Product Recurrence for Weighted Backward Shifts

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Abstract: We study product recurrence properties for weighted backward shifts on sequence spaces. The backward shifts that have non-zero product recurrent points are characterized as Devaney chaotic shifts. We also give an example of weighted shift that admits points which are recurrent and distal, but not product recurrent, in contrast with the dynamics on compact sets. An example of a product recurrent point with unbounded orbit is also provided.

This paper is dedicated to the memory of Professor José Sousa Ramos.

1 Introduction

Our framework are unilateral weighted backward shifts on sequence spaces, for which we plan to study product recurrence. The dynamics of a linear operator $T : X \to X$ on a topological vector space X (in short, tvs) has been intensively studied in recent years. We recall that an operator T on a tvs X is called *hypercyclic* if there is a vector x in X such that its *orbit* $\mathrm{Orb}(x,T) = \{x, T(x), T^2(x), \dots\}$ is dense in X. T is *topologically transitive* if for every pair of non-empty open subsets U and V of X there exists an $n \in \mathbb{N} := \{1, 2, \dots\}$ such that $T^n(U) \cap V \neq \emptyset$. It was well known that a continuous map T on a separable and complete metric space without isolated points admits dense orbits if and only if it is topologically transitive (see for instance [20]). According to [8] an operator is *Devaney chaotic* if it is topologically transitive, the set of periodic points $\mathrm{Per}(T)$ is dense in X and is sensitive, that is there exists $\varepsilon > 0$ such that for each x and each $\delta > 0$ there are y with $d(x,y) < \delta$ and $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) > \varepsilon$. We remark that the first two conditions implies sensitivity in the general setting of continuous maps on infinite metric spaces [3] (see also [15]). The recent books [5] and [17] contain the theory

and most of the recent advances on hypercyclicity and linear dynamics.

The interest on recurrent properties for the dynamics on compact sets goes back to Furstenberg [10, 12, 11], where important characterizations and results were obtained. We also refer to the works [1, 9, 14, 19, 22, 23] for more on this topic. In linear dynamics, some recent advances have been produced on recurrence (see [6, 7, 24]). Given a metric space (X, d), a continuous map $f : X \to X$, a point $x \in X$, and a subset $A \subset X$, we denote by $N(x, A) = \{n \geq 0 \; ; \; f^n x \in A\}$. A point $x \in X$ is called *recurrent* if $N(x, U)$ is infinite for every neighbourhood U of x. When X is compact, a point $x \in X$ is said to be *product recurrent* if given any recurrent point y in any dynamical system (Y, g), with Y a compact metric space, then the pair (x, y) is a recurrent point for the dynamical system $(X \times Y, f \times g)$. A pair of points $x_1, x_2 \in X$ is called *proximal* if there exists an increasing sequence $(n_k)_k$ in $\mathbb{N}$ such that $\lim_k d(f^{n_k} x_1, f^{n_k} x_2) = 0$. A point $x \in X$ is *distal* if it is not proximal to any point in its orbit closure other than itself. Furstenberg [11] showed that, in a compact metric space, product recurrent points coincide with distal recurrent points. It is worth mentioning that many years later Auslander and Furstenberg in [2] asked about points x which are recurrent in pair with any uniformly recurrent point y (i.e. points such that $N(x, U)$ is syndetic for every neighborhood U of x) in any dynamical system on compact metric space. There is no full characterization of such points x, however it is known that such point x does

not have to be distal. It was first proved by Haddad and Ott in [19] (in fact x does not even have to be uniformly recurrent as shown in [19]). Later, some other sufficient conditions for this kind of product recurrence were provided in [13] and [18].

From now on we will be concerned with the dynamics of a linear operator $T : X \to X$ on a metrizable and complete topological vector space (in short, F-space) X. In this situation, X is never compact, not even for the finite dimensional case since it is easy to prove that finite dimensional topological vector spaces are isomorphic to $\mathbb{K}^n$, a finite product of the scalar field.

We will say that $x \in X$ is product recurrent (with respect to T) if (x, y) is recurrent for $(X \times Y, T \times R)$, for any recurrent point $y \in Y$ with respect to an operator $R : Y \to Y$ on an F-space Y.

More particularly, we are interested in the dynamics of weighted shifts on sequence spaces. By a *sequence space* we mean an F-space X which is continuously included in ω, the countable product of the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Given a sequence $w = (w_n)_n$ of positive weights, the associated *unilateral weighted backward shift* $B_w : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ is defined by $B_w(x_1, x_2, \dots) = (w_2 x_2, w_3 x_3, \dots)$. If a sequence F-space X is invariant under certain weighted backward shift T, then T is also continuous on X by the closed graph theorem (see for instance [17, Theorem A.13]). Chapter 4 of [17] contains more details about dynamical properties of weighted shifts on sequence Fréchet spaces (i.e., F-spaces whose topology is defined by a sequence of seminorms). The results on Devaney chaos for shift operators given in [17] (we refer the reader to [16] for the original results) remain valid for a unilateral weighted backward shift $T = B_w : X \to X$ on a sequence F-space X in which the canonical unit vectors $(e_n)_{n \in \mathbb{N}}$ form an unconditional basis. In particular, B_w is chaotic if, and only if,

$$\sum_{n=1}^{\infty} \left(\prod_{v=1}^{n} w_v \right)^{-1} e_n,$$

respectively,

$$\sum_{n=-\infty}^{0} \left(\prod_{v=n+1}^{0} w_v \right) e_n + \sum_{n=1}^{\infty} \left(\prod_{v=1}^{n} w_v \right)^{-1} e_n,$$

converges unconditionally. We recall that a series $\sum_n x_n$ in X *converges unconditionally* if it converges and, for any 0-neighbourhood U in X, there exists some $N \in \mathbb{N}$ such that $\sum_{n \in F} x_n \in U$ for every finite set $F \subset \{N, N+1, N+2, \dots\}$.

For a weight sequence $v = (v_i)_i$ the following Banach sequence spaces are considered: for $1 \le p < \infty$

$$\ell^p(v) := \left\{ (x_i)_i \in \mathbb{K}^{\mathbb{N}} : \|x\| := \left(\sum_{i=1}^{\infty} |x_i|^p v_i \right)^{1/p} < \infty \right\},$$

and also

$$c_0(v) := \left\{ (x_i)_i \in \mathbb{K}^{\mathbb{N}} : \lim_{i \to \infty} |x_i| v_i = 0, \ \|x\| := \sup_i |x_i| v_i \right\}.$$

In this situation, the required condition to have the operator $B_w : \ell^p(v) \to \ell^p(v)$ bounded (i.e., continuous) is

$$\sup_{i \in \mathbb{N}} |w_{i+1}^p| \frac{v_i}{v_{i+1}} < \infty, \tag{1}$$

condition that will always be assumed to hold (details are given for the unweighted case in [17, Example 4.4 a)]).

Some basic notions on combinatorial number theory are also needed. We recall that $S \subset \mathbb{N}$ is *syndetic* if there exists $N \in \mathbb{N}$ such that $[i, i+N] \cap S \ne \emptyset$. A subset $A \subset \mathbb{N}$ is an *IP-set* if there exists a sequence $(p_i)_i$ in $\mathbb{N}$ such that

$$p_{n_1} + \dots + p_{n_k} \in A,$$

whenever $0 \le n_1 < \dots < n_k$, $k \ge 2$, where $p_0 := 0$. A subset $D \subset \mathbb{N}$ is an *IP*-set* if it intersects any IP-set. It is well-known that IP*-sets are syndetic. Simply, it was proved by Furstenberg that every thick set contains an IP-set (e.g. see Lemma 9.1 in [11]). In connection with recurrence, if x is a recurrent point then $N(x, U)$ is an IP-set for every neighbourhood U of x (see [11, Theorem 2.17]). Also, for any IP-set A, we can find a recurrent point y for the backward shift on $\{0, 1\}^{\mathbb{N}}$ such that $N(y, V) \subset A$ for some neighbourhood V of y (see the proof of [11, Theorem 2.17]). For more detail on IP-sets and their connections with recurrence the reader is referred to [11].

Since the backward shift on $\{0, 1\}^{\mathbb{N}}$ can be naturally embedded in the backward shift on a sequence space $X = \ell^p(v)$ or $X = c_0(v)$ with the weight sequence so that the shift is Devaney chaotic (see, e.g., [4, 21]), the above observation remains true for linear operators. In particular, either for compact dynamical systems or for linear operators, product recurrent points x are exactly those such that $N(x, U)$ is an IP*-set for any neighbourhood U of x.

2 Product recurrent points for backward shifts

In this section we characterize the backward shifts that admit (non-zero) product recurrent points on the Banach sequence spaces $X = \ell^p(v)$ and $X = c_0(v)$. They are the Devaney chaotic shifts. The study of particular product recurrent points in the space $X = \mathbb{K}^{\mathbb{N}}$ is more surprising, as it will be shown at the end of the section.

We recall that the weighted backward shift on a sequence space is defined as

$$B_w(x_1, x_2, \dots) = (w_2 x_2, w_3 x_3, \dots),$$

and that we write B if the $w = (1)_i$. We recall a basic result on recurrence of backward shifts [6, 24], and we include a proof of it for the sake of completeness.

Theorem 1. [24] *Let $B : X \to X$ be the backward shift on $X = \ell^p(v)$ or $X = c_0(v)$ satisfying condition (1). Then B admits a non-zero recurrent point x if, and only if, B is transitive.*

Proof. Suppose that $0 \neq x \in X$ is a recurrent point for B. Let $i \in \mathbb{N}$ such that $x_i \neq 0$ and fix $U := \{y \in X \; ; \; |y_i| > |x_i|/2\}$, which is a neighbourhood of x. Since $N(x,U)$ should be infinite, we can find an increasing sequence $(n_k)_k$ in $\mathbb{N}$ such that

$$B^{n_k}x \in U, \quad \forall k \in \mathbb{N}.$$

In other words, $|x_{i+n_k}| > |x_i|/2$ for each $k \in \mathbb{N}$. Suppose that $X = \ell^p(v)$. Since $x \in X$, we have

$$\sum_{k \in \mathbb{N}} v_{i+n_k} \leq \frac{2^p}{|x_i|^p} \sum_{k \in \mathbb{N}} |x_{i+n_k}|^p v_{i+n_k} \leq \frac{2^p}{|x_i|^p} \|x\|^p < \infty.$$

In particular, $\lim_k v_{i+n_k} = 0$ and this implies transitivity of B (characterizations for transitivity of backward shifts on X are well know and are included in [16,17] for instance). The case $X = c_0(v)$ is analogous but using the supremum norm instead of the p-norm.

Conversely, let B be transitive. Then it is hypercyclic and every hypercyclic vector is, obviously, recurrent.

Remark. If x is a recurrent point for certain operator T, and there exists a neighbourhood U of x such that $N(x,U)$ is not syndetic then, as mentioned in the introduction, one can construct an IP-set $J \subset \mathbb{N} \setminus N(x,U)$ and a recurrent point y for the backward shift B on, e.g., $\ell^1(v)$ for $v_k = 1/2^k, k \in \mathbb{N}$, such that $N(y,V) \subset J$ for some neighbourhood V of y. We then have that x cannot be product recurrent.

Theorem 2. *Let $B : X \to X$ be the backward shift on $X = \ell^p(v)$ or $X = c_0(v)$ satisfying condition (1). Then B admits a non-zero product recurrent point x if, and only if, B is Devaney chaotic.*

Proof. Suppose that x is a non-zero product recurrent point for B. Let $i \in \mathbb{N}$ such that $x_i \neq 0$, and fix $U := \{y \in X \; ; \; |y_i| > |x_i|/2\}$. By Remark 2 we have that $N(x,U)$ is syndetic. Let $l > i$ such that $\mathbb{N} \subset N(x,U) - [0,l]$. We find then an increasing sequence $(n_k)_k$ in $\mathbb{N}$ with such that $n_k - n_{k-1} \leq l$ and $|x_{n_k}| > |x_i|/2$ for all $k \in \mathbb{N}$, where $n_0 := 1$. Suppose that $X = \ell^p(v)$. Now,

$$\sum_{j \in \mathbb{N}} v_j \leq \sum_{s=0}^{l} \sum_{k=1, \, n_k > s}^{\infty} v_{n_k - s}$$
$$< \frac{2^p}{|x_i|^p} \sum_{s=0}^{l} \sum_{k=1, \, n_k > s}^{\infty} |x_{n_k}|^p v_{n_k - s}$$
$$\leq \frac{2^p}{|x_i|^p} \sum_{s=0}^{l} \|B^s x\|^p < \infty.$$

This implies that B is Devaney chaotic (again, see [16,17] for the characterizations of Devaney chaotic shifts). As in the previous proof, the case $X = c_0(v)$ is analogous using the sup norm.

Conversely, if B is chaotic then it admits non-zero periodic points, which are product recurrent.

For general weighted backward shifts on our sequence spaces, we can establish characterizations of recurrence and product recurrence for non-zero points in terms of the weights of the shift, and the weights of the space. To do this we simply have to take into account that weighted backward shifts are conjugated to the unweighted shift by changing the weights in the space (see Chapter 4 in [17]).

Corollary 1. *For a bounded weighted backward shift operator B_w defined on $X = \ell^p(v)$, $1 \leq p < \infty$, (respectively, on $X = c_0(v)$) the following conditions are equivalent:*

(i) $\sum_{i=1}^{\infty} \dfrac{v_i}{\prod_{j=1}^{i} \left| w_j^p \right|} < \infty$ *(respectively, $\lim_{i \to \infty} \dfrac{v_i}{\prod_{j=1}^{i} |w_j|} = 0$),*

(ii) B_w *admits a product recurrent point.*

(iii) B_w *is Devaney chaotic.*

Also, the following conditions are equivalent:

(i) $\inf_{i \in \mathbb{N}} \dfrac{v_i}{\prod_{j=1}^{i} \left| w_j^p \right|} = 0,$

(ii) B_w *admits a recurrent point.*

(iii) B_w *is topologically transitive.*

Remark. With a little more effort, it can be shown that the existence of non-zero recurrent points is equivalent to transitivity for unilateral weighted backward shifts on sequence F-spaces, and the existence of non-zero product recurrent points is equivalent to Devaney chaos for unilateral weighted backward shifts on sequence Fréchet spaces in which the canonical unit vectors $(e_n)_{n \in \mathbb{N}}$ form an unconditional basis (for details on the techniques, see Chapter 4 in [17]). We kept the framework of weighted ℓ^p-spaces and c_0-spaces because they are the most usual sequence spaces, and the characterizations can be written down in terms of the weights. Also, for product recurrence, it turns out that the distal points characterize product recurrent points for weighted shifts on $X = \ell^p(v)$ or $X = c_0(v)$ since, in this case, the orbit of a product recurrent point x is relatively compact. Indeed, $N(x,U)$ is syndetic for every neighbourhood U of x and, in particular for $U := x + V$ where V is the unit ball of X, we get that $\mathrm{Orb}(x)$ is bounded by boundedness of the operator. Since the shift is Devaney chaotic, it is easy to show that bounded orbits are relatively compact (one only has to check it for the unweighted shift, and then proceed by conjugacy in the general case).

When it comes to the study of particular product recurrent points for shifts on the countable product of the scalar field $X = \mathbb{K}^{\mathbb{N}}$, the main problem is that most of the techniques used for compact dynamical systems are useless in this context, because we can even face the situation of unbounded orbits. This fact also affects distality. The following two examples illustrate these type of phenomena.

Example 1. Let $B : X \to X$ be the backward shift on the countable product of the scalar field $X = \mathbb{K}^{\mathbb{N}}$. Recall that (X, d) is a complete metric space with

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

There exist distal and product recurrent points x whose orbit is unbounded. Indeed, let us consider the blocks $P_1 = (1)$, $P_{n+1} = (n+1, P_1, \ldots, P_n)$, $n \in \mathbb{N}$, and the vector

$$x = (P_1, P_2, \ldots, P_n, \ldots)$$

that is,

$$P_1 = (1)$$
$$P_2 = (2, 1)$$
$$P_3 = (3, 1, 2, 1)$$
$$P_4 = (4, 1, 2, 1, 3, 1, 2, 1)$$
$$\vdots$$
$$x = (1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1,$$
$$5, 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, \ldots).$$

Since convergence in X is coordinatewise convergence, we have that the closure of $\mathrm{Orb}(x)$ coincides with $\mathrm{Orb}(x)$ itself. Otherwise, we would have a sequence of vectors such that for some fixed index the corresponding coordinates are unbounded, and the sequence of vectors could not converge. Therefore, the fact that x is distal is an easy consequence. Also, the construction of x yields that, for each neighbourhood U of x, there is $m \in \mathbb{N}$ such that $m\mathbb{N} \subset N(x, U)$. In particular, $N(x, U)$ is an IP*-set, therefore x is product recurrent.

Example 2. Again, let $B : X \to X$ be the backward shift on the countable product of the scalar field $X = \mathbb{K}^{\mathbb{N}}$ with the same usual metric as Example 1.

There exist distal points y which are not product recurrent. We slightly modify the previous construction so that $P_1 = (1)$, $P_{n+1} = (n+1, \overset{(n)}{\ldots}, n+1, P_1, \ldots, P_n)$, $n \in \mathbb{N}$, where the amount of $n+1$'s at the beginning of block P_{n+1} equals n, and define the vector

$$y = (P_1, P_2, \ldots, P_n, \ldots)$$

that is,

$$P_1 = (1)$$
$$P_2 = (2, 1)$$
$$P_3 = (3, 3, 1, 2, 1)$$
$$P_4 = (4, 4, 4, 1, 2, 1, 3, 3, 1, 2, 1)$$
$$\vdots$$
$$y = (1, 2, 1, 3, 3, 1, 2, 1, 4, 4, 4, 1, 2, 1, 3, 3, 1, 2, 1,$$
$$5, 5, 5, 5, 1, 2, 1, 3, 3, 1, 2, 1, 4, 4, 4, 1, 2, 1, 3, 3, \ldots).$$

As before, the closure of $\mathrm{Orb}(y)$ coincides with $\mathrm{Orb}(y)$ itself, and y is distal. But now $N(y, U)$ is not syndetic for, e.g., $U := \{z \in X \; ; \; |z_1| < 2\}$. By Remark 2 we have that y is not product recurrent.

Remark. Recall that if (X, T) is a compact dynamical system and $x \in X$ is distal then x is product recurrent. Then in Example 2 it is essential that $\mathrm{Orb}(y)$ is not compact.

Acknowledgements

References

[1] E. Akin, *Recurrence in topological dynamics*, The University Series in Mathematics, Plenum Press, New York, 1997, Furstenberg families and Ellis actions.

[2] J. Auslander and H. Furstenberg, *Product recurrence and distal points*, Trans. Amer. Math. Soc., **343** (1994), 221–232.

[3] J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey, *On Devaney's definition of chaos*, Amer. Math. Monthly **99** (1992), 4, 332–334.

[4] S. Bartoll, F. Martínez-Giménez, and A. Peris, *The specification property for backward shifts*, J. Difference Equ. Appl. **18** (2012), 4, 599–605.

[5] F. Bayart and É.Matheron, *Dynamics of linear operators*, Cambridge Tracts in Mathematics, vol. 179, Cambridge University Press, Cambridge, 2009.

[6] K. Chan and I. Seceleanu, *Hypercyclicity of shifts as a zero-one law of orbital limit points*, J. Operator Theory **67** (2012), no. 1, 257–277.

[7] G. Costakis, A. Manoussos and I. Parissis, *Recurrent linear operators*, Complex Anal. Oper. Theory (To appear).

[8] R. L. Devaney, *An introduction to chaotic dynamical systems*, second ed., Addison-Wesley Studies in Nonlinearity, Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, 1989.

[9] P. Dong, S. Shao, and X. Ye, *Product recurrent properties, disjointness and weak disjointness*, Israel J. Math. **188** (2012), 463–507.

[10] H. Furstenberg, *The structure of distal flows*, Amer. J. Math. **85** (1963), 477–515.

[11] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, Princeton, N.J., 1981, M. B. Porter Lectures.

[12] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory **1** (1967), 1–49.

[13] P. Dong, S. Shao and X. Ye, *Product recurrent properties, disjointness and weak disjointness*, Israel J. Math., **188** (2012), no. 1, 463–507.

[14] E. Glasner, *Classifying dynamical systems by their recurrence properties*, Topol. Methods Nonlinear Anal. **24** (2004), no. 1, 21–40.

[15] E. Glasner, B. Weiss, *Sensitive dependence on initial conditions*, Nonlinearity, **6** (1993), 1067–1075.

[16] K.-G. Grosse-Erdmann, *Hypercyclic and chaotic weighted shifts*, Studia Math. **139** (2000), no. 1, 47–68.

[17] K.-G. Grosse-Erdmann and A. Peris Manguillot, *Linear chaos*, Universitext, Springer, London, 2011.

[18] P. Oprocha, *Weak mixing and product recurrence*, Annales de l'Institut Fourier, **60** (2010), 1233–1257.

[19] K. Haddad and W. Ott, *Recurrence in pairs*, Ergodic Theory Dynam. Systems **28** (2008), no. 4, 1135–1143.

[20] S. Kolyada and L. Snoha, *Some aspects of topological transitivity - a survey*, Iteration theory (ECIT 94) (Opava), 3–35, Grazer Math. Ber., **334**, Karl-Franzens-Univ. Graz, Graz, 1997.

[21] M. Murillo-Arcila and A. Peris, *Strong mixing measures for linear operators and frequent hypercyclicity*, J. Math. Anal. Appl. **398** (2013), no. 2, 462–465.

[22] P. Oprocha, *Weak mixing and product recurrence*, Ann. Inst. Fourier (Grenoble) **60** (2010), no. 4, 1233–1257.

[23] P. Oprocha and G. Zhang, *On weak product recurrence and synchronization of return times*, Adv. Math. **244** (2013), 395–412.

[24] I. Seceleanu, *Hypercyclic operators and their orbital limit points*, ProQuest LLC, Ann Arbor, MI, 2010. Thesis (Ph.D.)Bowling Green State University.

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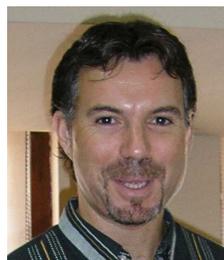
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