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This paper must be cited as:

Bivià-Ausina, C. (2015). Multiplicity and Lojasiewicz exponent of generic linear sections of monomial ideals. *Bulletin of the Australian Mathematical Society*. 91(2):191-201.
doi:10.1017/S0004972714001154.



The final publication is available at

<http://dx.doi.org/10.1017/S0004972714001154>

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Additional Information

MULTIPLICITY AND ŁOJASIEWICZ EXPONENT OF GENERIC LINEAR SECTIONS OF MONOMIAL IDEALS

C. BIVIÀ-AUSINA

ABSTRACT. We obtain a characterization of the monomial ideals $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ of finite colength that satisfy the condition $e(I) = \mathcal{L}_0^{(1)}(I) \cdots \mathcal{L}_0^{(n)}(I)$, where $\mathcal{L}_0^{(1)}(I), \dots, \mathcal{L}_0^{(n)}(I)$ is the sequence of mixed Lojasiewicz exponents of I and $e(I)$ is the Samuel multiplicity of I . These are the monomial ideals whose integral closure admits a reduction generated by homogeneous polynomials.

1. INTRODUCTION

Let (R, \mathbf{m}) denote a local ring of dimension n . Let I be an \mathbf{m} -primary ideal of R . There are two important numbers attached to I : the multiplicity of I , denoted by $e(I)$ (see for instance [9], [14] or [24]), and the Lojasiewicz exponent of I , that is usually denoted by $\mathcal{L}_0(I)$ (see [15], [22] and [23]). We shall also refer to \mathbf{m} -primary ideals as ideals of finite colength. We recall that $\mathcal{L}_0(I)$ is originally defined for ideals of the ring \mathcal{O}_n of analytic function germs $(\mathbb{C}^n, 0) \rightarrow \mathbb{C}$ around the origin. That is if I is generated by $g_1, \dots, g_r \in \mathcal{O}_n$, then $\mathcal{L}_0(I)$ is defined as the infimum between all positive real numbers α such that

$$\|x\|^\alpha \leq C \sup_i |g_i(x)|$$

for some constant $C > 0$ and all x that belongs to some open neighbourhood of the origin in \mathbb{C}^n . Lejeune and Teissier showed in [15] a relation between $\mathcal{L}_0(I)$ and the asymptotic Samuel function of I and, consequently, with the integral closure of I . This relation is the motivation of the definition of $\mathcal{L}_0(I)$ for an arbitrary ideal I of finite colength in a local ring (R, \mathbf{m}) . Let us explain this more precisely.

Let us fix a local ring (R, \mathbf{m}) . Let I be an ideal of R and let $h \in R$. Then the *order of h with respect to I* is defined as $\text{ord}_I(h) = \sup\{r \geq 1 : h \in I^r\}$. Therefore $\text{ord}_I(0) = +\infty$. It is proven in [15, Section 0.2] and [17] that the sequence $\{\frac{\text{ord}_I(h^r)}{r}\}_{r \geq 1}$ has a limit in $\mathbb{R}_{\geq 0} \cup \{+\infty\}$. The *asymptotic Samuel function* of I is defined as the function $\bar{\nu}_I : R \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$ given by

$$\bar{\nu}_I(h) = \lim_{r \rightarrow \infty} \frac{\text{ord}_I(h^r)}{r}$$

2010 *Mathematics Subject Classification*. Primary 13B22; Secondary 13H15.

Key words and phrases. Lojasiewicz exponents, integral closure of ideals, mixed multiplicities of ideals, monomial ideals, Newton polyhedra.

The first author was partially supported by DGICYT Grant MTM2012-33073.

for all $h \in R$, where we set $\bar{\nu}_I(0) = +\infty$. We remark that $\bar{\nu}_I(h) = 0$, for all $h \notin \sqrt{I}$. The number $\bar{\nu}_I(h)$ is also known as the *reduced order of h with respect to I* . It is known that the range of $\bar{\nu}_I$ is $\mathbb{Q}_{\geq 0} \cup \{+\infty\}$ (see for instance [14, § 10]).

If I is an ideal of \mathcal{O}_n such that \bar{I} is a monomial ideal (that is, \bar{I} is generated by monomials) then $\bar{\nu}_I$ can be expressed in terms of the Newton polyhedron of I (see [2]). If I and J are ideals of R , then we define

$$\bar{\nu}_I(J) = \min\{\bar{\nu}_I(h) : h \in J\}.$$

Thus the result of Lejeune-Teissier to which we referred before says that, if I is an ideal of \mathcal{O}_n of finite colength and \mathbf{m}_n denotes the maximal ideal of \mathcal{O}_n , then

$$(1) \quad \mathcal{L}_0(I) = \frac{1}{\bar{\nu}_I(\mathbf{m}_n)}.$$

The above equality is used as the definition of Lojasiewicz exponent of an arbitrary ideal I of finite colength in a local ring (R, \mathbf{m}) . We also remark that equality (1) is equivalent to $\mathcal{L}_0(I) = \inf\{r/s : r, s \in \mathbb{Z}_{\geq 1}, \mathbf{m}_n^r \subseteq \bar{I}^s\}$ (see [15, Section 7]).

The notion of multiplicity of an ideal was extended by Risler and Teissier [22] to sequences of \mathbf{m} -primary ideals thus leading to the notion of *mixed multiplicity* $e(I_1, \dots, I_n)$ of n \mathbf{m} -primary ideals in R (see [14, § 17.4]). The motivation of this generalization has its origin in the study developed by Teissier of the Milnor number of the restriction of a given function germ $f \in \mathcal{O}_n$ to generic subspaces of \mathbb{C}^n of different dimensions [22]. The study of mixed multiplicities of ideals was further developed by Rees in [19]. Let (R, \mathbf{m}) be a local ring and let I_1, \dots, I_n be ideals of R . Then we define

$$(2) \quad \sigma(I_1, \dots, I_n) = \sup_{r \in \mathbb{Z}_{\geq 1}} e(I_1 + \mathbf{m}^r, \dots, I_n + \mathbf{m}^r).$$

In general $\sigma(I_1, \dots, I_n)$ can be infinite. In [4, p. 393] we characterized the finiteness of $\sigma(I_1, \dots, I_n)$. From (2) it is clear that if each ideal has finite colength then $\sigma(I_1, \dots, I_n)$ exists and it is equal to the usual mixed multiplicity $e(I_1, \dots, I_n)$. We remark that $\sigma(I_1, \dots, I_n)$ coincides with the mixed multiplicity defined by Rees in [18, p. 181]. Then we also refer to $\sigma(I_1, \dots, I_n)$ as the *Rees' mixed multiplicity of I_1, \dots, I_n* .

Analogous to the generalization of the notion of multiplicity leading to mixed multiplicities, we introduced in [3] the notion of Lojasiewicz exponent of n ideals I_1, \dots, I_n in a local ring (R, \mathbf{m}) of dimension n (see Section 2 and [5] for details). We denote this number by $\mathcal{L}_0(I_1, \dots, I_n)$. In order to define $\mathcal{L}_0(I_1, \dots, I_n)$, the ideals I_1, \dots, I_n are not assumed to have finite colength but the condition $\sigma(I_1, \dots, I_n) < \infty$ is needed. Therefore, if I denotes an ideal of finite colength of R and $i \in \{1, \dots, n\}$, then we define the *i -th relative Lojasiewicz exponent of I* as

$$\mathcal{L}_0^{(i)}(I) = \mathcal{L}_0(I, \dots, I, \mathbf{m}, \dots, \mathbf{m}),$$

where I is repeated i times and \mathbf{m} is repeated $n - i$ times. In particular we have $\mathcal{L}_0^{(n)}(I) = \mathcal{L}_0(I)$ and $\mathcal{L}_0^{(1)}(I) = \text{ord}(I)$.

Let (R, \mathbf{m}) denote an equicharacteristic regular local ring of dimension $n \geq 2$ with residue field \mathbf{k} , $\text{char}(\mathbf{k}) = 0$. Let I be an ideal of R of finite colength and let us fix an index $i \in \{1, \dots, n\}$. M. Hickel proved in [12, Théorème 1.1] that there exists a Zariski open set $U^{(i)}$ of the Grassmannian $G_{\mathbf{k}}(i, n)$ of subspaces of dimension i of \mathbf{k}^n such that $\bar{\nu}_{IR_H}(\mathbf{m}_H)$ does not depend on H , for all $H \in U^{(i)}$, where we assume that H is the zero set of the collection of \mathbf{k} -linear forms h_1, \dots, h_{n-i} , the quotient ring $R/\langle h_1, \dots, h_{n-i} \rangle$ is denoted by R_H and \mathbf{m}_H is the maximal ideal of R_H . By [6, Lemma 4.9], we have $\mathcal{L}_0^{(i)}(I) = (\bar{\nu}_{IR_H}(\mathbf{m}_H))^{-1}$, for all $i = 1, \dots, n$. We remark that $(\bar{\nu}_{IR_H}(\mathbf{m}_H))^{-1}$ is denoted by $\nu_I^{(i)}$ in [12], for all $i = 1, \dots, n$.

Moreover Hickel proved in [12] that

$$(3) \quad e(I) \leq \mathcal{L}_0^{(1)}(I) \cdots \mathcal{L}_0^{(n)}(I).$$

We remark that this inequality was generalized in [6, Theorem 4.7]. Then there appears the problem of characterizing when equality holds in (3) and to understand the structure of the ideals satisfying that equality. This is already done by Hickel in dimension $n = 2$ in [12, Proposition 5.1].

In this article we consider this problem when we suppose that I is a monomial ideal of \mathcal{O}_n or $\mathbb{C}[[x_1, \dots, x_n]]$ (see Theorem 3.5). We prove that the only monomial ideals that satisfy the equality $e(I) = \mathcal{L}_0^{(1)}(I) \cdots \mathcal{L}_0^{(n)}(I)$ are those such that \bar{I} admits a reduction generated by homogeneous polynomials. This conditions reduces considerably the possibilities for the shape of the Newton polyhedron of the ideal. As is seen in Section 3, we translate this problem into a combinatorial problem, that, at first sight is independent from Łojasiewicz exponents and captures a special class of monomial ideals.

2. MIXED ŁOJASIEWICZ EXPONENTS

In this section we recall briefly the notion of mixed Łojasiewicz exponent and some basic facts about this concept.

Let (R, \mathbf{m}) denote a Noetherian local ring of dimension $n \geq 1$ and let I_1, \dots, I_n be ideals of R such that $\sigma(I_1, \dots, I_n) < \infty$. Let J be a proper ideal of R . Let us define

$$(4) \quad r_J(I_1, \dots, I_n) = \min \{r \in \mathbb{Z}_{\geq 0} : \sigma(I_1, \dots, I_n) = \sigma(I_1 + J^r, \dots, I_n + J^r)\}.$$

We recall that $\sigma(I_1, \dots, I_n)$ denotes the Rees' mixed multiplicity of I_1, \dots, I_n , defined in (2).

If we suppose that $I_1 = \dots = I_n = I$, for some ideal I of R of finite colength and we assume that R is formally equidimensional, then we can apply the Rees's multiplicity theorem (see [11, p. 147] or [14, p. 222]) to deduce that

$$r_J(I, \dots, I) = \min \{r \in \mathbb{Z}_{\geq 0} : J^r \subseteq \bar{I}\}.$$

Definition 2.1. [5] Under the above conditions, we define the *Łojasiewicz exponent of I_1, \dots, I_n with respect to J* , denoted by $\mathcal{L}_J(I_1, \dots, I_n)$, as

$$(5) \quad \mathcal{L}_J(I_1, \dots, I_n) = \inf_{s \geq 1} \frac{r_J(I_1^s, \dots, I_n^s)}{s}.$$

We also refer to the number $\mathcal{L}_J(I_1, \dots, I_n)$ as the *mixed Lojasiewicz exponent* of I_1, \dots, I_n with respect to J . When $J = \mathbf{m}$ then we denote this number by $\mathcal{L}_0(I_1, \dots, I_n)$.

Let us observe that in order to define $\mathcal{L}_J(I_1, \dots, I_n)$ we only need the ring R to be local and Noetherian, no additional condition on R is assumed. As mentioned in the Introduction, if I is an ideal of finite colength of R , then we can associate to I the vector $\mathcal{L}_0^*(I) = (\mathcal{L}_0^{(n)}(I), \dots, \mathcal{L}_0^{(1)}(I))$, where $\mathcal{L}_0^{(i)}(I) = \mathcal{L}_0(I, \dots, I, \mathbf{m}, \dots, \mathbf{m})$, with I repeated i times and \mathbf{m} repeated $n - i$ times, $i = 1, \dots, n$. The number $\mathcal{L}_J^{(i)}(I)$ is defined analogously, for all $i = 1, \dots, n$, and any ideal J of R of finite colength.

The following result is proven in [6, Corollary 4.11].

Theorem 2.2. *Let (R, \mathbf{m}) be a quasi-unmixed Noetherian local ring and let I, J be ideals of R of finite colength. Let us suppose that the residue field $k = R/\mathbf{m}$ is infinite. Then $\mathcal{L}_J^{(1)}(I) \leq \dots \leq \mathcal{L}_J^{(n)}(I)$.*

Let us fix coordinates x_1, \dots, x_n in \mathbb{C}^n . If $k \in \mathbb{Z}_{\geq 0}^n$, then we write x^k to denote the monomial $x_1^{k_1} \cdots x_n^{k_n}$. If $h \in \mathcal{O}_n$ and the Taylor expansion of h around the origin is given by $h = \sum_k a_k x^k$, then the *support* of h , denoted by $\text{supp}(h)$, is defined as the set of those $k \in \mathbb{Z}_{\geq 0}^n$ such that $a_k \neq 0$. If $h \neq 0$, then we define the *Newton polyhedron* of h , denoted by $\Gamma_+(h)$, as the convex hull in $\mathbb{R}_{\geq 0}^n$ of $\{k + v : k \in \text{supp}(h), v \in \mathbb{R}_+^n\}$. If $h = 0$, then we set $\Gamma_+(h) = \emptyset$. If I is an ideal of \mathcal{O}_n , then the *Newton polyhedron* of I , denoted by $\Gamma_+(I)$, is defined as the convex hull of $\Gamma_+(g_1) \cup \dots \cup \Gamma_+(g_s)$, where we assume that g_1, \dots, g_s is a generating system of I . It is immediate to see that the definition of $\Gamma_+(I)$ does not depend on the chosen generating system of I .

Let us fix a subset $L \subseteq \{1, \dots, n\}$, $L \neq \emptyset$. Then we denote by \mathbb{R}_L^n the set of those $k \in \mathbb{R}^n$ such that $k_j = 0$, for all $j \notin L$. If A denotes any subset of \mathbb{R}^n then we set $A^L = A \cap \mathbb{R}_L^n$. The cardinal of L will be denoted by $|L|$.

If I is a monomial ideal of \mathcal{O}_n , then we denote by I^L the ideal of \mathcal{O}_n generated by the monomials $x^k \in I$ such that $k \in \mathbb{R}_L^n$. If $\text{supp}(I) \cap \mathbb{R}_L^n = \emptyset$, then we set $I^L = 0$. If I is a monomial ideal of \mathcal{O}_n of finite colength, then we have $I^L \neq 0$, for all $L \subseteq \{1, \dots, n\}$, $L \neq \emptyset$.

The next result gives a description of the sequence $\mathcal{L}_0^*(I)$ in terms of $\Gamma_+(I)$ when I is a monomial ideal of finite colength of \mathcal{O}_n .

Theorem 2.3. [6] *Let I be a monomial ideal of \mathcal{O}_n of finite colength. Let $i \in \{1, \dots, n\}$. Then*

$$\mathcal{L}_0^{(i)}(I) = \max \{ \text{ord}(I^L) : L \subseteq \{1, \dots, n\}, |L| = n - i + 1 \}.$$

The following result is motivated by [12, Théorème 1.1] and in turn, the case $J = \mathbf{m}$ is the motivation of the problem considered in this article. This result can be seen as a particular case of [6, Theorem 4.7] (see [6, Corollary 4.8]). We also refer to [12, Remarque 4.3 (3)] for the deduction of inequality (6) using different techniques in a slightly different context.

Proposition 2.4. [6] *Let (R, \mathfrak{m}) be a quasi-unmixed Noetherian local ring and let I and J be ideals of R of finite colength. Then*

$$(6) \quad \frac{e(I)}{e(J)} \leq \mathcal{L}_J^{(1)}(I) \cdots \mathcal{L}_J^{(n)}(I).$$

In the main result, Theorem 3.5, we obtain a characterization of when equality holds in (6) considering that I is a monomial ideal of \mathcal{O}_n and J is the maximal ideal of \mathcal{O}_n . As we will see, Theorem 3.5 can be considered as a purely combinatorial result.

3. MAIN RESULT

In this section we expose the concepts and results from [21] that we need in the proof of the main result.

If A is a subset of \mathbb{R}^n , then we denote by $\text{Conv}(A)$ the convex hull of A in \mathbb{R}^n . If $P \subseteq \mathbb{R}^n$, then we say that P is a *polytope* when there exists a finite subset $A \subseteq \mathbb{R}^n$ such that $P = \text{Conv}(A)$. If A is contained in \mathbb{Z}^n , then $\text{Conv}(A)$ is said to be *lattice polytope*.

If P is a polytope in \mathbb{R}^n , then the *dimension* of P is defined as the minimum dimension of an affine subspace containing P .

If P is any subset of \mathbb{R}^n , then we denote by $\mathbb{C}[P]$ the family of polynomial maps $h \in \mathbb{C}[x_1, \dots, x_n]$ such that $\text{supp}(h) \subseteq P$. Let $\mathbf{P} = (P_1, \dots, P_n)$ be an n -tuple of subsets of \mathbb{R}^n . We denote by $\mathbb{C}_n[\mathbf{P}]$ the set of polynomial maps $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $\text{supp}(F_i) \subseteq P_i$, for all $i = 1, \dots, n$. We can identify $\mathbb{C}_n[\mathbf{P}]$ with a finite dimensional vector space \mathbb{C}^N , for a big enough positive integer N , by associating to each map $F \in \mathbb{C}_n[\mathbf{P}]$ the vector formed by the coefficients of F . Under this identification, we say that a property holds for a generic $F \in \mathbb{C}_n[\mathbf{P}]$ when the said property holds in a dense Zariski open subset of \mathbb{C}^N .

Given a lattice polytope $P \subseteq \mathbb{R}^n$, then we say that P is *cornered* when for all $j = 1, \dots, n$ there exists some $k \in P$ such that $k_j = 0$ (see [21, p. 119]). If $P \subseteq \mathbb{R}_{\geq 0}^n$ and f_P denotes the polynomial obtained as the sum of all terms x^k such that $k \in P \cap \mathbb{Z}_{\geq 0}^n$, then we observe that P is cornered if and only if f_P is not divisible by x_j , for all $j = 1, \dots, n$.

Let $\mathbf{P} = (P_1, \dots, P_n)$ be an n -tuple of lattice polytopes in \mathbb{R}^n , then \mathbf{P} is said to be *cornered* when P_i is cornered, for all $i = 1, \dots, n$. We say that \mathbf{P} is *nice* when $F^{-1}(0)$ is finite, for a generic map $F \in \mathbb{C}_n[\mathbf{P}]$.

If $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a polynomial map such that $F^{-1}(0)$ is finite, then we denote by $m(F)$ the number of roots of F counted with multiplicities. That is, if $I(F)$ denotes the ideal of $\mathbb{C}[x_1, \dots, x_n]$ generated by the component functions of F , then applying [7, Chapter 4, Corollary 2.5] we have

$$m(F) = \dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{I(F)}.$$

If $K \subseteq \mathbb{R}^n$, then we denote by $\text{Vol}_n(K)$ the n -dimensional volume of K . Let C_1, \dots, C_n be n polytopes of \mathbb{R}^n and let $\lambda_1, \dots, \lambda_n \in \mathbb{R}_{\geq 0}$. Let $\lambda_1 C_1 + \dots + \lambda_n C_n = \{\lambda_1 k_1 + \dots + \lambda_n k_n : k_i \in C_i\}$.

$C_i, i = 1, \dots, n\}$. It is a classical result from convex geometry that $\text{Vol}_n(\lambda_1 C_1 + \dots + \lambda_n C_n)$ is a homogeneous polynomial of degree n in the variables $\lambda_1, \dots, \lambda_n$ (see for instance [7, p. 337]). The n -dimensional mixed volume of C_1, \dots, C_n is defined as the coefficient of $\lambda_1 \cdots \lambda_n$ in the polynomial $\text{Vol}_n(\lambda_1 C_1 + \dots + \lambda_n C_n)$. We denote this number by $M(C_1, \dots, C_n)$. Let us recall some elementary properties of this number (we take them from [20, p. 112], see also [7, Chapter 7, § 4]):

- (1) $M(C_1, \dots, C_n)$ is symmetric and linear in each variable.
- (2) $M(C_1, \dots, C_n) \geq 0$ and $M(C_1, \dots, C_n) = 0$ if and only if $\dim(\sum_{i \in I} C_i) < |I|$, for some non-empty subset $I \subseteq \{1, \dots, n\}$, where $|I|$ denotes the cardinal of I .
- (3) $M(C_1, \dots, C_n) \in \mathbb{Z}_{\geq 0}$, if C_i is a lattice polytope, for all $i = 1, \dots, n$.
- (4) $M(C, \dots, C) = n! \text{Vol}_n(C)$, for any polytope $C \subseteq \mathbb{R}^n$.

We refer to [7, 10, 20] for more information about $M(C_1, \dots, C_n)$.

If P is a polytope in \mathbb{R}^n , then we denote $\text{Conv}(P \cup \{0\})$ by P^0 . If $\mathbf{P} = (P_1, \dots, P_n)$ is an n -tuple of polytopes of \mathbb{R}^n , then we define $\mathbf{P}^0 = (P_1^0, \dots, P_n^0)$. In particular, it makes sense to speak about the mixed volumes $M(\mathbf{P})$ and $M(\mathbf{P}^0)$. Let us remark that \mathbf{P}^0 is always cornered. By [16, Theorem 2.4], if $F \in \mathbb{C}_n[\mathbf{P}]$, \mathbf{P} is a lattice polytope and $F^{-1}(0)$ is finite, then $m(F) \leq M(\mathbf{P}^0)$. As remarked in [21, p. 119], the conditions nice and cornered on \mathbf{P} are independent conditions. The following results tells us that both properties together in \mathbf{P} imply $m(F) = M(\mathbf{P}^0)$, for a generic $F \in \mathbb{C}_n[\mathbf{P}]$.

Theorem 3.1. [21, p. 119] *Let $\mathbf{P} = (P_1, \dots, P_n)$ be an n -tuple of lattice polytopes of $\mathbb{R}_{\geq 0}^n$. Let us suppose that \mathbf{P} is nice and cornered. Then a generic polynomial map $F \in \mathbb{C}_n[\mathbf{P}]$ has exactly $M(\mathbf{P}^0)$ roots in \mathbb{C}^n , counting multiplicities.*

The previous theorem is proven in [21] in a more general context (i.e. for polynomial maps with coefficients in a given algebraically closed field of any characteristic).

Let $\mathbf{P} = (P_1, \dots, P_n)$ and $\mathbf{Q} = (Q_1, \dots, Q_n)$ be n -tuples of polytopes in \mathbb{R}_+^n . We write $\mathbf{P} \subseteq \mathbf{Q}$ to denote that $P_i \subseteq Q_i$, for all $i = 1, \dots, n$. We also define the n -tuples of subsets $\mathbf{P} \cap \mathbf{Q} = (P_1 \cap Q_1, \dots, P_n \cap Q_n)$ and $\mathbf{Q} \setminus \mathbf{P} = (Q_1 \setminus P_1, \dots, Q_n \setminus P_n)$. If $L \subseteq \{1, \dots, n\}$, $L \neq \emptyset$, then we set $\mathbf{P}^L = (P_1 \cap \mathbb{R}_L^n, \dots, P_n \cap \mathbb{R}_L^n)$.

Next we recall a particular case of a definition introduced in [21, p. 120].

Definition 3.2. Let \mathbf{P} and \mathbf{Q} be n -tuples of polytopes in $\mathbb{R}_{\geq 0}^n$ such that \mathbf{Q} is nice and cornered. We say that \mathbf{P} counts \mathbf{Q} when

- (1) $\mathbf{P} \subseteq \mathbf{Q}$;
- (2) \mathbf{P} is nice
- (3) For any $F \in \mathbb{C}_n[\mathbf{Q} \setminus \mathbf{P}]$, the map $F + F'$ has a finite zero set and $m(F + F') = M(\mathbf{Q}^0)$, for a generic $F' \in \mathbb{C}_n[\mathbf{P}]$.

In particular, if \mathbf{P} counts \mathbf{Q} , then $m(F) = M(\mathbf{Q}^0)$, for a generic $F \in \mathbb{C}_n[\mathbf{P}]$, and therefore $M(\mathbf{P}^0) = M(\mathbf{Q}^0)$ provided that \mathbf{P} is also cornered, by Theorem 3.1.

Definition 3.3. [21, p.124] Let $\mathbf{P} = (P_1, \dots, P_n)$ be an n -tuple of polytopes in \mathbb{R}^n . The *support* of \mathbf{P} is defined as the set of indices $i \in \{1, \dots, n\}$ such that $P_i \neq \emptyset$. We denote this set by $\text{supp}(\mathbf{P})$. Let $J \subseteq \{1, \dots, n\}$. Then J is said to be *essential* for \mathbf{P} when the following conditions hold:

- (1) $J \subseteq \text{supp}(\mathbf{P})$;
- (2) $\dim(\sum_{j \in J} P_j) = |J| - 1$;
- (3) for all nonempty proper subset $J' \subset J$ we have $\dim(\sum_{j \in J'} P_j) \geq |J'|$.

Given a closed subset $P \subseteq \mathbb{R}_{\geq 0}^n$ and a vector $w \in \mathbb{R}^n$, then we define $\ell(w, P) = \min\{\langle w, k \rangle : k \in P\}$, where $\langle \cdot, \cdot \rangle$ stands for the standard inner product in \mathbb{R}^n . If $\ell(w, P) > -\infty$, then we denote by P^w , or by $\Delta(w, P)$, the subset of P formed by those $k \in P$ such that $\langle w, k \rangle = \ell(w, P)$. The sets of the form $\Delta(w, P)$, for some $w \in \mathbb{R}^n$, are called *faces* of P . If $\mathbf{P} = (P_1, \dots, P_n)$ is an n -tuple of lattice polytopes contained in $\mathbb{R}_{\geq 0}^n$, then we denote the n -tuple (P_1^w, \dots, P_n^w) by \mathbf{P}^w .

Next we state a particular case of [21, Theorem 7] that we need for our purposes (we remark that the mentioned theorem is stated for polynomials with coefficients in any algebraically closed field). Given two n -tuples of polytopes \mathbf{P} and \mathbf{Q} of \mathbb{R}^n such that $\mathbf{P} \subseteq \mathbf{Q}$, the next result gives a purely combinatorial characterization of when \mathbf{P} counts \mathbf{Q} .

Theorem 3.4. [21, p.127] *Let \mathbf{P} and \mathbf{Q} be n -tuples of lattice polytopes contained in $\mathbb{R}_{\geq 0}^n$ such that $\mathbf{P} \subseteq \mathbf{Q}$. Let us suppose that \mathbf{Q} is nice and cornered and $M(\mathbf{Q}^0) > 0$. Then \mathbf{P} counts \mathbf{Q} if and only if $\text{supp}(\mathbf{P} \cap \mathbf{Q}^w)$ contains an essential subset for \mathbf{Q}^w , for all $w \in \mathbb{R}^n \setminus \mathbb{R}_{\geq 0}$.*

If I is a monomial ideal of \mathcal{O}_n of finite colength then we define

$$a_i(I) = \max \{ \text{ord}(I^L) : L \subseteq \{1, \dots, n\}, |L| = n - i + 1 \}$$

for any $i \in \{1, \dots, n\}$. Let us observe that $a_1(I) \leq \dots \leq a_n(I)$. The us remark that the definition of $a_i(I)$ only depends on $\Gamma_+(I)$. Therefore $a_i(I) = a_i(\bar{I})$, for all $i = 1, \dots, n$. We also define the vector $\mathbf{a}(I) = (a_1(I), \dots, a_n(I))$. For instance, if $I = \langle xyz, x^a, y^b, z^c \rangle \subseteq \mathcal{O}_3$, where $3 < a \leq b \leq c$, then $\mathbf{a}(I) = (3, b, c)$.

We recall that $a_i(I) = \mathcal{L}_0^{(i)}(I)$, for all $i = 1, \dots, n$, by Theorem 2.3, however this equality is not used in the main result.

If $k \in \mathbb{R}^n$ then we denote by $|k|$ the sum of the coordinates of k .

Theorem 3.5. *Let I be a monomial ideal of finite colength of \mathcal{O}_n . Then*

$$(7) \quad e(I) \leq a_1(I) \cdots a_n(I)$$

and equality holds if and only if there exist polynomials $g_1, \dots, g_n \in \mathbb{C}[x_1, \dots, x_n]$ such that g_i is homogeneous of degree $a_i(I)$, for all $i = 1, \dots, n$, and $\bar{I} = \overline{\langle g_1, \dots, g_n \rangle}$.

Proof. Since $e(I) = e(\bar{I})$ and $a_i(I) = a_i(\bar{I})$, for all $i = 1, \dots, n$, we can assume that I is integrally closed. Then $I = \langle x^k : k \in \Gamma_+(I) \rangle$ (see for instance [14, Proposition 1.4.6]). Let

$a_i = a_i(I)$, for all $i = 1, \dots, n$. Let us denote by D_i the convex hull in \mathbb{R}^n of the set

$$\{k \in \text{supp}(I^{\mathbf{L}}) : |k| = a_i, \mathbf{L} \subseteq \{1, \dots, n\}, |\mathbf{L}| = n - i + 1\},$$

for all $i = 1, \dots, n$. By the definition of a_i we have $a_i = \text{ord}(I^{\mathbf{L}})$, for some $\mathbf{L} \subseteq \{1, \dots, n\}$ such that $|\mathbf{L}| = n - i + 1$, for all $i = 1, \dots, n$. In particular $D_i \neq \emptyset$, for all $i = 1, \dots, n$. Let \mathbf{D} denote the n -tuple of polytopes (D_1, \dots, D_n) .

If $\alpha \in \mathbb{R}_{\geq 0}$, let $\Delta(\alpha)$ denote the convex hull in \mathbb{R}^n of the set $\{k \in \mathbb{Z}_{\geq 0}^n : |k| = \alpha\}$ and let $\mathbf{\Delta}$ denote the n -tuple of polytopes $(\Delta(a_1), \dots, \Delta(a_n))$. It is clear that $\mathbf{\Delta}$ is nice and cornered and $M(\mathbf{\Delta}^0) = a_1 \cdots a_n > 0$. Clearly we have $\mathbf{D} \subseteq \mathbf{\Delta}$. We claim that \mathbf{D} counts $\mathbf{\Delta}$. To see this we will apply Theorem 3.4.

Let us fix a vector $w = (w_1, \dots, w_n) \in \mathbb{R}^n \setminus \mathbb{R}_{\geq 0}^n$ and let $w_0 = \min\{w_1, \dots, w_n\}$. Let \mathbf{L}_w denote the set of indices $\{i : w_i = w_0\}$. Then it is immediate to see that $\ell(w, \Delta(a_j)) = a_j w_0$ and $\Delta(a_j)^w = \Delta(a_j) \cap \mathbb{R}_{\mathbf{L}_w}^n$, for all $w \in \mathbb{R}^n \setminus \mathbb{R}_{\geq 0}^n$ and all $j = 1, \dots, n$. Then $\mathbf{\Delta}^w = \mathbf{\Delta}^{\mathbf{L}_w}$, for all $w \in \mathbb{R}^n \setminus \mathbb{R}_{\geq 0}^n$. In particular we have the equality

$$\{\mathbf{\Delta}^w : w \in \mathbb{R}^n \setminus \mathbb{R}_{\geq 0}^n\} = \{\mathbf{\Delta}^{\mathbf{L}} : \mathbf{L} \subseteq \{1, \dots, n\}, \mathbf{L} \neq \emptyset\}.$$

Let us fix a subset $\mathbf{L} \subseteq \{1, \dots, n\}$, $\mathbf{L} \neq \emptyset$. Let $\alpha = |\mathbf{L}|$ and let us consider the set of indices $J_{\mathbf{L}} = \{n + 1 - \alpha, \dots, n\}$. Let us show that $J_{\mathbf{L}} \subseteq \text{supp}(\mathbf{D} \cap \mathbf{\Delta}^{\mathbf{L}})$ and $J_{\mathbf{L}}$ is an essential set for $\mathbf{\Delta}^{\mathbf{L}}$.

If $i \in J_{\mathbf{L}}$, then $\alpha \geq n - i + 1$ and thus $\text{ord}(I^{\mathbf{L}'}) \leq \text{ord}(I^{\mathbf{L}}) \leq a_i$, for all $\mathbf{L}' \subseteq \mathbf{L}$ such that $|\mathbf{L}'| = n - i + 1$. In particular, if $\mathbf{L}' \subseteq \mathbf{L}$ is any subset such that $|\mathbf{L}'| = n - i + 1$, there exists some $k \in \text{supp}(I^{\mathbf{L}'}) \subseteq \text{supp}(I^{\mathbf{L}})$ such that $|k| = a_i$. Then $D_i \cap \Delta(a_i)^{\mathbf{L}} \neq \emptyset$, for all $i \in J_{\mathbf{L}}$. That is, we have $J_{\mathbf{L}} \subseteq \text{supp}(\mathbf{D} \cap \mathbf{\Delta}^{\mathbf{L}})$. We observe that $\dim \Delta(a)^{\mathbf{L}} = |\mathbf{L}| - 1$, for all $a \in \mathbb{R}_{\geq 0}$. Moreover it is immediate to see that $\sum_{j \in J_{\mathbf{L}}} \Delta(a_j)^{\mathbf{L}} = \Delta(\sum_{j \in J_{\mathbf{L}}} a_j)^{\mathbf{L}}$. In particular we have $\dim \sum_{j \in J_{\mathbf{L}}} \Delta(a_j)^{\mathbf{L}} = |\mathbf{L}| - 1$. Then we observe that $J_{\mathbf{L}}$ verifies immediately conditions (2) and (3) of the definition of essential subset for $\mathbf{\Delta}^{\mathbf{L}}$ (see Definition 3.3). Thus we deduce that \mathbf{D} counts $\mathbf{\Delta}$, by Theorem 3.4.

In particular, there exist homogeneous polynomials $g_i \in \mathbb{C}[D_i]$, $i = 1, \dots, n$, such that the map $G = (g_1, \dots, g_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ verifies that $G^{-1}(0)$ is finite and $m(G) = M(\mathbf{\Delta}^0) = a_1 \cdots a_n$. Since g_i is homogeneous, for all $i = 1, \dots, n$, and $G^{-1}(0)$ is finite, we conclude that $G^{-1}(0) = \{0\}$. Let $I(G)$ be the ideal of \mathcal{O}_n generated by g_1, \dots, g_n . Then $I(G)$ has finite colength and $e(I(G)) = a_1 \cdots a_n$. We assume that I is monomial and integrally closed, then $I(G) \subseteq I$. This implies $a_1 \cdots a_n = e(I(G)) \geq e(I)$.

Then by the Rees' multiplicity theorem (see for instance [11, p. 147] or [14, p. 222]), the equality $e(I(G)) = e(I)$ holds if and only if $I = \bar{I} = \overline{\langle g_1, \dots, g_n \rangle}$. \square

Let G denote a homogeneous polynomial map $\mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $G^{-1}(0) = \{0\}$. Let us denote by $I(G)$ the ideal of \mathcal{O}_n generated by the component functions of G . We remark that the integral closure of $I(G)$ is not always a monomial ideal, as is shown by the map $G : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ given by $g(x, y) = (xy + x^2, y^3)$.

Let I be a monomial ideal of \mathcal{O}_n of finite colength and let $\mathbf{v} = (1, \dots, 1) \in \mathbb{R}^n$. Then we denote the face $\Delta(\mathbf{v}, \Gamma_+(I))$ by $\Delta_0(I)$. Let us observe that the elements $k \in \Gamma_+(I)$ such that $|k| = \text{ord}(I)$ are contained in $\Delta_0(I)$.

Remark 3.6. Let I be a monomial ideal of \mathcal{O}_n of finite colength satisfying the condition $e(I) = a_1 \cdots a_n$, where $a_i = a_i(I)$, $i = 1, \dots, n$. Then, as we have seen in the proof of Theorem 3.5, the ideal \bar{I} admits a reduction $\langle g_1, \dots, g_n \rangle$, where g_i is a homogeneous polynomial of $\mathbb{C}[x_1, \dots, x_n]$ of degree a_i , for all $i = 1, \dots, n$. Let $d = \dim \Delta_0(I)$. Then, as a consequence of [1, Theorem 2.10], where all the reductions of monomial ideals are characterized, it follows that $a_1 = \cdots = a_d = a_{d+1}$. In particular, if $a_1 < a_2$, then $\dim \Delta_0(I) = 0$, that is, the face $\Delta(\mathbf{v}, \Gamma_+(I))$ is a vertex.

Let us also observe that, by [1, Theorem 2.10], the condition $e(I) = a_1 \cdots a_n$ forces the face $\Delta_0(I)$ to intersect all faces of $\Gamma_+(I)$ of dimension n . We conjecture that it is possible to obtain a characterization of the condition $e(I) = \prod_{i=1}^n a_i(I)$ in terms of some property of the tree determined by the vertexes and edges of $\Gamma_+(I)$.

Example 3.7. Let us consider the ideal of \mathcal{O}_3 given by $I = \langle x^a, y^b, z^c, xy, xz, yz \rangle$, where $2 \leq a \leq b \leq c$. Then $\mathbf{a}(I) = (2, 2, c)$. Moreover $e(I) = 2 + a + b + c$. Then we observe that $e(I) \leq 4c$ and equality holds if and only if $a = b = c = 2$.

Here we illustrate Remark 3.6. Let us observe that the face $\Delta_0(I)$, contains the convex hull of the supports of the monomials xy, xz, yz . Hence $\dim \Delta_0(I) = 2$. Then I does not satisfy the relation $e(I) = a_1(I)a_2(I)a_3(I)$ if $c > 2$, by Remark 3.6.

Example 3.8. Let I be the ideal of \mathcal{O}_n generated by $x^k, x_1^{a_1}, \dots, x_n^{a_n}$, where $k \in \mathbb{Z}_{\geq 0}^n$, $k \neq 0$, and a_1, \dots, a_n are integers such that $|k| \leq a_1 \leq \cdots \leq a_n$. We recall that $|k|$ denotes the sum of the coordinates of k . Then we have $\mathbf{a}(I) = (k_1 + \cdots + k_n, a_2, \dots, a_n)$ and $e(I) = k_1 a_2 \cdots a_n + \cdots + a_1 \cdots a_{n-1} k_n$. Therefore it is immediate to see that $e(I) = \prod_{i=1}^n a_i(I)$ if and only if $a_1 = \cdots = a_n$.

Let $\text{lct}(I)$ denote the log canonical threshold of an ideal I of \mathcal{O}_n and let $\mu(I)$ denote the inverse $1/\text{lct}(I)$, which is also known as the Arnold index of I . By a result of de Fernex-Ein-Mustața [8, Theorem 1.4], if I is an ideal of \mathcal{O}_n of finite colength, then $e(I) \geq (n\mu(I))^n$ and equality holds if and only if $\bar{I} = \mathbf{m}_n^{\text{ord}(I)}$. In particular, using (3), we have

$$(8) \quad (n\mu(I))^n \leq e(I) \leq \mathcal{L}_0^{(1)}(I) \cdots \mathcal{L}_0^{(n)}(I).$$

We recall that, by a result of Howald [13], if I is a monomial ideal then

$$(9) \quad \mu(I) = \min\{\alpha > 0 : \alpha \mathbf{v} \in \Gamma_+(I)\},$$

where $\mathbf{v} = (1, \dots, 1) \in \mathbb{R}^n$. Let us denote the number on the right hand of (9) by $\alpha(I)$. Then, as a consequence of (7) and [8, Theorem 1.4] we obtain the following conclusion, which is a combinatorial result.

Corollary 3.9. *Let I be a monomial ideal of \mathcal{O}_n of finite colength. Then*

$$a_1(I) \cdots a_n(I) \geq (n \alpha(I))^n$$

and equality holds if and only if $\bar{I} = \mathbf{m}_n^{\text{ord}(I)}$.

Acknowledgement. Part of this work was developed during the stay of the author at the Max Planck Institute for Mathematics, Bonn (Germany), in April 2011 and the Department of Mathematics of Saitama University (Japan) in March 2012. The author wishes to thank these institutions for their hospitality and financial support.

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