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Behaviour of fixed and critical points of the (α, c) −family of iterative methods

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Abstract In this paper we study the dynamical behavior of the (α, c)-family of iterative methods for solving nonlinear equations, when we apply the fixed point operator associated to this family on quadratic polynomials. This is a family of third-order iterative root-finding methods depending on two parameters; so, as we show throughout this paper, its dynamics is really interesting, but complicated.

In fact, we have found in the real (α, c)-plane a line in which the corresponding elements of the family have a lower number of free critical points. As this number is directly related with the quantity of basins of attraction, it is probable to find more stable behavior between the elements of the family in this region.

Keywords Non linear equations · iterative methods · dynamics of rational functions · parameter planes.

1 Introduction

Many problems from Engineering or Science lead to nonlinear equations that may have analytical roots although we are not capable of finding them. In this case, the numerical methods are needed. For the case of problems coming from Chemistry, nonlinear equations regularly appear; for example, iterative

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methods can be applied in the reaction-diffusion equations that arise in autocatalytic chemical reactions (see [10]) or in the analysis of electronic structure of the hydrogen atom in strong magnetic fields (see [9]). Moreover, numerical treatments of some chemical problems allow to check the models of observable phenomena [8]. Even more, many problems from Chemistry consist in finding chemical potentials that are basic for studying other thermodynamic properties; the modeling of such potentials leads to nonlinear integral equations that can be reduced to a set of nonlinear algebraic equations (see [11] for example). This encourages the mathematicians to study and improve the numerical methods implied.

A dynamical study of the operators defined by the iterative methods help us to know more widely the regions where these methods have a good behavior [1].

In some previous papers, we have considered the dynamical study of Chebyshev-Halley family [7], the King's class [6], the c-family [4] and, finally, the (α, c)-class which includes Chebyshev-Halley and $c-$families. In the study that we are conducting about (α, c)-family (see [2], [3]), we note that the dynamical behavior of this family is much more complicated because it includes two parameters .

As we have said, iterative methods are used for finding roots of a nonlinear equation and, from a dynamical point of view, these roots are fixed points of the operator R associated to the method.

The dynamic studies the asymptotic behavior of the orbits depending on the initial condition z_0:

$$\{z_0,\, R(z_0)\,,\, R^2(z_0)\,, ..., R^n(z_0)\,, ...\}$$

and classifies the starting points from the asymptotic behavior of their orbits (see [12] for example).

A point $z_0 \in \hat{\mathbb{C}}$ is called a *fixed point* if it satisfies: $R(z_0) = z_0$. A *periodic point* z_0 of period $p > 1$ is a point such that $R^p(z_0) = z_0$ and $R^k(z_0) \neq z_0$, $k < p$. A *pre-periodic point* is a point z_0 that is not periodic but there exists a $k > 0$ such that $R^k(z_0)$ is periodic.

Moreover, a fixed point z_0 is called *attractor* if $|R'(z_0)| < 1$, *superattractor* if $|R'(z_0)| = 0$, *repulsor* if $|R'(z_0)| > 1$ and *parabolic* if $|R'(z_0)| = 1$.

The basin of attraction of an attractor z^* is defined as the set of pre-images of any order:

$$\mathcal{A}(z^*) = \{z_0 \in \hat{\mathbb{C}} \;:\; R^n(z_0) \to z^*,\; n \to \infty\}.$$

It is known that each basin of attraction needs at least one critical point inside [1]. A point z_0 is a *critical point* of a map R if R fails to be injective in any neighborhood of z_0.

From the results stated in Section 2 we present in Section 3 a complete study of the critical points of the (α, c)-family. From this analysis we found some interesting values of the parameters, whose dynamical planes are analyzed in Section 4, that provide stable elements of the class.

2 Previous results

The (α, c)-family is a two-parametric class of third-order iterative root-finding methods defined by:

$$z_{n+1} = z_n - \left(1 + \frac{1}{2}\frac{L_f(z_n)}{1-\alpha L_f(z_n)} + cL_f(z_n)^2\right)\frac{f(z_n)}{f'(z_n)}, \tag{1}$$

where

$$L_f(z) = \frac{f(z)\,f''(z)}{f'(z)^2}$$

and α and c are complex parameters. As we have pointed before, this class includes Chebyshev-Halley family for $c = 0$ and $c-$family when $\alpha = 0$.

We apply (1) on quadratic polynomials $p(z) = z^2 + a$. For this polynomial, the operator $M_p(z, a, \alpha, c)$ associated to (1) is a rational function depending on three complex parameters: a, α and c. Due to the Scaling theorem, the parameter a can be obviated and the roots of the polynomial $p(z)$ became 0 and ∞; so, the operator associated to the iterative method has only two parameters when the Möbius transformation is applied.

For this operator we obtain the following fixed points: $0, \infty$ (that correspond to the roots of $p(z)$), $z = 1$ and six fixed points that are the roots of a 6-degree polynomial. These last seven points are called strange fixed points, as they do not correspond to any root of $p(z)$.

In this section we present the explicit expressions of the strange fixed points and the critical points. An exhaustive proof can be found in [2], and here we only show an sketch of it.

The associated operator of (1), after the Möbius transformation $h(z) = \frac{z+i\sqrt{a}}{z-i\sqrt{a}}$, is:

$$O_p(z,\alpha,c) = z^3\frac{(1+z)^4(-2+2\alpha-z)+4c(1+z(2-2\alpha+z))}{(1+z)^4(2\alpha z-1-2z)+4cz^3(1+z)^2-8\alpha cz^4} \tag{2}$$

and the relation $O_p(z,\alpha,c) - z$ can be written as:

$$O_p(z,\alpha,c) - z = -z(z-1)\frac{P(z,\alpha,c)}{(1+z)^4(2\alpha z-1-2z)+4cz^3(1+z)^2-8\alpha cz^4}$$

where $P(z,\alpha,c)$ is the 6-degree polynomial:

$$\begin{aligned} P(z,\alpha,c) = {} & z^6 + (7-2\alpha)z^5 + (19-8\alpha+4c)z^4 + (26-12\alpha+8c-8\alpha c)z^3 \\ & + (19-8\alpha+4c)z^2 + (7-2\alpha)z + 1. \end{aligned} \tag{3}$$

So, the fixed points of $O_p(z,\alpha,c)$ are 0, ∞, 1 and the six roots of the 6-degree symmetric polynomial $P(z,\alpha,c)$. The fixed points are given in the following result (see [2]).

Theorem 1 *The fixed points of the operator (2), associated to the bi-parametric family of iterative methods (1) on quadratic polynomials are:*

- $z = 0$ *and* $z = \infty$*, corresponding to the roots of the polynomial* $p(z)$.
- *Seven strange fixed points:* $z = 1$ *and the following six points:*

$$\begin{aligned} z_1(\alpha,c) &= \frac{x_1(\alpha,c) + \sqrt{x_1(\alpha,c)^2 - 4}}{2}, & z_2(\alpha,c) &= \frac{x_1(\alpha,c) - \sqrt{x_1(\alpha,c)^2 - 4}}{2}, \\ z_3(\alpha,c) &= \frac{x_2(\alpha,c) + \sqrt{x_2(\alpha,c)^2 - 4}}{2}, & z_4(\alpha,c) &= \frac{x_2(\alpha,c) - \sqrt{x_2(\alpha,c)^2 - 4}}{2}, \\ z_5(\alpha,c) &= \frac{x_3(\alpha,c) + \sqrt{x_3(\alpha,c)^2 - 4}}{2}, & z_6(\alpha,c) &= \frac{x_3(\alpha,c) - \sqrt{x_3(\alpha,c)^2 - 4}}{2}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} x_1(\alpha,c) &= \frac{1}{3}(2\alpha - 7) + \frac{1}{3}\left(\sqrt[3]{f(\alpha,c)} + \sqrt[3]{g(\alpha,c)}\right), \\ x_2(\alpha,c) &= \frac{1}{3}(2\alpha - 7) - \frac{1}{6}\left(\sqrt[3]{f(\alpha,c)} + \sqrt[3]{g(\alpha,c)}\right) - i\frac{\sqrt{3}}{6}\left(\sqrt[3]{f(\alpha,c)} - \sqrt[3]{g(\alpha,c)}\right), \\ x_3(\alpha,c) &= \frac{1}{3}(2\alpha - 7) - \frac{1}{6}\left(\sqrt[3]{f(\alpha,c)} + \sqrt[3]{g(\alpha,c)}\right) + i\frac{\sqrt{3}}{6}\left(\sqrt[3]{f(\alpha,c)} - \sqrt[3]{g(\alpha,c)}\right) \end{aligned}$$

and

$$\begin{aligned} f(\alpha,c) &= (-1+2\alpha)^3 + 18(1+4\alpha)c + 6\sqrt{3}\sqrt{c\left(2\alpha(-1+2\alpha)^3 + c(-1+40\alpha+32\alpha^2+16c)\right)}, \\ g(\alpha,c) &= (-1+2\alpha)^3 + 18(1+4\alpha)c - 6\sqrt{3}\sqrt{c\left(2\alpha(-1+2\alpha)^3 + c(-1+40\alpha+32\alpha^2+16c)\right)}. \end{aligned}$$

Now, we present the expressions of the critical points. The critical points are the solutions of $O'_p(z,\alpha,c) = 0$, where the prime means the derivative of $O_p(z,\alpha,c)$ with respect to z. A critical point is called free if it does no correspond to any root of $p(z)$. The expression of $O'_p(z,\alpha,c)$ can be written as

$$O'_p(z,\alpha,c) = \frac{-2z^2(1+z)^4 Q(z,\alpha,c)}{\left((1+z)^4(2\alpha z - 1 - 2z) + 4cz^3(1+z)^2 - 8\alpha c z^4\right)^2}, \tag{5}$$

where

$$Q(z,\alpha,c) = b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_2 z^4 + b_1 z^5 + b_0 z^6 \tag{6}$$

and

$$\begin{aligned} b_0 &= -3 + 3\alpha + 6c, \\ b_1 &= -18 + 20\alpha - 4\alpha^2 + 16c - 24\alpha c, \\ b_2 &= -45 + 53\alpha - 16\alpha^2 + 10c - 16\alpha c + 24\alpha^2 c, \\ b_3 &= -60 + 72\alpha - 24\alpha^2 + 16\alpha c - 32\alpha^2 c. \end{aligned}$$

Then, the critical points are $z = 0$, $z = \infty$, $z = -1$ and the roots of the 6-degree symmetric polynomial $Q(z,\alpha,c)$. The critical points are given in the following result (see [3]).

Theorem 2 *The critical points of the operator (2), associated to the bi-parametric family of iterative methods (1) are $z=0$ and $z=\infty$, that are associated to the non strange fixed points, and the following free critical points:*

– *For $\alpha \neq 1-2c$, the points $z=-1$ and*

$$
\begin{aligned}
z_1(\alpha,c) &= \frac{x_1(\alpha,c)+\sqrt{x_1(\alpha,c)^2-4}}{2}, & z_2(\alpha,c) &= \frac{x_1(\alpha,c)-\sqrt{x_1(\alpha,c)^2-4}}{2},\\
z_3(\alpha,c) &= \frac{x_2(\alpha,c)+\sqrt{x_2(\alpha,c)^2-4}}{2}, & z_4(\alpha,c) &= \frac{x_2(\alpha,c)-\sqrt{x_2(\alpha,c)^2-4}}{2},\\
z_5(\alpha,c) &= \frac{x_3(\alpha,c)+\sqrt{x_3(\alpha,c)^2-4}}{2}, & z_6(\alpha,c) &= \frac{x_3(\alpha,c)-\sqrt{x_3(\alpha,c)^2-4}}{2},
\end{aligned}
\tag{7}
$$

where

$$
\begin{aligned}
x_1(\alpha,c) &= -\frac{b_1}{3b_0}+s_1+s_2,\\
x_2(\alpha,c) &= -\frac{b_1}{3b_0}-\frac{1}{2}(s_1+s_2)+i\frac{\sqrt{3}}{2}(s_1-s_2),\\
x_3(\alpha,c) &= -\frac{b_1}{3b_0}-\frac{1}{2}(s_1+s_2)-i\frac{\sqrt{3}}{2}(s_1-s_2)
\end{aligned}
$$

and

$$
\begin{aligned}
s_1 &= \sqrt[3]{-\frac{q+\sqrt{D}}{2}}, & p &= -3+\frac{b_2}{b_0}-\frac{b_1^2}{3b_0^2},\\
s_2 &= \sqrt[3]{-\frac{q-\sqrt{D}}{2}}, & q &= \frac{b_3-b_1}{b_0}-\frac{b_1b_2}{3b_0^2}+2\frac{b_1^3}{27b_0^3},\\
D &= \frac{4}{27}p^3+q^2.
\end{aligned}
$$

– *For $\alpha=1-2c$ and $c\neq\pm\frac{\sqrt{5}}{2}$, the points $z=-1$ and*

$$
\begin{aligned}
z_1(\alpha,c) &= \frac{x_1(\alpha,c)+\sqrt{x_1(\alpha,c)^2-4}}{2}, & z_2(\alpha,c) &= \frac{x_1(\alpha,c)-\sqrt{x_1(\alpha,c)^2-4}}{2},\\
z_3(\alpha,c) &= \frac{x_2(\alpha,c)+\sqrt{x_2(\alpha,c)^2-4}}{2}, & z_4(\alpha,c) &= \frac{x_2(\alpha,c)-\sqrt{x_2(\alpha,c)^2-4}}{2},
\end{aligned}
\tag{8}
$$

where

$$
x_{1,2}(\alpha,c) = -\frac{c_1}{2c_0}\pm\frac{\alpha}{c_0}\sqrt{\alpha(\alpha-1)(40-29\alpha+9\alpha^2)}.
$$

– *For $\alpha=1-2c$ and $c=\frac{\sqrt{5}}{2}$, the point $z=-1$ and the two complex points:*

$$
z_{1,2}(\alpha,c) = \frac{1}{71}\left(-86+20\sqrt{5}\pm i\sqrt{5\left(-871+688\sqrt{5}\right)}\right).
$$

– *For* $\alpha = 1-2c$ *and* $c = -\frac{\sqrt{5}}{2}$, *the point* $z=-1$ *and the two real points:*

$$z_{1,2}\left(\alpha ,c\right) = \frac{1}{71}\left(-86-20\sqrt{5} \pm \sqrt{5\left(871+688\sqrt{5}\right)}\right).$$

Let us notice that the polynomials (3) and (6) that provide the strange fixed points and the free critical points, respectively, are 6-degree symmetric polynomials. This symmetry allows us to obtain the exact analytical solutions. In the following we give a brief sketch of the proof of Theorem 2; the same sketch is valid for Theorem 1 considering $b_0 = 1$.

For $b_0 \neq 0$, the roots of $Q\left(z,\alpha ,c\right)$ are the solutions of the equation

$$1 + a_1 z + a_2 z^2 + a_3 z^3 + a_2 z^4 + a_1 z^5 + z^6 = 0,$$

where $a_i = \dfrac{b_i}{b_0}$, $i = 1,2,3$ and $\alpha \neq 1-2c$. The change of variable $z + \dfrac{1}{z} = x$ leads to the cubic equation

$$x^3 + \overline{a}x^2 + \overline{b}x + \overline{c} = 0,$$

where

$$\overline{a} = a_1, \quad \overline{b} = -3 + a_2 \quad \text{and} \quad \overline{c} = a_3 - 2a_1.$$

The quadratic term is eliminated by means of the change

$$x = y - \frac{\overline{a}}{3}$$

and the final equation is

$$y^3 + py + q = 0, \tag{9}$$

where

$$\begin{aligned} p &= \overline{b} - \frac{\overline{a}^2}{3} = -3 + \frac{b_2}{b_0} - \frac{b_1^2}{3b_0^2}, \\ q &= \frac{2\overline{a}^3}{27} - \frac{\overline{a}\overline{b}}{3} + \overline{c} = \frac{b_3 - b_1}{b_0} - \frac{b_1 b_2}{3b_0^2} + \frac{2b_1^3}{27b_0^3}. \end{aligned}$$

By using the change $y = s_1 + s_2$, then $y^3 = s_1^3 + s_2^3 + 3s_1 s_2 y$ is obtained. By identifying the coefficients of this equation with the coefficients of equation (9) we obtain

$$s_1^3 + s_2^3 = -q, \tag{10}$$

$$s_1^3 s_2^3 = -\frac{p^3}{27}. \tag{11}$$

So, s_1^3 and s_2^3 are solutions of the quadratic equation $W^2 + qW - \frac{p^3}{27} = 0$, that is,

$$s_1^3, s_2^3 = \frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}.$$

Therefore, the three solutions of (9) are

$$
\begin{aligned}
x_1 &= \frac{\overline{a}}{3} + s_1 + s_2, \\
x_2 &= \frac{\overline{a}}{3} - \frac{1}{2}(s_1 + s_2) + i\frac{\sqrt{3}}{2}(s_1 - s_2), \\
x_3 &= \frac{\overline{a}}{3} - \frac{1}{2}(s_1 + s_2) - i\frac{\sqrt{3}}{2}(s_1 - s_2),
\end{aligned}
$$

Undoing all the changes, the six roots of the symmetric polynomial are obtained from

$$
z = \frac{x \pm \sqrt{x^2 - 4}}{2}, \tag{12}
$$

for the different values of variable x.

The independent term is 1 for the polynomial (3) associated to the fixed points, but we must divide by b_0 for the polynomial (6) associated to the critical points. So, we must study what happens with the critical points when $b_0 = 0$, that is, on the line $c = \frac{1-\alpha}{2}$.

Let us obtain the roots of the polynomial $Q(z, \alpha, c)$ in this case. If we make the substitution $c = \frac{1-\alpha}{2}$ in $Q(z, \alpha, c)$, as $b_0 = 0$ we have the polynomial

$$
Q\left(z, \alpha, \frac{1-\alpha}{2}\right) = c_0 z + c_1 z^2 + c_2 z^3 + c_1 z^4 + c_0 z^5 = z\,(c_0 + c_1 z + c_2 z^2 + c_1 z^3 + c_0 z^4),
$$

where

$$
\begin{aligned}
c_0 &= -10 + 8\alpha^2, \\
c_1 &= 4(-10 + 10\alpha + \alpha^2 - 3\alpha^3), \\
c_2 &= -60 + 80\alpha - 48\alpha^2 + 16\alpha^3.
\end{aligned}
$$

Then, the critical points are $z = 0$, $z = \infty$ and the four roots of the four-degree symmetric polynomial:

$$
q_4(z, \alpha) = c_0 + c_1 z + c_2 z^2 + c_1 z^3 + c_0 z^4.
$$

If $c_0 \neq 0$, we can divide $q_4(z, \alpha)$ by c_0 and consider the polynomial

$$
1 + \frac{c_1}{c_0} z + \frac{c_2}{c_0} z^2 + \frac{c_1}{c_0} z^3 + z^4.
$$

As we know that $z = 0$ is not a root of this polynomial, we make the change

$$
x = \frac{1}{z} + z.
$$

Then, it is transformed in

$$
x^2 + \frac{c_1}{c_0} x + \frac{c_2}{c_0} = 0,
$$

whose roots are

$$x_{1,2}(\alpha) = -\frac{c_1}{2c_0} \pm \frac{\alpha}{c_0}\sqrt{\alpha(\alpha-1)(40-29\alpha+9\alpha^2)}.$$

Then, the roots of $Q\left(z,\alpha,\frac{1-\alpha}{2}\right)$ are $z=0$ and the four values given by

$$z = \frac{x_{1,2} \pm \sqrt{x_{1,2}^2-4}}{2}$$

Taking into account the range of α where x_1^2 and x_2^2 are lower than 4, we obtain the number of real and complex roots of $q_4\left(z,\alpha\right)$ on the line $c=\frac{1-\alpha}{2}$.

- If $\alpha < \frac{1}{2}(-5-\sqrt{65})$, the polynomial $q_4\left(z,\alpha\right)$ has 2 real and 2 complex roots.
- For $\frac{1}{2}(-5-\sqrt{65}) \le \alpha \le 0$, the polynomial $q_4\left(z,\alpha\right)$ has 4 real roots.
- For $0<\alpha<1$, the polynomial $q_4\left(z,\alpha\right)$ has 4 complex roots.
- For $\alpha=1$, the polynomial $q_4\left(z,\alpha\right)$ has 1 real root -1 with multiplicity 4.
- For $1<\alpha\le\frac{1}{2}(-5+\sqrt{65})$, the polynomial $q_4\left(z,\alpha\right)$ has 2 real and 2 complex roots.
- For $\frac{1}{2}(-5+\sqrt{65}) < \alpha < 2$, the polynomial $q_4\left(z,\alpha\right)$ has 4 complex roots.
- For $\alpha\ge 2$, the polynomial $q_4\left(z,\alpha\right)$ has 2 real and 2 complex roots.

Finally, we must to analyze the case $c_0=0$, that is when $\alpha=\pm\frac{\sqrt{5}}{2}$. The polynomial $q_4\left(z,\alpha\right)$ becomes

$$c_1z+c_2z^2+c_1z^3 = z(c_1+c_2z+c_1z^2),$$

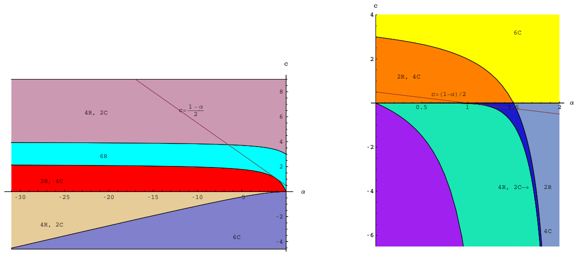
so, the critical points are $z=0$ and the two roots of the symmetric 2-degree polynomial

$$q_2\left(z,\alpha\right) = c_1+c_2z+c_1z^2.$$

Then,

- If $\alpha=-\frac{\sqrt{5}}{2}$, the polynomial $q_2\left(z,\alpha\right)$ has 2 real roots.
- If $\alpha=\frac{\sqrt{5}}{2}$, the polynomial $q_2\left(z,\alpha\right)$ has 2 complex roots.

The (α,c)-plane is divided into different regions where the number and nature of the critical points change, as was stated in [2] (see Figure 1). But the number of critical points different from 0 and ∞ reduces to 4 on the line $c=\frac{1-\alpha}{2}$; and on this line, the number of critical points different from 0 and ∞ reduces to 2 on the points $(-\frac{\sqrt{5}}{2},\frac{2+\sqrt{5}}{4})$ and $(\frac{\sqrt{5}}{2},\frac{2-\sqrt{5}}{4})$.



(a) $\alpha < 0$ (b) $\alpha \geq 0$

Fig. 1: Bifurcation regions of critical points

3 Study of the critical points

From the stated in Section 2 we can determine the number of complex and real and critical points depending on the values of parameters α and c.

For $-1 + \alpha + 2c \neq 0$ the values x_1, x_2 and x_3 are real if $D \leq 0$ and one of these values is real and the other two are conjugated complex values if $D > 0$. So, the expression $D = 0$ give us a bifurcation curve formed by the two functions:

$$C^{\pm} = \frac{-\alpha\left(\left(-3575 + 5360\alpha - 2696\alpha^2 + 384\alpha^3 + 36\alpha^4\right) \pm \alpha\sqrt{3\left(5 - 6\alpha + 2\alpha^2\right)\left(95 - 58\alpha + 6\alpha^2\right)^3}\right)}{16\left(-2 + \alpha\right)\left(-5 + 3\alpha\right)^3}$$

Moreover, as we have to evaluate $\sqrt{x_i\left(\alpha, c\right)^2 - 4}$ in 12, from $x_i = 2$, we obtain the bifurcations curves $\alpha = 2$, $c = 0$ and $c = \frac{2(2\alpha-3)}{\alpha-2}$ (the detail of these calculations can be seen in [2]).

The bifurcation curves separate the $(\alpha, c)-$plane into different regions. In order to visualize the regions properly they must be shown in different figures (see [2]). In this paper we show two of them including the line $c = \frac{1-\alpha}{2}$ (see Figure 1).

Now, let us analyze the bifurcations of the 6 critical points z_i in the $(\alpha, c)-$plane when crossing these curves. As we made with the fixed points, we consider different fixed values for the parameter α and we move the value of the parameter c in order to cover all regions. In the bifurcation diagrams the critical points z_1, z_2, z_3, z_4, z_5 and z_6 defined in (7) are depicted in different colors.

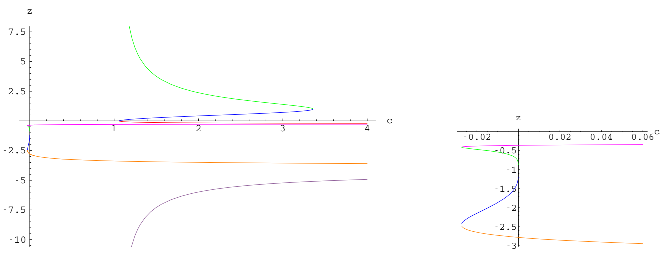
On the line $c = \frac{1-\alpha}{2}$ the number of critical points different from 0 and ∞ is reduced to four; and it is reduced to two for $\alpha = \pm\frac{\sqrt{5}}{2}$. Moreover, these points are bifurcation points because the line $c = \frac{1-\alpha}{2}$ is tangent to the bifurcation curves C^+ and C^-, respectively.

The bifurcation of critical points give us information about the number of attractive basins, because each attractive basin needs at least one critical point inside. So, a change of the critical points can produce a change in the dynamical behavior of the system.

In the following we analyze the bifurcation diagrams for $\alpha = \pm\frac{\sqrt{5}}{2}$.

3.1 Bifurcation diagram for $\alpha = -\frac{\sqrt{5}}{2}$

The bifurcation diagram of critical points for $\alpha = -\frac{\sqrt{5}}{2}$ is shown in Figure 2. We describe the bifurcations by increasing the value of c.



(a) $\alpha = -\frac{\sqrt{5}}{2}$

(b) A detail

Fig. 2: Bifurcation diagram of critical points for $\alpha = -\frac{\sqrt{5}}{2}$.

For $c < C^+\left(-\frac{\sqrt{5}}{2}\right) \approx -0.0275036$ the six critical points are complex. A first bifurcation occurs when crossing the curve C^+, two pairs of complex roots of $Q(z,\alpha,c)$ become two double real roots while z_1 and z_2 remain complex. For $c = 0$ two of the real roots reach the value -1 and afterwards become a pair of complex conjugated; moreover, the two complex z_1 and z_2 take the value -1 but they continue being complex. So, at the bifurcation point there are six real roots, one is -1 with multiplicity four. After this bifurcation there are two real and four complex roots. We can see the detail of these two bifurcations in Figure 2b.

For $c = \frac{2+\sqrt{5}}{4} \approx 1.059016994$ a new bifurcation occurs. At this point the line $c = \frac{1-\alpha}{2}$ is tangent to curve C^- so both curves are simultaneously crossed. We show that a pair of complex conjugated roots goes to zero and the other one goes to infinity. This is a remarkable case because the number of critical points has been reduced: there are only two critical points different from zero and infinity and they are inverse; so, there is only one free independent critical point. After the bifurcation point there are six different real critical points.

The last bifurcation occurs when crossing the hyperbola $c = \frac{2(2\alpha-3)}{\alpha-2}$ for $c = \frac{4}{11}\left(7+\sqrt{5}\right) \approx 3.3585701736$. In this case, the two real roots reach the value 1 and became a pair of complex conjugated roots. Then, for $c > \frac{4}{11}\left(7+\sqrt{5}\right)$ there are four real roots.

3.2 Bifurcation diagram for $\alpha = \frac{\sqrt{5}}{2}$

We show the bifurcation diagram of critical points for $\alpha = \frac{\sqrt{5}}{2}$ in Figure 3. As above, we describe the bifurcations by increasing the value of c.

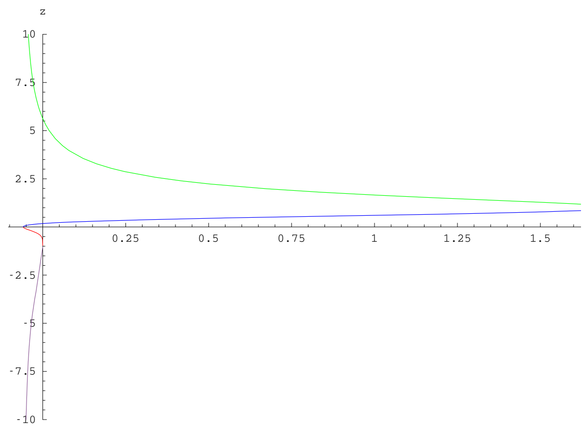


Fig. 3: Bifurcation diagram of critical points for $\alpha = \frac{\sqrt{5}}{2}$.

For $c < C^-\left(\frac{\sqrt{5}}{2}\right) \approx -12.7148$ there are six complex critical points, after crossing the curve C^- there are also six complex critical points and two of them have reached multiplicity two on the curve C^-.

For $c = \frac{2-\sqrt{5}}{4} \approx -0.05\,9017$ both C^+ and the line $c = \frac{1-\alpha}{2}$ are crossed. At this bifurcation point, two critical points become zero and other two are infinity while the other two remain complex (see Figure 3). In this case the reduction of the number of critical points is interesting from a numerical point of view because there are no real critical points different from zero and infinity. After this bifurcation, there are four real and two complex critical point.

At $c = 0$ two real roots reach the value -1 and became complex. For positive values of c we have two real and four complex critical points up to the value $c = \frac{4(7-\sqrt{5})}{11}$ corresponding to the crossing of the hyperbola $c = \frac{2(2\alpha-3)}{\alpha-2}$; at this point the two real roots reach the value 1 and there are six complex critical points for bigger values of c.

On the other hand, a change in the stability of the fixed points produces bifurcations in the dynamics. In the next section, we study this stability for given values of the parameter α.

4 Dynamical planes

In this section, we analyze the dynamical behavior of some particular methods lying on the line $c = \frac{1-\alpha}{2}$, along the intervals described in Section 2. The interest of these regions is that, for these elements of the (α, c)-family, there exist a lower number of free critical points. Therefore, a more stable behavior can be expected.

In the following we analyze the behavior of the iterative methods by using analytical tools as well as the dynamical planes associated with the scheme on quadratic polynomials. These planes has been generated by slightly modifying the routines described in [5]. In them, a mesh of 400×400 points has been used, 40 has been the maximum number of iterations involved and 10^{-3} the tolerance used as a stopping criterium. Then, if an starting point of this mesh converges to one of the fixed points of the operator, that is, it is at a distance to the fixed point (in norm) lower than 10^{-3}, it is painted in the color assigned to the point which has converged to (marked as a white star in the figures). The color used is brighter when the number of iterations is lower. If it reaches the maximum number of iterations without converging to any of the roots, it is painted in black.

4.1 $\alpha < \frac{-5-\sqrt{65}}{2}$

Let us study the case $\alpha = -9$; then, the fixed point operator is

$$Op(z, -9, 5) = -\frac{z^4\left(-319 + 104z + 86z^2 + 24z^3 + z^4\right)}{-1 - 24z - 86z^2 - 104z^3 + 319z^4}$$

and the set of fixed points is

$\{ -20.8994, -1.94675{+}4.54476i, -1.94675{-}4.54476i, 1, -0.079639{+}0.18592i, -0.079639 - 0.18592, -0.0478483, 0, \infty \}$,

whose stability is respectively given by the derivative of the operator as

$$\{26.4403, 3.66872, 3.66872, 6.76923, 3.66872, 3.66872, 26.4403, 0, 0\}.$$

Then, only 0 and ∞ are superattracting and the rest of fixed points are repulsive. Nevertheless, there exist in this case two periodic orbits of period 2,

$$\{0.2670 - 0.9637i, 0.8007 + 0.5992i\} \text{ and } \{0.2670 + 0.9637i, 0.8007 - 0.5992i\}.$$

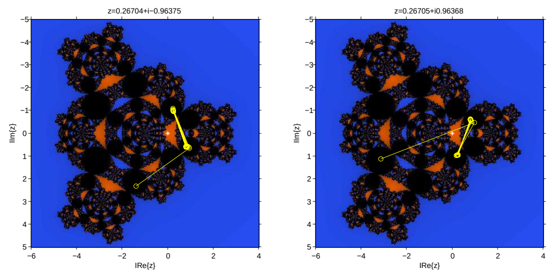
On the other hand, the list of critical points is

$$\{-15.3209, -1, 0.896866 + 0.442302i, 0.896866 - 0.442302i, -0.0652702, 0, \infty\}.$$

Then, it is clear that two free critical points yield near the elements of the periodic orbit, so it will be attractive, as it is seen in Figure 4 and can be checked by calculating the multiplier of the elements of the orbit.

$$|Op'(0.2670 + 0.9637i, -9, 5) \cdot Op'(0.8007 - 0.59927i, -9, 5)| = 0.71399 < 1.$$

Similar performance can be found for other values of α in this interval.



(a) Periodic orbit for $\alpha = -9$

(b) Periodic orbit for $\alpha = -9$

Fig. 4: Dynamical planes for $\alpha < \frac{-5\pm\sqrt{65}}{2}$

4.2 $\frac{-5-\sqrt{65}}{2} \leq \alpha \leq -\frac{\sqrt{5}}{2}$

In this range of values of α, on $c = \frac{1-\alpha}{2}$, different kinds of behavior can be found but with a common fact: the absence of attracting strange fixed points. This does not imply directly stable behavior, although it appears in a wide range of this interval. However, chaos and attracting periodic orbits can be found for specific values of parameter α.

As it can be observed in Figure 5a, for $\alpha = \frac{-5-\sqrt{65}}{2}$ there exist only two attracting fixed points, 0 and ∞. This fact is easily checked by analyzing the fixed point operator

$$Op(z, \frac{-5-\sqrt{65}}{2}, \frac{7+\sqrt{65}}{4}) = -\frac{z^4\left(-319+104z+86z^2+24z^3+z^4\right)}{-1-24z-86z^2-104z^3+319z^4}.$$

The set of strange fixed points is

$\{ -15.928, -1.93584{+}3.95386i, -1.93584{-}3.95386i, -0.0998862{+}0.204013i, -0.0998862-0.204013i, -0.0627824 \}$.

Let us notice that, in this case, $z = 1$ is not a fixed point. In fact, it can be checked that $\{-1, 1\}$ is a neutral periodic orbit, that acts as a repulsive one (chaotic area around the points of the orbit). It is a big region of unstable points that belong to the Julia set.

It can be also seen in Figure 5b and 5c, obtained for $\alpha = -6$, that small black regions appear, besides the basins of 0 and ∞. In this case, the set of fixed points is

$\{ -14.8553, -1.93295{+}3.81401i, -1.93295{-}3.81401i, 1, -0.105724{+}0.20861i, -0.105724-0.20861, -0.0673159, 0, \infty \}$

and the value of derivative of $Op(z)$ at these fixed points gives us their stability:

$$\{20.6594, 3.97736, 3.97736, 32, 3.97736, 3.97736, 20.6594, 0, 0\}.$$

Then, it is clear that all the strange fixed points are repulsive. So, what is the origin of the black regions? If we solve the equation

$$Op(Op(z, -6, \frac{7}{2}), -6, \frac{7}{2})) = z,$$

a great amount of 2-periodic orbits appear, but only nine of them are attractive, specifically

$\{-3.412851, 1.599830\}$, $\{-2.872085, 1.256418\}$,
$\{-1.202891 + 1.248838i, 1.050520 + 0.105422i\}$, $\{-1.202891 - 1.248838i, 1.050520 - 0.105422i\}$,
$\{0.795914, -0.348179\}$, $\{0.625066, -0.293010\}$,
$\{-0.400091 + 0.415373i, 0.942418 + 0.094574i\}$, $\{-0.400091 - 0.415373i, 0.942418 - 0.094574i\}$,
$\{0.305437, -0.072694\}$,

being the value of the stability function at each of them, respectively,

$$\{0.812332, 0.469646, 0.343289, 0.343289, 0.348759, 0.671555, 0.925905, 0.925905, 0.53973\}.$$

Only two of these orbits are drawn in Figures 5b and 5c.

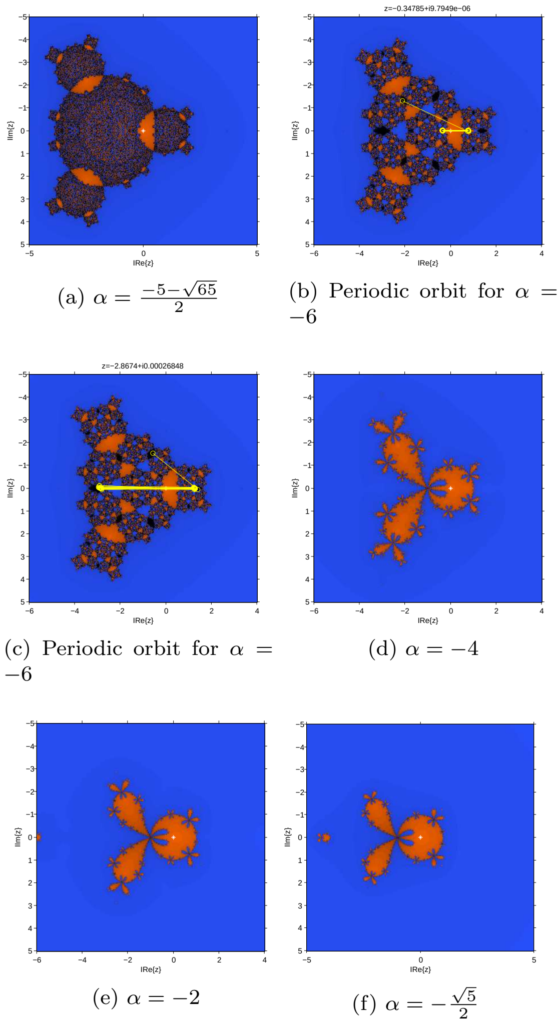
The situation is different for $\alpha = -4$, $\alpha = -2$, even $\alpha = -\frac{\sqrt{5}}{2}$ (Figures 5d to 5f, respectively); the observed stable behavior comes from the absence of attractive strange fixed or periodic points. In fact, the set of strange fixed points for $\alpha = -\frac{\sqrt{5}}{2}$ is
$\{ -4.68798, -1.92501 + 2.05826i, -1.92501 - 2.05826i,$
$1, -0.24238 + 0.259159i, -0.24238 - 0.259159i, -0.213311 \}$
and the associate multipliers are

$$\{14.201, 6.67849, 6.67849, 3.47894, 6.67849, 6.67849, 14.201\}.$$

Moreover, there are no periodic orbits and the free critical points are:

$$\{-3.38705, -1, -0.295242\}.$$

Let us remark that this value of parameter α reduces the number of free critical points to three. This provides the stable behavior in its neighborhood.



(a) $\alpha = \frac{-5-\sqrt{65}}{2}$

(b) Periodic orbit for $\alpha = -6$

(c) Periodic orbit for $\alpha = -6$

(d) $\alpha = -4$

(e) $\alpha = -2$

(f) $\alpha = -\frac{\sqrt{5}}{2}$

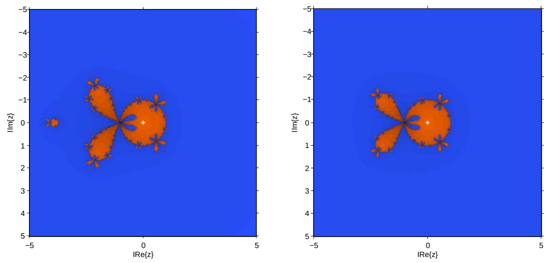
Fig. 5: Dynamical planes for $\frac{-5-\sqrt{65}}{2} \leq \alpha \leq -\frac{\sqrt{5}}{2}$

4.3 $\frac{-\sqrt{5}}{2} < \alpha \leq 0$

A very stable behavior is also observed for this range of α-values on the line $c = \frac{1-\alpha}{2}$, see Figure 6. The case $\alpha = 0$ corresponds to a stable element of the Chebyshev-Halley family, meanwhile $\alpha = -1$ (and $c = 1$), corresponds to a good element of the (α, c)-family. In this case, the associate fixed point operator is

$$Op(z, -1, 1) = z^4 \frac{1 + 24z + 22z^2 + 8z^3 + z^4}{1 + 8z + 22z^2 + 24z^3 + z^4}.$$

All the strange fixed points are repulsive and the free critical points lie on the two basins of 0 and ∞. The free critical points are $\{-28.4549, -3.19723, -1, -0.31277, -0.0351434\}$.



(a) $\alpha = -1$ (b) $\alpha = 0$

Fig. 6: Dynamical planes for $\frac{-\sqrt{5}}{2} < \alpha \leq 0$

4.4 $0 < \alpha \leq 1$

The same stable behavior is obtained for values of the parameter α in the interval $(0, 1]$. In Figure 7a we show the dynamical plane corresponding to a stable element of the (α, c)-family, for $\alpha = \frac{1}{2}$. There are only two basins of attraction, associated to the solutions of the problem.

In case $\alpha = 1$, the behavior observed in Figure 7b corresponds to operator

$$Op(z, 1, 0) = z^4,$$

that is, as stable as Newton's method but with fourth-order of convergence. This scheme is known as super-Halley's method.

Unlike Newton's scheme, this one has strange fixed points:

$$\{-0.5 - 0.866025i, -0.5 + 0.866025i, 1\},$$

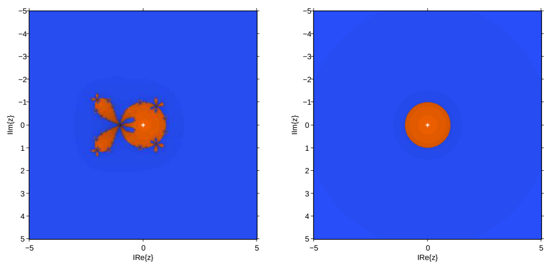
but they are repulsive as it is stated by the value of the stability function on them, $\{4, 4, 4\}$.

4.5 $1 < \alpha \leq \frac{-5+\sqrt{65}}{2}$

In this interval, we can found another value of the parameter, $\alpha = \frac{\sqrt{5}}{2}$, such that the number of free critical points is reduced to three (see Figure 8a),

$$\{-1, -0.581389 + 0.813626i, -0.581389 - 0.813626i\}.$$

The strange fixed points are



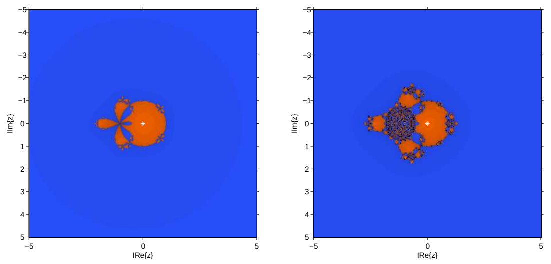
(a) $\alpha = \frac{1}{2}$ (b) $\alpha = 1$

Fig. 7: Dynamical planes for $0 < \alpha \leq 1$

$\{ -2.13533, -0.593699 + 0.931668i, -0.593699 - 0.931668i,$
$1, -0.486445 + 0.76336i, -0.486445 - 0.76336i, -0.468311 \}$
that are repulsive, as it can be stated at the sight of their values at the stability function

$$\{11.4525, 1.83595, 1.83595, 4.46589, 1.83595, 1.83595, 11.4525\}.$$

Moreover, there are no periodic orbits and the general behavior is very stable.



(a) $\alpha = \frac{\sqrt{5}}{2}$ (b) $\alpha = \frac{-5+\sqrt{65}}{2}$

Fig. 8: Dynamical planes for $1 < \alpha \leq \frac{-5+\sqrt{65}}{2}$

As in case $\alpha = \frac{-5-\sqrt{65}}{2}$, the strange fixed points are repulsive and there exists one neutral periodic orbit at $\{-1, 1\}$. This causes the chaotic region around these points (see Figure 8b).

4.6 $\frac{-5+\sqrt{65}}{2} < \alpha < 2$

For $\alpha = \frac{9}{5}$, the fixed point operator is

$$Op\left(z, \frac{9}{5}, -\frac{2}{5}\right) = z^4 \frac{199 + 100z + 10z^2 - 60z^3 - 25z^4)}{-25 - 60z + 10z^2 + 100z^3 + 199z^4},$$

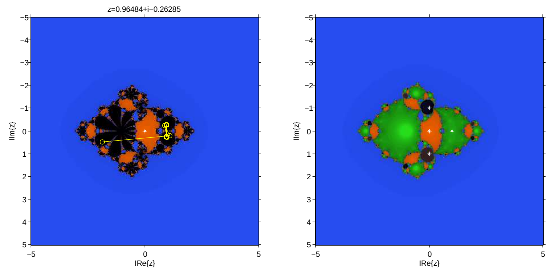
the only attracting fixed points are 0 and ∞, as the strange fixed points

$\{-3.0901, 0.0038{+}1.1140i, 0.0038{-}1.1140i, 1, 0.0030{+}0.8976i, 0.0030{-}0.8976i, -0.3236\}$

are all repulsive. The free critical points are, in this case

$\{-1, -0.165793{+}0.986161i, -0.165793{-}0.986161i, 0.951723{+}0.306959i, 0.951723{-}0.306959i\}$.

The two last free critical points are close to the 2-periodic orbit $\{0.96484 + 0.26285i, 0.96484 - 0.26285i\}$ that is then attractive. It can be observed in Figure 9a.



(a) Periodic orbit for $\alpha = \frac{9}{5}$ (b) $\alpha = 1.9$

Fig. 9: Dynamical planes for $\frac{-5+\sqrt{65}}{2} < \alpha < 2$

When $\alpha = 1.9$ is considered, there exist three attracting strange fixed points, as the value of the stability function at
$\{\ -3.17486, 0.121291 + 0.992617i, 0.121291 - 0.992617i,$
$1, 0.0236248 + 0.999721i, 0.0236248 - 0.999721i, -0.314975\ \}$
is

$$\{7.17274, 1.16358, 1.16358, 0.514469, 0.816345, 0.816345, 7.17274\}.$$

This yields to a dynamical plane, Figure 9b, with five different basins of attraction.

4.7 $\alpha \geq 2$

The case $\alpha = 2$ is specially interesting; the operator associated to the method is

$$Op\left(z, 2, -\frac{1}{2}\right) = -z^4 \frac{-11 - 6z - 2z^2 + 2z^3 + z^4}{-1 - 2z + 2z^2 + 6z^3 + 11z^4}$$

and, by solving the equation $Op(z, 2, -\frac{1}{2}) = z$, the following fixed points are found:

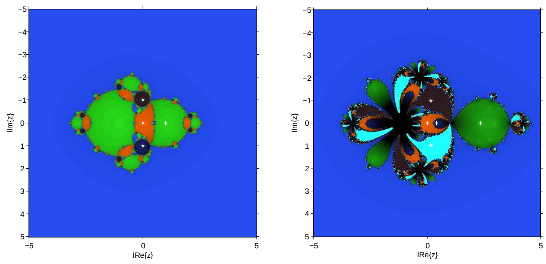
$\{-3.25426, 1, i, -i, 0.280776{+}0.959773i, 0.280776{-}0.959773i, -0.307289, 0, \infty\}$.

By evaluating them at the stability function, the following values are obtained:

$$\{7.02214, 0, 0.5, 0.5, 1.35286, 1.35286, 7.02214, 0, 0\}.$$

We observe that three superattracting points appear, 0, ∞ and 1. Moreover, $z = i$ and $z = -i$ are also attracting and they will have also their own basin of attraction (see Figure 10a), as there exist two free critical points near them, as can be observed in the list:

$$\{1, -0.0909091 + 0.995859i, -0.0909091 - 0.995859i\}.$$



(a) $\alpha = 2$ (b) $\alpha = 3$

Fig. 10: Dynamical planes for $\alpha \geq 2$

The existence of several basins of attraction corresponding to strange fixed points is a common behavior in this interval, as can be seen in Figure 10b, for $\alpha = 3$.

5 Conclusions

In a search of the most stable elements of (α, c)-family, we have found a region of the real (α, c)-plane such that the number of free critical points is reduced. As any basin of attraction must include a critical point, the existence of values of the parameters with a lower number of critical points gives us a hint about the existence of stable behavior. Iterative methods corresponding to values of (α, c) in the line $c = \frac{1-\alpha}{2}$, $\alpha \in \left[-5.9, \frac{-5+\sqrt{(54)}}{2}\right)$ present a stable behavior. This performance is due to the absence of basins of attraction of strange fixed points or attracting periodic orbits.

The study of the stability of these methods when they are applied to more complicated nonlinear equations is a subject that is still starting.

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