Solving Riccati time-dependent models with random quadratic coefficient

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Abstract

This paper deals with the construction of approximate solutions of random logistic differential equation whose nonlinear coefficient is assumed to be an analytic stochastic process and the initial condition is a random variable. Applying p-mean stochastic calculus, the nonlinear equation is transformed into a random linear equation whose coefficients keep analyticity. Next, an approximate solution of the nonlinear problem is constructed in terms of a random power series solution of the associate linear problem. Approximations of the average and variance of the solution are provided. The proposed technique is illustrate through an example where comparisons with respect to Monte Carlo simulations are shown.

Key words: Random logistic differential equation, random power series solution, p-mean stochastic calculus

1 Introduction

Deterministic differential equation $x(t) = ax(t) - b(x(t))^2$, constitutes the basic pattern of logistic model which appears in modeling problems (population models in Biology, widespread of illness in Epidemiology, diffusion of new products and technologies in Marketing, etc). In the Verhulst's population model, $a > 0$ represents the maximum per capita growth rate and $b > 0$ is interpreted in terms of the maximum sustainable population (carrying capacity). In many cases, biologists can fix the parameter a depending on the

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species under study, however unsettled environment and inherent complexity of the surrounding medium suggest to consider b as a random variable (r.v.) rather than a numerical constant. In addition, in practice both quantities vary with time t, thus it is more realistic consider them as functions of time. These considerations lead us to study the random nonlinear differential equation:

$$\dot{X}(t) + a(t)X(t) + B(t)(X(t))^2 = 0, \quad X(0) = X_0, \tag{1}$$

where $a(t)$ is an analytic function and $B(t)$ is an analytic process,

$$a(t) = \sum_{n \geq 0} a_n t^n, \quad B(t) = \sum_{n \geq 0} B_n t^n, \quad t \in D = \{t : |t| \leq c\}, \, c > 0, \tag{2}$$

being B_n, X_0 r.v.'s satisfying certain conditions to be specified later. These type of models have been considered in different scenarios. In [1], a random logistic model with time-independent coefficients is studied using the so-called sample approach [2]. Recently, a logistic model with nonlinear perturbations and randomness modeled by means of the white noise process has been studied taking advantage of Wiener-Hermite method [3]. In [4,5] the stochastic logistic model is studied by considering Itô calculus and the generalized polynomial chaos approaches, respectively.

2 Preliminaries about p-mean stochastic calculus

In this section we summarize the main concepts related to the so-called p-mean stochastic calculus that will be required along this paper. For $p = 2$, usually they are referred to as mean square (m.s.) and mean fourth (m.f.) calculus. For further details, we refer to [2, chap.4] for m.s. calculus, and [6], for m.f. calculus. Let $(\Omega, \mathcal{F}, P)$ be a probability space, throughout this article we will work in the Banach spaces $(L_p, \|X\|_p)$, with $p = 2, 4, 8$, whose elements $X = X(\omega)$ are called p-order real random variables (p-r.v.'s), i.e., r.v.'s $X : \Omega \to \mathbb{R}$ such that $\mathrm{E}[|X|^p] < +\infty$, where $\mathrm{E}[\cdot]$ denotes the expectation operator and, $\|X\|_p = (\mathrm{E}[|X|^p])^{1/p}$. In this space a stochastic process (s.p.) will be denoted by $X(t) = X(t, \omega)$, with t lying in a set T. By applying Schwarz inequality [2, p.43], one obtains the following inequality between p-norms:

$$\|XY\|_p \leq \|X\|_{2p} \|Y\|_{2p}, \quad \forall X, Y \in L_{2p}, \tag{3}$$

that will be applied later for $p = 2, 4, 8$. As usual, in L_p spaces, p-mean convergence is referred to as the corresponding p-norm; when $p = 2$, it is called m.s. convergence, for $p = 4$, m.f. convergence and so on. Two important features related to L_p spaces and p-mean convergence is that every p_2-r.v. is a p_1-r.v. and, p_2-mean convergence entails p_1-convergence, whenever $p_2 > p_1$. M.s. differentiation product rule for m.s. differentiable processes holds when one of the factors is deterministic or independent of the other (see [2, p.96] and proposition 2.5 in [6]). Lemma 3.14 in [6] proves a product m.s. differentiation rule with no such restrictions. Instead, we require that both processes be m.f. differentiable. From that, it is straightforward to obtain the following m.s. differentiation quotient rule:

Lemma 1 *Let $\{W(t) : t \in T\}$ and $\{Z(t) : t \in T\}$ be 4-s.p.'s having 4-derivatives $\frac{dW(t)}{dt}$ and $\frac{dZ(t)}{dt}$, respect. and such that $Z(t,\omega) \neq 0$ for all $\omega \in \Omega$. Then $W(t)/Z(t)$ is m.s. differentiable at $t \in T$ and*

$$\frac{d}{dt}\left(\frac{W(t)}{Z(t)}\right) = \frac{\frac{dW(t)}{dt}Z(t) - W(t)\frac{dZ(t)}{dt}}{(Z(t))^2}.$$

Note that if the m.f. derivative exists, so does the corresponding m.s. derivative. Therefore, we can use indistinct notations for the m.s. and the m.f. derivative. Even more, this also remains true for p-derivatives of higher order, i.e., if $\{X(t) : t \in T\}$ is p_2-mean differentiable then it is also p_1-mean differentiable whenever $p_2 > p_1$ and in this case both p-derivatives coincide.

Now, we address to extend the so-called (deterministic) Cauchy's inequalities that satisfy the coefficients of an analytic function in the case that we consider a p-mean analytic s.p. By following an analogous development as in the m.s. calculus, we can characterize the p-mean analyticity of a s.p. $\{H(t) : |t| < c\}$, i.e.,

$$H(t) = \sum_{n=0}^{\infty} H_n t^n, \quad H_n = \frac{H^{(n)}(0)}{n!}, \qquad 0 \leq |t| < c, \tag{4}$$

(where convergence and derivatives are considered in the p-mean sense) in terms of the (deterministic) analyticity of its correlation function $\Gamma_H(t_1, \ldots, t_p)$ at the diagonal points, i.e., $t_1 = \ldots = t_p$, where

$$\Gamma_H(t_1, \ldots, t_p) = \sum_{n_1,\ldots,n_p=0}^{\infty} \frac{\gamma_{n_1\ldots n_p}}{(n_1)! \times \cdots \times (n_p)!}(t_1)^{n_1} \times \cdots \times (t_p)^{n_p}, \ |t_i| < c, \ 1 \leq i \leq p, \tag{5}$$

being

$$\gamma_{n_1\ldots n_p} = \left.\frac{\partial^{n_1+\cdots+n_p}\Gamma_H(t_1, \ldots, t_p)}{\partial t_1^{n_1} \cdots \partial t_p^{n_p}}\right|_{(t_1,\ldots,t_p)=(0,\ldots,0)}. \tag{6}$$

Since $\Gamma_H(t_1, \ldots, t_p)$ is analytic on $(-c, c)^p$, by the deterministic Cauchy's inequalities one gets

$$\exists\, M_{\Gamma_H} > 0 : \left|\gamma_{n_1\ldots n_p}\right| \leq \frac{M_{\Gamma_H}}{\rho^{n_1+\cdots+n_p}}, \quad 0 < \rho < c. \tag{7}$$

Now, taking into account (4)-(7) and, formula (4.133) of [2] extended to the p-mean calculus following an analogous reasoning as it was done for the m.f.

calculus in paper [6], one obtains

$$\left|\gamma_{n_1 \ldots n_p}\right| = \left|\mathrm{E}\Big[H^{(n_1)}(0) \times \cdots \times H^{(n_p)}(0)\Big]\right| \leq \frac{M_{\Gamma_H}}{\rho^{n_1+\cdots+n_p}}, \quad 0 < \rho < c. \tag{8}$$

By the definition of the p-norm, the expression of coefficients $H_n(0)$ given by (4) and expression (8) for $n = n_1 = \ldots = n_p$, one gets

$$\|H_n\|_p = \sqrt[p]{\mathrm{E}\left[\left(\frac{H^{(n)}(0)}{n!}\right)^p\right]} \leq \sqrt[p]{\mathrm{E}\Big[(H^{(n)}(0))^p\Big]} \leq \sqrt[p]{\frac{M_{\Gamma_H}}{\rho^{pn}}} = \frac{\sqrt[p]{M_{\Gamma_H}}}{\rho^n}.$$

Thus setting $M = \sqrt[p]{M_{\Gamma_H}} > 0$, the following result has been established:

Proposition 2 *Let $\{H(t) : 0 \leq |t| < c\}$ be a p-analytic s.p. given by (4). Then there exists $M > 0$ such that*

$$\|H_n\|_p \leq \frac{M}{\rho^n}, \quad 0 < \rho < c, \ \forall n \geq 0.$$

3 Random logistic differential model: approximate solution, statistical properties and example. Conclusions

Let us consider the linear random initial value problem

$$\dot{Y}(t) = a(t)Y(t) + B(t), \qquad Y(0) = (X_0)^{-1}. \tag{9}$$

Following an analogous development as in [7], one can construct a random power series solution s.p. of (9) whose expression is given by

$$Y(t) = \sum_{n\geq 0} Y_n t^n, \ Y_n = \frac{1}{n}\left\{B_{n-1} + \sum_{k=0}^{n-1} a_{n-k-1}Y_k\right\}, n \geq 1, \ Y_0 = (X_0)^{-1}, \tag{10}$$

which is $p = 8$-mean convergent (hence $p = 4$-mean convergent) and $p = 8$-mean differentiable on certain domain T_δ. At this point, we emphasize an important aspect regarding the paper [7]: the construction of solution (10) involves the use of random Cauchy's inequalities whose proof was not provided in [7] but we have already established in Section 2. Notice that, as $X_0(\omega) \neq 0$ for each $\omega \in \Omega$, r.v. Y_0 is well-defined and it also satisfies that $Y_0(\omega) \neq 0$ for each $\omega \in \Omega$. Thus, as a consequence of $p = 8$-mean continuity of s.p. $Y(t)$ in T_δ, the s.p. $X(t) = 1/Y(t)$ is well-defined on this domain. Since $Y(t)$ is $p = 8$-mean differentiable, so it is also m.f. differentiable and by Lemma 1 one gets $\dot{X}(t) = -(Y(t))^{-2}\dot{Y}(t)$. This change of variable transforms nonlinear random differential equation (1) into linear problem (9) which solution s.p. $Y(t)$ is given by (10). However, from a practical standpoint the solution s.p. $X(t) = 1/Y(t)$ is not suitable since it involves through $Y(t)$ an infinite series, so its truncation $Y_N(t)$ is demanded in order to obtain computable approximations not only for the solution s.p., but also for average and variance. This motivation leads us to define the approximation $X_N(t) = 1/Y_N(t)$ where

$$Y_N(t) = \sum_{n\geq 0}^{N} Y_n t^n,\ Y_n = \frac{1}{n}\left\{B_{n-1} + \sum_{j=0}^{n-1} a_{n-j-1} Y_j\right\}, N \geq 1,\ Y_0 = (X_0)^{-1}. \tag{11}$$

Let us justify that this approximation $X_N(t)$ is m.s. convergent to solution s.p. $X(t)$. First, note that $Y(t)$ is $p = 8$-mean differentiable so $p = 8$-mean continuous and, since $Y(0,\omega) = Y_0 \neq 0$ for all $\omega \in \Omega$, by a usual continuity argument in the Banach space L_8, we can assure that both $\left\|(Y(t))^{-1}\right\|_8$ and $\left\|(Y_N(t))^{-1}\right\|_8$ are bounded in a certain neighborhood T_δ about $t = 0$. Then applying inequality (3) and the m.f. convergence of $Y_N(t)$ to $Y(t)$ one gets:

$$\begin{aligned}\|X_N(t) - X(t)\|_2 &= \left\|(Y(t) - Y_N(t))\,(Y_N(t)Y(t))^{-1}\right\|_2 \\ &\leq \|(Y(t) - Y_N(t))\|_4 \left\|(Y_N(t)Y(t))^{-1}\right\|_4 \\ &\leq \|(Y(t) - Y_N(t))\|_4 \left\|(Y_N(t))^{-1}\right\|_8 \left\|(Y(t))^{-1}\right\|_8 \xrightarrow[N\to\infty]{} 0.\end{aligned}$$

Therefore, the following result has been established:

Theorem 3 *Let be the random differential equation* (1) *whose coefficients* $a(t)$ *and* $B(t)$ *defined by* (2) *are assumed to be an analytic deterministic function and a* $p = 8$*-mean analytic stochastic process, respectively. If initial condition* X_0 *is a 8-r.v. such that* $X_0(\omega) \neq 0$ *for each* $\omega \in \Omega$ *and it is independent of* $B(t)$ *for each* t*, then there exists* $\delta > 0$ *such that its m.s. solution is* $X(t) = 1/Y(t)$ *for* $t \in T_\delta = \{t \in D : |t| \leq \delta\}$*, where* $Y(t)$ *is given by* (10).

Notice that by applying theorem 4.3.1 [2, p.88] to $X_N(t) = 1/Y_N(t)$, we can compute approximations of the mean and variance of the solution s.p. $X(t)$.

Example 4 *Let us suppose that* $a(t) = -e^{at}$ *and* $B(t) = Bt$*, being* B *a 8-r.v in (1). Thus* $a_n = -a^n/n!$*,* $\forall n \geq 0$*;* $B(t)$ *is a* $p = 8$*-mean analytic s.p. and* $B_0 = 0$*,* $B_1 = B$*,* $B_n = 0$*,* $\forall n \geq 2$*. In this case does not exist a closed-form solution of (1). Assuming that* B *and* X_0 *are independent r.v.'s with p.d.f.* $f_B(b)$*,* $f_{X_0}(x_0)$*, respectively, one gets*

$$E\left[(X_N(t))^i\right] = \int_{\mathbb{R}^2} \frac{1}{(g_N(a,b,x_0;t))^i} f_B(b) f_{X_0}(x_0)\, db\, dx_0,\ i = 1,2,$$

$$g_N(a,b,x_0;t) = \sum_{n=0}^{N} Y_n t^n,\ Y_n = \begin{cases} Y_0 = (X_0)^{-1},\ Y_1 = -(X_0)^{-1},\ Y_2 = \left(B + (X_0)^{-1}(1-a)\right)/2, \\ Y_n = -\dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1} \frac{a^{n-k-1}}{(n-k-1)!} Y_k,\ n \geq 3. \end{cases}$$

Table 1 shows approximations for both average and standard deviation according to truncation method and Monte Carlo simulations at some selected points of the interval $0 \le t \le 5$*. We have taken* $a = -0.01$*,* $B \approx Exp(\lambda = 0.05)$ *and* $X_0 \approx Be(\alpha = 3; \beta = 2)$*. Computations have been carried out taking as truncation order* $N = 30$ *in the series method, whereas* $m = 200000$ *simulations were needed to get reliable numerical results by Monte Carlo approach.*

t	$\mu_{X_N}(t)\,, N = 30$	$\mu_{X_m}^{MC}(t)\,, m = 200000$	$\sigma_{X_N}(t)\,, N = 30$	$\sigma_{X_m}^{MC}(t)\,, m = 200000$
0.00	0.6	0.599558	0.2	0.20005
1.00	0.286393	0.286663	0.315007	0.316199
2.00	0.175139	0.176816	0.389404	0.397917
3.00	0.133116	0.135886	0.50605	0.524485
4.00	0.111498	0.114902	0.689718	0.725205
5.00	0.0982849	0.102281	0.969874	1.0395

Table 1
Comparison of the average and standard deviation functions in Example 4

In this paper we have obtained an approximate solution of random logistic equation (1)-(2) taking advantage of the p-mean stochastic calculus. We have also computed reliable approximations for its main statistical functions.

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