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SPECTRAL ANALYSIS OF THE COMPLEX HILL OPERATOR ON THE STAR GRAPH

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In memory of M. G. Gasymov on his 75th birthday

Abstract. The analysis of Hill operator is extended from the real line to a star graph. Generalizing the problem, a detailed analysis of the Hill operator on $L_2(G)$ is provided. An explicit description of the resolvent is given and the spectrum is described exactly, solved inverse problem.

1. Introduction

The main goal of the present paper is to solve direct and inverse problems on a star graph for the Sturm-Liouville operator with complex, periodic potential.

The solution of the inverse problems in spectral analysis is the recovery of the operator from its spectral data. Most of the works in this field are devoted to the so-called direct problems that study the properties of the spectrum. Inverse spectral problems, because of their nonlinearity, are more difficult to investigate, and nowadays there are only isolated fragments, not constituting a general picture, in the inverse problem theory for differential operators. These factors do not allow us to carry out necessary mathematical transformations and get the desirable results.

Difficulties become more apparent when we consider inverse problems on graphs. Graphs, mathematical structures used to model pairwise relations between objects from a certain collection. A "graph" in this context is a collection of "vertices" or "nodes" and a collection of edges that connect pairs of vertices. For this reason, until now, the inverse problems for differential operators on graphs are not solved fully (see [1, 2, 9, 10, 16, 17]).

This problem can be considered as a generalization of the classical inverse spectral problem for the Schrodinger operator on the line. More complete investigations for spectral theory of differential operators was obtained for Sturm -Liouville equation.

In this work we will consider generalization of results obtained by Gasymov [8], for direct and inverse problems for Sturm- Liouville operator with complex, periodic potential of the type $q(x) = \sum_{n=1}^{\infty} q_n e^{inx}$ in $L_2(-\infty, +\infty)$ on star graph.

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Despite the fact that these potential has complex-valued form there are specific feature: they giving rise to a real energy spectrum and more suitable for explanation of process in optical lattice.

A Sturm-Liouville problem on a graph arises, for example, when one is calculating the electronic vibrations of a complicated molecule in the framework of the free-electron model [15]. Adjoining to the compact graph infinite rays, we obtain a flexible mathematical construction that, although one dimensional reproduces some features of multidimensional objects. The models which can be obtained investigating differential operators on graphs have both features of ordinary and partial differential operators. Many of the problems can be solved exactly. Such models have already been used by physicists; therefore this area is progressive and new [11-13]. As a final remark let us mention some works [3-6].

For the first time, Gerasimenko [10] consider direct and inverse problems on graphs for Schrodinger operator. Currently, there is increasing interest solvable models on graphs in particular, as a reaction to a great deal of progress in fabricating graph-like structures of a semiconductor material, for which graph Hamiltonians represent a natural model. Among the systems that were successfully modeled by quantum graphs we mention e.g., singlemode acoustic and electro-magnetic waveguide networks.

The inverse eigenvalue problem on graphs was recently studied by Pivovarchik, Carlson and Yurko [1, 2, 7, 16]. It appears that this problem is much more complicated than the inverse spectral problem for the Sturm-Liouville operator on an interval. Therefore one can expect that the inverse spectral problem on noncompact graphs has several new features compared with the inverse problem on the line.

All the above once again underlines the urgency of the considered problem.

The paper consists of introduction and three sections. In Section 2 we introduce the main notions and give a formulation of the direct problem. In Section 3 properties of the spectrum are studied. In Section 4 we give a formulation of the inverse problem and provide a constructive procedure for the solution of the inverse problem.

2. The Complex Hill Operator on Star Graph

We will consider the special case of a metric graph so called star graph with single vertex in which a finite number of edges N_j are joined.

We assume throughout that all edges N_j are infinite and we identify $N_j = [0, \infty)$. Let us introduce some notation which will be used throughout of paper.

Domain and functions:

Let N_1, N_2, N_3 be 3 disjoint sets identified with $[0, \infty)$, and put $N := \bigcup_{k=1}^{3} N_k$.

For the notation of functions two viewpoints are used:

1. Functions y on the object N and y_k is the restriction of y to N_k;
2. 3- tuples of functions on branches N_k then sometimes we write

$$y_k(x_k, \lambda) = (y_{1k}(x_k, \lambda), y_{2k}(x_k, \lambda), y_{3k}(x_k, \lambda)).$$

Definition of operator:

We define the spaces $L_2(G)$

$$L_2(G) = \sum_{k=1}^{3} \oplus L_2(N_k),$$

with scalar product

$$(f,g)_{L_2(G)} = \sum_{k=1}^{3} (f_k, g_k)_{L_2(N_k)}$$

and $D(L_G)$ by the way

$$D(L_G) = \sum_{k=1}^{3} \oplus C^{\infty}(N_k), \;\; D(L_G) \subset L_2(G),$$

and consider the operator L_G on $D(L_G)$

$$L_G = \sum_{k=1}^{3} \oplus L_k,$$

where

$$L_k = -d^2/dx_k^2 + q_k(x_k) \tag{2.1}$$

with

$$D(L_k) = \{f(x_k) : f \in C^{\infty}(N_k)\}$$

and

$$q_k(x_k) = \sum_{n=1}^{\infty} q_{nk} e^{inx_k}, \;\; \sum_{n=1}^{\infty} |q_{nk}| < \infty. \tag{2.2}$$

Formulation of the Direct Problem. The main goal of this work is to extend the theory of complex Hill's equation

$$-y'' + q\,(x)\,y = \lambda^2 y,$$

$$q\,(x) = \sum_{n=1}^{\infty} q_n e^{inx}$$

to star graphs. This equation will be interpreted as a system of equations on $[0, \infty)$ with certain transition conditions satisfied at the vertices. The domains of these operators will be determined by a set of boundary conditions at a graph vertex.

We study the L_G on the noncompact graph defined by the following system of boundary conditions at the nodes of the original graph:

a) y is continuous at the nodes of the graph, i. e., in particular for our graph

$$y_1(0) = y_2(0) = y_3(0); \tag{2.3}$$

b) the sum of the derivatives over all the branches emanating from a node, calculated for each node, is zero

$$y_1'(0) + y_2'(0) + y_3'(0) = 0. \tag{2.4}$$

The spectral problem can be described as follows:

Find vector (y_1, y_2, y_3) satisfying the Sturm-Liouville equation

$$-y_k'' + q_k(x_k) y_k = \lambda^2 y_k, \;\; k = 1, 2, 3, \tag{2.5}$$

on N_1, N_2, N_3 coupled at zero by usually Kirchhoff conditions and complemented with initial conditions for the functions y_k.

It is known (see [8]) that for each fixed $k = 1, 2, 3$ on the edge N_k, there exists a fundamental system of solutions of equations (2.5) $\{f_{1k}(x_k, \lambda), f_{1k}(x_k, -\lambda)\}$ with properties:

$$f_{1k}(x_k, \lambda) = e^{i\lambda x_k}(1 + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}^{(k)}}{n + 2\lambda} e^{i\alpha x_k}), \tag{2.6}$$

where the numbers $V_{n\alpha}^{(k)}$ satisfy the following recurrence system

$$\begin{aligned} &\alpha(\alpha - n)V_{n\alpha}^{(k)} + \sum_{s=n}^{\alpha-1} q_{\alpha-s}^{(k)} V_{\alpha s}^{(k)} = 0, \quad 1 \le n < \alpha, \\ &\alpha \sum_{n=1}^{\alpha} V_{n\alpha}^{(k)} + q_{\alpha}^{(k)} = 0 \end{aligned} \tag{2.7}$$

and the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\alpha=n}^{\infty} \alpha \left| V_{n\alpha}^{(k)} \right|$$

converges.

Then by the solution of problem we will understand a matrix

$$Y(x, \lambda) = [y_{jk}(x_k, \lambda)]_{k,j=1,2,3}$$

on the noncompact graph on the basis of the following requirements:

1.
$$L_G Y = \lambda^2 Y;$$

2. $y_{jk}(x_k, \lambda)$ is a solution on the ray $N_k = [0, \infty)$, $k = 1, 2, 3$;

3.
$$y_{jk}(x_k, \lambda) = T_{jk}(\lambda) f_{1k}(x_k, \lambda), \; k \ne j \tag{2.8}$$

and

$$y_{kk}(x_k, \lambda) = f_{1k}(x_k, -\lambda) + R_{kk}(\lambda) f_{1k}(x_k, \lambda), \; k = 1, 2, 3, \tag{2.9}$$

where $T_{kj}(\lambda)$ the transmission coefficients and $R_{kk}(\lambda)$ the reflection coefficients.

The coefficients $T_{kj}(\lambda)$ and $R_{kk}(\lambda)$ can be found by writing down the boundary conditions (2.3), (2.4) for the solution $y_{jk}(x_k, \lambda)$.

To be specific, suppose $k = 1$, then

$$f_{11}(0, -\lambda) + R_{11}(\lambda) f_{11}(0, \lambda) = T_{12}(\lambda) f_{12}(0, \lambda) = T_{13}(\lambda) f_{13}(0, \lambda),$$

$$f'_{11}(0, -\lambda) + R_{11}(\lambda) f'_{11}(0, \lambda) + T_{12}(\lambda) f'_{12}(0, \lambda) + T_{13}(\lambda) f'_{13}(0, \lambda) = 0.$$

We solve these equations for $R_{11}(\lambda)$, $T_{12}(\lambda)$ and $T_{13}(\lambda)$. Noting that for the Wronskian of solutions, $W[f_{11}(0, \lambda), f_{11}(0, -\lambda)] = 2i\lambda$, we obtain

$$R_{11}(\lambda) = -\frac{f_{11}(0, -\lambda)}{f_{11}(0, \lambda)} + \frac{2i\lambda}{f_{11}(0, \lambda) f_{11}(0, -\lambda) A(\lambda)},$$

$$T_{12}(\lambda) = \frac{2i\lambda}{f_{12}(0, \lambda) f_{11}(0, -\lambda) A(\lambda)},$$

$$T_{13}(\lambda) = \frac{2i\lambda}{f_{13}(0, \lambda) f_{11}(0, -\lambda) A(\lambda)},$$

where

$$A(\lambda) = \left[\frac{f_{11}'(0,\lambda)}{f_{11}(0,\lambda)} + \frac{f_{12}'(0,\lambda)}{f_{12}(0,\lambda)} + \frac{f_{13}'(0,\lambda)}{f_{13}(0,\lambda)}\right] f_{11}(0,\lambda) f_{12}(0,\lambda) f_{13}(0,\lambda).$$

3. The Properties of Spectrum

Theorem 1. *L_G has no real eigenvalues. It's continuous spectrum filling the axis $[0,\infty)$, and there may be second-order spectral singularities on the continuous spectrum, which must coincide with numbers of the form $(\frac{n}{2})^2$, $n \in N$.*

Proof. As we introduce above the solution of the problem on the graph for $x \in [0,\infty)$, is

$$\{ y_{1k}(x_k,\lambda), y_{2k}(x_k,\lambda), y_{3k}(x_k,\lambda)\},$$

where $y_{jk}(x_k,\lambda)$ determines by (2.8), (2.9).

So, the solutions of the spectral problem on the ray N_k, $k = 1,2,3$, will be

$$y_{kk}(x_k,\lambda) = f_{1k}(x_k,-\lambda) + R_{kk}(\lambda) f_{1k}(x_k,\lambda),$$
$$y_{jk}(x_k,\lambda) = T_{jk}(\lambda) f_{1k}(x_k,\lambda),\ k \neq j, j = 1,2,3.$$

First, we shall prove that the operator L_G has no positive eigenvalues. We recall that in each edge the equation (2.5) has fundamental solutions $f_{1k}(x_k,\lambda)$, $f_{1k}(x_k,-\lambda)$.

Then in case $\lambda^2 > 0$, the solution of equation (2.5) can be written in the form

$$y_k(x_k,\lambda) = C_1 e^{i|\lambda| x_k}\left(1 + \sum_{n=1}^{\infty} \tfrac{1}{n+2|\lambda|} \sum_{\alpha=n}^{\infty} V_{n\alpha}^{(k)} e^{i\alpha x_k}\right) +$$
$$+ C_2 e^{-i|\lambda| x_k}\left(1 + \sum_{n=1}^{\infty} \tfrac{1}{n-2|\lambda|} \sum_{\alpha=n}^{\infty} V_{n\alpha}^{(k)} e^{i\alpha x_k}\right).$$

So, $y_k(x_k,\lambda) \notin L_2(0,+\infty)$ since the principle parts of the solutions are periodic.

We turn to the construction of a "spectral" Green's function on a graph. By the spectral Green's function $G(x,t,\lambda)$ we shall understand a matrix

$$G(x,t,\lambda) = [G_{jk}(x_k,t_j,\lambda)]_{j,k=1,2,3}$$

which satisfies an equation with a δ function on the right-hand side,

$$-\frac{d^2 G_{jk}}{dx_k^2} + q_k(x_k) G_{jk} - \lambda^2 G_{jk} = \delta_{jk}\delta(x_j - t_j)$$

and the (2.3), (2.4) boundary conditions:

$$G_{j1}(0,t_j,\lambda) = G_{j2}(0,t_j,\lambda) = G_{j3}(0,t_j,\lambda),$$
$$G_{j1}'(0,t_j,\lambda) + G_{j2}'(0,t_j,\lambda) + G_{j3}'(0,t_j,\lambda) = 0.$$

Now, by using standard method we will construct Green function of considered problem.

We will seek the solution of problem on the ray x_k, $y_k = (y_{1k}, y_{2k}, y_{3k})$ as

$$y_k(x,\lambda) = C_{1k}(x_k,\lambda) y_{kk}(x_k,\lambda) + C_{2k}(x_k,\lambda) y_{j^*k}(x_k,\lambda),$$

where without of loss generality we can take as y_{j^*k} one of linearly depending solutions

$$T_{jk}(\lambda) y_{jk}(x_k,\lambda),\ k \neq j$$

on the ray N_k. Then by the standard method, taking into account that fulfilled the conditions (2.3), (2.4), we obtained that

$$y_k(x_k,\lambda)=\int_0^\infty G(x,t,\lambda)f(t_k)\,dt_k+\frac{A_k(\lambda)}{A(\lambda)}f_{1k}(x_k,\lambda),$$

where

$$G(x,t,\lambda)=\begin{cases} G_{kk}(x_k,t_k,\lambda),\\ G_{jk}(x_k,t_j,\lambda).\end{cases}$$

Here

$$G_{kk}(x_k,t_k,\lambda)=\frac{1}{2i\lambda}\begin{cases} [f_{1k}(x_k,-\lambda)+R_{kk}(\lambda)\,f_{1k}(x_k,\lambda)]f_{1k}(t_k,\lambda)\,, & t_k\le x_k,\\ [f_{1k}(t_k,-\lambda)+R_{kk}(\lambda)\,f_{1k}(t_k,\lambda)]f_{1k}(x_k,\lambda)\,, & t_k\ge x_k,\end{cases}$$

and

$$G_{jk}(x_k,t_j,\lambda)=\frac{1}{2i\lambda}T_{jk}(\lambda)\,f_{1k}(x_k,\lambda)\,f_{1j}(t_j,\lambda)\,,\quad for\ k\ne j,\ k,j=1,2,3.$$

The Green function $G(x,t,\lambda)$ has poles of first order at the points $\lambda_0=\frac{n}{2}$, $n\in N$. Therefore $\lambda_0^2=(\frac{n}{2})^2$ is a spectral singularity of the operator L_G in the sense of Naimark [14]. In order to all numbers $\lambda^2>0$ belong to the continuous spectra it suffices to show that domain of value $R_{L_G-\lambda^2 I}$ of the operator $\left(L_G-\lambda^2 I\right)$ is dense in $L_2(G)$), so that the orthogonal complement of the set $R_{L_G-\lambda^2 I}$ consists of only zero element. However, the orthogonal complement of the set $R_{L_G-\lambda^2 I}$ coincide with space of the solutions of equation $L_G^* f=\lambda^2 f$. It is easy to see that, the operator L_G^* is adjoint to operator L_G.

Let $\psi(x)\in L_2(0,+\infty)$, $\psi(x)\ne 0$ and

$$\int_0^{+\infty}\left(L_G f-\lambda^2 f\right)\overline{\psi(x)}dx=0 \tag{3.1}$$

be satisfied for any $f(x)\in D(L_G)$.

From (3.1) it follows that $\psi(x)\in D(L_G^*)$ and $\psi(x)$ is an eigenfunction of operator L_G^* corresponding to eigenvalues λ. More exactly $\overline{\psi(x)}$ is the solution of the equation

$$-z_k''+q_k(x_k)\,z_k=\lambda^2 z_k \tag{3.2}$$

belonging to $L_2(G)$. We obtained that $\psi(x)=0$, since the operator generated by expression standing at the left hand side of (3.2), is an operator of type L_G. This contradiction shows that domain of value $R_{L_G-\lambda^2 I}$ of the operator $\left(L_G-\lambda^2 I\right)$ is everywhere dense in $L_2(G)$. The theorem is proved.

Corollary 1. *The eigenvalues of L_G are finite and coincide with zeros of function $A(\lambda)$.*

4. The Inverse Spectral Problem

We consider inverse problems of recovering differential operator on each fixed edge.

Since the coefficients $T_{kj}(\lambda)$ and $R_{kk}(\lambda)$, $k,j=1,2,3$ can be found by using matching conditions (2.3), (2.4) at central vertex, it is right to formulate and to solve the inverse problem recovering of the potentials $q_k(x_k)$ at each edge by reflection coefficients $R_{kk}(\lambda)$.

Inverse Problem: Given reflection coefficients $R_{kk}(\lambda)$ on each edge N_k, construct the potentials $q_k(x_k)$, $k = 1, 2, 3$.

Theorem 2. *In each fixed edges* $k = 1, 2, 3$ *satisfied*

$$\lim_{\lambda \to n/2} (n - 2\lambda) R_{kk}(\lambda) = \mathrm{V}_{\mathrm{nn}}^{(k)}.$$

Proof. As mentioned above the

$$R_{kk}(\lambda) = \frac{f_{1k}(0, -\lambda)}{f_{1k}(0, \lambda)} + \frac{2i\lambda}{f_{1k}(0, \lambda) f_{1k}(0, -\lambda) A(\lambda)}.$$

Easy to obtain that (see [7], remark2)

$$\frac{\lim\limits_{\lambda \to n/2} (n - 2\lambda) f_{1k}(0, -\lambda)}{\lim\limits_{\lambda \to n/2} f_{1k}(0, \lambda)} = \frac{V_{nn}^{(k)} f_{1k}(0, n/2)}{f_{1k}(0, n/2)} = V_{nn}^{(k)}.$$

It follows from representation (2.6) that $f_{1k}(x, -\lambda)$ is meromorphic function on λ and has first order poles at points $\frac{n}{2}$, $n \in N$ and

$$f_{1k}(x, -\lambda) = \frac{f_{nk}(x)}{n - 2\lambda} + \sum_{\gamma=1}^{\infty} \sum_{\alpha=\gamma}^{n-1} \frac{V_{\gamma\alpha}^{(k)}}{\gamma - 2\lambda} e^{i(\alpha-\lambda)x} + \sum_{\gamma=1}^{\infty} \sum_{\alpha=n+1}^{\infty} \frac{V_{\gamma\alpha}^{(k)}}{\gamma - 2\lambda} e^{i(\alpha-\lambda)x}.$$

It means that the function $\frac{1}{f_{1k}(x, -\lambda)}$ has not poles at points $\frac{n}{2}$, $n \in N$. Therefore

$$\lim_{\lambda \to n/2} (n - 2\lambda) \frac{2i\lambda}{f_{1k}(0, \lambda) f_{1k}(0, -\lambda) A(\lambda)} = 0.$$

Then, we obtain that

$$\lim_{\lambda \to n/2} (n - 2\lambda) R_{kk}(\lambda) = \lim_{\lambda \to n/2} (n - 2\lambda) \tfrac{f_{1k}(0, -\lambda)}{f_{1k}(0, \lambda)} + \\ + \lim_{\lambda \to n/2} (n - 2\lambda) \tfrac{2i\lambda}{f_{1k}(0, \lambda) f_{1k}(0, -\lambda) A(\lambda)} = V_{nn}^{(k)}.$$

The theorem is proved.

We obtain a constructive procedure for solution of inverse problem.

Our plan is following.

Step 1: Computing residues of $R_{kk}(\lambda)$ for each fixed $k = 1, 2, 3$ we find the $V_{nn}^{(k)}$, $n = 1, 2, 3, ...$, $k = 1, 2, 3$.

Step 2: Using the $V_{nn}^{(k)}$ we find all rest numbers $V_{n\alpha}^{(k)}$ from

$$V_{m,\alpha+m}^{(k)} = V_{nn}^{(k)} \sum_{n=1}^{\alpha} \frac{V_{n\alpha}^{(k)}}{n + m}, \quad \alpha = 1, 2, ...$$

Step 3: Reconstruct the potentials $q_k(x_k)$ uniquely and effectively by (2.7).

Following Theorem give us the answer of question: When the sequence $V_{nn}^{(k)}$ may be sequence "norming" numbers for an operator of type L.

Theorem 3. *The numbers* $V_{nn}^{(k)}$ *are "norming" numbers for each fixed* k *for an operator of type L with potential of the form (2.2) if following condition are* *satisfied:*

$$\sum_{m=1}^{\infty} m\left|V_{mm}^{(k)}\right| = \rho_1 < \infty,$$

$$\sum_{m=1}^{\infty} \frac{\left|V_{mm}^{(k)}\right|}{m+1} = \rho < 1.$$

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