pseudo-)metric $(M, *)$ can be considered as a KM-fuzzy quasi-(pseudo-)metric by defining $M(x, y, 0) = 0$ for all $x, y \in X$. Hence, each GV-fuzzy quasi-pseudo-metric space generates a topology τ_M defined as in the KM-case.

Therefore, if $(X, M, *)$ is a GV-fuzzy quasi-pseudo-metric space, then $(H_M^-, *)$, $(H_M^+, *)$ and $(H_M, *)$ are fuzzy quasi-pseudo-metrics on $\mathcal{P}_0(X)$, and $(H_M, *)$ is a fuzzy quasi-metric on $\mathcal{C}_\cap(X)$ ~~whenever $(X, M, *)$ is a GV-fuzzy quasi-metric space.~~

The next example, given in [40], shows that, however, $(H_M, *)$ is not a GV-fuzzy quasi-metric on $\mathcal{C}_\cap(X)$ even in the case that $(X, M, *)$ is a GV-fuzzy metric space.

Example 6. Denote by $*_L$ the Lukasiewicz t-norm, i.e., $a *_L b = \max\{a + b - 1, 0\}$, for all $a, b \in [0, 1]$.

Now let $(x_n)_{n \geq 2}$ and $(y_n)_{n \geq 2}$ be two sequences of distinct points such that $A \cap B = \emptyset$, where $A = \{x_n : n \geq 2\}$ and $B = \{y_n : n \geq 2\}$.

Put $X = A \cup B$ and define a fuzzy set M in $X \times X \times (0, +\infty)$ by:

$$\begin{aligned} M(x_n, x_m, t) &= M(y_n, y_m, t) = 1 - \left[\frac{1}{n \wedge m} - \frac{1}{n \vee m} \right], \\ M(x_n, y_m, t) &= M(y_m, x_n, t) = \frac{1}{n} \wedge \frac{1}{m}, \end{aligned}$$

for all $n, m \geq 2$.

Then $(X, M, *_L)$ is a GV-fuzzy metric space. Since τ_M is the discrete topology on X , it follows that $A, B \in \mathcal{C}_0(X)$. From the fact that for each $n \geq 2$ and each $s > 0$, $M(x_n, B, s) = M(A, y_n, s) = 1/n$, we deduce that $H_M^-(A, B, t) = H_M^+(A, B, t) = 0$ and thus $H_M(A, B, t) = 0$ for all $t > 0$. Consequently $(H_M, *_L)$ is not a GV-fuzzy (quasi-)metric on $\mathcal{C}_0(X)$.

Despite the above example, it is shown in [40] that the formula given immediately before of Example 4 in Section 3 provides a suitable Hausdorff GV-fuzzy metric on $\mathcal{K}_0(X)$ for any GV-fuzzy metric space $(X, M, *)$. **SE PODRIA PONER LA FORMULA OTRA VEZ?**

In the rest of this section we discuss the corresponding situation to GV-fuzzy quasi-metric spaces.

We start this study with an example of a GV-fuzzy quasi-metric space $(X, M, *)$ whose induced topology is compact and metrizable, but for which $(H_M, *)$ is not a GV-fuzzy quasi-pseudo-metric on $\mathcal{K}_0(X)$.

Example 7. Let $X = \mathbb{N} \cup \{0\}$ and let d be the quasi-metric on X given by $d(x, x) = 0$ for all $x \in X$, $d(0, n) = 1/n$ for all $n \in \mathbb{N}$, and $d(n, x) = n$

for all $n \in \mathbb{N}$ and $x \in X \setminus \{n\}$. Clearly (X, d) is a quasi-metric space such that τ_d is a compact and metrizable topology. Consider the standard fuzzy quasi-metric $(M_d, *)$ of (X, d) as given in Example 3, and denote also by $(M_d, *)$ its restriction to $X \times X \times (0, +\infty)$. It is clear that this restriction is a GV-fuzzy quasi-metric on X .

Now put $A = X \setminus \{1\}$ and $B = \{1\}$. Then $A, B \in \mathcal{K}_0(X)$, and, by one of the formulas obtained in Example 5, we have that

$$H_{M_d}^-(A, B, t) = \frac{t}{t + H_d^-(A, B)},$$

for all $t > 0$. Therefore

$$H_{M_d}^-(A, B, t) = \frac{t}{t + \sup_{a \in A} d(a, \{1\})} = 0,$$

for all $t > 0$, so that $H_{M_d}(A, B, t) = 0$ for all $t > 0$. We conclude that $(H_{M_d}, *)$ is not a GV-fuzzy quasi-pseudo-metric on $\mathcal{K}_0(X) \cap \mathcal{C}_\cap(X)$.

The next is an example of a GV-fuzzy quasi-metric space $(X, M, *)$ for which $(H_M, *)$ is not a GV-fuzzy quasi-pseudo-metric on $\mathcal{K}_D(X)$, where $\mathcal{K}_D(X)$ denotes the collection of all nonempty subsets of X that are τ_M -compact and $\tau_{M^{-1}}$ -compact.

Example 8. Let $X = \mathbb{N} \cup \{0\} \cup \{+\infty\}$ and let d be the function defined on $X \times X$ by $d(x, x) = 0$ for all $x \in X$, $d(0, +\infty) = 1$, $d(0, n) = n$ for all $n \in \mathbb{N}$, $d(x, 0) = 1$ for all $x \in X \setminus \{0\}$, $d(x, y) = 0$ whenever $x \in \mathbb{N}$ and $x \leq y$, and $d(x, y) = y$ otherwise (we assume that \leq is the usual order on X).

It is a routine to show that d is a quasi-metric on X . As in Example 7 let $(M_d, *)$ be the GV-fuzzy quasi-metric on X obtained by restricting the standard fuzzy quasi-metric on (X, d) to $X \times X \times (0, +\infty)$.

Now observe that $X \in \mathcal{K}_D(X)$. Indeed, since the only τ_{M_d} -open set different from X that contains 1 is $\mathbb{N} \cup \{+\infty\}$, we obtain that X is τ_{M_d} -compact, and since the only $\tau_{M_d^{-1}}$ -open set different from X that contains $+\infty$ is $\mathbb{N} \cup \{+\infty\}$, it follows that X is $\tau_{M_d^{-1}}$ -compact.

Finally, put $A = \{0\}$ and $B = X$. Then

$$H_{M_d}^+(A, B, t) = \frac{t}{t + \sup_{b \in B} d(\{0\}, b)} = \frac{t}{t + \sup_{n \in \mathbb{N}} n} = 0,$$

for all $t > 0$. So $H_{M_d}(A, B, t) = 0$ for all $t > 0$. We conclude that $(H_{M_d}, *)$ is not a GV-fuzzy quasi-pseudo-metric on $\mathcal{K}_D(X)$.

The rest of the section is devoted to prove that for a GV-fuzzy quasi-metric space $(X, M, *)$, $(H_M, *)$ is a GV-fuzzy quasi-pseudo-metric on the collection \mathcal{K}_0^i of all nonempty subsets of X that are compact in the GV-fuzzy metric space $(X, M^i, *)$. In this way we extend the main result of [40] to the fuzzy quasi-metric framework.

To this end, we first generalize several auxiliary results of [40, Section 2] to GV-fuzzy quasi-metric spaces (although the main part of the proofs are similar to the ones given in [40], we include such proofs in order to help the readers).

Proposition 4. *Let $(X, M, *)$ be a GV-fuzzy quasi-metric space. Then M is a continuous function on $X \times X \times (0, +\infty)$ for the product topology $\tau_{M^i} \times \tau_{M^i} \times \tau_E$, where by τ_E we denote the Euclidean topology on $(0, +\infty)$.*

Proof. Let $x, y \in X$ and $t > 0$, and let $(x'_n, y'_n, t'_n)_n$ be a sequence in $X \times X \times (0, +\infty)$ that converges to (x, y, t) with respect to $\tau_{M^i} \times \tau_{M^i} \times \tau_E$.

Since $(M(x'_n, y'_n, t'_n))_n$ is a sequence in $(0, 1]$, there is a subsequence $(x_n, y_n, t_n)_n$ of $(x'_n, y'_n, t'_n)_n$ such that the sequence $(M(x_n, y_n, t_n))_n$ converges to an element of $[0, 1]$.

Fix $\delta > 0$ such that $\delta < t/2$. Then, there is $n_0 \in \mathbb{N}$ such that $|t - t_n| < \delta$ for all $n \geq n_0$. Hence

$$M(x_n, y_n, t_n) \geq M(x_n, x, \delta/2) * M(x, y, t - 2\delta) * M(y, y_n, \delta/2),$$

and

$$M(x, y, t + 2\delta) \geq M(x, x_n, \delta/2) * M(x_n, y_n, t_n) * M(y_n, y, \delta/2),$$

for all $n \geq n_0$.

Since $\lim_n M^i(x, x_n, \delta/2) = \lim_n M^i(y, y_n, \delta/2) = 1$, we obtain, by taking limits when $n \rightarrow \infty$, that

$$\lim_n M(x_n, y_n, t_n) \geq 1 * M(x, y, t - 2\delta) * 1 = M(x, y, t - 2\delta),$$

and

$$M(x, y, t + 2\delta) \geq 1 * \lim_n M(x_n, y_n, t_n) * 1 = \lim_n M(x_n, y_n, t_n),$$

respectively.

So, by continuity of the function $t \mapsto M(x, y, t)$, we immediately deduce that $M(x, y, t) = \lim_n M(x_n, y_n, t_n)$. Therefore M is continuous for $\tau_{M^i} \times \tau_{M^i} \times \tau_E$. ■

Lemma 3. *Let $(X, M, *)$ be a GV-fuzzy quasi-metric space. Then, for each $a \in X$, $B \in \mathcal{K}_0^i(X)$ and $t > 0$, there is $b_0 \in B$ such that*

$$M(a, B, t) = M(a, b_0, t).$$

Proof. Let $a \in X$, $B \in \mathcal{K}_0^i(X)$ and $t > 0$. By Proposition 4, the function $y \mapsto M(a, y, t)$ is continuous on X for τ_{M^i} . Thus, by compactness of B , there exists $b_0 \in B$ such that $\sup_{b \in B} M(a, b, t) = M(a, b_0, t)$, i.e., $M(a, B, t) = M(a, b_0, t)$. ■

Lemma 4. *Let $(X, M, *)$ be a GV-fuzzy quasi-metric space. Then, for each $a \in X$ and $B \in \mathcal{K}_0^i(X)$ the function*

$$t \mapsto M(a, B, t)$$

is continuous on $(0, +\infty)$.

Proof. Since $M(a, B, t) = \sup_{b \in B} M(a, b, t)$ and for each $b \in B$ the function $t \mapsto M(a, b, t)$ is continuous on $(0, +\infty)$, it follows that the function $t \mapsto M(a, B, t)$ is lower semicontinuous on $(0, +\infty)$.

We shall prove that $t \mapsto M(a, B, t)$ is upper semicontinuous on $(0, +\infty)$. To this end, let $t > 0$ and let $(t_n)_n$ be a sequence in $(0, +\infty)$ that converges to t . By Lemma 3, for each $n \in \mathbb{N}$ there is $b_n \in B$ such that $M(a, B, t_n) = M(a, b_n, t_n)$. Since $B \in \mathcal{K}_0^i(X)$, there exists a subsequence $(b_{n_k})_k$ of $(b_n)_n$ and a point $b_0 \in B$ such that $b_{n_k} \rightarrow b_0$ in $(X, M^i, *)$. Hence $\lim_k M(a, b_{n_k}, t_{n_k}) = M(a, b_0, t)$, by Proposition 4, and thus

$$\lim_k M(a, B, t_{n_k}) = M(a, b_0, t) \leq M(a, B, t).$$

Consequently, the function $t \mapsto M(a, B, t)$ is upper semicontinuous on $(0, +\infty)$. This concludes the proof. ■

Lemma 5. *Let $(X, M, *)$ be a GV-fuzzy quasi-metric space. Then, for each $A \in \mathcal{K}_0^i(X)$, $B \in \mathcal{P}_0(X)$ and $t > 0$, there is $a_0 \in A$ such that*

$$\inf_{a \in A} M(a, B, t) = M(a_0, B, t).$$

Proof. Put $\alpha = \inf_{a \in A} M(a, B, t)$. Then, there is a sequence $(a_n)_n$ in A such that $\alpha + 1/n > M(a_n, B, t)$ for all $n \in \mathbb{N}$. Since $A \in \mathcal{K}_0^i(X)$, there exists a subsequence $(a_{n_k})_k$ of $(a_n)_n$ and a point $a_0 \in A$ such that $a_{n_k} \rightarrow a_0$ in $(X, M^i, *)$.

Choose an arbitrary $b \in B$. By Proposition 4, $\lim_k M(a_{n_k}, b, t) = M(a_0, b, t)$. Since for each $k \in \mathbb{N}$, $\alpha + 1/n_k > M(a_{n_k}, b, t)$, it follows, taking limits when $k \rightarrow \infty$, that $\alpha \geq M(a_0, b, t)$. We conclude that $\alpha = M(a_0, B, t)$. ■

From Lemmas 3 and 5 we immediately deduce the following.

Corollary. *Let $(X, M, *)$ be a GV-fuzzy quasi-metric space and let $A, B \in \mathcal{K}_0^i(X)$ and $t > 0$. Then there exist $a_0 \in A$ and $b_0 \in B$ such that*

$$\inf_{a \in A} M(a, B, t) = M(a_0, b_0, t).$$

Proposition 5. *Let $(X, M, *)$ be a GV-fuzzy quasi-metric space. Then, for each $A, B \in \mathcal{K}_0^i(X)$ the function*

$$t \longmapsto \inf_{a \in A} M(a, B, t)$$

is continuous on $(0, +\infty)$.

Proof. By Lemma 4, the function $t \longmapsto M(a, B, t)$ is continuous on $(0, +\infty)$. Hence, the function $t \longmapsto \inf_{a \in A} M(a, B, t)$ is upper semicontinuous on $(0, +\infty)$.

We shall prove that $t \longmapsto \inf_{a \in A} M(a, B, t)$ is lower semicontinuous on $(0, +\infty)$. To this end, let $t > 0$ and let $(t_n)_n$ be a sequence in $(0, +\infty)$ that converges to t . By Lemma 5, for each $n \in \mathbb{N}$ there is $a_n \in A$ such that $M(a_n, B, t_n) = \inf_{a \in A} M(a, B, t_n)$. Since $A \in \mathcal{K}_0^i(X)$, there exists a subsequence $(a_{n_k})_k$ of $(a_n)_n$ and a point $a_0 \in A$ such that $a_{n_k} \rightarrow a_0$ in $(X, M^i, *)$. Then, by Lemma 3, there is $b_0 \in B$ such that $M(a_0, b_0, t) = M(a_0, B, t)$, and thus $\lim_k M(a_{n_k}, b_0, t_{n_k}) = M(a_0, b_0, t)$, by Proposition 4. Therefore, given $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$, $M(a_0, b_0, t) < \varepsilon + M(a_{n_k}, b_0, t_{n_k})$. So

$$\inf_{a \in A} M(a, B, t) \leq M(a_0, b_0, t) < \varepsilon + M(a_{n_k}, B, t_{n_k}) = \varepsilon + \inf_{a \in A} M(a, B, t_{n_k}),$$

for all $k \geq k_0$. Consequently, the function $t \mapsto \inf_{a \in A} M(a, B, t)$ is lower semicontinuous on $(0, +\infty)$. This concludes the proof. ■

Remark 5. Note that Proposition 5 also shows that for $A, B \in \mathcal{K}_0^i(X)$ the function $t \mapsto \inf_{b \in B} M(A, b, t)$ is continuous on $(0, +\infty)$.

Now let $(X, M, *)$ be a GV-fuzzy quasi-metric **space** and let $(H_M, *)$ be the Hausdorff fuzzy quasi-pseudo-metric on $\mathcal{P}_0(X)$ constructed in Section 3. In order to prove that $(H_M, *)$ is actually a GV-fuzzy quasi-pseudo-metric on $\mathcal{K}_0^i(X)$, we first show that for each $A, B \in \mathcal{K}_0^i(X)$ and $t > 0$, we have

$$H_M^-(A, B, t) = \inf_{a \in A} M(a, B, t).$$

Indeed, since

$$H_M^-(A, B, t) = \sup_{0 < s < t} \inf_{a \in A} M(a, B, s),$$

there exists an increasing sequence $(s_n)_n$ convergent to t such that $H_M^-(A, B, t) = \lim_n \inf_{a \in A} M(a, B, s_n)$. So, by Proposition 5, $H_M^-(A, B, t) = \inf_{a \in A} M(a, B, t)$.

Similarly, we obtain that

$$H_M^+(A, B, t) = \inf_{b \in B} M(A, b, t).$$

Thus

$$H_M(A, B, t) = \min\{\inf_{a \in A} M(a, B, t), \inf_{b \in B} M(A, b, t)\},$$

for all $A, B \in \mathcal{K}_0^i(X)$ and $t > 0$.

Since $(M, *)$ is a GV-fuzzy quasi-metric, it follows from Lemma 5 that $H_M^-(A, B, t) > 0$ and $H_M^+(A, B, t) > 0$, so $H_M(A, B, t) > 0$ for all $A, B \in \mathcal{K}_0^i(X)$ and $t > 0$. Hence $(H_M, *)$ satisfies condition (i) of the definition of a GV-fuzzy quasi-pseudo-metric.

It is also clear that $H_M(A, A, t) = 1$ for all $A \in \mathcal{K}_0^i(X)$ and all $t > 0$. Moreover, for each $A, B, C \in \mathcal{K}_0^i(X)$ and $t, s \geq 0$, we have, by Lemma 2 (2a) and (2b) that $H_M(A, C, t + s) \geq H_M(A, B, t) * H_M(B, C, s)$.

Finally, given $A, B \in \mathcal{K}_0^i(X)$, continuity of $H_M(A, B, -) : (0, +\infty) \rightarrow (0, 1]$ is an immediate consequence of Proposition 5 and Remark 5.

Thus, we have shown the following

Theorem 8. *Let $(X, M, *)$ be a GV-fuzzy quasi-metric space. Then $(H_M, *)$ is a GV-fuzzy quasi-pseudo-metric on $\mathcal{K}_0^i(X)$. Furthermore, we have*

$$H_M(A, B, t) = \min\left\{\inf_{a \in A} M(a, B, t), \inf_{b \in B} M(A, b, t)\right\},$$

for all $A, B \in \mathcal{K}_0^i(X)$ and $t > 0$.

6 A fuzzy approach to the domain of words

In this section we apply the results obtained in the preceding sections to **model** a typical example of theoretical computer science from a fuzzy quasi-metric point of view. This will be done with the help of the parameter t which provides a useful additional ingredient to construct such models.

Let us recall that the domain of words Σ^∞ ([24, 33, 35, 45, 51, 58, etc]) consists of all finite and infinite sequences (“words”) over a nonempty set (“alphabet”) Σ , ordered by the so-called information order \sqsubseteq on Σ^∞ ([9, Example 1.6]), i.e., $x \sqsubseteq y \Leftrightarrow x$ is a prefix of y , where we assume that the empty sequence ϕ is an element of Σ^∞ . If $x \sqsubseteq y$ with $x \neq y$, we shall write $x \sqsubset y$.

For each $x, y \in \Sigma^\infty$ denote by $x \sqcap y$ the longest common prefix of x and y , and for each $x \in \Sigma^\infty$ denote by $\ell(x)$ the length of x . Thus $\ell(x) \in [1, \infty]$ whenever $x \neq \phi$, and $\ell(\phi) = 0$.

In theory of computation the fact that $x \sqsubseteq y$ is interpreted as the element y contains all the information provided by x , and thus the partially defined objects (finite words) customary represent stages of a computational process for which the totally defined objects (infinite words) contain exactly the amount of information provided by **the process**.

Given a nonempty alphabet Σ , Smyth introduced in [58] a quasi-metric d_{\sqsubseteq} on Σ^∞ given by $d_{\sqsubseteq}(x, y) = 0$ if $x \sqsubseteq y$, and $d(x, y) = 2^{-\ell(x \sqcap y)}$ otherwise (see also [24, 42, 43, etc]).

This quasi-metric has the advantage that its specialization order coincides with the order \sqsubseteq , and thus the quasi-metric space (Σ^∞, d) **preserves** the information provided by \sqsubseteq (compare Remark 1). Moreover, the metric $(d_{\sqsubseteq})^s$ is given by $(d_{\sqsubseteq})^s(x, y) = 0$ if $x = y$, and $(d_{\sqsubseteq})^s(x, y) = 2^{-\ell(x \sqcap y)}$ otherwise; so that $(d_{\sqsubseteq})^s$ is **exactly** the celebrated Baire metric on Σ^∞ .

However, the quasi-metric d_{\sqsubseteq} is unable to give us information on the degree of approximation to a word z from two different prefix x, y of z . For

instance, if we consider the totally defined object π and the partially defined ones $x = 3.14$ and $y = 3.141$, then it is clear that y contains more information on π than x , but $d_{\sqsubseteq}(x, \pi) = d_{\sqsubseteq}(y, \pi) = 0$, so d_{\sqsubseteq} is not sensitive to this amount of information.

Motivated by this fact, we shall construct a fuzzy quasi-metric on Σ^∞ that preserves the advantages of d_{\sqsubseteq} and that, in addition, permits us to measure, with the help of the parameter t , the degree of approximation to a given word of each **one of its prefixes**. Finally, we shall apply this construction to measure, in some representative cases, (fuzzy) distances between elements of $\mathcal{P}_0(\Sigma^\infty)$ via the Hausdorff fuzzy quasi-(pseudo-)metric.

Define a fuzzy set M in $\Sigma^\infty \times \Sigma^\infty \times [0, +\infty)$ by

$$\begin{aligned} M(x, y, 0) &= 0 \text{ for all } x, y \in \Sigma^\infty, \\ M(x, x, t) &= 1 \text{ for all } x \in \Sigma^\infty \text{ and } t > 0, \\ M(x, y, t) &= 1 \text{ if } x \sqsubset y \text{ and } t > 2^{-\ell(x)}, \\ M(x, y, t) &= 1 - 2^{-\ell(x \sqcap y)} \text{ otherwise.} \end{aligned}$$

We wish to show that (M, \wedge) is a fuzzy quasi-metric on Σ^∞ . To this end we only prove that for each $x, y, z \in \Sigma^\infty$ and $t, s \geq 0$, one has $M(x, z, t+s) \geq M(x, y, t) \wedge M(y, z, s)$, because the rest of conditions in the definition of a fuzzy quasi-metric are obviously true.

Indeed, if $M(x, z, t+s) = 1$, the conclusion is obvious. Assume now that $M(x, z, t+s) = 1 - 2^{-\ell(x \sqcap z)}$. We distinguish two cases: (a) x is a prefix of z ; (b) x is not a prefix of z . In case (a) we have $M(x, z, t+s) = 1 - 2^{-\ell(x)}$ and $t+s \leq 2^{-\ell(x)}$, and thus $M(x, y, t) = 1 - 2^{-\ell(x \sqcap y)}$ because $t \leq 2^{-\ell(x)}$. Since $\ell(x) \geq \ell(x \sqcap y)$, it follows that $M(x, z, t+s) \geq M(x, y, t)$. In case (b) we have $M(x, z, t+s) = 1 - 2^{-\ell(x \sqcap z)}$, and the conclusion follows immediately from the well-known facts that for each $x, y, z \in \Sigma^\infty$, one has: (i) $\ell(x \sqcap z) \geq \min\{\ell(x \sqcap y), \ell(y \sqcap z)\}$, and (ii) $\ell(x \sqcap z) = \ell(y \sqcap z)$ whenever x is a prefix of y but not a prefix of z .

We conclude that (M, \wedge) is a fuzzy quasi-metric on Σ^∞ .

Now, observe that if x, y are **prefixes** of z , with $x \neq y$, and one obtains for some $t_0 > 0$, $M(x, z, t_0) < 1$ and $M(y, z, t_0) = 1$, then $2^{-\ell(y)} < t_0 \leq 2^{-\ell(x)}$, so that $\ell(x) \leq \ell(y)$, i.e., $x \sqsubset y$; which shows that y is a better approximation to z than x .

Then, for each $z \in \Sigma^\infty \setminus \{\phi\}$, and each $x \sqsubset z$ we can define the degree of approximation of x to z , associated to (M, \wedge) , as the number $DA(x, z) = 1/t_x$ where $t_x = \inf\{t > 0 : M(x, z, t) = 1\}$. It is clear that $DA(x, z) = 2^{\ell(x)}$.

In particular, for $x = 3.14$ and $y = 3.141$ as given above, one obtains

for each $t \in (2^{-4}, 2^{-3}]$, $M(x, \pi, t) < 1$ and $M(y, \pi, t) = 1$, which agrees with the fact that y contains more information on π than x . Furthermore $DA(x, \pi) = 2^3$ and $DA(y, \pi) = 2^4$, which provides reasonable (and desirable) values on the degree of approximation of x and y to π , respectively.

Finally, we apply this approach to compute the distance between some interesting subsets of $\mathcal{P}_0(\Sigma^\infty)$ via the (lower) Hausdorff fuzzy quasi-pseudo-metric of $(\Sigma^\infty, M, \wedge)$.

Let $z \in \Sigma^\infty$ such that $\ell(z) = \infty$ and let x be a prefix of z different from z , i.e., $x \sqsubset z$. Put $x_\rightarrow = \{y \in \Sigma^\infty : x \sqsubseteq y \sqsubset z\}$. Since $z \in \overline{x_\rightarrow}^M \cap \overline{x_\rightarrow}^{M^{-1}}$, it follows from Lemma 2 (1a) that for each $t > 0$,

$$H_M^-(\{z\}, x_\rightarrow, t) = 1.$$

Furthermore, it is easy to see that

$$H_M^-(x_\rightarrow, \{z\}, t) = 1 \iff t > 2^{-\ell(x)}.$$

Therefore (compare Remark 3), one has $\{z\} \leq_{H_M^-} x_\rightarrow$ and $x_\rightarrow \leq_{H_M^-} \{z\}$.

The last relation is not a surprise because it can be computationally interpreted as that z contains at least the same amount of information of z than x_\rightarrow . However, the first relation seems certainly interesting because it can be computationally interpreted as that x_\rightarrow contains at least the same amount of information of z than z , which is true because actually x_\rightarrow has exactly the same amount of information of z than z .

Now let $x \sqsubset y \sqsubset z$. Then

$$H_M^-(y_\rightarrow, x_\rightarrow, t) = 1,$$

for all $t > 0$, by Lemma 2 (1a). Moreover

$$H_M^-(x_\rightarrow, y_\rightarrow, t) = 1 \iff t > 2^{-\ell(x)},$$

so $y_\rightarrow \leq_{H_M^-} x_\rightarrow$ and $x_\rightarrow \leq_{H_M^-} y_\rightarrow$, as we could expect, because x_\rightarrow and y_\rightarrow contain the same amount of information of z . Thus, the preorder $\leq_{H_M^-}$ provides a better computational interpretation than the preorder $\leq_{H_M^-, t}$ on $\mathcal{P}_0(\Sigma^\infty)$.

7 Conclusion

We have established connections between the theory of Hausdorff fuzzy distances and the theory of **asymmetric** topology by means of fuzzy quasi-metrics. We have obtained several properties of Hausdorff fuzzy quasi-metric spaces and we have exploited the information given by the parameter $t > 0$ to **model** the domain of words by means of our approach; some advantages of our model are discussed.

References

- [1] C. Alegre, Continuous operators of asymmetric normed spaces, Acta Math. Hungar., in press.
- [2] S. Andima, R. Kopperman, P. Nickolas, An asymmetric Ellis theorem, Topology Appl. 155 (2007), 146-160.
- [3] J. Andres, J. Fišer, G. Gabor, K. Leśniak, Multivalued fractals, Chaos, Solitons and Fractals 24 (2005), 665-700.
- [4] J.W. de Bakker, E.P. de Vink, Control Flow Semantics, The MIT Press, Cambridge, Massachusetts, 1996.
- [5] J. W. de Bakker, E. P. de Vink, Denotational models for programming languages: applications of Banach's fixed point theorem, Topology Appl. 85 (1998), 35-52.
- [6] G. Beer, Topologies on Closed and Closed Convex Sets, Kluwer Academic Publishers, Dordrecht, 1993.
- [7] G. Berthiaume, On quasi-uniformities in hyperspaces, Proc. Amer. Math. Soc. 66 (1977), 335-343.
- [8] Y.J. Cho, M. Grabiec, V. Radu, On Nonsymmetric Topological and Probabilistic Structures, Nova Sci. Publ., New York, 2006.
- [9] B.A. Davey, H.A. Priestley, Introduction to Lattices and Order, Cambridge Univ. Press 1990.
- [10] A. Di Concilio, G. Gerla, Quasi-metric spaces and point-free geometry, Math. Struct. Comput. Sci. 16 (2006), 115-137.

- [11] A. Fedeli, On chaotic set-valued discrete dynamical systems, *Chaos, Solitons and Fractals* 23 (2005), 1381-1384.
- [12] G.A. Edgar, *Measure, Topology and Fractal Geometry*, Springer, Berlin, New York, 1992.
- [13] R.J. Egbert, Products and quotients of probabilistic metric spaces, *Pacific J. Math.* 24 (1968), 437-455.
- [14] R. Engelking, *General Topology*, PWN-Polish Sci. Publ., Warsaw, 1977.
- [15] P. Fletcher, W.F. Lindgren, *Quasi-Uniform Spaces*, Marcel Dekker, New York, 1982.
- [16] L.M. García-Raffi, S. Romaguera, M. Schellekens, Applications of the complexity space to the General Probabilistic Divide and Conquer Algorithms, *J. Math. Anal. Appl.* 348 (2008), 346-355.
- [17] A. George, P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets and Systems* 64 (1994), 395-399.
- [18] A. George, P. Veeramani, On some results of analysis of fuzzy metric spaces, *Fuzzy Sets and Systems* 90 (1997), 365-368.
- [19] G. Gerla, Fuzzy submonoids, fuzzy preorders and quasi-metrics, *Fuzzy Sets and Systems* 157 (2006), 2356-2370.
- [20] V. Gregori, S. Romaguera, Fuzzy quasi-metric spaces, *Appl. Gen. Topology* 5 (2004), 129-136.
- [21] C. Guerra, V. Pascucci, Line-based object recognition using Hausdorff distance: from range images to molecular secondary structure, *Image and Vision Comput. J.* 23 (2005), 405-415.
- [22] D.P. Huttenlocher, G.A. Klanderman, W.J. Rucklidge, Comparing images using the Hausdorff distance, *IEEE Trans. Pattern Anal. Mach. Intelligence* 15 (1993), 850-863.
- [23] I. Kramosil, J. Michalek, Fuzzy metrics and statistical metric spaces, *Kybernetika* 11 (1975), 326-334.

- [24] H.P.A. Künzi, Nonsymmetric topology, in: Proc. Szekszárd Conference, Bolyai Soc. Math. Studies 4 1993 Hungary (Budapest 1995), 303-338.
- [25] H.P.A. Künzi, Nonsymmetric distances and their associated topologies: About the origins of basic ideas in the area of asymmetric topology, in: Handbook of the History of General Topology, C.E. Aull, R. Lowen (Eds.), vol. 3, pp. 853-968, Kluwer, Dordrecht, 2001.
- [26] H.P.A. Künzi, An Introduction to Quasi-Uniform Spaces, in the book: Beyond Topology, F. Mynard and E. Pearl editors; Contemporary Mathematics, American Mathematical Society, 2008, in press.
- [27] H.P.A. Künzi, V. Kreinovich, Static space-times naturally lead to quasipseudometrics, Theoret. Comput. Sci. 405 (2008), 64-72.
- [28] H.P.A. Künzi, C. Ryser, The Bourbaki quasi-uniformity, Topology Proc. 20 (1995), 161-183.
- [29] Y. Liu, Z. Li, Coincidence point theorems in probabilistic and fuzzy metric spaces, Fuzzy Sets and Systems 158 (2007), 58-70.
- [30] Y. Lu, C.L.Tan, W. Huang, L. Fan, An approach to word image matching based on weighted Hausdorff distance, Sixth Internat. Conf. Document Analysis and Recognition (ICDAR'01), 2001, pp. 921-925.
- [31] R. Lucchetti, Convexity and Well-Posed Problems, CMS Books in Mathematics, Springer, 2006.
- [32] J. Marín, An extension of Alaoglu's theorem for topological semicones, Houston J. Math. 34 (2008), 795-806.
- [33] S.G. Matthews, Partial metric topology, Ann. New York Acad. Sci. 728 (1994), 183-197.
- [34] A. Peris, Set-valued discrete chaos, Chaos, Solitons and Fractals, 26 (2005), 19-23.
- [35] D. Perrin, J.E. Pin, Infinite Words. Automata, Semigroups, Logic and Games. Pure and Appl. Math. Series, vol. 141, Elsevier Acad. Press. 2004.

- [36] I.L. Reilly, P.V. Subrahmanyam and M.K. Vamanamurthy, Cauchy sequences in quasi-pseudo-metric spaces, *Monatsh. Math.* 93 (1982), 127-140.
- [37] R.T. Rockafellar, R.J.B. Wets, *Variational Analysis*, Comprehensive Studies in Mathematics, vol. 317, Springer, 1998.
- [38] J. Rodríguez-López, S. Romaguera, The relationship between the Vietoris topology and the Hausdorff quasi-uniformity, *Topology Appl.* 124 (2002), 451-464.
- [39] J. Rodríguez-López, S. Romaguera, Wijsman and hit-and-miss topologies of quasi-metric spaces, *Set-Valued Analysis* 11 (2003), 323-344.
- [40] J. Rodríguez-López, S. Romaguera, The Hausdorff fuzzy metric on compact sets, *Fuzzy Sets and Systems* 147 (2004), 273-283.
- [41] J. Rodríguez-López, S. Romaguera, Closedness of bounded convex sets of asymmetric normed linear spaces and the Hausdorff quasi-metric, *Bull. Belgian Math. Soc. - Simon Stevin* 13 (2006), 551-562.
- [42] J. Rodríguez-López, S. Romaguera, O. Valero, Denotational semantics for programming languages, balanced quasi-metrics and fixed points, *Internat. J. Comput. Math.* 85 (2008), 623-630.
- [43] S. Romaguera, A. Sapena, P. Tirado, The Banach fixed point theorem in fuzzy quasi-metric spaces with application to the domain of words, *Topology Appl.* 154 (2007), 2196-2203.
- [44] S. Romaguera, A. Sapena, O. Valero, Quasi-uniform isomorphisms in fuzzy quasi-metric spaces, bicompletion and D-completion, *Acta Math. Hungar.* 114 (2007), 49-60
- [45] S. Romaguera, M. Schellekens, Partial metric monoids and semivaluation spaces, *Topology Appl.* 153 (2005), 948-962.
- [46] S. Romaguera, O. Valero, On the structure of the space of complexity partial functions, *Internat. J. Comput. Math.* 85 (2008), 631-640.
- [47] H. Román-Flores, A note on transitivity in set-valued discrete systems, *Chaos, Solitons and Fractals*, 17 (2003), 99-104.

- [48] M.A. Sánchez-Granero, Weak completeness of the Bourbaki quasi-uniformity, *Appl. Gen. Topology* 2 (2001), 101-112.
- [49] M. Sanchis, M. Tkachenko, Totally Lindelöf and totally ω -narrow paratopological vector spaces, *Topology Appl.* 155 (2008), 322-334.
- [50] M. Schellekens, The Smyth completion: a common foundation for denotational semantics and complexity analysis, *Electronic Notes in Theoretical Computer Science* 1 (1995), 535-556.
- [51] M. Schellekens, The correspondence between partial metrics and semi-valuations, *Theoret. Comput. Sci.* 315 (2004), 135-149.
- [52] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, North-Holland, 1983.
- [53] A.K. Seda, P. Hitzler, Generalized distance functions in the theory of computation, *The Computer Journal*, in press.
- [54] C. Sempì, Hausdorff distance and the completion of probabilistic metric spaces, *Boll. U.M.I. 6-B* (1992), 317-327.
- [55] B. Sendov, Hausdorff distance and image processing, *Russian Math. Surveys* 59 (2004), 319-328.
- [56] J. Serra, *Image Analysis and Mathematical Morphology*, Academic Press, 1982.
- [57] R. Sikora, S. Piramuthu, Efficient genetic algorithm based data mining using feature selection with Hausdorff distance, *Inf. Tech. Management* 6 (2005), 315-331.
- [58] M.B. Smyth, Quasi-uniformities: Reconciling domains with metric spaces, In: M. Main *et al.* (Eds), *Mathematical Foundations of Programming Language Semantics, Third Workshop, Tulane, 1987*, Lecture Notes in Computer Science, vol. 298 (Berlin: Springer), 236-253.
- [59] R.M. Tardiff, Topologies for probabilistic metric spaces, *Pacific J. Math.* 65 (1976), 233-251.
- [60] P. Veeramani, Best approximation in fuzzy metric spaces, *J. Fuzzy Math.* 9 (2001), 75-80.

- [61] Y. Wang, G. Wei, Characterizing mixing, weak mixing and transitivity of induced hyperspace dynamical systems, *Topology Appl.* 155 (2007), 56-68.
- [62] P. Waszkiewicz, Partial metrisability of continuous posets, *Math. Struct. Comput. Sci.* 16 (2006), 359-372.
- [63] C. Zhao, W. Shi, Y. Deng, A new Hausdorff distance for image matching, *Pattern Recogn. Letters* 26 (2005), 581-586.