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This paper must be cited as:

Hernández, AE.; Lattanzi, MB.; Thome, N. (2015). Weighted binary relations involving the Drazin inverse. Applied Mathematics and Computation. 253:215-223. doi:10.1016/j.amc.2014.12.102.

The final publication is available at

https://dx.doi.org/10.1016/j.amc.2014.12.102

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Additional Information

Weighted binary relations involving the Drazin inverse

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Abstract

The Drazin inverse of a matrix has been used in the literature to define a pre-order on the set of square complex matrices. In this paper we analyze new binary relations defined on the set of rectangular complex matrices and some relationships to the W-support idempotent. We introduce the class of weighted Drazin equal projectors and analyze the pre-orders on this class. Moreover, adjacent matrices are studied under the considered relations. Finally, some observations on weighted partial orders are given.

AMS Classification: 15A09, 06A06

Keywords: Drazin inverse; Drazin pre-order; sharp partial order.

1 Introduction

The symbol $\mathbb{C}^{m\times n}$ denotes the set of $m \times n$ complex matrices. For a given $A \in \mathbb{C}^{m\times n}$, the notation A^* stands for the conjugate transpose of A. As usual, I_n and O_n denote the $n \times n$ identity and zero matrices, respectively. The subscripts will be deleted when no confusion is caused. Given two matrices $A \in \mathbb{C}^{t\times t}$ and $B \in \mathbb{C}^{(m-t)\times(n-t)}$ we will denote

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by $A \oplus B$ the $m \times n$ matrix where A is in the N-W corner, B is in the S-E corner and the other two blocks correspond to rectangular zero matrices of adequate sizes.

Let $A \in \mathbb{C}^{n\times n}$. The index of A, denoted by $ind(A)$, is the smallest nonnegative integer k such that A^k and A^{k+1} have the same rank. The only matrix $X \in \mathbb{C}^{n\times n}$ satisfying $XAX = X$, $AX = XA$, and $A^{k+1}X = A^k$, with $k = ind(A)$, is called the Drazin inverse of A [3]. It always exists and is denoted by $X = A^D$. It is clear that $A^{r+1}A^D = A^r$, for all integer $r \geq ind(A)$ and $A^{r+1}A^D = A^D A^{r+1}$, for all integer $r \geq 0$. We recall that if A has index at most 1, the Drazin inverse of A is called the group inverse of A and is denoted by $A^{\#}$. The Drazin inverse of a matrix can be computed via the core-nilpotent decomposition. Indeed, for a nonzero given matrix $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$, there exist nonsingular matrices $P \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{a\times a}$ such that $A = P(C \oplus N)P^{-1}$ where $N \in \mathbb{C}^{(n-a)\times(n-a)}$ is absent ($k = 0$), $N = O_{n-a}$ ($k = 1$) or $N \in \mathbb{C}^{(n-a)\times(n-a)}$ is a nonzero nilpotent matrix with nilpotence index equals $k > 1$. By abuse of language, we will say that N is nilpotent for each one of these three possibilities. In this case, $A^D = P(C^{-1} \oplus O)P^{-1}$. We can write A in the core-nilpotent decomposition as $A = A_1 + A_2$ where $A_1 = P(C \oplus O)P^{-1}$ and $A_2 = P(O \oplus N)P^{-1}$ and A_1, A_2 are the unique matrices in these conditions (see [3]).

The following result will be used in the sequel.

Theorem 1.1 [4, Lemma 1.1] *If $W \in \mathbb{C}^{n\times m}$ is a nonzero matrix, $A \in \mathbb{C}^{m\times n}$, $k_1 = ind(AW)$, and $k_2 = ind(WA)$ then there exist four nonsingular matrices $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{n\times n}$, $A_1, W_1 \in \mathbb{C}^{t\times t}$, and two matrices $A_2 \in \mathbb{C}^{(m-t)\times(n-t)}$ and $W_2 \in \mathbb{C}^{(n-t)\times(m-t)}$ such that A_2W_2 and W_2A_2 are nilpotent of indices k_1 and k_2, respectively, where*

$$A = P(A_1 \oplus A_2)Q^{-1} \qquad \text{and} \qquad W = Q(W_1 \oplus W_2)P^{-1}. \tag{1}$$

We observe that Theorem 1.1 can also be established for $k = \max\{k_1, k_2\}$. Moreover, the matrix W can be seen as a weight needed to transform the rectangular matrix A into two square ones, namely, AW and WA.

On the other hand, we recall that a binary relation is a pre-order if it is reflexive and transitive and, a partial order, if it is also antisymmetric. Partial orders have been widely studied (see, for example, [8] and the references therein). Interesting applications of partial orders and pre-orders were investigated, for instance, in [1, 2]. In those papers, properties on the distribution of quadratics forms in normal variables are dealt in the

Cochran's Theorem environment. The utility of a pre-order was studied, for instance, in [1]. The authors generalized a property on the independence of two quadratic forms that involves the Löwner partial order (i.e., $A \leq_L B$ if $B - A$ is nonnegative definite where A and B are $n \times n$ symmetric matrices), replacing $\leq_L$ by the column space pre-order, which is simpler to be verified. For more applications, we refer the reader to [12].

The following binary relations are well known.

Let $A, B \in \mathbb{C}^{n\times n}$ be matrices with index at most 1. It is said that A is below B under the sharp partial order, and is denoted by $A \leq^{\#} B$, if $A^{\#}A = A^{\#}B$ and $AA^{\#} = BA^{\#}$.

Suppose that $A, B \in \mathbb{C}^{n\times n}$ are matrices of arbitrary index, and they are written in the respective core-nilpotent decompositions as $A = A_1 + A_2$ and $B = B_1 + B_2$. It is said that A is related to B under the Drazin pre-order, and is denoted by $A \preceq^d B$, if $A_1 \leq^{\#} B_1$. Observe that $A \preceq^d B$ is equivalent to $A^D A = A^D B$ and $AA^D = BA^D$.

The main aim of this paper is to investigate some new binary relations defined on the set of rectangular matrices $\mathbb{C}^{m\times n}$.

This paper is organized as follows. Section 2 introduces and characterizes three relations considered on rectangular matrices: $\preceq^{d,W,r}$, $\preceq^{d,W,\ell}$, and $\preceq^{d,W}$. Moreover, the concept of W-support idempotent is recalled and some links to these relations are given. For each rectangular matrix A and a fixed weight W, it is possible to define two projectors involving the Drazin inverse of AW and WA. Section 3 analyzes the class of all matrices for which those two projectors coincide. Additionally, the relation $\preceq^{d,W}$ is studied on this class. In Section 4, we characterize the adjacent matrices under the three considered relations. Finally, in Section 5, some considerations on weighted partial orders are given.

2 Weighted binary relations and the Drazin inverse

The fact that Drazin inverse exists for all square matrix allowed to define the Drazin pre-order on square matrices [8]. However, it is not possible to define this pre-order for rectangular matrices in the same way. In order to do that, we are going to consider a weight matrix and define some binary relations on the set of rectangular matrices by means of the Drazin inverse of certain square matrices.

Definition 2.1 *Let $W \in \mathbb{C}^{n\times m}$ a nonzero matrix and $A, B \in \mathbb{C}^{m\times n}$. It is said that*

(a) $A \preceq^{d,W,r} B$ *if* $AW \preceq^{d} BW$,

(b) $A \preceq^{d,W,\ell} B$ *if* $WA \preceq^{d} WB$,

(c) $A \preceq^{d,W} B$ *if* $A \preceq^{d,W,r} B$ *and* $A \preceq^{d,W,\ell} B$,

where $\preceq^{d}$ *is adequately considered on* $\mathbb{C}^{m\times m}$ *or* $\mathbb{C}^{n\times n}$.

Using that $\preceq^{d}$ is a pre-order we obtain the following result.

Lemma 2.1 *The binary relations* $\preceq^{d,W,r}$, $\preceq^{d,W,\ell}$, *and* $\preceq^{d,W}$ *define a pre-order on* $\mathbb{C}^{m\times n}$.

Next we characterize the relation $\preceq^{d,W,r}$ in terms of block decompositions of the involved matrices. Before doing that, notice that $\preceq^{d,W,r}$, $\preceq^{d,W,\ell}$, and $\preceq^{d,W}$ pre-orders do not preserve equivalences (that is, in general that $A \preceq^{\diamond} B$ implies $\Gamma_1 A \Gamma_2 \preceq^{\diamond} \Gamma_1 B \Gamma_2$ is not valid for all nonsingular Γ_1, Γ_2 and for each $\preceq^{\diamond} \in \{\preceq^{d,W,r}, \preceq^{d,W,\ell}, \preceq^{d,W}\}$). In order to illustrate this situation we give the following example.

Example 2.1 *The matrices*

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \Gamma_1 = I_2, \quad \text{and} \quad \Gamma_2 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

satisfy $AW = BW = I_2$; *however* $\Gamma_1 A \Gamma_2 \not\preceq^{d,W,r} \Gamma_1 B \Gamma_2$ *because* $(\Gamma_1 A \Gamma_2 W)(\Gamma_1 A \Gamma_2 W)^D = I_2$ *and*

$$(\Gamma_1 B \Gamma_2 W)(\Gamma_1 A \Gamma_2 W)^D = \begin{bmatrix} \frac{2}{3} & 0 \\ \frac{1}{3} & 1 \end{bmatrix}.$$

This fact tells us that we can not remove matrices P and Q when using Theorem 1.1 for characterizing the pre-orders of Lemma 2.1.

Theorem 2.1 *Let* $W \in \mathbb{C}^{n\times m}$ *be a nonzero matrix and* $A, B \in \mathbb{C}^{m\times n}$. *The following conditions are equivalent:*

(a) $A \preceq^{d,W,r} B$.

(b) $(AW)^D(AW) = (AW)^D(BW) = (BW)(AW)^D$.

(c) $(AW)^{k_1}(BW) = (BW)(AW)^{k_1} = (AW)^{k_1+1}$, *where* $k_1 = ind(AW)$.

(d) There exist nonsingular matrices $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{n\times n}$, $A_1, W_1^A \in \mathbb{C}^{t_A\times t_A}$, *and* $(B_2)_1, (W_2^A)_1 \in \mathbb{C}^{t\times t}$, *and there exist matrices* $A_2 \in \mathbb{C}^{(m-t_A)\times(n-t_A)}$, $B_3 \in \mathbb{C}^{t_A\times(n-t_A)}$, $(B_2)_2 \in \mathbb{C}^{(m-t_A-t)\times(n-t_A-t)}$, *and* $(W_2^A)_2 \in \mathbb{C}^{(n-t_A-t)\times(m-t_A-t)}$ *satisfying*

$$A = P(A_1 \oplus A_2)Q^{-1}, \qquad B = P\begin{bmatrix} A_1 & B_3 \\ O & (B_2)_1 \oplus (B_2)_2 \end{bmatrix} Q^{-1},$$

$$W = Q(W_1^A \oplus ((W_2^A)_1 \oplus (W_2^A)_2))P^{-1},$$

where $A_2((W_2^A)_1 \oplus (W_2^A)_2)$, $((W_2^A)_1 \oplus (W_2^A)_2)A_2$, $(B_2)_2(W_2^A)_2$, *and* $(W_2^A)_2(B_2)_2$ *are nilpotent matrices and* $B_3((W_2^A)_1 \oplus (W_2^A)_2) = O$.

Proof. Items (a), (b) and (c) are equivalent taking into account the equality of the projectors $(AW)(AW)^D$ and $(AW)^D(AW)$ and using the definition of Drazin inverse.

(b) $\Longrightarrow$ (d) Suppose that $A, B \in \mathbb{C}^{m\times n}$ satisfy (b). By Theorem 1.1, there are nonsingular matrices $P_A \in \mathbb{C}^{m\times m}$, $Q_A \in \mathbb{C}^{n\times n}$, $A_1, W_1^A \in \mathbb{C}^{t_A\times t_A}$, and matrices $A_2' \in \mathbb{C}^{(m-t_A)\times(n-t_A)}$, $W_2^A \in \mathbb{C}^{(n-t_A)\times(m-t_A)}$ satisfying

$$A = P_A(A_1 \oplus A_2')Q_A^{-1}, \qquad W = Q_A(W_1^A \oplus W_2^A)P_A^{-1},$$

where $A_2'W_2^A$ and $W_2^AA_2'$ are k_1-nilpotent and k_2-nilpotent, respectively, where $k_1 = ind(AW)$ and $k_2 = ind(WA)$.

Now, we consider the following partition of B

$$B = P_A\begin{bmatrix} B_1 & B_3' \\ B_4 & B_2 \end{bmatrix} Q_A^{-1}$$

according to the size of the blocks of A. Then,

$$BW = P_A\begin{bmatrix} B_1W_1^A & B_3'W_2^A \\ B_4W_1^A & B_2W_2^A \end{bmatrix} P_A^{-1}$$

and using the equality $(AW)^D = P_A((A_1W_1^A)^{-1} \oplus O)P_A^{-1}$ we get

$$(BW)(AW)^D = P_A\begin{bmatrix} B_1A_1^{-1} & O \\ B_4A_1^{-1} & O \end{bmatrix} P_A^{-1},$$

$$(AW)^D(BW) = P_A \begin{bmatrix} (A_1W_1^A)^{-1}B_1W_1^A & (A_1W_1^A)^{-1}B_3'W_2^A \\ O & O \end{bmatrix} P_A^{-1},$$

and $(AW)^D(AW) = P_A(I_{t_A} \oplus O)P_A^{-1}$. From (b), we obtain $B_1 = A_1$, $B_4 = O$, and $B_3'W_2^A = O$, that is

$$B = P_A \begin{bmatrix} A_1 & B_3' \\ O & B_2 \end{bmatrix} Q_A^{-1}.$$

Suppose that $W_2^A \neq O$. Applying Theorem 1.1 to matrices $B_2 \in \mathbb{C}^{(m-t_A)\times(n-t_A)}$ and $W_2^A \in \mathbb{C}^{(n-t_A)\times(m-t_A)}$, there exist nonsingular matrices $R \in \mathbb{C}^{(m-t_A)\times(m-t_A)}$, $S \in \mathbb{C}^{(n-t_A)\times(n-t_A)}$, $(B_2)_1, (W_2^A)_1 \in \mathbb{C}^{t\times t}$, and matrices $(B_2)_2 \in \mathbb{C}^{(m-t_A-t)\times(n-t_A-t)}$, $(W_2^A)_2 \in \mathbb{C}^{(n-t_A-t)\times(m-t_A-t)}$, satisfying

$$B_2 = R((B_2)_1 \oplus (B_2)_2)S^{-1} \qquad \text{and} \qquad W_2^A = S((W_2^A)_1 \oplus (W_2^A)_2)R^{-1}, \tag{2}$$

where $(B_2)_2(W_2^A)_2$ and $(W_2^A)_2(B_2)_2$ are nilpotent.

Consider the matrices $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ defined by

$$P = P_A(I_{t_A} \oplus R) \qquad \text{and} \qquad Q = Q_A(I_{t_A} \oplus S). \tag{3}$$

Replacing (2) and (3) in the expressions of A, B, and W and setting $A_2 = R^{-1}A_2'S$ and $B_3 = B_3'S$ we arrive at

$$A = P\,(A_1 \oplus A_2)\,Q^{-1}, \qquad B = P \begin{bmatrix} A_1 & B_3 \\ O & R^{-1}B_2S \end{bmatrix} Q^{-1},$$

and

$$W = Q\left(W_1^A \oplus S^{-1}W_2^AR\right)P^{-1}.$$

Observe that the case $W_2^A = O$ can also be written as in (d) with $(W_2^A)_1 = O$ and $(W_2^A)_2 = O$.

(d) $\Longrightarrow$ (b) It is straightforward. ∎

A similar characterization is established for the inequality $\preceq^{d,W,\ell}$.

Theorem 2.2 *Let $W \in \mathbb{C}^{n\times m}$ be a nonzero matrix and $A, B \in \mathbb{C}^{m\times n}$. The following conditions are equivalent:*

(a) $A \preceq^{d,W,\ell} B$.

(b) $(WA)^D(WA) = (WA)^D(WB) = (WB)(WA)^D$.

(c) $(WA)^{k_2}(WB) = (WB)(WA)^{k_2} = (WA)^{k_2+1}$, *where* $k_2 = ind(WA)$.

(d) There exist nonsingular matrices $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{n\times n}$, $A_1, W_1^A \in \mathbb{C}^{t_A\times t_A}$, *and* $(B_2)_1, (W_2^A)_1 \in \mathbb{C}^{t\times t}$, *and there exist matrices* $A_2 \in \mathbb{C}^{(m-t_A)\times(n-t_A)}$, $B_4 \in \mathbb{C}^{(m-t_A)\times t_A}$, $(B_2)_2 \in \mathbb{C}^{(m-t_A-t)\times(n-t_A-t)}$, *and* $(W_2^A)_2 \in \mathbb{C}^{(n-t_A-t)\times(m-t_A-t)}$ *satisfying*

$$A = P(A_1 \oplus A_2)Q^{-1}, \qquad B = P\begin{bmatrix} A_1 & O \\ B_4 & (B_2)_1 \oplus (B_2)_2 \end{bmatrix} Q^{-1},$$

$$W = Q(W_1^A \oplus ((W_2^A)_1 \oplus (W_2^A)_2))P^{-1},$$

where $((W_2^A)_1 \oplus (W_2^A)_2)A_2$, $A_2((W_2^A)_1 \oplus (W_2^A)_2)$, $(W_2^A)_2(B_2)_2$, *and* $(B_2)_2(W_2^A)_2$ *are nilpotent matrices and* $((W_2^A)_1 \oplus (W_2^A)_2)B_4 = O$.

Proof. It follows from the fact that $A \preceq^{d,W,\ell} B$ is equivalent to $A^* \preceq^{d,W^*,r} B^*$ and after application Theorem 2.1. ∎

While in Theorem 2.2 we use the same matrix names as in Theorem 2.1, we remark that they are not necessarily the same ones. That is, for example, matrix P in Theorem 2.1 may be different from matrix P in Theorem 2.2.

Theorem 2.3 *Let* $W \in \mathbb{C}^{n\times m}$ *be a nonzero matrix and* $A, B \in \mathbb{C}^{m\times n}$. *The following conditions are equivalent:*

(a) $A \preceq^{d,W} B$.

(b) $(AW)^D(AW) = (AW)^D(BW) = (BW)(AW)^D$ *and* $(WA)^D(WA) = (WA)^D(WB) = (WB)(WA)^D$.

(c) $(AW)^{k_1}(BW) = (BW)(AW)^{k_1} = (AW)^{k_1+1}$ *and* $(WA)^{k_2}(WB) = (WB)(WA)^{k_2} = (WA)^{k_2+1}$, *where* $k_1 = ind(AW)$ *and* $k_2 = ind(WA)$.

(d) *There exist nonsingular matrices* $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{n\times n}$, $A_1, W_1^A \in \mathbb{C}^{t_A\times t_A}$, *and* $(B_2)_1, (W_2^A)_1 \in \mathbb{C}^{t\times t}$, *and there exist matrices* $A_2 \in \mathbb{C}^{(m-t_A)\times(n-t_A)}$, $(B_2)_2 \in \mathbb{C}^{(m-t_A-t)\times(n-t_A-t)}$, *and* $(W_2^A)_2 \in \mathbb{C}^{(n-t_A-t)\times(m-t_A-t)}$ *satisfying*

$$A = P(A_1 \oplus A_2)Q^{-1}, \qquad B = P\left(A_1 \oplus ((B_2)_1 \oplus (B_2)_2)\right)Q^{-1},$$

$$W = Q(W_1^A \oplus ((W_2^A)_1 \oplus (W_2^A)_2))P^{-1},$$

where $((W_2^A)_1 \oplus (W_2^A)_2)A_2$, $A_2((W_2^A)_1 \oplus (W_2^A)_2)$, $(W_2^A)_2(B_2)_2$, *and* $(B_2)_2(W_2^A)_2$ *are nilpotent.*

Proof. Suppose that $A, B \in \mathbb{C}^{m\times n}$ satisfy $A \preceq^{d,W} B$. This is, $A \preceq^{d,W,r} B$ and $A \preceq^{d,W,\ell} B$. By Theorem 2.1,

$$A = P(A_1 \oplus A_2)Q^{-1}, \qquad B = P\begin{bmatrix} A_1 & B_3 \\ O & (B_2)_1 \oplus (B_2)_2 \end{bmatrix}Q^{-1},$$

$$W = Q(W_1^A \oplus ((W_2^A)_1 \oplus (W_2^A)_2))P^{-1},$$

where all the blocks satisfy the conditions found in item (d) of Theorem 2.1. Using that $WA \preceq^d WB$ and making some computations we get $B_3 = O$. This shows (a) $\Longrightarrow$ (d). The remaining implications follow directly from Theorem 2.1 and Theorem 2.2. ■

For a rectangular matrix A and a weight W of adequate sizes, Castro-González and Vélez-Cerrada considered in [4] the W-support idempotent $A^{\sigma,W} = A(WA)^D = (AW)^D A$. The authors established conditions under which the projectors $A^{\sigma,W}W$ and $B^{\sigma,W}W$ coincide. Similarly, for the projectors $WA^{\sigma,W}$ and $WB^{\sigma,W}$. In addition, they characterized matrices B such that $A^{\sigma,W} = B^{\sigma,W}$. We reconcile results in Theorems 2.1, 2.2 and 2.3 above with Theorem 2.1, Theorem 2.4, and Corollary 2.7 in [4]. In order to do that, we first observe that if $A \preceq^{d,W,r} B$ then, from Theorem 2.1 above, we can write

$$P^{-1}BQ = \begin{bmatrix} A_1 & X & Y \\ O & (B_2)_1 & O \\ O & O & (B_2)_2 \end{bmatrix}$$

where B_3 has been partitioned as $\begin{bmatrix} X & Y \end{bmatrix}$ according to the blocks of $(B_2)_1 \oplus (B_2)_2$. Clearly, the matrix

$$\widetilde{B} = \begin{bmatrix} A_1 & X \\ O & (B_2)_1 \end{bmatrix}$$

is nonsingular and setting $\widetilde{Y} = \begin{bmatrix} Y^* & O \end{bmatrix}^*$ we get

$$B = P \begin{bmatrix} \widetilde{B} & \widetilde{Y} \\ O & (B_2)_2 \end{bmatrix} Q^{-1}. \tag{4}$$

Since $B_3((W_2^A)_1 \oplus (W_2^A)_2) = O$, some algebraic manipulations yields $X = O$ and $\widetilde{Y}(W_2^A)_2 = O$. So, if $A \preceq^{d,W,r} B$ then B can be written as in (4) where $\widetilde{B} \in \mathbb{C}^{(t_A+t)\times(t_A+t)}$ is nonsingular, $\widetilde{Y}(W_2^A)_2 = O$, and $(B_2)_2(W_2^A)_2$ is nilpotent. Nevertheless, we remark that the matrix B_1 in [4, Theorem 2.1 (ii)] and the matrix A_1 in [4, Lemma 1.1] have the same size. This shows that conditions in our Theorem 2.1 does not imply those conditions in [4, Theorem 2.1]. That is, Theorem 2.1 above is essentially different from [4, Theorem 2.1]. The same occurs with Theorem 2.2 and [4, Theorem 2.4] and also with Theorem 2.3 and [4, Corollary 2.7]. However, it follows that

$$A^{\sigma,W}W = B^{\sigma,W}W \qquad \text{implies} \qquad A \preceq^{d,W,r} B \text{ and } B \preceq^{d,W,r} A.$$

We can get similar results for left and both sides relations as well. The converse of this result is also true as we show in the following theorem.

Theorem 2.4 *Let $W \in \mathbb{C}^{n\times m}$ be a nonzero matrix and $A, B \in \mathbb{C}^{m\times n}$.*

(a) If $A \preceq^{d,W,r} B$ and $B \preceq^{d,W,r} A$ then $A^{\sigma,W}W = B^{\sigma,W}W$.

(b) If $A \preceq^{d,W,\ell} B$ and $B \preceq^{d,W,\ell} A$ then $WA^{\sigma,W} = WB^{\sigma,W}$.

(c) If $A \preceq^{d,W} B$ and $B \preceq^{d,W} A$ then $A^{\sigma,W} = B^{\sigma,W}$.

Proof. It is enough to prove only item (a) because the second one can be obtained in a similar way and the third one is immediate from [4]. In fact, from $A \preceq^{d,W,r} B$ and Theorem 2.1 we have that

$$A = P(A_1 \oplus A_2)Q^{-1}, \qquad B = P \begin{bmatrix} A_1 & B_3 \\ O & (B_2)_1 \oplus (B_2)_2 \end{bmatrix} Q^{-1},$$

$$W = Q(W_1^A \oplus ((W_2^A)_1 \oplus (W_2^A)_2))P^{-1},$$

where $A_1 W_1^A$ and $(B_2)_1(W_1^A)_1$ are nonsingular, $(B_2)_2(W_2^A)_2$, $(W_2^A)_2(B_2)_2$, $A_2((W_2^A)_1 \oplus (W_2^A)_2)$ and $((W_2^A)_1 \oplus (W_2^A)_2)A_2$ are nilpotent matrices and $B_3((W_2^A)_1 \oplus (W_2^A)_2) = O$. It then follows that $BW = P(A_1 W_1^A \oplus ((B_2)_1(W_1^A)_1 \oplus (B_2)_2(W_2^A)_2))P^{-1}$. Thus, $(BW)^D = P((A_1 W_1^A)^{-1} \oplus (((B_2)_1(W_1^A)_1)^{-1} \oplus O))P^{-1}$ leads to $BW(BW)^D = P(I_{t_A} \oplus (I_t \oplus O))P^{-1}$.

If we now partition

$$A_2 = \begin{bmatrix} (A_2)_1 & (A_2)_3 \\ (A_2)_4 & (A_2)_2 \end{bmatrix},$$

we can get

$$A_2((W_2^A)_1 \oplus (W_2^A)_2) = \begin{bmatrix} (A_2)_1(W_2^A)_1 & (A_2)_3(W_2^A)_2 \\ (A_2)_4(W_2^A)_1 & (A_2)_2(W_2^A)_2 \end{bmatrix}.$$

Using that $B \preceq^{d,W,r} A$, we have that $(BW)^D BW = (BW)^D AW = AW(BW)^D$ and replacing by the expressions of BW, $(BW)^D$, and AW it can be easily obtained that $(A_2)_1(W_2^A)_1 = (B_2)_1(W_1^A)_1$, $(A_2)_3(W_2^A)_2 = O$, and $(A_2)_4 = O$. Hence, the nilpotent matrix $A_2((W_2^A)_1 \oplus (W_2^A)_2) = (B_2)_1(W_1^A)_1 \oplus (A_2)_2(W_2^A)_2$. Since $(B_2)_1(W_1^A)_1$ is nonsingular, $(B_2)_1$ must be absent in the decomposition of matrix B. So, by [4, Theorem 2.1], $A^{\sigma,W}W = B^{\sigma,W}W$. ■

From Theorem 2.1 (d), it follows directly that $A \preceq^{d,W,r} B$ implies $A^{\sigma,W} \preceq^{d,W,r} B^{\sigma,W}$. Analogously, for left and both sides relations similar implications can be stated.

We close this section emphasizing that the relations $\preceq^{d,W,r}$, $\preceq^{d,W,\ell}$, and $\preceq^{d,W}$ are pairwise different. It is enough to show that $A \preceq^{d,W,r} B$ does not imply $A \preceq^{d,W,\ell} B$. In fact, matrices A, B, and W given in Example 2.1 satisfy $A \preceq^{d,W,r} B$ and $A \not\preceq^{d,W,\ell} B$.

3 Equal Weighted Drazin Projectors

We recall that a square matrix A is said to be *EP* if $AA^\dagger = A^\dagger A$, where $A^\dagger$ denotes the Moore-Penrose inverse of A (that is, $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(AA^\dagger)^* = AA^\dagger$, and $(A^\dagger A)^* = A^\dagger A$ hold). These and similar matrices have been widely studied in different environments [5, 6, 9, 10].

Let $W \in \mathbb{C}^{n\times m}$. We now observe that, if $A \in \mathbb{C}^{m\times n}$, the matrices $(AW)^D AW$ and $(WA)^D WA$ are projectors of size $m \times m$ and $n \times n$, respectively. The following definition considers the case where both weighted Drazin projectors are equal and is inspired in the definition of EP matrix.

Definition 3.1 *Let $W \in \mathbb{C}^{n\times n}$ be a nonzero matrix. A matrix $A \in \mathbb{C}^{n\times n}$ is called EDP_W if satisfies $(AW)^D AW = (WA)^D WA$.*

The class of all EDP_W matrices will be denoted by $\mathcal{EDP}_W$. Notice that if $A \in \mathcal{EDP}_W$ then $PAP^{-1} \in \mathcal{EDP}_{PWP^{-1}}$ for all nonsingular $P \in \mathbb{C}^{n\times n}$ because $(PAP^{-1})^D = PA^D P^{-1}$. Our next aim is to characterize EDP_W matrices.

Theorem 3.1 *Let $A, W \in \mathbb{C}^{n\times n}$ with $W \neq O$. The following conditions are equivalent:*

(a) $A \in \mathcal{EDP}_W$.

(b) There exist nonsingular matrices $P \in \mathbb{C}^{n\times n}$, $A_1, W_1 \in \mathbb{C}^{t\times t}$, and there exist matrices $A_2, W_2 \in \mathbb{C}^{(n-t)\times(n-t)}$ such that

$$A = P(A_1 \oplus A_2)P^{-1} \qquad \text{and} \qquad W = P(W_1 \oplus W_2)P^{-1},$$

where A_2W_2 and W_2A_2 are nilpotent.

Proof. Assume that $A \in \mathbb{C}^{n\times n}$ is EDP_W. Since $W \neq O$, we can write

$$A = P(A_1' \oplus A_2')Q^{-1} \qquad \text{and} \qquad W = Q(W_1' \oplus W_2')P^{-1}, \tag{5}$$

where the block matrices have the properties indicated in Theorem 1.1. It is easy to see that

$$(AW)^D AW = P(I_t \oplus O)P^{-1}, \qquad \text{and} \qquad (WA)^D WA = Q(I_t \oplus O)Q^{-1}.$$

Equating and partitioning

$$P^{-1}Q = \begin{bmatrix} M & N \\ R & S \end{bmatrix}$$

according to the blocks of A we get $N = O$ and $R = O$, which implies that $Q = P(M \oplus S)$. The result follows by replacing Q in the expression (5) of A and W and setting $A_1 = A_1'M^{-1}$, $A_2 = A_2'S^{-1}$, $W_1 = MW_1'$ and $W_2 = SW_2'$. The converse is trivial. ■

We now study the pre-order $\preceq^{d,W}$ on the class of matrices with equal weighted Drazin projectors.

Theorem 3.2 *Let $W \in \mathbb{C}^{n\times n}$ be a nonzero matrix and $A \in \mathcal{EDP}_W$. The following conditions are equivalent:*

(a) There exists $B \in \mathcal{EDP}_W$ such that $A \preceq^{d,W} B$.

(b) There exist nonsingular matrices $V \in \mathbb{C}^{n\times n}$, $A_1, W_1, \in \mathbb{C}^{t\times t}$, and $(B_2)_1, (W_2)_1 \in \mathbb{C}^{r\times r}$, and matrices $A_2 \in \mathbb{C}^{(n-t)\times(n-t)}$, $(B_2)_2, (W_2)_2 \in \mathbb{C}^{(n-t-r)\times(n-t-r)}$ such that

$$A = V(A_1 \oplus A_2)V^{-1}, \quad B = V(A_1 \oplus ((B_2)_1 \oplus (B_2)_2))V^{-1} \tag{6}$$

and

$$W = V(W_1 \oplus ((W_2)_1 \oplus (W_2)_2))V^{-1}, \tag{7}$$

where $(B_2)_2(W_2)_2$, $(W_2)_2(B_2)_2$, $A_2((W_2)_1 \oplus (W_2)_2)$, and $((W_2)_1 \oplus (W_2)_2)A_2$ are nilpotent and $(B_2)_2 \in \mathcal{EDP}_{(W_2)_2}$.

Proof. Since $A \in \mathcal{EDP}_W$, applying Theorem 3.1 we can write

$$A = P(A_1 \oplus A_2')P^{-1} \qquad \text{and} \qquad W = P(W_1 \oplus W_2')P^{-1} \tag{8}$$

where the block matrices have the properties indicated there. Partition

$$B = P \begin{bmatrix} B_1 & B_3 \\ B_4 & B_2 \end{bmatrix} P^{-1}$$

with the blocks of appropriate sizes accordingly to A. The equalities $(AW)^D AW = BW(AW)^D$ and $(WA)^D WA = (WA)^D WB$ are equivalent to $B_1 = A_1$, $B_4 = O$ and $B_3 = O$. So,

$$B = P(A_1 \oplus B_2)P^{-1}. \tag{9}$$

Now, $B \in \mathcal{EDP}_W$ if and only if the equality $(BW)^D BW = (WB)^D WB$ holds, from we have $B_2 \in \mathcal{EDP}_{W_2'}$. Again, Theorem 3.1 applied to the matrix B_2 and the weight W_2' asserts that $B_2 = P_1((B_2)_1 \oplus (B_2)_2)P_1^{-1}$ and $W_2' = P_1((W_2)_1 \oplus (W_2)_2)P_1^{-1}$, where $(B_2)_2(W_2)_2$ and $(W_2)_2(B_2)_2$ are nilpotent matrices.

Setting $V = P(I_t \oplus P_1)$, $A_2 = P_1^{-1}A_2'P_1$ and replacing in expressions (8) and (9) we obtain matrices A and B of (6), and matrix W of (7). Moreover, $(B_2)_2 \in \mathcal{EDP}_{(W_2)_2}$ and $A_2((W_2)_1 \oplus (W_2)_2)$ and $((W_2)_1 \oplus (W_2)_2)A_2$ are nilpotent.Hence, (a) $\Longrightarrow$ (b) holds. The converse is trivial. ■

Of course that, right and left relations can also be analyzed on the class of matrices with equal weighted Drazin projectors obtaining similar results.

4 Adjacent matrices under the $\preceq^{d,W,r}$, $\preceq^{d,W,\ell}$, and $\preceq^{d,W}$ relations

For two given matrices $A, B \in \mathbb{C}^{m\times n}$, it is said that A and B are adjacents if $rank(B-A) = 1$ (see, for example, [7, 11]). In what follows, we investigate the expressions for two matrices to be adjacent under $\preceq^{d,W,r}$, $\preceq^{d,W,\ell}$, and $\preceq^{d,W}$ relations.

Theorem 4.1 *Let $W \in \mathbb{C}^{n\times m}$ be a nonzero matrix and $A, B \in \mathbb{C}^{m\times n}$ such that $A \preceq^{d,W,r} B$. The following conditions are equivalent:*

(a) A and B are adjacent.

(b) There exist nonsingular matrices $P \in \mathbb{C}^{m\times m}$, and $Q \in \mathbb{C}^{n\times n}$, and nonzero vectors $u \in \mathbb{C}^{m\times 1}$ and $v \in \mathbb{C}^{(n-t_A)\times 1}$ such that

$$B = A + P \begin{bmatrix} O & uv^* \end{bmatrix} Q^{-1}.$$

Proof. Let $A, B \in \mathbb{C}^{m\times n}$ with $A \preceq^{d,W,r} B$. From Theorem 2.1,

$$A = P(A_1 \oplus A_2)Q^{-1} \qquad \text{and} \qquad B = P \begin{bmatrix} A_1 & B_3 \\ O & (B_2)_1 \oplus (B_2)_2 \end{bmatrix} Q^{-1}.$$

It can be shown that $rank(B - A) = 1$ if and only if

$$rank\left(\begin{bmatrix} B_3 \\ ((B_2)_1 \oplus (B_2)_2) - A_2 \end{bmatrix}\right) = 1$$

which is equivalent to

$$\begin{bmatrix} B_3 \\ (B_2)_1 \oplus (B_2)_2 \end{bmatrix} = \begin{bmatrix} O \\ A_2 \end{bmatrix} + uv^*,$$

for some nonzero vectors $u \in \mathbb{C}^{m\times 1}$ and $v \in \mathbb{C}^{(n-t_A)\times 1}$. Replacing in the expression above we have $B = A + P\begin{bmatrix} O & uv^* \end{bmatrix} Q^{-1}$. Observe that $A \neq B$ in item (a) is equivalent to $u \neq 0$ and $v \neq 0$ in item (b). ■

Analogously, we can give the following similar results.

Theorem 4.2 *Let $W \in \mathbb{C}^{n\times m}$ be a nonzero matrix and $A, B \in \mathbb{C}^{m\times n}$ such that $A \preceq^{d,W,\ell} B$. The following conditions are equivalent:*

(a) A and B are adjacent.

(b) There exist nonsingular matrices $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{n\times n}$, and nonzero vectors $u \in \mathbb{C}^{(m-t_A)\times 1}$ and $v \in \mathbb{C}^{n\times 1}$ such that

$$B = A + P\begin{bmatrix} O \\ uv^* \end{bmatrix} Q^{-1}.$$

Theorem 4.3 *Let $W \in \mathbb{C}^{n\times m}$ be a nonzero matrix and $A, B \in \mathbb{C}^{m\times n}$ such that $A \preceq^{d,W} B$. The following conditions are equivalent:*

(a) A and B are adjacent.

(b) There exist nonsingular matrices $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{n\times n}$, and nonzero vectors $u \in \mathbb{C}^{(m-t_A)\times 1}$ and $v \in \mathbb{C}^{(n-t_A)\times 1}$ such that

$$B = A + P(O \oplus uv^*)Q^{-1}.$$

Remark 4.1 If A and B are adjacent matrices then

(i) either $A \preceq^{d,W,r} B$ and $B \preceq^{d,W,r} A$ hold, or AW and BW are adjacent matrices,

(ii) either $A \preceq^{d,W,\ell} B$ and $B \preceq^{d,W,\ell} A$ hold, or WA and WB are adjacent matrices.

Indeed, in case (i), $rank(BW - AW) \leq rank(B - A) = 1$ if A and B are adjacent. That is, either $AW = BW$ holds or AW and BW are adjacent matrices. Similarly for the case (ii).

5 Further considerations on weighted partial orders

For a nonzero matrix $W \in \mathbb{C}^{n\times m}$, we now consider the set

$$\mathcal{M}_{W,r} = \{A \in \mathbb{C}^{m\times n} : ind(AW) \le 1\}.$$

Observe that $\mathcal{M}_{W,r} \neq \emptyset$ since $rank((W^*W)^2) = rank((W^*W)(W^*W)^*) = rank(W^*W)$ yields $ind(W^*W) \le 1$ and then $W^* \in \mathcal{M}_{W,r}$.

Definition 5.1 *Let $A, B \in \mathcal{M}_{W,r}$. It is said that $A \preceq^{\#,W,r} B$ if $AW \le^{\#} BW$.*

We remark that the relation $\preceq^{\#,W,r}$ is a pre-order and coincides with the restriction of $\preceq^{d,W,r}$ on $\mathcal{M}_{W,r}$. Moreover, we stand out that the representation Theorem 2.1 is also true for the relation in Definition 5.1. In addition, taking into account that $A \in \mathcal{M}_{W,r}$, it then holds that either $A_2W_2 = O$ or A_2W_2 is absent. Furthermore, since $k_1 \in \{0,1\}$ and $|k_1 - k_2| \le 1$, it results $k_2 \in \{0,1,2\}$ by [13, Theorem 11.1.2]. Then, W_2A_2 is absent if $k_2 = 0$, $W_2A_2 = O$ if $k_2 = 1$ or $(W_2A_2)^2 = O \neq W_2A_2$ if $k_2 = 2$.

Since $\le^{\#}$ is a partial order on index at most one matrices in $\mathbb{C}^{m\times m}$, we establish the following result.

Theorem 5.1 *The relation $\preceq^{\#,W,r}$ is a partial order on $\mathcal{M}_{W,r}$ provided that W has full row rank.*

Next we consider the set $\mathcal{P}_{W,r} = \{\mathcal{Z} \subseteq \mathbb{C}^{m\times n} : \preceq^{d,W,r}$ is a partial order on $\mathcal{Z}\}$ ordered by set inclusion.

Theorem 5.2 *If $W \in \mathbb{C}^{n\times m}$ has full row rank then $\mathcal{M}_{W,r}$ is a maximal element of $\mathcal{P}_{W,r}$.*

Proof. We first observe that $\mathcal{M}_{W,r} \in \mathcal{P}_{W,r}$. Assume that there exists a subset $\mathcal{Z} \in \mathcal{P}_{W,r}$ such that $\mathcal{M}_{W,r} \subseteq \mathcal{Z}$. If we suppose that $A \in \mathcal{Z} - M_{W,r}$ then $ind(AW) > 1$ and Theorem 1.1 assures that $A = P(A_1 \oplus A_2)Q^{-1}$ and $W = Q(W_1 \oplus W_2)P^{-1}$ as indicated in (1). Since $ind(A_2W_2) > 1$, we get $A_2W_2 \neq O$ [3]. On the other hand, set $B = P(A_1 \oplus O)Q^{-1}$. It can easily be seen that $B \in \mathcal{M}_{W,r} \subseteq \mathcal{Z}$. By Theorem 2.1, $A \preceq^{d,W,r} B$ holds. Now, by definition, it is easy to that $B \preceq^{d,W,r} A$. Since $A, B \in \mathcal{Z}$ and $\preceq^{d,W,r}$ is antisymmetric on $\mathcal{Z}$, we get $A = B$. Hence, $A_2 = O$, which is a contradiction. ■

Similarly, if $A, B \in \mathcal{M}_{W,\ell} = \{A \in \mathbb{C}^{m\times n} : ind(WA) \leq 1\}$, we define $A \preceq^{\#,W,\ell} B$ if $WA \leq^{\#} WB$. It is obtained that $\preceq^{\#,W,\ell}$ is a partial order on $\mathcal{M}_{W,\ell}$ provided that W has full column rank. It is also valid that $\mathcal{M}_{W,\ell}$ is a maximal element among all the subsets $\mathcal{Z}$ of $\mathbb{C}^{m\times n}$ satisfying that $\preceq^{d,W,\ell}$ is a partial order on $\mathcal{Z}$.

Finally, defining $A \preceq^{\#,W} B$ if $A \preceq^{\#,W,r} B$ and $A \preceq^{\#,W,\ell} B$ for $A, B \in \mathcal{M}_{W,r} \cap \mathcal{M}_{W,\ell}$, it can also be established that $\preceq^{\#,W}$ is a partial order provided that W has full rank.

6 Acknowledgements

This paper was partially supported by Universidad Nacional de La Pampa, Facultad de Ingeniería of Argentina (grant Resol. N^o 049/11) and the third author was partially supported by Ministerio de Economía y Competitividad of Spain (grant DGI MTM2010-18228).

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