Inverse eigenvalue problem for normal J-hamiltonian matrices

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Abstract

A complex square matrix A is called J-hamiltonian if AJ is hermitian where J is a normal real matrix such that $J^2 = -I_n$. In this paper we solve the problem of finding J-hamiltonian normal solutions for the inverse eigenvalue problem.

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1. Introduction

Inverse eigenvalue problems arise as important tools in several research subjects, including structural design, parameter identification and modeling [3, 5, 11], etc. The main goal of the inverse eigenvalue problem is to construct a matrix A with a determined structure and a specified spectrum. In the literature, this kind of problems has been studied under certain constraints on A. For instance, the case when A is hermitian reflexive or anti-reflexive with respect to a tripotent hermitian matrix was analyzed in [7]. Subsequently, that problem was generalized to matrices that are hermitian reflexive with respect to a normal $\{k+1\}$-potent matrix [4]. By using hamiltonian matrices, in [1] Bai solved the inverse eigenvalue problem for hermitian and generalized skew-hamiltonian matrices.

It is remarkable that hamiltonian matrices play an important role in several engineering areas such as optimal quadratic linear control [8, 10], H_∞ optimization [12] and the solution of Riccati algebraic equations [6], among others.

The symbols M^* and $M^\dagger$ will denote the conjugate transpose and the Moore-Penrose inverse of a matrix M, respectively. As is standard, I_n will stand for the $n \times n$ identity

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matrix. We remind the reader that for a given complex rectangular matrix $M \in \mathbb{C}^{m\times n}$, its Moore-Penrose inverse is the unique matrix $M^\dagger \in \mathbb{C}^{n\times m}$ that satisfies $MM^\dagger M = M$, $M^\dagger MM^\dagger = M^\dagger$, $(MM^\dagger)^* = MM^\dagger$ and $(M^\dagger M)^* = M^\dagger M$. This matrix always exists [2]. We also need the following notation for both specified orthogonal projectors: $W_M^{(l)} = I_n - M^\dagger M$ and $W_M^{(r)} = I_m - MM^\dagger$.

It is well known that a matrix $A \in \mathbb{C}^{2k\times 2k}$ is called hamiltonian if it satisfies $(AJ)^* = AJ$ for

$$J = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix}.$$

We extend this concept by considering the following matrices.

Definition 1. *Let $J \in \mathbb{R}^{n\times n}$ be a normal matrix such that $J^2 = -I_n$. A matrix $A \in \mathbb{C}^{n\times n}$ is called J-hamiltonian if $(AJ)^* = AJ$.*

From now on, we will consider a fixed normal matrix $J \in \mathbb{R}^{n\times n}$ such that $J^2 = -I_n$. It is clear that $n = 2k$ for some positive integer k. For a given matrix $X \in \mathbb{C}^{n\times m}$ and a given diagonal matrix $D \in \mathbb{C}^{m\times m}$, we are looking for solutions of the matrix equation

$$AX = XD \tag{1}$$

where the unknown $A \in \mathbb{C}^{n\times n}$ must be normal and J-hamiltonian.

2. Inverse eigenvalue problem

2.1. General expression for matrices A

Let $J \in \mathbb{R}^{n\times n}$ be a normal matrix satisfying $J^2 = -I_n$. It is easy to see that J is skew-hermitian and its spectrum is included in $\{-i, i\}$ where both eigenvalues i and $-i$ have the same multiplicity, $k = n/2$. Then, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$J = U \begin{bmatrix} iI_k & 0 \\ 0 & -iI_k \end{bmatrix} U^*. \tag{2}$$

Using block matrices, we can analyze the structure of matrices A as follows. We partition

$$U^*AU = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{3}$$

according to the partition of J. From (2) and (3), equality $(AJ)^* = AJ$ yields

$$U \begin{bmatrix} -iA_{11}^* & -iA_{21}^* \\ iA_{12}^* & iA_{22}^* \end{bmatrix} U^* = U \begin{bmatrix} iA_{11} & -iA_{12} \\ iA_{21} & -iA_{22} \end{bmatrix} U^*$$

from where we deduce

$$A_{11}^* = -A_{11}, \qquad A_{22}^* = -A_{22}, \qquad A_{21}^* = A_{12}. \tag{4}$$

Since A must be normal, using expressions (4) we get that

$$AA^* = U \left[\begin{array}{cc} -A_{11}^2 + A_{12}A_{12}^* & A_{11}A_{12} - A_{12}A_{22} \\ -A_{12}^*A_{11} + A_{22}A_{12}^* & A_{12}^*A_{12} - A_{22}^2 \end{array} \right] U^*$$

and

$$A^*A = U \left[\begin{array}{cc} -A_{11}^2 + A_{12}A_{12}^* & -A_{11}A_{12} + A_{12}A_{22} \\ A_{12}^*A_{11} - A_{22}A_{12}^* & A_{12}^*A_{12} - A_{22}^2 \end{array} \right] U^*$$

imply $A_{11}A_{12} = A_{12}A_{22}$. We have obtained the following result.

Theorem 1. *Let $J \in \mathbb{R}^{n\times n}$ be partitioned as in (2). Then $A \in \mathbb{C}^{n\times n}$ is a normal J-hamiltonian matrix if and only if*

$$A = U \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{array} \right] U^* \tag{5}$$

where $A_{11}^* = -A_{11}$, $A_{22}^* = -A_{22}$, and $A_{11}A_{12} = A_{12}A_{22}$.

2.2. Existence and explicit solution

In order to solve the inverse eigenvalue problem we need the next result.

Lemma 1. *Let $M, N \in \mathbb{C}^{n\times m}$. Then $YM = N$ has a skew-hermitian solution Y if and only if*

$$NW_M^{(l)} = 0 \quad \textit{and} \quad M^*N \quad \textit{is skew-hermitian.}$$

In this case, the general solution is given by

$$Y = NM^\dagger - (NM^\dagger)^*W_M^{(r)} + W_M^{(r)}ZW_M^{(r)} \tag{6}$$

where $Z \in \mathbb{C}^{n\times n}$ is skew-hermitian.

Proof. By Theorem 1, [2, pp. 52], the equation $YM = N$ has a solution if and only if $N = NM^\dagger M$. Let Y be a skew-hermitian solution of $YM = N$. It is easy to see that M^*N is skew-hermitian. Now, if M^*N is skew-hermitian then N^*M and $Y_0 = NM^\dagger - (NM^\dagger)^* + (NM^\dagger)^*MM^\dagger$ are skew-hermitian as well since $NM^\dagger - (NM^\dagger)^*$ and $(NM^\dagger)^*MM^\dagger = (M^\dagger)^*(N^*M)M^\dagger$ are skew-hermitian. Moreover, it can be easily shown that Y_0 is a solution of $YM = N$.

The general skew-hermitian solution can be obtained adding to Y_0 the general skew-hermitian solution of the homogeneous equation $YM = 0$. Hence, by Corollary 1 of Lemma 2.3.1 of [9] we deduce that in fact the solution is (6). ■

Now we consider the following partition of X

$$X = U \left[\begin{array}{c} X_1 \\ X_2 \end{array} \right] \tag{7}$$

where $X_1, X_2 \in \mathbb{C}^{k\times m}$.

Substituting (5) and (7) in $AX = XD$ we get

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} D.$$

This matrix equation can be equivalently written as

$$\left\{ \begin{array}{rcl} A_{11}\,X_1 + A_{12}\,X_2 & = & X_1 D \\ A_{12}^*\,X_1 + A_{22}\,X_2 & = & X_2 D \end{array} \right. . \tag{8}$$

Clearly, from the first equation we have

$$A_{11}\,X_1 = X_1 D - A_{12}\,X_2. \tag{9}$$

By Theorem 1 in [2, pp. 52], equation (9) has a solution in A_{11} if and only if

$$(X_1 D - A_{12}\,X_2) W_{X_1}^{(l)} = 0. \tag{10}$$

The condition (10) is equivalent to

$$A_{12}\,X_2 W_{X_1}^{(l)} = X_1 D W_{X_1}^{(l)}. \tag{11}$$

Again, by Theorem 1 in [2, pp. 52], equation (11) has a solution in A_{12} if and only if

$$X_1 D W_{X_1}^{(l)} W_{X_2 W^{(l)}(X_1)}^{(l)} = 0. \tag{12}$$

In this case, the general expression for A_{12} is

$$A_{12} = X_1 D W_{X_1}^{(l)} (X_2 W_{X_1}^{(l)})^\dagger + Y_{12} W^{(r)}_{X_2 W_{X_1}^{(l)}} \tag{13}$$

for arbitrary $Y_{12} \in \mathbb{C}^{k \times k}$.

If we now substitute A_{12} by (13) in equation (9) we obtain

$$A_{11}\,X_1 = X_1 D - X_1 D W_{X_1}^{(l)} (X_2 W_{X_1}^{(l)})^\dagger\,X_2 - Y_{12} W^{(r)}_{X_2 W^{(l)}(X_1)}\,X_2. \tag{14}$$

Using Lemma 1, equation (14) has a skew-hermitian solution in A_{11} if and only if

$$\left[X_1 D - X_1 D W_{X_1}^{(l)} (X_2 W_{X_1}^{(l)})^\dagger\,X_2 - Y_{12} W^{(r)}_{X_2 W_{X_1}^{(l)}}\,X_2 \right] W_{X_1}^{(l)} = 0 \tag{15}$$

and

$$X_1^* \left[X_1 D - X_1 D W_{X_1}^{(l)} (X_2 W_{X_1}^{(l)})^\dagger\,X_2 - Y_{12} W^{(r)}_{X_2 W_{X_1}^{(l)}}\,X_2 \right] \tag{16}$$

is skew-hermitian. In this case the general solution of (14) is given by

$$\begin{aligned} A_{11} \quad = \quad & \left[X_1 D - X_1 D W_{X_1}^{(l)} (X_2 W_{X_1}^{(l)})^\dagger\,X_2 - Y_{12} W^{(r)}_{X_2 W_{X_1}^{(l)}}\,X_2 \right] X_1^\dagger - \\ & -(X_1^\dagger)^* \left[X_1 D - X_1 D W_{X_1}^{(l)} (X_2 W_{X_1}^{(l)})^\dagger\,X_2 - Y_{12} W^{(r)}_{X_2 W_{X_1}^{(l)}}\,X_2 \right]^* W_{X_1}^{(r)} + W_{X_1}^{(r)} Y_{11} W_{X_1}^{(r)}, \end{aligned}$$

for arbitrary skew-hermitian $Y_{11} \in \mathbb{C}^{k\times k}$. The properties of the Moore-Penrose inverse provide the following expression:

$$\begin{aligned} A_{11} &= \left[X_1 D - X_1 D W^{(l)}_{X_1} (X_2 W^{(l)}_{X_1})^\dagger X_2 - Y_{12} W^{(r)}_{X_2 W^{(l)}_{X_1}} X_2\right] X_1^\dagger + \\ &\quad + (X_1^\dagger)^* X_2^* W^{(r)}_{X_2 W^{(l)}_{X_1}} Y_{12}^* W^{(r)}_{X_1} + W^{(r)}_{X_1} Y_{11} W^{(r)}_{X_1}. \end{aligned} \tag{17}$$

In order to determine A_{22}, we substitute expression A_{12} given by (13) in the second equation of (8) and we obtain

$$A_{22} X_2 = X_2 D - (W^{(l)}_{X_1} X_2^*)^\dagger W^{(l)}_{X_1} D^* X_1^* X_1 - W^{(r)}_{X_2 W^{(l)}_{X_1}} Y_{12}^* X_1. \tag{18}$$

Equation (18) has a skew-hermitian solution in A_{22} if and only if

$$\left[X_2 D - (W^{(l)}_{X_1} X_2^*)^\dagger W^{(l)}_{X_1} D^* X_1^* X_1 - W^{(r)}_{X_2 W^{(l)}_{X_1}} Y_{12}^* X_1\right] W^{(l)}_{X_2} = 0 \tag{19}$$

and

$$X_2^* \left[X_2 D - (W^{(l)}_{X_1} X_2^*)^\dagger W^{(l)}_{X_1} D^* X_1^* X_1 - W^{(r)}_{X_2 W^{(l)}_{X_1}} Y_{12}^* X_1\right] \tag{20}$$

is skew-hermitian. In this case, its general solution is given by

$$\begin{aligned} A_{22} &= \left[X_2 D - (W^{(l)}_{X_1} X_2^*)^\dagger W^{(l)}_{X_1} D^* X_1^* X_1 - W^{(r)}_{X_2 W^{(l)}_{X_1}} Y_{12}^* X_1\right] X_2^\dagger - \\ &\quad - (X_2^\dagger)^* \left[X_2 D - (W^{(l)}_{X_1} X_2^*)^\dagger W^{(l)}_{X_1} D^* X_1^* X_1 - W^{(r)}_{X_2 W^{(l)}_{X_1}} Y_{12}^* X_1\right]^* W^{(r)}_{X_2} + W^{(r)}_{X_2} Y_{22} W^{(r)}_{X_2}, \end{aligned}$$

for arbitrary skew-hermitian $Y_{22} \in \mathbb{C}^{k\times k}$, which can be also written as

$$\begin{aligned} A_{22} &= \left[X_2 D - (W^{(l)}_{X_1} X_2^*)^\dagger W^{(l)}_{X_1} D^* X_1^* X_1 - W^{(r)}_{X_2 W^{(l)}_{X_1}} Y_{12}^* X_1\right] X_2^\dagger + \\ &\quad + (X_2^\dagger)^* \left[X_1^* X_1 D W^{(l)}_{X_1} (X_2 W^{(l)}_{X_1})^\dagger - X_1^* Y_{12} W^{(r)}_{X_2 W^{(l)}_{X_1}}\right] W^{(r)}_{X_2} + W^{(r)}_{X_2} Y_{22} W^{(r)}_{X_2}. \end{aligned} \tag{21}$$

Summarizing, we have obtained the following result.

Theorem 2. *Let $X \in \mathbb{C}^{n\times m}$, $D \in \mathbb{C}^{m\times m}$ be a diagonal matrix and $J \in \mathbb{R}^{n\times n}$ be a normal matrix such that $J^2 = -I_n$. Consider the partition $X = U \begin{bmatrix} X_1^* & X_2^* \end{bmatrix}^*$ as in (7) where $X_1, X_2 \in \mathbb{C}^{k\times m}$. Then there exists a J-hamiltonian, normal matrix $A \in \mathbb{C}^{n\times n}$ such that $AX = XD$ if and only if conditions (12), (15), (16), (19), (20) and $A_{11}A_{12} = A_{12}A_{22}$ hold. In this case, the general solution can be written as*

$$A = U \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix} U^*$$

where A_{11}, A_{12} and A_{22} are given by (17), (13) and (21), respectively.

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