Beltrán, A.; Felipe Román, MJ.; Melchor, C. (2015). Graphs associated to conjugacy classes of normal subgroups in finite groups. Journal of Algebra. 443:335-348. doi:10.1016/j.jalgebra.2015.06.040.

Graphs associated to conjugacy classes of normal subgroups in finite groups

Antonio Beltrán
Departamento de Matemáticas,
Universidad Jaume I, 12071 Castellón, Spain
e-mail: abeltran@mat.uji.es

María José Felipe
Instituto Universitario de Matemática Pura y Aplicada,
Universidad Politécnica de Valencia, 46022 Valencia, Spain
e-mail: mfelipe@mat.upv.es

Carmen Melchor
Departamento de Educación,
Universidad Jaume I, 12071 Castellón, Spain
e-mail: cmelchor@uji.es

Abstract

Let G be a finite group and let N be a normal subgroup of G. We attach to N two graphs $\Gamma_G(N)$ and $\Gamma^*_G(N)$ associated to the conjugacy classes of G contained in N and to the set of primes dividing the sizes of these classes respectively. We prove that the number of connected components of both graphs is at most 2, determine the diameter of these graphs and characterize the structure of N when these graphs are disconnected.

Keywords. Finite groups, conjugacy classes, normal subgroups, graphs.
Mathematics Subject Classification (2010): 20E45, 20D15.

1 Introduction

Let G be a finite group and let N be a normal subgroup of G, we denote $N \trianglelefteq G$. For each element $x \in N$ the G-conjugacy class is $x^G = \{x^g \mid g \in G\}$. We will denote by $\mathrm{Con}_G(N)$ the set of conjugacy classes in G of elements of N and by $\mathrm{cs}_G(N)$ the set of G-conjugacy classes sizes of N.

Definition 1.1 Let G be a finite group and N a normal subgroup in G, we define the graph $\Gamma_G(N)$ in the following way: the set of vertices are the non-central elements of $\text{Con}_G(N)$, and two vertices x^G and y^G are joined by and edge if and only if $|x^G|$ and $|y^G|$ have a common prime divisor, what is the same, $(|x^G|, |y^G|) \neq 1$. We call this graph the ordinary graph of G-conjugacy classes of N.

If $G = N$ then $\Gamma_G(N) = \Gamma(G)$ where $\Gamma(G)$ is the graph defined in [4] by Bertram, Herzon and Mann.

Notice that $\Gamma(N)$ is not subgraph of $\Gamma_G(N)$ because the set of vertices of $\Gamma(N)$ not necessarily have to be included in the set of vertices of $\Gamma_G(N)$. For instance, if $G = S_3$ and $N = A_3$, $\Gamma(N)$ is empty because N is abelian, whereas it is not true in the case of $\Gamma_G(N)$ and it has a single vertex, the G-conjugacy class $\{(1, 2, 3), (1, 3, 2)\}$.

The first property that we can prove is that $\Gamma_G(N)$ is a subgraph of $\Gamma(N)$. Then, at first we can think that the number of connected components of $\Gamma_G(N)$ is unbounded. We denote this number as $n(\Gamma_G(N))$ and we prove the following theorem.

Definition 1.2 Let G be a finite group and N a normal subgroup in G, we define the "dual" graph of $\Gamma_G(N)$ denoted $\Gamma^*_G(N)$ as it follows: the set of vertex are de elements of the set:

$$\sigma_G(N) = \{q \quad | \; q \text{ is prime that divides } |B| \text{ with } B \in \text{Con}_G(N)\}$$

Two vertex r and q are joined by an edge if there exists a non-central G-conjugacy class $C \in \text{Con}_G(N)$ such that rq divides $|C|$. We call this graph the dual graph of G-conjugacy classes of N.

In case of $N = G$ we have $\Gamma^*_G(N) = \Gamma^*(G)$. Furthermore, its easy to prove that $\Gamma^*_G(N)$ is subgraph of $\Gamma^*(G)$.

Theorem A. *Let G a finite group and N a normal subgroup of G then $n(\Gamma_G(N)) \leq 2$.*

Theorem B. *Let G a finite group and N a normal subgroup of G.*

1. *If $n(\Gamma_G(N)) = 1$ then $d(\Gamma_G(N)) \leq 3$.*
2. *If $n(\Gamma_G(N)) = 2$, each connected component is a complete graph.*

Also for $\Gamma^*_G(N)$ we prove that de number of connected components is at most 2 and the number of connected components of $\Gamma_G(N)$ and $\Gamma^*_G(N)$ is exactly the same.

Theorem C. *If G is a finite group and $N \trianglelefteq G$ then, $n(\Gamma_G^*(N)) \leq 2$ and $n(\Gamma_G^*(N)) = n(\Gamma_G(N))$.*

Theorem D. *Let G a finite group and N a normal subgroup of G.*

1. *If $n(\Gamma_G^*(N)) = 1$ then $d(\Gamma_G^*(N)) \leq 3$.*
2. *If $n(\Gamma_G^*(N)) = 2$, each connected component is a complete graph.*

We give a characterization of normal subgroups whose associated graph has exactly two connected components.

Theorem E. *Let G a finite group and N a normal subgroup of G. If $\Gamma_G(N)$ has two connected components then, either N is quasi-Frobenius with abelian kernel and complement, or $N = P \times A$ where P is a p-group and $A \leqslant \mathbf{Z}(G)$.*

If A is a finite group, $\pi(A)$ denotes the set of primes that divide the order of A.

2 Number of connected components of $\Gamma_G(N)$ and $\Gamma_G^*(N)$

In this section we prove Theorems A and C. We start with the following lemma, which is basic for our development.

Lemma 2.1 *Let G be a finite group and N a normal subgroup of G. Let $B = b^G$ y $C = c^G$ non-central elements in $\mathrm{Con}_G(N)$. If $(|B|, |C|) = 1$. Then*

1. *$\mathbf{C}_G(b)\mathbf{C}_G(c) = G$.*
2. *$BC = CB$ is a non-central element of $\mathrm{Con}_G(N)$ and $|BC|$ divides $|B||C|$.*
3. *Suppose that $d(B, C) \geq 3$ and $|B| < |C|$. Then $|BC| = |C|$ and $CBB^{-1} = C$. Furthermore, $C\langle BB^{-1}\rangle = C$, $\langle BB^{-1}\rangle \subseteq \langle CC^{-1}\rangle$ and $|\langle BB^{-1}\rangle|$ divides $|C|$.*

Proof. It is enough to mimic the proof of Lemma 1 of [9] taking into account that the product of two classes of $\mathrm{Con}_G(N)$ is contained in N again. □

Proof of Theorem A. Suppose that $\Gamma_G(N)$ has at least three connected components and take three non-central classes $B = b^G$, $C = c^G$ and $D = d^G$ in $\mathrm{Con}_G(N)$ each of which belongs to a different connected component. Then, it holds that $(|B|, |C|) = 1$, $(|B|, |D|) = 1$ and $(|C|, |D|) = 1$. We can assume without loss of generality $|B| < |C| < |D|$. Therefore, by applying Lemma 2.1, we get that $|\langle BB^{-1}\rangle|$ divides $|D|$ and $|\langle BB^{-1}\rangle|$ divides $|C|$. Then, $(|C|, |D|) > 1$, which is a contradiction. □

Proof of Theorem C. Suppose that $n(\Gamma_G^*(N)) \geq 3$. We take three primes r, s and l each of which belongs to a different connected component and let A, B and C be elements of $\mathrm{Con}_G(N)$ such that r divides $|A|$, s divides $|B|$ and l divides $|C|$. Without loss of generality we suppose that $|C| < |B| < |A|$. We have $d(A, C) \geq 3$ and $d(A, B) \geq 3$ and by applying Lemma 2.1, we obtain that $|\langle CC^{-1}\rangle|$ divides $|A|$ and also $|\langle CC^{-1}\rangle|$ divides $|B|$, and this leads to a contradiction, because A and B would have a common prime divisor and $d(r, s)$ would be less or equal than 2. This proves that $n(\Gamma_G^*(N)) \leq 2$.

Suppose now that $n(\Gamma_G(N)) = 1$ and $n(\Gamma_G^*(N)) = 2$. Let r and s be primes such that each of them belongs to a distinct connected component of $\Gamma_G^*(N)$. Then there exist $B_r, B_s \in \Gamma_G(N)$ such that r divides $|B_r|$ and s divides $|B_s|$. Let us consider the following path in $\Gamma_G(N)$ that joins B_r and B_s, which exists because $n(\Gamma_G(N)) = 1$:

$$B_r \overset{p_1}{\longleftrightarrow} B_1 \overset{p_2}{\longleftrightarrow} B_2 \overset{p_3}{\longleftrightarrow} \cdots \overset{p_l}{\longleftrightarrow} B_s$$

This leads to a contradiction, because r and s are connected in $\Gamma_G^*(N)$ by the following path:

$$r \overset{B_r}{\longleftrightarrow} p_1 \overset{B_1}{\longleftrightarrow} p_2 \overset{B_2}{\longleftrightarrow} \cdots \overset{B_{l-1}}{\longleftrightarrow} p_s \overset{B_l}{\longleftrightarrow} s$$

So, we have proved that $n(\Gamma_G(N)) = 1$ implies that $n(\Gamma_G^*(N)) = 1$. Now, if $n(\Gamma_G(N)) = 2$ and $n(\Gamma_G^*(N)) = 1$ we can get a contradiction by arguing in a similar way. This shows that $n(\Gamma_G(N)) = n(\Gamma_G^*(N))$. □

3 Diameter of $\Gamma_G(N)$

The following two lemmas, one for the disconnected case and the other for the connected case, summarize important structural properties of the normal subgroup concerning the graph $\Gamma_G(N)$, which will be needed for determining the diameters of the graphs. We start with the disconnected case.

Lemma 3.1 *Let G a finite group and let N be a normal subgroup of G. Suppose that $n(\Gamma_G(N)) = 2$ and let X_1 and X_2 be the connected components of $\Gamma_G(N)$. Let B_0 be a non-central element of $\mathrm{Con}_G(N)$ of maximal size and assume that $B_0 \in X_2$. We define*

$$S = \langle C \mid C \in X_1\rangle \text{ and } T = \langle CC^{-1} \mid C \in X_1\rangle.$$

Then

1. *S is a normal subgroup of G and every element in S, either is central, or its G-conjugacy class is in X_1.*

2. *If C is a G-conjugacy class of N out of S, then $|T|$ divides $|C|$.*

3. *T is normal in G, $T = [S, G]$ and $T \leq \mathbf{Z}(S)$.*

4. *S is abelian.*

5. *$\mathbf{Z}(G) \cap N \subseteq S$ and $\pi(S/(\mathbf{Z}(G) \cap N)) \subseteq \pi(T) \subseteq \pi(B_0)$.*

6. *Let $b^G = B \in X_1$. Then $\mathbf{C}_G(b)/S$ is a q-group for some prime $q \in \pi(B_0)$.*

Proof. 1. The fact that S is normal is elementary. Let $C \in X_2$ and $B \in X_1$. We know that BC is a G-conjugacy class of $\mathrm{Con}_G(N)$ of maximal size between $|B|$ and $|C|$ by Lemma 2.1. Assume that $|BC| = |B|$. By Lemma 2.1 again, it follows that $|\langle CC^{-1}\rangle|$ divides $|B|$ and that $\langle CC^{-1}\rangle \subseteq \langle BB^{-1}\rangle$. On the other hand, $|B_0B| = |B_0|$ again by Lemma 2.1, and also $|\langle BB^{-1}\rangle|$ divides $|B_0|$. From these facts, we deduce that $(|B|, |B_0|) > 1$, which is a contradiction. Thus, $|BC| = |C|$ for all $C \in X_2$ and $B \in X_1$. Now, let A be the union of all G-conjugacy classes of size $|C|$ in S and assume that $A \neq \emptyset$. We have that if $B \in X_1$, then $BA \subseteq A$. Hence, $SA = A$, and consequently, since A is a normal subset, $|S|$ divides $|A|$. This is not possible because $A \subseteq S - \{1\}$. This contradiction shows that $A = \emptyset$ and hence S does not contain any class of size $|C|$. Therefore, since S is normal in G, then S does not contain elements whose classes are in X_2.

2. Let $B \in X_1$. Since C is in X_2, it follows by Lemma 2.1 that CB is a G-conjugacy class. If we suppose that $CB \in X_1$, as $B^{-1} \in X_1$, then $CBB^{-1} \subseteq S$, so in particular $C \subseteq S$, which contradicts (1). Thus, $CB \in X_2$ and its size must be $|C|$ by Lemma 2.1. Again by this lemma, we have $C\langle BB^{-1}\rangle = C$, and consequently, $CT = C$. Therefore, $|T|$ divides $|C|$, as wanted.

3. By definition, $T = [S, G]$ and so, it is a normal subgroup of G. Let us prove that $T \leq \mathbf{Z}(S)$. Indeed, if $B = b^G \in X_1$, then $(|T|, |G : \mathbf{C}_G(b)|) = (|T|, |B|) = 1$, because $|T|$ divides every class size in X_2 by (1). Now, since $|T : \mathbf{C}_T(b)|$ divides $(|T|, |B|) = 1$, we deduce that $T = \mathbf{C}_T(b)$. As the classes in X_1 generate S, we conclude that T is central in S.

4. Since $T = [S, G]$, then $[S/T, G/T] = 1$ and $S/T \subseteq \mathbf{Z}(G/T)$. In particular, S/T is abelian and as a result, S is nilpotent. We can write $S = R \times Z$ where Z is the largest Hall subgroup of S which is contained in $\mathbf{Z}(G)$. Let p be a prime divisor of $|R|$ and let P be a Sylow p-subgroup of R. It follows that $P \trianglelefteq G$ and $T = [S, G] = [R, G] \geqslant [P, G] > 1$. Hence p divides $|T|$ and by applying (1) and (2), $|T|$ divides $|B_0|$. Thus, we have $\pi(R) \subseteq \pi(T) \subseteq \pi(B_0)$. We can show now that $R \leq \mathbf{Z}(S)$. In fact, let $b^G = B \in X_1$. Since $(|B|, |B_0|) = 1$, we obtain in particular, $(|B|, |R|) = 1$. Thus, $|R : \mathbf{C}_R(b)| = 1$ since this index trivially divides $|R|$ and $|B|$ because $R \trianglelefteq G$. This means that $R = \mathbf{C}_R(b)$ for every generating element b of S. So, R is contained in $\mathbf{Z}(S)$ as wanted, and S is abelian.

5. Let $z \in \mathbf{Z}(G) \cap N$ and let $B = b^G \in X_1$. Note that $b^G z = (bz)^G$. Moreover $bz \in N$, because both elements lie in N. As $|(bz)^G| = |Bz| = |B|$,

then $bz \in S$ and so $z \in S$. This proves that $\mathbf{Z}(G) \cap N \subseteq S$. Let us prove now that $\pi(S/(\mathbf{Z}(G) \cap N)) \subseteq \pi(T) \subseteq \pi(B_0)$. First, let $p \in \pi(T)$. We know that B_0 is out of S by (1) and that $|T|$ divides $|B_0|$ by (2), so $p \in \pi(B_0)$. On the other hand, let q be a prime divisor of $|S : \mathbf{Z}(G) \cap N|$. As we have proved in (4), that $S = R \times Z$ and $\pi(R) \subseteq \pi(T)$, we conclude that q divide $|R|$, so $q \in \pi(T)$.

6. By considering the primary decomposition of b, it is clear that we can write $b = b_q b_{q'}$ where b_q and $b_{q'}$ are the q-part and the q'-part of b, and q is a prime such that $b_q \notin \mathbf{Z}(G) \cap N$. Hence, $q \in \pi(B_0)$ by (5). Furthermore, it is elementary that $\mathbf{C}_G(b) \subseteq \mathbf{C}_G(b_q)$, and as a result, $|(b_q)^G|$ divides $|B|$. We claim that any element $xS \in \mathbf{C}_G(b)/S$ is a q-element. For any $x \in \mathbf{C}_G(b)$, write $x = x_q x_{q'}$ (it is possible $x_q = 1$). It is clear that x_q and $x_{q'}$ belong to $\mathbf{C}_G(b)$. We consider $a = b_q x_{q'}$ and observe that $\mathbf{C}_G(a) = \mathbf{C}_G(b_q) \cap \mathbf{C}_G(x_{q'}) \subseteq \mathbf{C}_G(b_q)$, so $|(b_q)^G|$ divides $|a^G|$. Since $(b_q)^G \in X_1$, this forces that $a^G \in X_1$, and we conclude that $x_{q'} \in S$, that is, xS is a q-element, as wanted. This shows that $\mathbf{C}_G(b)/S$ is a q-group. □

Lemma 3.2 *Let G be a finite group and $N \trianglelefteq G$ with $\Gamma_G(N)$ connected. Let B_0 be a G-conjugacy class of $\mathrm{Con}_G(N)$ of maximal size. Let*

$$M = \langle D \mid D \in \mathrm{Con}_G(N) \text{ and } d(B_0, D) = 2\rangle$$

$$K = \langle D^{-1}D \mid D \in \mathrm{Con}_G(N) \text{ and } d(B_0, D) = 2\rangle$$

Then

1. *M and K are normal subgroups of G. Furthermore, $K = [M, G]$, $K \leq \mathbf{Z}(M)$.*

2. *M is abelian.*

3. *$\mathbf{Z}(G) \cap N \subseteq M$ and $\pi(M/(\mathbf{Z}(G) \cap N)) \subseteq \pi(K) \subseteq \pi(B_0)$.*

4. *Let $B = b^G \in \mathrm{Con}_G(N)$ such that $d(B, B_0) \geq 3$. Then $\mathbf{C}_G(b)/M$ is a q-group for some $q \in \pi(B_0)$.*

Proof. 1. We easily see that M and K are normal subgroups of G and $K = [M, G]$. Let us prove that $K \leq \mathbf{Z}(M)$. If $C \in \mathrm{Con}_G(N)$ satisfies that $d(B_0, C) = 2$, we have $(|B_0|, |C|) = 1$ and then, $|B_0| = |B_0 C|$. Moreover, by Lemma 2.1, it follows that $B_0 C C^{-1} = B_0$ and as a result $|K|$ divides $|B_0|$. Now, since $d(B_0, C) = 2$ and $|K|$ divides $|B_0|$, then $(|K|, |C|) = 1$. We have that $|K : \mathbf{C}_K(c)|$ divides $(|K|, |C|) = 1$. Thus, $K = \mathbf{C}_K(c)$ and consequently, $K \leq \mathbf{Z}(M)$.

2. We show first that M is nilpotent. As $K = [M, G]$, then $M/K \leq \mathbf{Z}(G/K)$ and since $K \leq \mathbf{Z}(M)$ by (1), hence M is nilpotent. We can write $M = R \times Z$ where Z is the largest Hall subgroup of M that is contained in $\mathbf{Z}(G)$. Let q be a prime divisor of $|R|$ and let Q be the Sylow q-subgroup of R. Then $Q \trianglelefteq G$

and $K = [M, G] \geqslant [R, G] \geqslant [Q, G]$. We have $[Q, G] \neq 1$, so, q divides $|K|$. We conclude that $\pi(R) \subseteq \pi(K)$. On the other hand, let $C \in \mathrm{Con}_G(N)$ such that $d(B_0, C) = 2$. Then $(|B_0|, |C|) = 1$ and by Lemma 2.1 (2), $|B_0C|$ divides $|B_0||C|$. The maximality of $|B_0|$ and the fact that $d(B_0, C) = 2$ imply that $|B_0C| = |B_0|$, and we obtain $B_0CC^{-1} = B_0$. We deduce that that $|\langle CC^{-1}\rangle|$ divides $|B_0|$. This implies that $\pi(K) \subseteq \pi(B_0)$. Thus, $q \in \pi(B_0)$. Hence, given $B = b^G$ a generating class of M, we know that $d(B, B_0) = 2$. Thus, we have $(|B|, |B_0|) = 1$. Hence $(|Q|, |B|) = 1$. Since $|Q : \mathbf{C}_Q(b)|$ divides $(|Q|, |B|) = 1$, we have $\mathbf{C}_G(b) = Q$ and $Q \leq \mathbf{Z}(M)$. Thus, $R \leq \mathbf{Z}(M)$ and M is abelian.

3. We prove that $\mathbf{Z}(G) \cap N \subseteq M$. Let $z \in \mathbf{Z}(G) \cap N$ and let $C = c^G \in \mathrm{Con}_G(N)$ such that $d(B_0, C) = 2$. Notice that $c^G z = (cz)^G$. Moreover $cz \in N$, because $C \subseteq N$ and $z \in N$. As $|(cz)^G| = |c^G z|$, then $d(B_0, |(cz)^G|) = 2$. Thus, $cz \in M$ and $\mathbf{Z}(G) \cap N \subseteq M$. We only have to show that $\pi(M/(\mathbf{Z}(G) \cap N)) \subseteq \pi(K)$, because we know $\pi(K) \subseteq \pi(B_0)$ by the proof of (2). Let q be a prime divisor of $|M : \mathbf{Z}(G) \cap N|$. Then there is a q-element $y \in M \setminus (\mathbf{Z}(G) \cap N)$. Since $M = R \times Z$ we have that q divides $|R|$ and thus $q \in \pi(K)$ because $\pi(R) \subseteq \pi(K)$.

4. By considering the primary decomposition of b, we can write $b = b_q b_{q'}$ where b_q and $b_{q'}$ are the q-part and the q'-part of b, and q is a prime such that $b_q \notin \mathbf{Z}(G) \cap N$. Hence, $q \in \pi(B_0)$ by (3). Also, $\mathbf{C}_G(b) \subseteq \mathbf{C}_G(b_q)$, implies that $|(b_q)^G|$ divides $|B|$ and then any class which is connected to $(b_q)^G$ must be connected to B. This implies that $d((b_q)^G, B_0) \geq 3$ too. We claim that any element $x \in \mathbf{C}_G(b_q) \setminus M$ satisfies that xM is a q-element in $\mathbf{C}_G(b_q)/M$. Write $x = x_q x_{q'}$ and suppose that $x_{q'} \notin M$. Set $a = x_{q'} b_q$ and notice that $a \notin M$. By definition of M, we have $d(a^G, B_0) \leq 1$ and since $\mathbf{C}_G(a) = \mathbf{C}_G(x_{q'}) \cap \mathbf{C}_G(b_q)$, it follows that $|(b_q)^G|$ divides $|a^G|$. These facts show that $d((b_q)^G, B_0) \leq 2$, a contradiction. Therefore, $x_{q'} \in M$ and xM is a q-element. In conclusion, $\mathbf{C}_G(b_q)/M$ is a q-group. $\square$

The following consequence, which has interest on its own, is the key to bound the diameter of $\Gamma_G(N)$, which we denote by $d(\Gamma_G(N))$.

Theorem 3.3. *Let G a finite group and N a normal subgroup of G and suppose that $\Gamma_G(N)$ is connected. Let B_0 a non-central conjugacy class of $\mathrm{Con}_G(N)$ with maximal size. Then $d(B, B_0) \leq 2$ for every non-central $B \in \mathrm{Con}_G(N)$.*

Proof. Let $B = b^G \in \mathrm{Con}_G(N)$ such that $d(B_0, B) = 3$ and let

$$B_0 \longleftrightarrow B_1 \longleftrightarrow B_2 \longleftrightarrow B$$

be a shortest chain linking B and B_0 of length 3. As $\mathbf{Z}(G) \cap N \subseteq \mathbf{C}_G(c)$ for any class $c^G \in \mathrm{Con}_G(N)$, then the size of any class of $\mathrm{Con}_G(N)$ divides $|G : \mathbf{Z}(G) \cap N|$. Specifically, if M is the subgroup defined in Lemma 3.2, then $|B_2|$ divides

$$|G : \mathbf{Z}(G) \cap N| = |G : \mathbf{C}_G(b)|\,|\mathbf{C}_G(b) : M|\,|M : \mathbf{Z}(G) \cap N|.$$

Also, we know by Lemma 3.2(3) and (4) that $\pi(M/\mathbf{Z}(G) \cap N) \subseteq \pi(B_0)$ and that $\mathbf{C}_G(b)/M$ is a q-group for some $q \in \pi(B_0)$. Thus, $|B_2|$ must divide $|B|$. This is a contradiction, since B_1 and B would be joined by an edge. □

Proof of Theorem B. 1. Suppose that D_1 and D_2 are classes of $\mathrm{Con}_G(N)$ such that $d(D_1, D_2) = 4$. Let B_0 be a class of maximal size in $\mathrm{Con}_G(N)$. By Theorem 3.3 we know that $d(B_0, D_i) \leq 2$ for $i = 1, 2$. We can suppose then that $d(B_0, D_i) = 2$ for $i = 1, 2$. Furthermore, without loss of generality, $|D_1| > |D_2|$. Then, by Lemma 2.1 it is true that $|\langle D_2 D_2^{-1} \rangle|$ divides $|D_1|$. In addition, $B_0 D_2$ is a conjugacy class of $\mathrm{Con}_G(N)$ and $|B_0 D_2| = |B_0|$ by Lemma 2.1(2) and maximality of B_0. It follows that $B_0 D_2 D_2^{-1} = B_0$ and $|\langle D_2 D_2^{-1} \rangle|$ divides $|B_0|$. Thus, B_0 and D_1 are joined by an edge, which is a contradiction.

2. Let $B_1 = b_1^G$ and $B_2 = b_2^G$ in X_1. Notice that $b_1, b_1 \in S$ where S is the subgroup defined in Lemma 3.1. Then $|b_2^G|$ divides

$$|G : \mathbf{Z}(G) \cap N| = |G : \mathbf{C}_G(b_1)||\mathbf{C}_G(b_1) : S||S : \mathbf{Z}(G) \cap N|$$

and we know by Lemma 3.1 that the primes dividing $|\mathbf{C}_G(b_1) : S|$ and $|S : \mathbf{Z}(G) \cap N|$ are in $\pi(B_0)$. So, we have that $|b_1^G|$ divides $|b_2^G|$. By arguing symmetrically we get that $|b_2^G|$ divides $|b_1^G|$, so we conclude that all classes in X_1 have the same size. Therefore, X_1 is a complete graph. Now, we show that X_2 is also a complete graph. Let us consider again S and T defined in Lemma 3.1 and observe that every $C \in X_2$ is out of S and that $|T|$ divides $|C|$ by Lemma 3.1 (1) and (2). This proves that X_2 is a complete graph. □

4 Diameter of $\Gamma_G^*(N)$

Proof of Theorem D. 1. Suppose that there exist two primes r and s in $\Gamma_G^*(N)$ such that $d(r, s) = 4$ and we will get a contradiction. This means that the primes r and s are connected by a path of length 4, say

$$r \overset{B_1}{\longleftrightarrow} p_1 \overset{B_2}{\longleftrightarrow} p_2 \overset{B_3}{\longleftrightarrow} p_3 \overset{B_4}{\longleftrightarrow} s$$

where $B_i \in \mathrm{Con}_G(N)$ for $i = 1, \ldots, 4$ and $p_i \in \Gamma_G^*(N)$ for $i = 1, 2, 3$. By Theorem 3.3 we know that $d(B_i, B_0) \leqslant 2$ for $i = 1, \ldots, 4$ where B_0 is a non-central G-conjugacy class of maximal size. Notice that $d(B_1, B_4) = 3$. We distinguish two possible cases:

Case 1. $d(B_0, B_1) = 2 = d(B_0, B_4)$. By symmetry, we can assume for instance that $|B_1| > |B_4|$. Since $d(B_1, B_4) = 3$, by Lemma 2.1 we have that $|\langle B_4 B_4^{-1} \rangle|$ divides $|B_1|$. Moreover, $B_0 B_4$ is an element of $\mathrm{Con}_G(N)$ such that $|B_0 B_4| = |B_0|$ and by Lemma 2.1, $|\langle B_4 B_4^{-1} \rangle|$ divides $|B_0|$. Therefore, $d(B_0, B_1) = 1$, because their cardinalities have a prime common divisor. This is a contradiction.

Case 2. Either $d(B_0, B_1) = 2$ and $d(B_0, B_4) = 1$ or $d(B_0, B_1) = 1$ and $d(B_0, B_4) = 2$. Without loss we assume for instance the latter case. We consider the subgroup M defined in Lemma 3.2 and $B_4 = b^G$. Since $d(B_0, B_4) = 2$, then $b \in M$ by definition of M. Moreover, $|B_1|$ divides

$$|G : \mathbf{Z}(G) \cap N| = |G : \mathbf{C}_G(b)| \cdot |\mathbf{C}_G(b) : M| \cdot |M : \mathbf{Z}(G) \cap N|.$$

Now, notice that $r \notin \pi(B_0)$, otherwise $d(r, s) \leq 3$, a contradiction, and certainly $r \notin \pi(B_4)$. Also, $\pi(M/\mathbf{Z}(G) \cap N) \subseteq \pi(B_0)$, so we have that r (which divides $|B_1|$) must divide $|\mathbf{C}_G(b) : M|$. Therefore, there exists an r-element $y \in \mathbf{C}_G(b) \setminus M$. On the other hand, $b \in M$, and by Lemma 3.2 (3) we can assume that b is an r'-element, by replacing b by its r'-part. Therefore

$$\mathbf{C}_G(yb) = \mathbf{C}_G(y) \cap \mathbf{C}_G(b) \subseteq \mathbf{C}_G(b).$$

Consequently, $|B_4|$ divides $|(yb)^G|$. Furthermore, since $yb \notin M$, by definition of M we have that $d((yb)^G, B_0) \leq 1$. This provides a contradiction because B_4 and B_0B_4 and B_0 would be joined by an edge ???? (ESTE FINAL NO ESTA CLARO).

2. Let X_1 and X_2 be the connected components of $\Gamma_G(N)$ where X_2 is the component that contains de G-conjugacy class with the biggest size class. Let us prove first that X_1^*, X_2^* are the connected components of $\Gamma_G^*(N)$, where $X_i^* = \{p \in \pi(B) \,|\, B \in X_i\}$, and secondly, that X_1^* y X_2^* are complete graphs.

Let X be a connected component of $\Gamma_G(N)$ and let $r, s \in X^*$. Then there exist B_r, B_s such that r divides $|B_r|$ and s divides $|B_s|$. Let us consider de path in $\Gamma_G(N)$ that joins B_r and B_s:

$$B_r \overset{p_1}{\longleftrightarrow} B_1 \overset{p_2}{\longleftrightarrow} B_2 \overset{p_3}{\longleftrightarrow} \cdots \overset{p_l}{\longleftrightarrow} B_s$$

We have that r and s are connected in $\Gamma_G^*(N)$ in the following way:

$$r \overset{B_r}{\longleftrightarrow} p_1 \overset{B_1}{\longleftrightarrow} p_2 \overset{B_2}{\longleftrightarrow} \cdots \overset{B_{l-1}}{\longleftrightarrow} p_s \overset{B_l}{\longleftrightarrow} s$$

So X^* is contained in a connected component $Y \in \Gamma_G^*(N)$. Now, we take $q \in Y$ which is connected by an edge to some $r \in X^*$. Then there exists $B \in \mathrm{Con}_G(N)$ such that $qr||B|$. It follows that $B \in X$ and $q \in X^*$. Thus, $X^* = Y$ and X^* is a connected component of $\Gamma_G^*(N)$ as wanted.

We show now that X_1^* is a complete graph. If $B, B' \in X_1$, we have proved in Theorem B that $|B| = |B'|$. Thus $X_1^* = \pi(B)$ which trivially implies that it is a complete graph. Let us show that X_2^* is a complete graph too. Suppose that B_0 is a conjugacy class with maximal size, which is in X_2 and let $B_1 = b_1^G \in X_1$. Thus, the subgroup S defined in Lemma 3.1 is abelian, and $S \subseteq \mathbf{C}_G(b_1)$. Now, if $p \in X_2^*$, then there exists $D \in X_2$ such that p divides $|D|$. Notice that $|D|$ divides

$$|G : \mathbf{Z}(G) \cap N| = |B_1||\mathbf{C}_G(b_1) : S||S : \mathbf{Z}(G) \cap N|,$$

and by Lemma 2.1 we know that $|\mathbf{C}_G(b_1) : S|$ is a q-number with $q \in \pi(B_0)$ and $\pi(S/(\mathbf{Z}(G) \cap N)) \subseteq \pi(B_0)$. It follows that all primes in $\pi(D)$ are in $\pi(B_0)$. Therefore, all primes in X_2^* are in $\pi(B_0)$ and it trivially is a complete graph. □ it

5 Structure of N in the disconnected case

Proof of Theorem E. Let S the subgroup defined in Lemma 3.1.

Step 1: If $S \leq \mathbf{Z}(N)$ then $N = P \times A$ with $A \leq \mathbf{Z}(G) \cap N$ and P a p-group.

We can take x a p-element and y a q-element of N, for some primes p and q, such that $x^G \in X_1$ and $y^G \in X_2$. If $p = q$ for every election of x and y, then $N = P \times A$ with $A \leq \mathbf{Z}(G) \cap N$. We consider that $p \neq q$. Since $x \in S \leq \mathbf{Z}(N)$, we obtain that $N/S = \mathbf{C}_N(x)/S$ and this group has prime power order for Lemma 3.1(6). As a consequence, N is nilpotent and $[x, y] = 1$. Thus $\mathbf{C}_G(xy) = \mathbf{C}_G(x) \cap \mathbf{C}_G(y)$ and $|(xy)^G|$ divides $|x^G|$ and $|y^G|$, a contradiction.

Notice that we can assume that $S\mathbf{Z}(N) < N$, because if $S\mathbf{Z}(N) = N$ then N is abelian and $S \leq \mathbf{Z}(N)$. For the following, we assume $\mathbf{Z}(N) < S\mathbf{Z}(N) < N$ and we will prove that N is quasi-Forbenius with abelian kernel and complement. We divide the proof into several steps.

We denote by $\pi = \{p \text{ prime} \mid p \text{ divides } |B| \text{ with } B \in X_1\}$.

Step 2: N has a normal π-complement and abelian Hall π-subgroups.

Let us prove that N is p-nilpotent and has abelian Sylow p-subgroups for $p \in \pi$. Let $a \in N \setminus S$, then $a^G \in X_2$ by Lemma 3.1(1) and $|a^G|$ is a π'-number. If $P \in \mathrm{Syl}_p(N)$, then there exists $g \in N$ such that $P^g \subseteq \mathbf{C}_N(a)$ and $a \in \mathbf{C}_N(P^g) = \mathbf{C}_N(P)^g$. Thus, we have

$$N = S \cup \bigcup_{g \in N} \mathbf{C}_N(P)^g$$

and it follows that

$$|N| \leq (|S| - 1) + |N : \mathbf{N}_N(\mathbf{C}_N(P))|(|\mathbf{C}_N(P)| - 1) + 1$$

hence

$$1 \leq \frac{|S|}{|N|} + \frac{|\mathbf{C}_N(P)|}{|\mathbf{N}_N(\mathbf{C}_N(P))|} - \frac{1}{|\mathbf{N}_N(\mathbf{C}_N(P))|}$$

If $\mathbf{C}_N(P) <\mathbf{N}_N(\mathbf{C}_N(P)$, as $S < N$, we have

$$1 \leq \frac{1}{2} + \frac{1}{2} - \frac{1}{|\mathbf{N}_N(\mathbf{C}_N(P))|}$$

and this is a contradiction. Therefore $\mathbf{C}_N(P) =\mathbf{N}_N(\mathbf{C}_N(P))$ and

$$P \leq \mathbf{N}_N(P) \leq \mathbf{N}_N(\mathbf{C}_N(P)) \leq \mathbf{C}_N(P)$$

Then $\mathbf{C}_N(P) =\mathbf{N}_N(P)$ and P is abelian. By Burside's Theorem (see for instance 17.9 of [8]) we obtain that N is p-nilpotent for all $p \in \pi$ and N has normal π-complement. In particular, N is π-separable. Let H be a Hall π-subgroup of N. Reasoning in the same way that in case of P for H, we obtain that $\mathbf{C}_N(H) =\mathbf{N}_N(H)$ and H is abelian too.

Let $K/\mathbf{Z}(N)$ be the normal π-complement of $N/\mathbf{Z}(N)$. By Lemma 3.1(5) we have that $S\mathbf{Z}(N)/\mathbf{Z}(N)$ is a normal $\pi'-$subgroup of $N/\mathbf{Z}(N)$, so $S \leq K$.

Step 3: $K =\mathbf{C}_N(x)$ for all $x \in S\setminus \mathbf{Z}(G) \cap N$ and $S \leq \mathbf{Z}(K)$.

Let $x \in S\setminus \mathbf{Z}(G) \cap N$, then $x^G \in X_1$ and by Lemma 3.1(1) and $\mathbf{C}_G(x)/S$ is a π'-group by Lemma 3.1(6. Since $|x^N|$ is a π-number, we obtain that $\mathbf{C}_N(x)/\mathbf{Z}(N)$ is a π'-Hall of $N/\mathbf{Z}(N)$. Thus $K = \mathbf{C}_N(x)$ for all $x \in S\setminus \mathbf{Z}(G)\cap N$ and, in particular, $S \leq \mathbf{Z}(K)$.

Step 4: $K = S$.

Let H be an abelian Hall π-subgroup of N. We have seen in the proof of Step 2 that

$$N = S \cup \bigcup_{g\in N} \mathbf{C}_N(H)^g$$

and this implies that

$$N = \bigcup_{g\in N} S\mathbf{C}_N(H)^g$$

Then $N =\mathbf{C}_N(H)S$ and $HS \trianglelefteq N$.

Let $a \in K \setminus S$, then $a^G \in X_2$ by Lemma 3.1(1) and, as $|a^G|$ is a π'-number, we have $a \in \mathbf{C}_K(H^g) = \mathbf{C}_K(H)^g$ with $g \in N$. Moreover $S \leq \mathbf{Z}(K)$ by Step 3, so we have the following equalities

$$\mathbf{C}_K(H^g) = \mathbf{C}_K(H^gS) = \mathbf{C}_K(HS) = \mathbf{C}_K(H)$$

Thus $a \in \mathbf{C}_K(H)$ for $a \in K \setminus S$ ande $K = \langle K \setminus S\rangle \subseteq \mathbf{C}_N(H)$. As H is abelian and $N = HK$, we have $H \leq \mathbf{Z}(N) \leq K$. This implies that $N = HK = K$ and $S \leq \mathbf{Z}(N)$ by Step 3, a contradiction.

Step 5: N is quasi-Frobenius with abelian kernel and complement.

Let $\overline{N} = N/\mathbf{Z}(N)$ and let $\overline{K} = K/\mathbf{Z}(N)$. If $\overline{K} = \mathbf{C}_{\overline{N}}(\overline{x})$ for all $\overline{x} \in \overline{K} \setminus \{1\}$ then N is quasi-Frobenius with abelian kernel K and abelian complement H. Suppose that $\overline{K} < \mathbf{C}_{\overline{N}}(\overline{x})$ for some $\overline{x} \in \overline{K} \setminus \{1\}$. We can suppose that $o(\overline{x})$ is an r-number for a prime $r \in \pi'$. Now, let $\overline{y} \in \mathrm{C}_{\overline{N}}(\overline{x}) \setminus \overline{K}$ such that $o(\overline{y})$ is a q-number for some $q \in \pi$. We can suppose without loss of generality that $o(y)$ is a q-number. We have $[y, x] \in \mathbf{Z}(N)$ because $\overline{y} \in \mathbf{C}_{\overline{N}}(\overline{x})$ and, since $(o(x), o(y)) = 1$, it follows that $[x, y] = 1$. Then $y \in \mathbf{C}_N(x) = K$ and $\overline{y} \in \overline{K}$ and this is a contradiction. $\square$

We remark that the converse of the previous theorem is false.

Example 5.1

We know that the special linear group $H = SL(2, 5)$ acts Frobenius on $K = C_{11} \times C_{11}$. Then, it is clear that any subgroup of H acts Frobenius on K. We consider in particular P be a Sylow 5-subgroup of H. We know that $\mathbf{N}_H(P)$ is a cyclic group of order 20 and also acts Frobenius on K. We define $N = KP$, that is trivially a normal subgroup in $G = K\mathbf{N}_H(P)$.

We have N is Frobenius with abelian kernel and abelian complement. Moreover,

$$N = 1 \cup (K - 1) \cup (\bigcup_{k \in K} P^k \setminus \{1\})$$

and K is decomposed into the trivial class and classes with cardinal 5, while the elements of $\bigcup_{k \in K}(P^k \setminus \{1\})$ are decomposed in N-conjugacy classes of cardinal 121. The set of conjugacy class sizes of N is $\{1, 5, 121\}$.

Now we consider the G-conjugacy classes of N. As $G = K\mathbf{N}_H(P)$ is also a Frobenius group with kernel K and complement $\mathbf{N}_H(P)$, it follows that K is decomposed exactly in the trivial class and classes with cardinal $|\mathbf{N}_H(P)| = 20$. That is, the N-conjugacy classes contained in K have been grouped 4 by 4 to form G-classes. And, on the other hand, the N-conjugacy classes contained in $\bigcup_{k \in K} P^k \setminus \{1\}$, which have size 121, are grouped in pairs, and become two G-conjugacy classes of size 121×2. Then the set of G-conjugacy class sizes of N is $\{1, 4 \times 5, 2 \times 121\}$ and the graph $\Gamma_G(N)$ has only one component.

On the other hand, the following provides an example of the other case about the structure of N in Theorem E.

Example 5.2

Let G be the group of the library of the small groups of GAP (see [7]) with number Id(324,8) and with presentation

$$\langle x, y, z | x^3 = y^4 = z^9 = 1, [x, y] = 1, z^y = z^{-1}, z^2 = xzxzx = x^{-1}zx^{-1}zx^{-1} \rangle$$

Using GAP, it is easy to check that G has an abelian normal subgroup $N \cong C_3 \times C_3$ and the set of G-class sizes of N is $\{1, 2, 3\}$.

Acknowledgements

The research of the first and second authors is supported by the Valencian Government, Proyecto PROMETEO/2011/30 and by the Spanish Government, Proyecto MTM2010-19938-C03-02.

References

[1] Z. Akhlagi, A. Beltrán, M.J. Felipe, M. Khatami, Structure of normal subgroups with three G-class sizes. Monatsh. Math. **167** (2012), 1-12.

[2] A. Beltrán and M.J.Felipe, On the diameter of p-regular conjugacy class graph of finite groups. Comm. Algebra **30**(12) (2003), 5861-5873.

[3] A. Beltrán and M.J.Felipe, On the diameter of p-regular conjugacy class graph of finite groups II. Comm Algebra **31**(9) (2003), 4393-4403.

[4] E.A. Bertram, M.Herzog and A.Mann, On a graph related to conjugacy classes of groups. Bull. London Math. Soc. **22** (6) (1990), 569-575.

[5] D. Chillag, M. Herzog and A. Mann, On the diameter of a graph related to conjugacy classes of groups. Bull. London Math. Soc. **25** (1993), 255-262.

[6] S. Dolfi. Arithmetical conditions on the length of the conjugacy classes of a finite group. J. Algebra **174** (1995), 753-771.

[7] The GAP Group, GAP - Groups, Algorithms and Programming, Vers. 4.4.12 (2008). http://www.gap-system.org

[8] B. Huppert, Character Theory of Finite groups, Vol. 25, De Gruyter Expositions in Mathemathics, Berlin, New York, 1998.

[9] Z. Lu and J. Zhang. On the diameter of a grah related to p-regular conjugacy classes of finite groups. J.Algebra **231** (2000), 705-712.