Unified formulation for Reissner-Mindlin plates: a comparison with numerical results

Luca TASSINARI*, Salvador MONLEÓN CREMADES[a], Cristina GENTILINI[b]

*Ecole polytechnique fédérale de Lausanne
Bât. GC-Station 18, 1015 Lausanne, Switzerland
luca.tassinari@epfl.ch
[a] Universidad Politécnica de Valencia
[b] Università di Bologna – Facoltà di Ingegneria

Abstract

A non-standard formulation is applied to find analytical solutions for elastic plates considering shear deformation using a computer-aided approach. Classical formulations of mechanics of elastic body solve each particular problem by obtaining standard differential equations. In contrast, the unified formulation used in this paper, which is described in Tassinari *et al.* [9], is based on a matrix framework inspired by the finite element method. For this reason, an implementation using mathematical software can be efficiently applied. The aim of this study is to use the proposed computer-aided method to solve analytically the problem and to compare the results with the finite element model predictions.

In order to illustrate this approach, the case of the Reissner-Mindlin plate problem with simply support on two opposite edges is investigated (Reissner [6], Mindlin [7]). In addition, the general validity of the method and its applicability to several problems of structural mechanics is discussed. The analytical solution for a particular load case is presented showing the differences between the Reissner-Mindlin solution and the predictions given by the Love-Kirchhoff model (Kirchhoff [4], Love [5]). Finally, the vertical displacement field obtained analytically for different thickness-to-side ratios is compared with finite element method predictions. The results obtained from the proposed analytical method and numerical models presented in this study are comparable.

Keywords: analytical solution, unified formulation, shell, plate, Reissner-Mindlin, Love-Kirchhoff, shear strain, finite element.

1. Introduction

Analytical solutions to problems of solids mechanics are useful to test the accuracy and the precision of the numerical approaches. Problems such as shear locking or boundary layer can be better understood with the help of analytical solutions. However, it is known that only simple cases can be solved by hand. Furthermore, some cases which are theoretically solvable can also result in rather tedious analysis. Nowadays computer and mathematical software development allow to avoid this problem if they are associated with a suitable formulation. The problem of the Reissner-Mindlin plate is a typical example of this problem (Reissner [6], Mindlin [7]). The Reissner-Mindlin model takes into account the shear deformation and should be used for moderately thick plates whereas Love-Kirchhoff model is recommended for thin plates (Kirchhoff [4], Love [5]).

The system of field equilibrium equations for the bending of the Reissner-Mindlin plate model is (Timoshenko and Woinowsky-Krieger [10]):

$$\begin{aligned} &\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q_z \\ &\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{yx}}{\partial y} = Q_x - m_x \\ &\frac{\partial M_{yx}}{\partial x} + \frac{\partial M_{yy}}{\partial y} = Q_y - m_y \end{aligned} \tag{1}$$

where the quantities (Q_x, Q_y) are the transverse shear forces, (M_{xx}, M_{yy}) are the bending moments and M_{yx} is the torsion moment; (q_z, m_y, m_y) are the external loads reduced to the middle surface. The quantities $(Q_x, Q_y, M_{xx}, M_{yy}, M_{xy})$ are the generalized stresses. The constitutive laws of the generalized stresses resulting from the RM kinematical assumption can be expressed as:

$$\begin{aligned} &Q_x = Gh\left(\frac{\partial w}{\partial x} + \psi_x\right) \qquad Q_y = Gh\left(\frac{\partial w}{\partial x} + \psi_y\right) \\ &M_{xx} = D\left(\frac{\partial \psi_x}{\partial x} + \nu\frac{\partial \psi_y}{\partial y}\right) \quad M_{yy} = D\left(\nu\frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y}\right) \quad M_{yx} = \tfrac{(1-\nu)}{2} D\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right) \end{aligned} \tag{2}$$

where w denotes the transverse displacement of the midplane $z = 0$ (Figure 1) and (ψ_x, ψ_y) the rotation of the transverse normal to the midplane about the $y-$ and $x-$ axes, respectively. The quantities (w, ψ_x, ψ_y) are the generalized displacements. The constants D and G are defined as:

$$D = \frac{Eh^3}{12(1-\nu^2)} \quad , \quad G = \frac{E}{2(1+\nu)} \tag{3}$$

where h denotes the thickness of the plate (Figure 1).

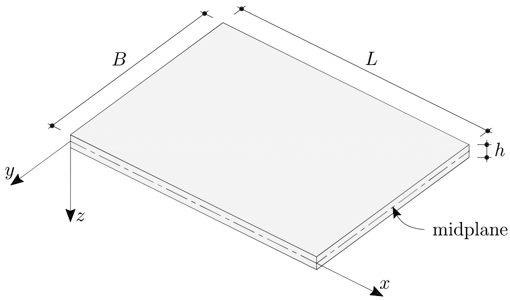

Figure 1: Geometry and reference system of a rectangular plate.

2. Unified formulation of the solution

In case of a rectangular plate with known boundary conditions on opposite two opposite edges and arbitrary boundary conditions on the remaining edges, the generalized displacement can be expressed as a series (Timoshenko and Woinowsky-Krieger [10], Szilard [8]). When hard simply supported edges ($w = \psi_y = 0$) are considered on the $x = 0$ and $x = L$, the series become:

$$\begin{aligned} w(x,y) &= \sum_{N=1}^{\infty} \sin \alpha_N x \, w_N(y) \\ \psi_x(x,y) &= \sum_{N=1}^{\infty} \cos \alpha_N x \, \psi_{xN}(y) \\ \psi_y(x,y) &= \sum_{N=1}^{\infty} \sin \alpha_N x \, \psi_{yN}(y) \end{aligned} \tag{4}$$

where $\alpha_N = \pi N/L, N \in \mathbb{N}$ and $w_N(y), \psi_{xN}(y), \psi_{yN}(y)$ are unknown functions, which have to be determined so that the equilibrium is satisfied everywhere in the domain of the plate. The generalized displacements represented by Eq. (4) satisfy the boundary conditions on the edges $x = 0, L$ independently from the unknown functions of y .

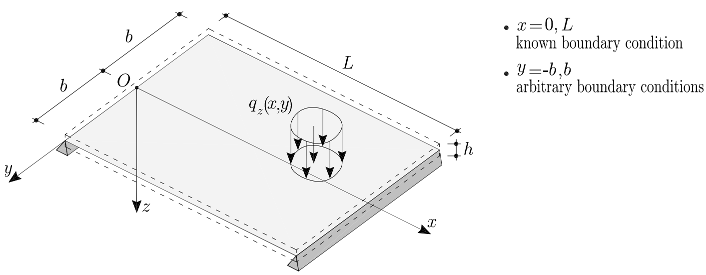

Figure 2: Plate with boundary given condition on two opposite edges.

Furthermore, suppose that the vertical load $q_z(x,y)$ can be expressed in by the following series:

$$q_z(x,y) = \sum_{N=1}^{\infty} \sin \alpha_N x \, q_{zN}(y) \tag{5}$$

where:

$$q_{zN}(y) = \frac{2}{L}\int_0^L \sin \alpha_N x q_z(x,y)dx \tag{6}$$

Similar series can be used for the loads $m_x(x,y)$ and $m_y(x,y)$.

By using the series forms (4) and (5), the equilibrium equations and the constitutive laws can be combined and rearranged to obtain the following system of first order linear ordinary differential equations (Tassinari *et al.* [9]):

$$\begin{aligned}
\frac{dw_N}{dy} &= -\psi_{yN} + \frac{Q_{yN}}{Gh} \\
\frac{d\psi_{xN}}{dy} &= -\alpha_N \psi_{yN} + \frac{2M_{yxN}}{(1-\nu)D} \\
\frac{d\psi_{yN}}{dy} &= \alpha_N \nu \psi_{xN} + \frac{M_{yxN}}{D} \\
\frac{dQ_{yN}}{dy} &= \alpha_N Gh(\alpha_N w_N + \psi_{xN}) - q_{zN} \\
\frac{dM_{yxN}}{dy} &= Gh(\alpha_N w_N + \psi_{xN}) + \alpha_N^2 (1-\nu^2) D\psi_{xN} - \alpha_N \nu M_{yyN} - m_{xN} \\
\frac{dM_{yyN}}{dy} &= Q_{yN} + \alpha_N M_{yxN} - m_{yN}
\end{aligned} \tag{7}$$

(8)

By introducing the state and driving vectors defined, respectively, as:

$$\mathbf{E}_N(y) = \left\{ w_N, \psi_{xN}, \psi_{yN}, Q_{yN}, M_{yxN}, M_{yyN} \right\}^T \quad , \quad \mathbf{F}_N(y) = \left\{ 0,0,0,q_{zN}, m_{xN}, m_{yN} \right\}^T \tag{9}$$

the system above can be written in the matrix form:

$$\frac{d}{dy}\mathbf{E}_N(y) = \mathbf{W}_N(y)\mathbf{E}_N(y) - \mathbf{F}_N(y) \tag{10}$$

where the matrix $\mathbf{W}_N(y)$ is the system matrix defined as:

$$\mathbf{W}_N(y) = \begin{bmatrix} 0 & 0 & -1 & \frac{1}{Gh} & 0 & 0 \\ 0 & 0 & -\alpha_N & 0 & \frac{2}{(1-\nu)D} & 0 \\ 0 & \alpha_N \nu & 0 & 0 & 0 & \frac{1}{D} \\ \alpha_N^2 Gh & \alpha_N Gh & 0 & 0 & 0 & 0 \\ \alpha_N Gh & Gh + \alpha_N^2 (1-\nu^2) D & 0 & 0 & 0 & -\alpha_N \nu \\ 0 & 0 & 0 & 1 & \alpha_N & 0 \end{bmatrix} \tag{11}$$

The previous derivation suggests that each problem whose governing equation can be reduced to a first order ODEs system, can be represented by Eq. (10) written in term of the consistent state vector, driving vector and system matrix.

The general solution of the homogeneous system $\mathbf{E}'_N = \mathbf{W}_N(y)\mathbf{E}_N$ is (Arnold [2], Dennis [3]):

$$\mathbf{E}_N(y) = \mathbf{\Phi}_N(y)\mathbf{k}_N \tag{12}$$

where $\mathbf{\Phi}_N(y)$ is fundamental (solution) operator and $\mathbf{k}_N$ denote an arbitrary constant vector. A fundamental operator is a matrix whose columns are a set of linear independent solutions of the homogeneous system and thus it is not unique.

Imposing the initial condition $\mathbf{E}_N(y=0) = \mathbf{E}_{0N}$, Eq. (10) leads to an initial value problem with the following solution:

$$\begin{aligned} &\mathbf{E}_N(y) = \mathbf{G}_N(y)\mathbf{E}_{0N} - \mathbf{\Phi}_N(y)\int_0^y \mathbf{\Phi}_N^{-1}(t)\mathbf{F}_N(t)dt \\ &\mathbf{G}_N(y) = \mathbf{\Phi}_N(y)\mathbf{\Phi}_{0N}^{-1} \quad , \quad \mathbf{\Phi}_{0N} = \mathbf{\Phi}_N(0) \end{aligned} \tag{13}$$

$\mathbf{G}_N$ is transition operator and is the unique fundamental operator that solves the IVP.

In general, the matrix $\mathbf{W}_N$ for shells depends on y since the thickness and the radii of curvature can be variable. On the other hand, in the case of a plate with constant thickness the elements of $\mathbf{W}_N$ are constant and thus the ODE system has constant coefficients. In this particular case the system is denoted as autonomous and the solution of the IVP becomes:

$$\mathbf{E}_N(y) = \mathbf{G}_N(y)\mathbf{E}_{0N} - \int_0^y \mathbf{G}(y-t)\,\mathbf{F}_N(t)dt \tag{14}$$

In addition, the autonomous systems a fundamental operator $\mathbf{\Phi}_N$ can be constructed by a pure algebraic method. In this context, the spectral properties of the system matrix $\mathbf{W}_N$ are significant. More specifically, in the case of nondeficient eigenspace, the eigenvectors of matrix $\mathbf{W}_N$ are a set of linear independent solutions of the homogeneous system. If the system matrix has deficient eigenspace then the concept of eigenvector can be generalized to obtain independent solutions. In this case, the exponential matrix formalism is also useful.

However, the manual resolution of the ODE system of the RM model implies rather difficult calculus because the matrix $\mathbf{W}_N$ has dimension 6×6 and deficient eigenspace. In fact, the characteristic polynomial is given by:

$$\det(\mathbf{W}_N - \lambda\mathbf{I}) = (\lambda^2 - \alpha_N^2)(\lambda^2 - \alpha_N^2 - 12/h^2) \tag{15}$$

Consequently, the eigenvalues are the simple roots $\lambda = \pm\sqrt{\alpha_N^2 + 12/h^2}$ and the double roots $\lambda = \pm\alpha_N$.

The formulation of the governing equations and the solution presented in this work is derived to be consistent with the use of mathematical software that allows to find a fundamental operator in a short time. Hence, the solution is obtained by a computer-aided approach.

Another important characteristic of the proposed solution based on the transition operator $\mathbf{G}_N$ is that it allows to find generalized displacements and stresses at the same time and using an initial condition consistent to the state vector. Classical biharmonic formulation based solutions, allow to find only the vertical displacement whereas other quantities have to be assessed by secondary equations. Solutions based on the fundamental operator $\mathbf{\Phi}_N$ do not permit to find directly the integration constants vector $\mathbf{k}_N$ from an initial condition given on the state vector.

The explicit form of the transition operator $\mathbf{G}_N$ that defines the solution of the RM rectangular plate with two hard simply supported opposite edges is given in Tassinari *et al.*[9].

3. Analytical solution for vertical line load

In the following a study of the analytical solution is done for a square Reissner-Mindlin plate ($2b = L$) with constant thickness (Figure 3). The Poisson's ratio is taken as $\nu = 0.2$. The edges $x = 0$ and $x = L$ are hard simply supported whereas the edges $y = -b$ and $y = b$ are free. A vertical line load $q_z(x) = q_z$ is applied along the line $y = 0$. According to Figure 3 the initial point is taken at $y_0 = -b$.

In order to present the results, the nondimensional forms of the components of the state vector are introduced according to Eqs (16):

$$\begin{aligned}
&\tilde{w}_N(y) = \frac{D}{q_{zN}b^3} w_N(y) \qquad \tilde{\psi}_{xN}(y) = \frac{D}{q_{zN}b^2}\psi_{xN}(y) \qquad \tilde{\psi}_{yN}(y) = \frac{D}{q_{zN}b^2}\psi_{yN}(y) \\
&\tilde{Q}_{zN}(y) = \frac{1}{q_{zN}} Q_{zN}(y) \qquad \tilde{M}_{xyN}(y) = \frac{1}{q_{zN}b} M_{xyN}(y) \qquad \tilde{M}_{yyN}(y) = \frac{1}{q_{zN}b} M_{yyN}(y)
\end{aligned} \tag{16}$$

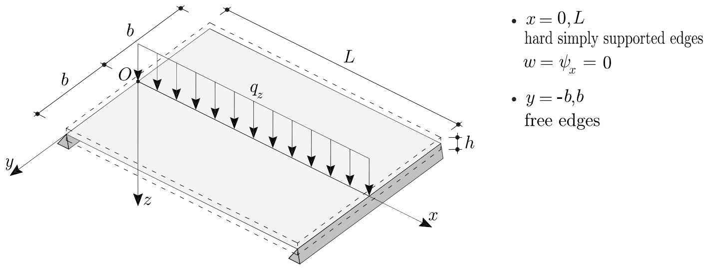

Figure 3: Analyzed case Reissner-Mindlin rectangular plate.

The nondimensional components are long mathematical expressions written in terms of hyperbolic functions and the eigenvalues of $\mathbf{W}_N$, as presented by Tassinari *et al.* [9]. In contrast to the Love-Kirchhoff model solution, the nondimensional components of the Reissner-Mindlin model depend also on the thickness of the plate. These can be expressed as a function of the thickness-to-side ratio $\delta = h / 2b$.

In this study, the LK solution is compared with the RM solution obtained by using four values of $\delta = 1/20, 1/10, 1/5$. The state vector of the LK plate was obtained by the same approach using a system matrix consistent to the LK model. Figure 4 presents the nondimensional components of the state vectors for $N = 1$. It is important to note that these results can also be considered as the exact solution of the sinusoidal line load $q_z(x) = q_z \sin \alpha_N(x)$.

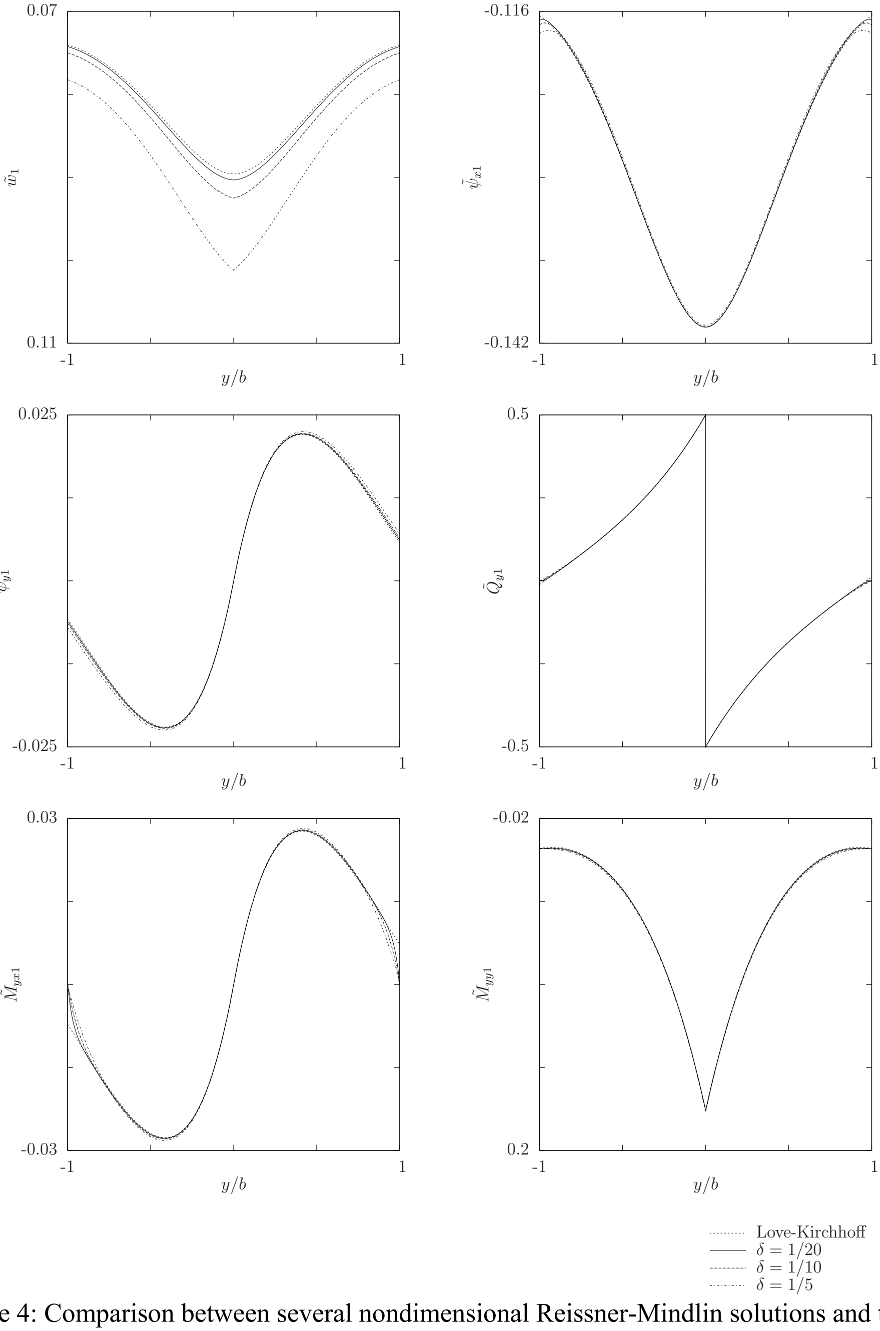

Figure 4: Comparison between several nondimensional Reissner-Mindlin solutions and the Love-Kirchhoff solution for $N = 1$

The weight of the higher order terms of the state vector is examined. Figure 5 shows a plot of the nondimensional vertical displacement $\widetilde{w}_N(y=0)$ versus the parameter δ for $N = 1,2,3$. The constant values of the LK theory are also shown for comparison. The effect of the transverse shear deformation is clearly evident in each term if the thickness-to-side ratios increase.

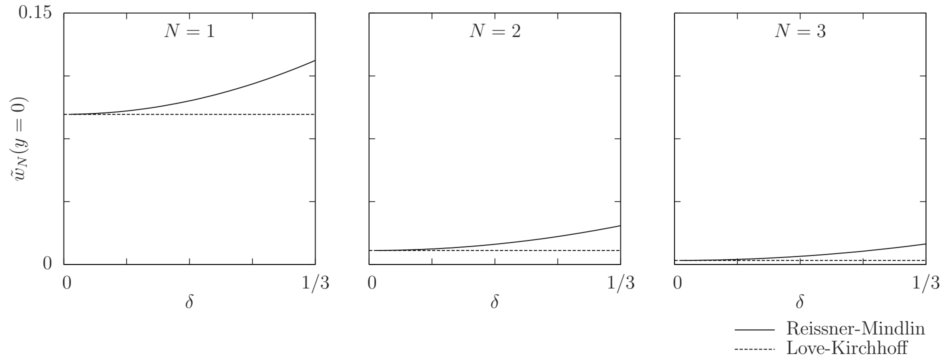

Figure 5: Mindlin rectangular plate. Reference system, boundary conditions and load case

The actual vertical displacement $w(x, y)$ can be evaluated by using Eq. (4). From the previous discussion, for a sinusoidal line load $q_z(x) = q_z \sin \alpha_N(x)$ the vertical displacement is given by:

$$w(x,y) = \frac{q_z b^3}{D} \sin \alpha_N(x) \widetilde{w}_N(y) \bigg|_{N=1} \tag{17}$$

whereas for a constant line load $q_z(x) = q_z$ the series should be developed:

$$w(x,y) = \frac{q_z b^3}{D} \sum_{N=1}^{\overline{N}=\infty} \frac{2}{\alpha_N L} (1 - \cos \alpha_N L) \sin \alpha_N(x) \widetilde{w}_N(y) \tag{18}$$

In order to compare the actual vertical displacement $w(x, y)$ with numerical results given by the finite element method, a study of the convergence of the series (18) is performed, see Figure 6. The nondimensional value of the truncated series in the central point of the plate $(x = L/2, y = 0)$ is plotted versus the truncation parameter $\overline{N}$ varying from 1 to 20. The convergence can be considered satisfactory if at last ten terms are considered.

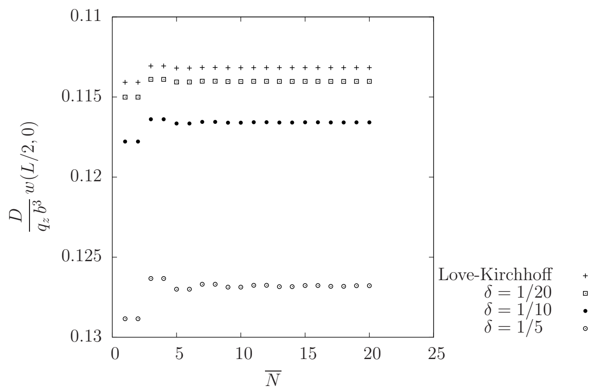

Figure 6: Study of the convergence of the series (18)

Finally, Figure 7 presents the nondimensional vertical displacement field considering 10 terms of the series for the considered values of δ and the LK solution.

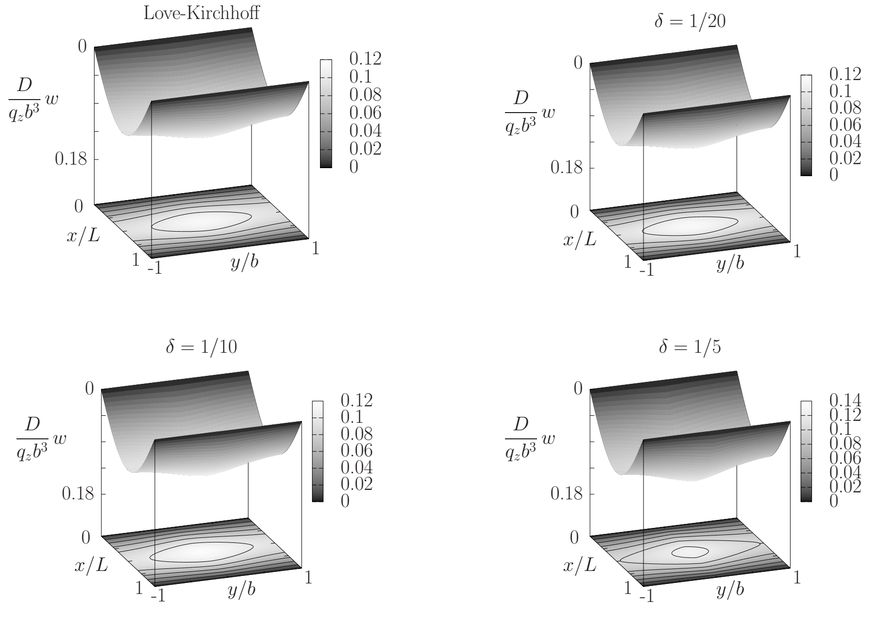

Figure 7: Nondimensional vertical displacement evaluated by using series (17) considering ten terms

4. Comparison between analytical and finite element results

A comparison between the vertical displacement in the central point of the plate predicted by the analytical solutions presented and a finite element model is shown in Figure 8. The finite element analysis was carried out using *ANSYS® Academic Research product (ANSYS Inc)*.

Two types of elements were applied; namely SHELL63 and SHELL181 (see *ANSYS release documentation* [12]). Both elements are 4-node shell elements with 6 degrees of freedom per node. SHELL181 is based on the first order shear deformation theory (RM model) as opposed to SHELL63 in which shear deformation is neglected (LK model). In addition, RM elements (SHELL181) allow to use either full or reduced integration.

Figure 8 shows the ratio between the vertical displacement w_{FEM} predicted by the FEM and w_{exact} given by the analytical solution. The results are shown for three different mesh densities. As mentioned earlier, the results from the LK model (element SHELL63) are independent of the thickness, since the deflection is normalized. For the RM model (element SHELL181) three thickness-to-side ratios were investigated: $\delta = 1/20, 1/10, 1/5$.

According to Figure 8 the LK element is less sensitive to mesh refinement than the RM element. The LK analysis can provide reasonable results even for a 2×2 mesh while at least a 4×4 mesh is required in the RM model. All the finite element models investigated converge to the analytical solution for the 8×8 mesh. This applies to both full and reduced integration, which provide similar predictions.

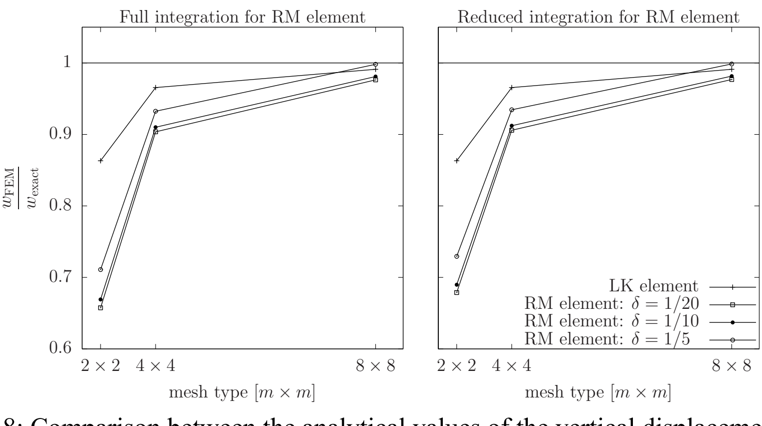

Figure 8: Comparison between the analytical values of the vertical displacement in the center of the plate and the numerical predictions

5. Conclusions

The main advantage of the proposed methodology is that it can be applied to problems that are radically different by means the same matrix formalism. For this reason, mathematical software can be employed efficiently to solve cases seldom considered in the literature. The application to the RM plate subjected to line load presented is a typical example. Owing to the mathematical complexity of the problems, analytical solution can be reached in a short time.

The analytical solution carried out shows the effect of the shear deformation in plate with small slenderness. In particular, the magnitude of the effect is significant for the vertical displacement. In general, the other components of the state vector are not affected significantly in the domain of the plate but exhibits perturbations close to the edges. This edge effect in known as boundary layer and more investigation should be carried out considering higher-order terms.

An important problem of the analytical solution in the form of a series is that it has to be evaluated numerically taking into account a reasonable number of terms. In the case analysed, the series for the vertical displacement converges if only ten terms are considered.

Finally, the finite element analysis carried out shows that for a reasonable mesh refinement the numerical models used give excellent approximation of the analytical solution.

Acknowledgement

The authors want to acknowledge the support and funding of École Polytechnique Fédérale de Lausanne, Universidad Politécnica de Valencia and Università di Bologna on this research.

References

[1] *ANSYS® Academic Research*, *Help System, Release 11.0 Documentation for ANSYS, ANSYS Inc*.

[2] Arnold V., *Ordinary differential equations*. Springer, 1992.

[3] Dennis G. Z., *A first course of differential equations.* (5th ed.), PWS publishing company, 1993.

[4] Kirchhoff G., Über das Gleichgewicht und die Bewegungen einer elastischen Scheibe. *Journal für reine und angewandte Mathematik*, 1850; **40**; 51–58.

[5] Love, A. E. H., *A treatise on the matematical theory of elasticity*, Dover, 1944.

[6] Mindlin R. D., Influence of rotatory inertia and shear on flexural vibration of isotropic, elastic plates. *Journal of Applied Mechanics*, 1951; **18**; 31–38.

[7] Reissner E., The effect of transverse shear deformation on the bending of elastic plates, *Journal of Applied Mechanics*, 1945; **67**; A69-A77.

[8] Szilard R., *Theories and applications of plate analysis: classical, numerical and engineering methods*. Wiley, 2004.

[9] Tassinari L, Monleon S, Gentilini C, Ubertini F. A unified approach for analytical solution of moderately thick shells and plates*. (Paper in preparation)*.

[10] Timoshenko S, Woinowsky-Krieger S. *Theory of plates and shells*. McGraw-Hill, 1959.