One-point Newton-type iterative methods: An unified point of view *

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Abstract

In this paper, an unified point of view that include the most of one-point Newton-type iterative methods for solving nonlinear equations is introduced. A simple idea to design iterative methods with quadratic or cubic convergence is also described. This idea is extended to construct one-point iterative methods of order four. In addition, several numerical examples are given to illustrate and compare different known methods and some ones introduced by using this unifying idea.

Keywords: Nonlinear equation, one-point method, Newton's method, order of convergence, efficiency index, optimal order, weight function procedure.

MSC 2000: 65H05.

1 Introduction

Solving nonlinear equations is a classical problem which has interesting applications in several branches of sciences and engineering. Many optimization problems such as searching for a local minimizer of function [1], the potential equations in the transonic regime of dense gases in gasdynamics [2] and the boundary value problems encountered in kinetic theory of gases [3], elasticity [4] and problems in other applied areas can be reduced to nonlinear equations. In general, to compute their roots we must drawn on to iterative methods.

This paper is concerned with iterative methods to find a simple root α of a nonlinear equation $f(x) = 0$, where f is a real function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, defined in an open interval I. There are many iterative methods such as Newton's method, Halley' and super-Halley's schemes, Chebyshev's method, etc. and their variants (see [5] and the references therein). In the following, some basics concepts are introduced, that can be found in [1, 5]. Newton's method is the best known and probably the most used algorithm for solving $f(x) = 0$. It is given by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, \ldots$$

which converges quadratically in some neighborhood of α, that is, there exists a positive constant C such that $\lim_{k\to\infty} \frac{|x_{k+1}-\alpha|}{|x_k-\alpha|^2} = C$. More generally, for the sequence $\{x_k\}_{k=0}^{\infty}$ generated by an

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iterative method, if there exist positive constants C and p such that

$$\lim_{k\to\infty} \frac{|x_{k+1}-\alpha|}{|x_k-\alpha|^p} = C,$$

then the method is said to converge to α with the local order of convergence p.

Commonly, the efficiency of an iterative method is measured by the efficiency index, defined by Ostrowski [6] as $p^{1/d}$, where p is the order of convergence and d is the number of functional evaluations per iteration. Kung and Traub conjectured in [7] that the order of convergence of any iterative method, without memory, can not exceed the bound 2^{d-1}, called the optimal order. Let us recall that an iterative method without memory is an scheme whose its $(k+1)$th iteration is obtained by using only the previous kth iteration. The efficiency index of Newton's method is 1.414, as it uses two functional evaluations (one of f and another one of f') and its order of convergence is two. So, it is an optimal scheme.

The construction of numerical methods for solving nonlinear equations is an interesting task, which has attracted the attention of many authors for more than three centuries. These schemes can be classified in two big families: one-point and multipoint schemes, depending on if $(k+1)$th iteration is obtained by using functional evaluations only of iteration k or also functional evaluations in other intermediate points, respectively. The classical methods mentioned before are one-point schemes. During the last years, numerous papers devoted to design one-point iterative methods for solving nonlinear equations, $f(x) = 0$, have appeared. These methods are developed from the classical algorithms by using Taylor interpolating polynomials, quadrature rules or some other techniques. However, due to the limitations and restrictions of the one-point iterative schemes, multipoint methods appeared in the literature. A good survey of these multipoint schemes can be found in [8].

In this work, a systematic treatment of the one-point iterative methods by using the weight function procedure is provided. We can include, under this unified point of view, all the one-point known methods, as far as we know, of order two and three.

The rest of the paper is organized as follows: in Section 2, many known one-point methods or classes of schemes of order two are shown, giving a general expression of them by using a real weight function H, that depends on a particular variable, and we analyze the conditions that function H must satisfy in order to obtain iterative methods of order two. Section 3 is devoted to the same idea as Section 2 but for iterative schemes of order three. In Section 4, the procedure carried out in the previous section for increasing the order of convergence is generalized. Several numerical examples are given in Section 5 to illustrate and compare the efficiency of the methods considered in the paper.

2 One-point methods of second order

In this section we are going to show some of the classical and more recent methods or families of schemes for finding a root of the equation $f(x) = 0$, with second order of convergence. Their iterative expressions will be described as a modified Newton's method with a weight function H, depending on variable $u(x) = \frac{f(x)}{f'(x)}$.

Kanwar and Tomar proposed in [9] the following parametric family of second order iterative methods

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)+\beta f(x_k)} = x_k - \frac{f'(x_k)}{f'(x_k)+\beta f(x_k)}\frac{f(x_k)}{f'(x_k)} = x_k - \frac{1}{1+\beta u(x_k)}\frac{f(x_k)}{f'(x_k)}, \quad (1)$$

where β is a parameter, derived by expanding a particular function in Taylor series. Let us observe that for $\beta = 0$ we obtain Newton's scheme.

From this idea, Kou and Li in [10] described the bi-parametric family of methods of order two

$$x_{k+1} = x_k - \left(1 + \frac{\lambda h(x_k)}{1+\beta h(x_k)}\right) h(x_k),$$

where λ, β are parameters and $h(x_k) = \frac{f(x_k)}{f'(x_k)+\beta f(x_k)}$. We can transform this iterative expression in

$$x_{k+1} = x_k - \left(1 + \frac{\lambda u(x_k)}{(1+\beta u(x_k))(1+2\beta u(x_k))}\right) \frac{f(x_k)}{f'(x_k)}. \tag{2}$$

In [11], Noor presented the following method with quadratic convergence:

$$\begin{aligned} x_{k+1} &= x_k - \frac{2f(x_k)}{f'(x_k) + \sqrt{f'(x_k)^2 + 4\beta^3 f(x_k)^3}} = x_k - \frac{2}{1+\sqrt{1+4\beta^3 f(x_k)^3/f'(x_k)^2}} \frac{f(x_k)}{f'(x_k)} \\ &= x_k - \frac{2}{1+\sqrt{1+4\beta^3 f(x_k) u(x_k)^2}} \frac{f(x_k)}{f'(x_k)} \end{aligned} \tag{3}$$

obtained by using Taylor polynomials along with an auxiliary equation.

All these methods are optimal in the sense of Kung-Traub's conjecture, since they have order two and use two functional evaluations per step. So, their efficiency index is the same, but not the number of floating point operations, which depends on the complexity of the iterative expression. We can observe that these methods have the general iterative expression

$$x_{k+1} = x_k - H(u(x_k)) \frac{f(x_k)}{f'(x_k)}, \tag{4}$$

where $H(u)$ is a function of variable $u(x) = f(x)/f'(x)$. For iterative methods described in the form (4) we can establish the following result.

Theorem 1 *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a real function with second derivative in I. Let $\alpha \in I$ a simple root of $f(x) = 0$. If we choose an initial guess close enough to α and a sufficiently differentiable function $H(u)$ such that $H(0) = 1$, then the methods described by (4) converge to α with quadratic order of convergence, being their error equation*

$$e_{k+1} = (-H'(0) + c_2)e_k^2 + O(e_k^3),$$

where $c_j = \frac{1}{j!}\frac{f^{(j)}(\alpha)}{f'(\alpha)}$, $j = 2, 3, \ldots$ and $e_k = x_k - \alpha$.

Proof: By using Taylor's expansion about α, we obtain

$$\begin{aligned} f(x_k) &= f'(\alpha)\left[e_k + c_2 e_k^2 + c_3 e_k^3\right] + O(e_k^4), \\ f'(x_k) &= f'(\alpha)\left[1 + 2c_2 e_k + 3c_3 e_k^2\right] + O(e_k^3). \end{aligned}$$

From these expressions

$$u(x_k) = -c_2 e_k^2 + (2c_2^2 - 2c_3)e_k^3 + O(e_k^4).$$

Then, we estimate function $H(u)$ about 0, since $u(x_k)$ tends to 0 when k tends to ∞.

$$H(u(x_k)) \approx H(0) + H'(0)u(x_k) = H(0) + H'(0)e_k - H'(0)c_2 e_k^2 + H'(0)\left(2c_2^2 - 2c_3\right)e_k^3 + O(e_k^4).$$

By using these expansions, we obtain the error equation

$$e_{k+1} = (1 - H(0))e_k + (H(0)c_2 - H'(0))e_k^2 + O(e_k^3).$$

By assuming $H(0) = 1$, the error equation is

$$e_{k+1} = (-H'(0) + c_2)e_k^2 + O(e_k^3)$$

and the proof is finished. □

In Table 1 we describe the function $H(u)$ that guarantees the quadratic convergence of the methods described in this section.

Method	Function $H(u)$
Newton	$H(u) = 1$
Kanwar-Tomar (1)	$H(u) = \dfrac{1}{1+\beta u}$, β parameter
Kou-Li (2)	$H(u) = 1 + \dfrac{\lambda u}{(1+\beta u)(1+2\beta u)}$, λ, β parameters
Noor (3)	$H(x) = \dfrac{2}{1+\sqrt{1+4\beta^3 f(x)x^2}}$, β parameter

Table 1: Weight functions $H(u)$ that express different quadratic methods

3 One-point methods of third order

In a similar way as in the previous section, we are going to give a general expression of many one-point iterative schemes of order three. In this case, this expression will be described as a modified Newton's method with a weight function G, depending on variable $w(x) = \dfrac{f(x)f''(x)}{f'(x)^2}$, which is called degree of logarithmic convexity.

In the literature, there are a lot of methods, or families of schemes, for solving a nonlinear equation $f(x) = 0$, with third order of convergence requiring the evaluation of the second derivative of function f. The most of the well-known one-point cubically convergent methods belong to the one-parameter class, called Chebyshev-Halley family

$$x_{k+1} = x_k - \left(1 + \frac{1}{2}\frac{f(x_k)f''(x_k)}{f'(x_k)^2 - \beta f(x_k)f''(x_k)}\right)\frac{f(x_k)}{f'(x_k)} = x_k - \left(1 + \frac{1}{2}\frac{w(x_k)}{1-\beta w(x_k)}\right)\frac{f(x_k)}{f'(x_k)}. \quad (5)$$

This family includes Chebyshev's method for $\beta = 0$, Halley's scheme for $\beta = 1/2$, super-Halley's method for $\beta = 1$ and Newton's method when β tends to $\pm\infty$.

Fang et al. [12] obtained the third order method

$$x_{k+1} = x_k - \frac{2f(x_k)}{f'(x_k) + \sqrt{f'(x_k)^2 - 2f(x_k)f''(x_k)}} = x_k - \frac{2}{1+\sqrt{1-2w(x_k)}}\frac{f(x_k)}{f'(x_k)}, \quad (6)$$

by expanding function f in Taylor series about the origin, dropping the third and higher order terms, and solving the obtained quadratic equation.

Abbasbandy [13] and Chun [14] proposed and studied several one-step iterative methods by using the decomposition technique of Adomian. Specifically, Abbasbandy rediscovered with this technique the third order Chebyshev's method, which was also obtained by Noor et al. [15] with another decomposition technique which does not involve the derivative of the Adomian polynomials. They designed the following predictor-corrector method with cubic convergence:
Predictor-step

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Corrector-step

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{(y_k - x_k)^2}{2f'(x_k)} f''(x_k) - \frac{(y_k + z_k - x_k)^2}{2f'(x_k)} f''(x_k),$$

where $z_k = -\dfrac{(y_k - x_k)^2}{2f'(x_k)} f''(x_k)$.

By algebraic manipulations, this method can be rewritten in the following form

$$\begin{aligned} x_{k+1} &= x_k - \frac{f(x_k)}{f'(x_k)} \left(1 + \frac{1}{2}\frac{f(x_k)f''(x_k)}{f'(x_k)^2}\left(1 + \frac{f(x_k)f''(x_k)}{f'(x_k)^2} + \frac{f(x_k)^2 f''(x_k)^2}{2f'(x_k)^4}\right)\right) \\ &= x_k - \left(1 + \frac{1}{2}w(x_k) + \frac{1}{2}w(x_k)^2 + \frac{1}{4}w(x_k)^3\right)\frac{f(x_k)}{f'(x_k)}. \end{aligned} \tag{7}$$

In [16], Hansen and Patrick presented a parametric family of iterative methods, with order of convergence three, whose iterative expression is

$$x_{k+1} = x_k - \frac{(\lambda+1)f(x_k)}{\lambda f'(x_k) + \sqrt{f'(x_k)^2 - (\lambda+1)f(x_k)f''(x_k)}} = x_k - \frac{\lambda+1}{\lambda + \sqrt{1-(\lambda+1)w(x_k)}}\frac{f(x_k)}{f'(x_k)}. \tag{8}$$

In particular, if $\lambda = 0$, we obtain the iterative method

$$x_{k+1} = x_k \pm \frac{1}{\sqrt{1-w(x_k)}}\frac{f(x_k)}{f'(x_k)}, \tag{9}$$

studied by Ostrowski in [6]. On the other hand, if $\lambda = 1$ we obtain the known Euler's method.

More recently, Chun and Kim, by using a geometric approach based on the circle of curvature, constructed in [17] a new iterative method of order three, whose expression is

$$x_{k+1} = x_k - \frac{f'(x_k)f(x_k)(f''(x_k)f(x_k) + 2 + 2f'(x_k)^2)}{2f'(x_k)^2(1+f'(x_k)^2)f(x_k)f''(x_k)}.$$

Algebraic manipulations allow us again to transform this iterative expression in

$$x_{k+1} = x_k - \frac{w(x_k) + 2s(x_k)}{2s(x_k) - w(x_k)/f'(x_k)^2}\frac{f(x_k)}{f'(x_k)}, \tag{10}$$

where $s(x_k) = 1 + \frac{1}{f'(x_k)^2}$.

Finally, we present the parametric family developed by Neta and Scott in [18], whose iterative formula is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{f(x_k)^2 f''(x_k)}{2f'(x_k)^3 - Af(x_k)f''(x_k)} = x_k - \left(1 + \frac{w(x_k)}{2 - Aw(x_k)}\right)\frac{f(x_k)}{f'(x_k)}, \tag{11}$$

where A is a parameter. Upon choosing $A = 1$ we have Halley's method. The choice $A = 0$ yields the well known Chebyshev's method. This latter scheme is also a special case of Hansen and Patrick's family (8) with $\lambda = 1$. The choice $A = 2$ gives the BSC method described by Basto et al. in [19].

As we have said, all these methods have order of convergence three and none of them is optimal in the sense of Kung-Traub's conjecture, since they use three functional evaluations per step. Although their efficiency index is $I = 3^{\frac{1}{3}}$, they use different number of floating point operations, being in general more efficient those that have easier iterative expressions.

We are going to present an unified and simple result for demonstrating the order of convergence of these methods. In fact, as Theorem 1, it is based on weight function procedure and allows us to establish the order of convergence of all one-point methods published, as far as we know, when their order is three. In addition, this result provides an easy procedure to design iterative methods with this order of convergence.

The methods described in this section have the general iterative expression

$$x_{k+1} = x_k - G(w(x_k))\frac{f(x_k)}{f'(x_k)}, \tag{12}$$

where $G(w)$ is a function of variable $w(x) = f(x)f''(x)/f'(x)^2$. For iterative methods described in form (12), we can establish the following result.

Theorem 2 *Let us assume that $f(x)$ and $G(w)$ are sufficiently differentiable functions and $f(x)$ has a simple zero $\alpha \in I$. If the initial estimation x_0 is close enough to α and function $G(w)$ satisfies $G(0) = 1$ and $G'(0) = 1/2$, then the methods described by (12) converge to α with cubic order of convergence, being their error equation*

$$e_{k+1} = -2((-1 + G''(0))c_2^2 + \frac{1}{2}c_3)e_k^3 + O(e_k^4),$$

where $c_j = \frac{1}{j!}\frac{f^{(j)}(\alpha)}{f'(\alpha)}$, $j = 2, 3, \dots$ and $e_k = x_k - \alpha$.

Proof: By using Taylor's expansion about α, we obtain

$$\begin{aligned}
f(x_k) &= f'(\alpha)\left[e_k + c_2e_k^2 + c_3e_k^3 + c_4e_k^4\right] + O(e_k^5),\\
f'(x_k) &= f'(\alpha)\left[1 + 2c_2e_k + 3c_3e_k^2 + 4c_4e_k^3 + 5c_5e_k^4\right] + O(e_k^5),\\
f''(x_k) &= f'(\alpha)\left[2c_2 + 6c_3e_k + 12c_4e_k^2 + 20c_5e_k^3\right] + O(e_k^4).
\end{aligned}$$

From these expressions

$$w(x_k) = 2c_2e_k + (-6c_2^2 + 6c_3)e_k^2 + 4(4c_2^3 - 7c_2c_3 + 3c_4)e_k^3 + O(e_k^4).$$

Now, we approximate function $G(w)$ about 0, since $w(x_k)$ tends to 0 when x_k tends to α.

$$\begin{aligned}
G(w(x_k)) &\approx G(0) + G'(0)w(x_k) + \frac{1}{2}G''(0)w(x_k)^2\\
&= G(0) + 2G'(0)c_2e_k + (2G''(0)c_2^2 + G'(0)(-6c_2^2 + 6c_3))e_k^2 + O(e_k^3).
\end{aligned}$$

By using these Taylor's expansions, we have the error equation

$$e_{k+1} = (1-G(0))e_k + (G(0)-2G'(0))c_2e_k^2 - 2((G(0)-4G'(0)+G''(0))c_2^2 - (G(0)-3G'(0))c_3)e_k^3 + O(e_k^4).$$

By assuming $G(0) = 1$, the error equation is

$$e_{k+1} = (1 - 2G'(0))c_2e_k^2 - 2((1 - 4G'(0) + G''(0))c_2^2 + (-1 + 3G'(0))c_3)e_k^3 + O(e_k^4)$$

and, if $G'(0) = 1/2$ we have

$$e_{k+1} = -2((1 + G''(0))c_2^2 + \frac{1}{2}c_3)e_k^3 + O(e_k^4)$$

and the proof is finished. □

From this error equation we can assure that if we also require $G''(0) = 1$, then any method described by (12) has order four for quadratic equations.

Ezquerro et al. studied in [20] the semilocal convergence of a particular family of methods described by (12), where

$$G(w(x_k)) = \sum_{j\geq 0} A_j w(x_k)^j,$$

being $A_0 = 1$, $A_1 = A_2 = 1/2$ and $\{A_j\}_{j\geq 0}$ a positive non-increasing real sequence such that $\sum_{j\geq 0} A_j t^j < +\infty$ for $|t| < r$.

In Table 2 we describe the functions $G(w)$ that guarantee the cubic convergence of the methods described in this section, since they satisfy the conditions of Theorem 2.

Method	Function $G(w)$
Halley	$G(w) = \dfrac{2}{2-w}$
super-Halley	$G(w) = \dfrac{w-2}{2(w-1)}$
Chebyshev	$G(w) = 1 + \dfrac{1}{2}w$
Fang et al. (6)	$G(w) = \dfrac{2}{1+\sqrt{1-w}}$
Noor et al. (7)	$G(w) = 1 + \dfrac{1}{2}w + \dfrac{1}{2}w^2 + \dfrac{1}{4}w^3$
Hansen-Patrick (8)	$G(w) = \dfrac{\lambda+1}{\lambda+\sqrt{1-(\lambda+1)w}}$, λ parameter
Euler	$G(w) = \dfrac{2}{1+\sqrt{1-2w}}$,
Ostrowski (9)	$G(w) = \dfrac{\pm 1}{\sqrt{1-w}}$
Chun-Kim (10)	$G(w) = \dfrac{w+2s(x)}{2s(x)-w/f'(x)^2}$, $s(x) = 1 + \frac{1}{f'(x)^2}$
Neta-Scott (11)	$G(w) = 1 + \dfrac{w}{2-Aw}$, A parameter
Basto et al.	$G(w) = 1 + \dfrac{w}{2(1-w)}$

Table 2: Weight functions $G(w)$ that express different cubic methods

From Theorem 2, it is easy to design iterative methods that converge cubically. It is sufficient to choose functions $G(w)$ such that $G(0) = 1$ and $G'(0) = 1/2$. We can select functions $G(w)$ more simple than those in Table 2. For example, by taking $G(w) = e^{w/2}$ we obtain the iterative method

$$x_{k+1} = x_k - e^{\frac{f(x_k)f''(x_k)}{2f'(x_k)^2}} \frac{f(x_k)}{f'(x_k)} \tag{13}$$

and with $G(w) = w^2 + w/2 + 1$ we obtain

$$x_{k+1} = x_k - \left(\frac{f(x_k)^2 f''(x_k)^2}{f'(x_k)^4} + \frac{f(x_k) f''(x_k)}{2f'(x_k)} + 1 \right) \frac{f(x_k)}{f'(x_k)}. \tag{14}$$

Both methods have order of convergence three and they are denoted in the numerical section by M1 and M2 respectively.

4 A new family of order of convergence at least four

If we want to increase the local order of convergence of the one-point methods, we must to have into account some known restrictions of them. In classical books of Numerical Analysis (see for example [1] or [5]) the following results can be found, which describe some limitations of the one-point methods in order to increase the order of convergence.

Theorem 3 *Let $x_{k+1} = P(x_k)$ be a one-point iterative method, which use d functional evaluations per step. Then its order of convergence is at most $p = d$.*

Theorem 4 *For designing a one-point method of order p, the iterative expression must contain derivatives of order at least $p - 1$.*

So, for constructing one-point methods of order four we must use the third derivative of function f. In fact, we consider the following general iterative expression with a weight function of two variables:

$$x_{k+1} = x_k - M(w(x_k), v(x_k)) \frac{f(x_k)}{f'(x_k)}, \tag{15}$$

where $w(x) = \dfrac{f(x) f''(x)}{f'(x)^2}$ and $v(x) = \dfrac{f(x) f'''(x)}{f'(x) f''(x)}$.

For this family of methods the following result can be established.

Theorem 5 *Let us suppose that $f(x)$ and $M(w, v)$ are sufficiently differentiable functions and $f(x)$ has a simple zero α. If the initial guess x_0 is close enough to α and function $M(w, v)$ satisfies $M(0, 0) = 1$, $M_w(0, 0) = 1/2$, $M_v(0, 0) = 0$, $M_{ww}(0, 0) = 1$, $M_{vv}(0, 0) = 0$ and $M_{wv}(0, 0) = -1/6$, then the methods described by (15) converge to α with order four, being their error equation*

$$\begin{aligned} e_{k+1} &= \Big[(5 - \frac{4}{3} M_{www}(0,0)) c_2^3 - (5 + 6M_{wwv}(0,0)) c_2 c_3 - 9 \frac{c_3^2}{c_2} M_{wvv}(0,0) \\ &\quad - \frac{9c_3^3}{2c_2^3} M_{vvv}(0,0) + c_4 \Big] e_k^4 + O(e_k^5), \end{aligned}$$

where $c_j = \frac{1}{j!} \frac{f^{(j)}(\alpha)}{f'(\alpha)}$, $j = 2, 3, \ldots$ and $e_k = x_k - \alpha$.

Proof: By using Taylor's expansion about α, we obtain

$$\begin{aligned} f(x_k) &= f'(\alpha) \left[e_k + c_2 e_k^2 + c_3 e_k^3 + c_4 e_k^4 \right] + O(e_k^5), \\ f'(x_k) &= f'(\alpha) \left[1 + 2c_2 e_k + 3c_3 e_k^2 + 4c_4 e_k^3 + 5c_5 e_k^4 \right] + O(e_k^5), \\ f''(x_k) &= f'(\alpha) \left[2c_2 + 6c_3 e_k + 12c_4 e_k^2 + 20c_5 e_k^3 \right] + O(e_k^4), \\ f'''(x_k) &= f'(\alpha) \left[6c_3 + 24c_4 e_k + 60c_5 e_k^2 \right] + O(e_k^3). \end{aligned}$$

From these expressions, we have

$$w(x_k) = 2c_2e_k + (-6c_2^2 + 6c_3)e_k^2 + 4(4c_2^3 - 7c_2c_3 + 3c_4)e_k^3 + O(e_k^4)$$

and

$$\begin{aligned} v(x_k) &= \frac{3c_3}{c_2}e_k + \left(-3c_3 - \frac{9c_3^2}{c_2^2} + \frac{12c_4}{c_2}\right)e_k^2 \\ &+ \frac{3(2c_2^4c_3 + 9c_3^3 - 4c_2^3c_4 - 18c_2c_3c_4 + c_2^2(c_3^2 + 10c_5))}{c_2^3}e_k^3 + O(e_k^4). \end{aligned}$$

Now, taking into account that $w(x_k)$ and $v(x_k)$ tend to zero when x_k tends to α, we approximate function $M(w, v)$ about $(0, 0)$,

$$\begin{aligned} M(w(x_k), v(x_k)) &\approx M(0,0) + M_w(0,0)w(x_k) + M_v(0,0)v(x_k) + \frac{1}{2}M_{ww}(0,0)w(x_k)^2 \\ &+\frac{1}{2}M_{vv}(0,0)v(x_k)^2 + M_{wv}(0,0)w(x_k)v(x_k) \\ &+\frac{1}{6}\left(M_{www}(0,0)w(x_k)^3 + M_{vvv}(0,0)v(x_k)^3\right. \\ &\left.+3M_{wvv}(0,0)w(x_k)v(x_k)^2 + 3M_{wwv}(0,0)w(x_k)^2v(x_k)\right). \end{aligned}$$

By replacing these Taylor's expansions in the iterative expression, we get

$$e_{k+1} = (1 - M(0,0))e_k + \left((M(0,0) - 2M_w(0,0))c_2 - \frac{3M_v(0,0)c_3}{c_2}\right)e_k^2 + O(e_k^3)$$

and if we assume $M(0,0) = 1$, $M_w(0,0) = 1/2$ and $M_v(0,0) = 0$, we have

$$e_{k+1} = \left(-2(-1 + M_{ww}(0,0))c_2^2 - (1 + 6M_{wv}(0,0))c_3 - \frac{9M_{vv}(0,0)c_3^2}{2c_2^2}\right)e_k^3 + O(e_k^4).$$

Finally, conditions $M_{ww}(0,0) = 1$, $M_{wv}(0,0) = -1/6$ and $M_{vv}(0,0) = 0$ give the final expression of the error equation

$$\begin{aligned} e_{k+1} &= \left[(5 - \frac{4}{3}M_{www}(0,0))c_2^3 - (5 + 6M_{wwv}(0,0))c_2c_3 - 9\frac{c_3^2}{c_2}M_{wvv}(0,0)\right. \\ &\left.-\frac{9c_3^3}{2c_2^3}M_{vvv}(0,0) + c_4\right]e_k^4 + O(e_k^5). \end{aligned}$$

□

The parametric function

$$M(w, v) = \frac{1 + w/2 + w^2}{1 + \beta v^3} - \frac{1}{6}wv - \frac{1}{2}w^2,$$

where β is a real parameter, satisfies the conditions of the previous result. So, it provides a parametric family of one-point iterative schemes of order four. As a particular case, $\beta = 0$, we obtain the iterative scheme

$$x_{k+1} = x_k - \left(1 + \frac{1}{2}L_f(x_k) + \frac{1}{2}L_f(x_k)^2 - \frac{1}{6}L_{f'}(x_k)L_f(x_k)^2\right)\frac{f(x_k)}{f'(x_k)},$$

expressed in terms of the degree of logarithmic convexity $L_f(x_k) = \frac{f(x_k)f''(x_k)}{f'(x_k)^2}$.

5 Numerical results

In this section, numerical examples for testing the effectiveness of some methods introduced previously, are given. We compare the numerical results obtained by applying the methods of Newton, Chebyshev, Ostrowski (9) and Chun-Kim, and the schemes denoted by M1 (13) and M2 (14), on the following test functions:

- $f_1(x) = \cos x - x, \quad \alpha \approx 0.739085,$
- $f_2(x) = \sin^2 x - x^2 + 1, \quad \alpha \approx 1.404492,$
- $f_3(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5, \quad \alpha \approx -1.207648,$
- $f_4(x) = \sin x + x\cos x, \quad \alpha = 0,$
- $f_5(x) = x^2 e^{x^2} - \sin^2 x + x, \quad \alpha = 0,$
- $f_6(x) = (x-1)^3 - 1, \quad \alpha = 2,$
- $f_7(x) = \frac{x^2-1}{x^2+1} + 1, \quad \alpha = 0.$

Numerical computations have been carried out using variable precision arithmetic, with 1000 digits, in MATLAB 7.13. The stopping criterion used is $|x_{k+1} - x_k| + |f(x_{k+1})| < 10^{-100}$. Therefore, we check that the sequence $\{x_k\}$ converges and that its limit is a solution of the nonlinear equation $f(x) = 0$. For every method and test function, we calculate the number of iterations, the value of $incr = |f(x_{k+1})|$ at the last iteration and the computational order of convergence $ACOC$, approximated by (see [21])

$$p \approx ACOC = \frac{\ln(|x_{k+1} - x_k| \,/\, |x_k - x_{k-1}|)}{\ln(|x_k - x_{k-1}| \,/\, |x_{k-1} - x_{k-2}|)}. \tag{16}$$

The value of $ACOC$ that appears in Table 3 is the last coordinate of vector (16) when the variation between its values is small. In some cases, the approximated order of convergence is not stable and it is not shown in the table.

Table 3 summarizes several results obtained by using the mentioned methods (Newton, Chebyshev, Ostrowski, Chun-Kim, M1 and M2) in order to estimate a root of nonlinear functions from f_1 to f_7. For every function we specify the initial estimation x_0 (chosen far enough from the solution for showing stability problems), the number of iterations, the value of function f in the last iteration and the value of ACOC.

It can be observed that, in general, the number of iterations and the value of $|f(x_{k+1})|$ for all the methods are in similar ranges, taking into account that Newton's method has order two while the rest of the schemes have order three. The computational order of convergence confirms the theoretical results, but some comments are needed. For function $f_4(x)$ Newton's method has order of convergence three, like the other methods, since $f_4'(\alpha) = 0$. In addition, as $f_5''(\alpha) = f_5'''(\alpha) = 0$, all convergent methods have order four. For $f_7(x)$ all methods converge linearly since the zero of this function is not simple, it has multiplicity 2.

Remark 1 *Let us note that some of the methods described in this work can be adapted for solving nonlinear systems $F(x) = 0$. In particular, the iterative expression of method M2 can be*

Function		$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$	$f_7(x)$
x_0		2.1	2.5	-3	0.5	3	4	0.8
iter	Newton	8	10	17	7	18	11	331
	Chebyshev	7	7	12	7	14	8	235
	Ostrowski	6	7	$> 10^3$	6	$> 10^3$	7	189
	Chun-Kim	7	7	12	6	13	8	211
	M1	7	7	11	6	13	8	225
	M2	6	7	10	7	11	7	167
incr	Newton	8e-266	6e-383	7e-217	2e-774	9e-504	9e-245	8e-201
	Chebyshev	1e-783	2e-551	2e-631	1e-834	0.0	7e-196	6e-201
	Ostrowski	3e-313	7e-780	-	1e-392	-	4e-595	4e-202
	Chun-Kim	2e-806	3e-559	1e-631	1e-301	1e-674	1e-592	7e-202
	M1	3e-834	4e-596	1e-443	3e-319	1e-954	3e-720	2e-201
	M2	2e-581	7e-652	0.0	1e-716	1e-856	5e-427	6e-202
ACOC	Newton	2.0	2.0	2.0	3.0	4.0	2.0	1.0
	Chebyshev	3.0	3.0	3.0	3.0	4.0	3.0	1.0
	Ostrowski	3.0	3.0	-	3.0	-	3.0	1.0
	Chun-Kim	3.0	3.0	3.0	3.0	4.0	3.0	1.0
	M1	3.0	3.0	3.0	3.0	4.0	3.0	1.0
	M2	3.0	3.0	3.0	3.0	4.0	3.0	1.0

Table 3: Numerical results for different methods and several test functions

rewritten as an iterative scheme for approximating the solution of $F(x) = 0$. If $x^{(k)}$ denotes the kth iteration in this context, we have

$$\begin{aligned} x^{(k+1)} &= x^{(k)} - \Big(F'(x^{(k)})^{-1}F(x^{(k)}) + (1/2)F'(x^{(k)})^{-1}F''(x^{(k)})(F'(x^{(k)})^{-1}F(x^{(k)}))^2 \\ &\quad + F'(x^{(k)})^{-1}F''(x^{(k)})(F'(x^{(k)})^{-1}F(x^{(k)}))(F'(x^{(k)})^{-1}F''(x^{(k)})(F'(x^{(k)})^{-1}F(x^{(k)}))^2)\Big), \end{aligned}$$

where $F'(x^{(k)})$ is the Jacobian matrix associated to function F, evaluated in $x^{(k)}$, and $F''(x^{(k)})$ is the bilinear operator that represents the second Fréchet derivative of F.

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