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A mixed 0–1 programming approach to topology-finding of tensegrity structures

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Abstract

In this paper we propose an optimization-based approach to finding a tensegrity structure based on the ground structure method. We first solve a problem which maximizes the number of struts over the self-equilibrium condition and the discontinuity condition of struts. Subsequently we solve the minimization problem of the number of cables in order to remove redundant self-equilibrium modes. The optimization problem at each step can be formulated as a mixed integer programming (MIP) problem. The method does not require any connectivity information of cables and struts to be known in advance, while the obtained tensegrity structure is guaranteed to satisfy the discontinuity condition of struts rigorously.

Keywords: Tensegrity; Shape design; Self-equilibrated configuration; Mixed integer program; Topology optimization.

1. Introduction

Tensegrity structure is a class of tension structures, which consists of pin-jointed members transmitting only axial forces. According to the definition given by Fuller [3], a tensegrity structure is a prestressed pin-jointed structure consisting of continuous tensile members (cables) and discontinuous compressive members (struts). Later, the concept of tensegrity has been generalized extensively; see, e.g., [7] and the references therein.

In this paper we propose an optimization-based approach to find a tensegrity structure which rigorously satisfies the discontinuity condition of struts. It is emphasized that our method does not require any information of the connectivity of cables and struts to be known in advance.

There have been many studies on form-finding of tensegrity structures; see, e.g., the review papers [5, 9] and the references therein. However, as input data, those methods require to specify the connectivity of members as well as the labeling indicating whether each member is to be a cable or strut. Based on the group representation theory, a systematic approach was presented to enumerate topologies, i.e. connectivities and labelings of members, of tensegrity structures which share a common group-theoretic symmetry property [1]. Particularly, for tensegrity structures with a rotational symmetry property, form-finding methods utilizing such a symmetry property have been proposed [6, 8, 10]. However, these methods based on the group theory assume that the group symmetry underlying a family of tensegrities is known in advance, i.e. it is necessary to specify a symmetry property of tensegrity structures before the form-finding process. Thus, it still remains as a challenging problem to find a completely new pair of the connectivity and the labeling of members of a tensegrity structure satisfying its definition rigorously.

We propose an approach to find a tensegrity structure based on the ground structure method. Given a pin-jointed structure with the specified locations of nodes and sufficiently many candidate members, our problem is to find a labeling of members which indicates whether each member is to be a cable, a strut, or removed, so that the resulting structure becomes a tensegrity structure. It is emphasized that our approach does not require any labeling of members or any underlying group symmetry property to be known in advance.

Our approach consists of two parts; at the first step we find a self-equilibrium mode of axial forces satisfying the discontinuous condition of struts, while at the second step we remove redundant cables from the structure obtained at the first step. At each step we solve a *mixed integer programming* (MIP) problem.

2. Maximization of number of struts

Based on the conventional ground structure method, consider a pin-jointed structure in the three-dimensional space, which consists of the nodes with the specified locations and sufficiently many members that can exist. Let V and E denote the set of nodes and the set of members, respectively. We denote by $\mathbf{q} = (q_i) \in \mathbb{R}^{|E|}$ the vector of the member axial forces. Note that a cable and a strut transmit only compressive and tensile forces, respectively, and hence we have $q_i > 0$ for a cable and $q_i < 0$ for a strut.

Although there exist various definitions of tensegrity structures [7], we employ the one which consists of the self-equilibrium condition and the discontinuity condition of struts. Let $E(n_j) \subset E$ denote the set of indices of the members which are connected to the node $n_j \in V$. We denote by $H \in \mathbb{R}^{3|V|\times|E|}$ the equilibrium matrix. Since any two struts do not share a common node, the self-equilibrium mode, $\hat{\mathbf{q}}$, should satisfy

$$\hat{\mathbf{q}} \neq \mathbf{0} : \ H\hat{\mathbf{q}} = \mathbf{0}, \tag{1}$$

$$|\{i \in E(n_j) \mid \hat{q}_i < 0\}| \leq 1, \quad \forall n_j \in V. \tag{2}$$

We next introduce a binary variable, $x_i \in \{0,1\}$, for each member in order to indicate whether the member i is a strut or not, i.e. $x_i = 1$ implies that the member i is to be a cable, while $x_i = 0$ implies that the member i is to be either a cable or a removed member. Let M and ε be positive constants, where M is sufficiently large, i.e. $0 < \varepsilon \ll M$. It is easy to see that the condition (2) is rewritten as

$$-Mx_i \leq q_i \leq M(1-x_i) - \varepsilon, \quad \forall i \in E, \tag{3}$$

$$\sum_{i \in E(n_j)} x_i \leq 1, \quad \forall n_j \in V, \tag{4}$$

where $x_i \in \{0,1\}$ $(\forall i \in E)$.

Note that the total number of struts is given by $\sum_{i \in E} x_i$. Since it is natural to attempt to choose as many struts as possible from the given ground structure, we consider the maximization problem of the number of struts. From (3) and (4), the maximization problem of the number of struts of a tensegrity structure is formulated as

$$\text{(MIP-1)} : \max_{\mathbf{q},\mathbf{x}} \quad \sum_{i \in E} x_i \tag{5a}$$

$$\text{s.t.} \quad H\mathbf{q} = \mathbf{0}, \tag{5b}$$

$$-Mx_i \leq q_i \leq M(1-x_i) - \varepsilon, \quad \forall i \in E, \tag{5c}$$

$$\sum_{i \in E(n_j)} x_i \leq 1, \quad \forall n_j \in V, \tag{5d}$$

$$\mathbf{x} \in \{0,1\}^{|E|}. \tag{5e}$$

Note that the problem (5) is a 0–1 mixed integer programming problem. Let $(\hat{\mathbf{q}}, \hat{\mathbf{x}})$ denote the optimal solution of (5). Observe that $\hat{\mathbf{q}} = \mathbf{0}$ implies $\hat{\mathbf{x}} = \mathbf{0}$. Since we attempt to maximize $\sum_{i \in E} x_i \geq 0$ in the problem (5), at its optimal solution the condition (1) is satisfied, unless $\mathbf{q} = \mathbf{x} = \mathbf{0}$ is a unique feasible solution of (5).

The characteristics of members are determined by

$$\begin{cases} \hat{q}_i > 0 & \Rightarrow \text{ the member } i \text{ is to be a cable;} \\ \hat{q}_i < 0 & \Rightarrow \text{ the member } i \text{ is to be a strut;} \\ \hat{q}_i = 0 & \Rightarrow \text{ the member } i \text{ is to be removed.} \end{cases} \tag{6}$$

Thus we can obtain a tensegrity structure satisfying the discontinuity condition of struts.

3. Minimization of number of cables

We have shown in section 2 that a tensegrity structure satisfying the discontinuity conditions of struts, as well as the self-equilibrium condition, can be obtained by solving (MIP-1) in (5). Since (MIP-1) requires only the existence of self-equilibrium mode of axial forces which satisfies the discontinuity condition of struts, the self-equilibrium mode of the obtained tensegrity structure is not unique in general. In such a case, there may exist some cables which can be removed from the tensegrity structure without changing the locations of struts. This motivates us to consider in this section the minimal tensegrity for the given set of struts.

Let $(\hat{\mathbf{q}}, \hat{\mathbf{x}})$ denote the optimal solution of (MIP-1). Define $E_{\mathrm{cable}}, E_{\mathrm{strut}} \subset E$ by

$$E_{\mathrm{cable}} = \{i \in E \mid \hat{q}_i > 0\}, \tag{7}$$
$$E_{\mathrm{strut}} = \{i \in E \mid \hat{q}_i < 0\}. \tag{8}$$

Let $\bar{E}_{\mathrm{cable}}$ be the set of candidates of cables, where $\bar{E}_{\mathrm{cable}} \supseteq E_{\mathrm{cable}}$ and $\bar{E}_{\mathrm{cable}} \cup E_{\mathrm{strut}} = \emptyset$. Consider a ground structure, which consists of the set of nodes V and set of members $E_{\mathrm{strut}} \cup \bar{E}_{\mathrm{cable}}$. We say that a tensegrity structure satisfying the discontinuity condition of struts is a minimal tensegrity if it includes no redundant cable. This condition is rigorously stated as

$$\left\{ \mathbf{q} \,\middle|\, \begin{array}{l} H\mathbf{q} = \mathbf{0} \\ q_{i'} = 0 \\ q_i \geq 0 \ (\forall i \in \bar{E}_{\mathrm{cable}} \setminus \{i'\}) \\ q_j < 0 \ (\forall j \in E_{\mathrm{strut}}) \end{array} \right\} = \emptyset, \quad \forall i' \in \bar{E}_{\mathrm{cable}}. \tag{9}$$

For each $i \in \bar{E}_{\mathrm{cable}}$, we introduce a binary variable $y_i \in \{0,1\}$ which indicates whether the cable i can be removed or not. Consider the linear inequalities

$$0 \leq q_i \leq M y_i, \tag{10}$$

where M is a sufficiently large positive constant. Since $y_i \in \{0,1\}$, we see that (10) is equivalent to

$$q_i \geq 0, \tag{11}$$

$$\begin{cases} q_i > 0 & \Rightarrow \quad y_i = 1, \\ q_i = 0 & \Leftarrow \quad y_i = 0. \end{cases} \tag{12}$$

We remove the member i if $y_i = 0$. From (12) it follows that by minimizing y_i over (10), y_i becomes equal to one if and only if $q_i > 0$. Consequently, the minimization problem of the number of cables with the specified $E_{\rm strut}$ is formulated as

$$\begin{aligned}
\text{(MIP-2)}: \min_{\mathbf{q},\mathbf{y}} \quad & \sum_{i \in \bar{E}_{\rm cable}} y_i && \text{(13a)}\\
\text{s.t.} \quad & Hq = 0, && \text{(13b)}\\
& q_i \leq -\varepsilon, \qquad \forall i \in E_{\rm strut}, && \text{(13c)}\\
& 0 \leq q_i \leq M y_i, \quad \forall i \in \bar{E}_{\rm cable}, && \text{(13d)}\\
& \mathbf{y} \in \{0,1\}^{|\bar{E}_{\rm cable}|}, && \text{(13e)}
\end{aligned}$$

where $0 < \varepsilon \ll M$. Note that (13) is a 0–1 mixed integer programming problem.

Let $(\mathbf{q}^*, \mathbf{y}^*)$ denote the optimal solution of the problem (13). Observe that in (13) we attempt to minimize the sum of y_i, from which and (12) we obtain

$$\begin{cases} q_i^* > 0 \quad \Leftrightarrow \quad y_i^* = 1, \\ q_i^* = 0 \quad \Leftrightarrow \quad y_i^* = 0, \end{cases}$$

at the optimal solution. Hence, the optimal value of the problem (13) is equal to the number of remaining cables. By removing cables corresponding to $y_i^* = 0$, we can obtain the minimal tensegrity which does not include any redundant cables.

4. Numerical examples

Consider a ground structure shown in Figure 1, where $|V| = 10$ and $|E| = 41$. This structure consists of three layers, where the top and bottom layers are in triangular shapes and the middle one in a rectangular shape. Note that the configuration of this structure is symmetric by the reflection with respect to the yz-plane and the rotation around the x-axis with the angle π. It is often that the symmetry of a tensegrity configuration causes the rank-deficiency of the equilibrium matrix, and hence the conventional Maxwell counting rule does not necessarily hold [2].

We solve (MIP-1) and (MIP-2) by using CPLEX Ver.11.2 [4] with the default settings. The optimal solution of (MIP-1) is shown in Figure 2(a), which consists of 5 struts and 18 cables. For finding the tensegrity with the minimum number of cables, we next solve (MIP-2), where the ground structure for (MIP-2) is given by Figure 1, i.e. $\bar{E}_{\rm cable} := E \setminus E_{\rm strut}$ in (13). The optimal solution of (MIP-2) is illustrated in Figure 2(b), which has 5 struts and 16 cables. We see that the tensegrity in Figure 2(b) satisfies $21 = 5 + 16 < 3|V| - 6 = 24$, which implies that the conventional Maxwell rule does not hold. However, this tensegrity can be stabilized by introducing prestresses, which is verified from an actually constructed model.

5. Conclusions

In this paper we have presented a numerical method for finding a tensegrity structure based

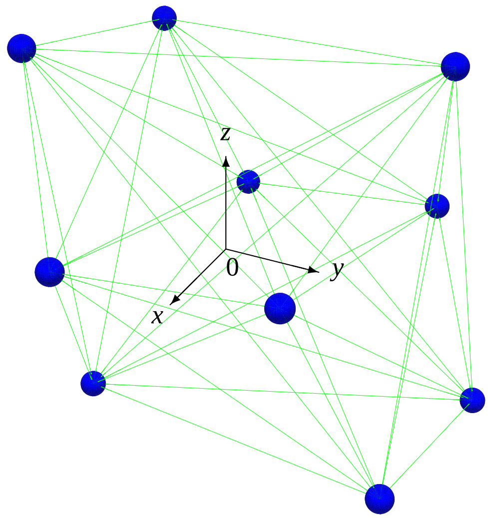

Figure 1: A ground structure with 10 nodes and 41 members.

on the ground structure method. In our method we solve the two MIPs (mixed integer programming problems) sequentially in order to find a tensegrity structure which satisfies the self-equilibrium condition as well as the discontinuity condition of struts.

At the first step we solve an MIP which maximizes the number of struts over the self-equilibrium condition and the discontinuity condition of struts. Note that it is very difficult to deal with the discontinuity condition of struts rigorously by existing methods for design of tensegrities. We have shown that this condition can be written as a system of linear inequalities in terms of the axial forces and some additional binary variables. Since the optimal solution obtained at the first step has some self-equilibrium modes in general, we solve an MIP which minimizes the number of cables as the second step.

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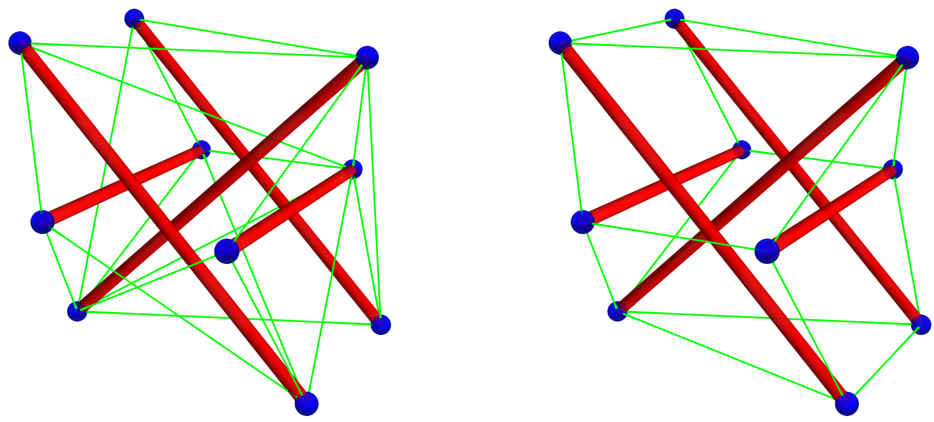

(a) The optimal solution of (MIP-1). (b) The optimal solution of (MIP-2).

Figure 2: The optimal solutions of a 10-node example.

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