Operators
and
Matrices
Volume 8, Number 3 (2014), 861–875
doi:10.7153/oam-08-48

NORMS OF POSITIVE DEFINITE TOEPLITZ MATRICES AND DETECTION OF ALMOST PERIODIC COMPONENTS IN RANDOM SIGNALS

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(*Communicated by A. Böttcher*)

Abstract. For positive definite Toeplitz matrices $Q_N = (b(j-k))_{j,k=0}^{N-1}$ generated by trigonometric moments $b(j)$ of a non-negative measure $d\sigma(\theta), \theta \in [-\pi,\pi]$, we note that the Hilbert-Schmidt norm $\|Q_N\|_2$ and the maximal eigenvalue $\lambda_m(N)$ satisfy the following relations

$$\lim_{N\to\infty} \frac{1}{N^2}\|Q_N\|_2^2 = \sum_\alpha \mathfrak{m}_\alpha^2, \quad \lim_{N\to\infty} \frac{1}{N}\lambda_m(N) = \max_\alpha \mathfrak{m}_\alpha,$$

where $\{\mathfrak{m}_\alpha\}$ is the set of jumps of $\sigma(\theta)$. Analogous relations hold for positive definite integral operators with difference kernels. The above relations are used in order to detect hidden almost periodic components in random signals.

1. Introduction

This paper is motivated by attempts to find indications of hidden instability in random neutron signals from boiling water nuclear reactors [4]. We assume that a signal from a monitored system forms, during a sufficiently long time interval, a real-valued stationary random process $\xi(t), \quad t \in \mathbb{Z}$, with discrete time, such that the means satisfy

$$\langle \xi(t) \rangle = 0, \quad \langle \xi^2(t) \rangle = 1.$$

The correlation function of such a process,

$$b(t) := \langle \xi(t)\xi(0) \rangle = \langle \xi(t+t')\xi(t') \rangle, \quad t,t' \in \mathbb{Z},$$

is a real-valued sequence admitting the representation

$$b(t) = \int_{-\pi}^{\pi} \exp(it\theta)\, d\sigma(\theta), \tag{1}$$

Mathematics subject classification (2010): Primary 47B35; Secondary 15B05, 45E10, 60G10, 60G35.
Keywords and phrases: Positive definite Toeplitz matrices, stationary processes, positive integral operators with difference kernels, signal processing.
The financial support of the Polytechnic University of Valencia is gratefully acknowledged.
The authors are thankful to Prof. Albrecht Böttcher for valuable remarks, suggestions and criticism.

where $\sigma(\theta)$ is a non-decreasing bounded function on $[-\pi,\pi]$ [3]. By our assumptions,

$$b(0) = \sigma(\pi) - \sigma(-\pi) = 1,$$

and for any $\theta_1,\ \theta_2$ such that $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant \pi$, we have

$$\sigma(\theta_2) - \sigma(\theta_1) = \sigma(-\theta_1) - \sigma(-\theta_2). \tag{2}$$

In general, the spectral distribution function $\sigma(\theta)$ which determines the process correlation function by (1), can be split into a sum

$$\sigma(\theta) = \sigma_c(\theta) + \sigma_d(\theta) \tag{3}$$

of a continuous non-decreasing function $\sigma_c(\theta)$ and a non-decreasing step function $\sigma_d(\theta)$. Herewith $\sigma_c(\theta)$ and $\sigma_d(\theta)$ in (3) are unique up to constants [3]. Notice that both these functions $\sigma_c(\theta)$ and $\sigma_d(\theta)$ satisfy the condition (2). Actually, the problem formulated in [4] was to find, in a real-time operation mode, whether the spectral distribution function of the random signal $\sigma(\theta)$ contains or does not contain a non-trivial component $\sigma_d(\theta)$. For brevity, we will call here the random process (signal) $\xi(t)$ *smooth* if its spectral distribution function $\sigma(\theta)$ is continuous and *non-smooth* otherwise. In other words, the process is non-smooth if and only if

$$\sigma_d(\pi) - \sigma_d(-\pi) > 0.$$

The main task of the present work is to find general criteria of smoothness of the process in terms of its correlation function $b(t)$ using as a tool the sequence of positive definite Toeplitz matrices $Q_N = (b(j-k))_{j,k=0}^{N-1}$. In doing so we do not involve the assumption (2).

This paper is organized in the following way. In Section 2 we find the asymptotic expression for the Hilbert-Schmidt norm $\|Q_N\|_2$ of Q_N as $N \to \infty$, and show that the process is smooth if and only if $\|Q_N\|_2 = o(N)$. Section 3 contains a similar criterion, but with the operator norm $\|Q_N\|$ (maximal eigenvalue $\lambda_m(Q_N)$) instead of $\|Q_N\|_2$. We prove here that if $N \to \infty$, then $\lambda_m(Q_N) = \mathfrak{m} \cdot N + o(N)$, where $\mathfrak{m}$ is the maximal jump of $\sigma(\theta)$. In Section 4 both criteria are generalized for continuous time processes or for positive integral operators with difference kernels. In Section 5 we demonstrate the validity of the above smoothness criteria for signal processing by application to real neutron signals emitted by the Forsmark 1&2 boiling water reactor.

2. Hilbert-Schmidt norm of truncated Toeplitz matrices and smoothness criterion

Let us denote by $\{Q_N\}$, $N = 1,2,\ldots$, the sequence of non-negative definite Toeplitz matrices $(b(j-k))_{j,k=0}^{N-1}$ and let $\|Q_N\|_2$ be the Hilbert-Schmidt norm of Q_N:

$$\|Q_N\|_2 = \left(\sum_{j,k=0}^{N-1} |b(j-k)|^2\right)^{\frac{1}{2}} = \left(\sum_{k=0}^{N-1}\sum_{p=-k}^{k} |b(p)|^2\right)^{\frac{1}{2}}. \tag{4}$$

It is appropriate to mention here that by definition of the Hilbert-Schmidt norm the numerical sequence $\|Q_N\|_2$ is non-decreasing. Our assumptions imply that

$$|b(k)|^2 = \left|\int_{-\pi}^{\pi} \exp(ik\theta)\, d\sigma(\theta)\right|^2 \leqslant b^2(0) = 1.$$

Therefore $\|Q_N\|_2 \leqslant N$ that is $\|Q_N\|_2$ may be either $O(N)$ or $o(N)$ as $N \to \infty$.

A baby version of the Szegő and Avram-Parter theorems states that *if* $\sigma(\theta)$ *is absolutely continuous,* $d\sigma(\theta) = \sigma'(\theta)d\theta$, *and* $\sigma' \in \mathrm{L}^2(-\pi,\pi)$, *then*

$$\lim_{N\to\infty} \frac{\|Q_N\|_2}{\sqrt{N}} = \sqrt{2\pi}\,\|\sigma'\|_2 := \sqrt{2\pi}\left(\int_{-\pi}^{\pi} \left|\left(\sigma'(\theta)\right)\right|^2 d\theta\right)^{1/2}, \tag{5}$$

(see, for example, [2], Proposition 4.14). We see that $\|Q_N\|_2 = O(\sqrt{N})$ *as* $N \to \infty$ *if* σ *is absolutely continuous and* $\sigma' \in \mathrm{L}^2(-\pi,\pi)$. One may ask whether the relation $\|Q_N\|_2 = o(N)$ is still true if σ is simply a continuous non-decreasing function on $[-\pi,\pi]$. The affirmative answer stems from the following theorem by N. Wiener [5] (see also [7], Chapter III).

THEOREM 1. *For any function* $\sigma(\theta)$ *of bounded variation on* $[-\pi,\pi]$ *all the jumps* $\{\mathfrak{m}_\alpha\}$ *and Fourier coefficients*

$$b(n) = \int_{-\pi}^{\pi} \exp(in\theta)\, d\sigma(\theta)\ , \quad n = 0, \pm 1, \ldots,$$

are related by the formula

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} |b(n)|^2 = \sum_{\alpha} |\mathfrak{m}_\alpha|^2. \tag{6}$$

Indeed, setting

$$c_k = \frac{1}{2k+1} \sum_{p=-k}^{k} |b(p)|^2$$

and representing $\|Q_N\|_2^2$ in the form

$$\|Q_N\|_2^2 = \sum_{k=0}^{N-1} (2k+1)c_k,$$

taking into account that for any convergent sequence $\{c_k\}$

$$\lim_{N\to\infty} c_N = \lim_{N\to\infty} \frac{1}{N^2} \sum_{k=0}^{N-1} (2k+1)c_k$$

and applying Wiener's theorem we see that

$$\lim_{N\to\infty} \frac{1}{N^2}\, \|Q_N\|_2^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} |b(n)|^2 = \sum_{\alpha} |\mathfrak{m}_\alpha|^2. \tag{7}$$

Hence

THEOREM 2. *The process $\xi(t)$ is smooth if and only if*

$$\lim_{N\to\infty}\frac{1}{N}\|Q_N\|_2=0.$$

3. Asymptotic form of the maximal eigenvalue of a truncated correlation matrix

Let us denote by $\lambda_m(N)$ the maximal eigenvalue of a positive (i.e., positive definite) matrix Q_N, $\lambda_m(N)=\|Q_N\|$. Theorem 2 implies the following.

THEOREM 3. *The process $\xi(t)$ is smooth if and only if*

$$\lim_{N\to\infty}\frac{\lambda_m(N)}{N}\left(=\lim_{N\to\infty}\frac{\|Q_N\|}{N}\right)=0.$$

Proof. If $\xi(t)$ is smooth, then $\|Q_N\|_2/N\to 0$ by Theorem 2, and since $\|Q_N\|\leqslant\|Q_N\|_2$, we conclude that $\|Q_N\|/N\to 0$. To show the inverse implication, note first that the operator norm $\|A\|$ of any square matrix (or any nuclear operator A) satisfies the inequality

$$\|A\|\geqslant\frac{\|A\|_2^2}{\|A\|_1}, \tag{8}$$

where $\|A\|_1$ and $\|A\|_2$ are the nuclear and Hilbert-Schmidt norms, respectively. As $\|Q_N\|_1=\mathrm{Tr}Q_N=Nb(0)=N$, it follows that $\|Q_N\|\geqslant\|Q_N\|_2^2/N$. Consequently, if $\|Q_N\|/N\to 0$, then $\|Q_N\|_2^2/N^2\to 0$, and Theorem 2 now yields that $\xi(t)$ is smooth. $\square$

Actually the limit of $\lambda_m(N)/N$ can be identified for general $\sigma(\theta)$.

THEOREM 4. *Given the sequence of maximal eigenvalues (norms) $\{\lambda_m(N)\}$ of positive definite Toeplitz matrices $\{Q_N\}$, generated by a non-negative measure $d\sigma(\theta)$ as in (3), it holds that*

$$\lim_{N\to\infty}\frac{\lambda_m(N)}{N}=\max_{\alpha}\mathfrak{m}_{\alpha}.$$

Proof. The Toeplitz matrix Q_N generated by the non-decreasing function (3) is the sum of non-negative Toeplitz matrices $Q_N^{(c)}$ and $Q_N^{(d)}$, generated by non-decreasing functions σ_c and σ_d, respectively. Let us denote by $\lambda_m^{(c)}(N)$ and $\lambda_m^{(d)}(N)$ the maximal eigenvalues (norms) of the matrices $Q_N^{(c)}$ and $Q_N^{(d)}$, respectively. Since $Q_N\geqslant Q_N^{(d)}$, then

$$\lambda_m^{(d)}(N)\leqslant\lambda_m(N)=\left\|Q_N^{(d)}+Q_N^{(c)}\right\|\leqslant\left\|Q_N^{(d)}\right\|+\left\|Q_N^{(c)}\right\|=\lambda_m^{(d)}(N)+\lambda_m^{(c)}(N).$$

By Theorem 3 $\lambda_m^{(c)}(N)=o(N)$. Hence it remains to prove that

$$\lim_{N\to\infty}\frac{\lambda_m^{(d)}(N)}{N}=\max_{\alpha}\mathfrak{m}_{\alpha}. \tag{9}$$

To this end let us consider first only $\sigma_d(\theta)$ with a finite number s of jumps at some points $\theta_1,...,\theta_s \subset [-\pi,\pi)$. We will not use in this proof the fact that the points θ_α, $1 \leqslant \alpha \leqslant s$, are located symmetrically with respect to $\theta = 0$. The Toeplitz matrix $Q_N^{(d)} = (b_d(j-k))_{j,k=0}^{N-1}$, $s < N$, generated by σ_d, can be represented in this case in the form

$$Q_N^{(d)} = \sum_{\alpha=1}^{s} \mathfrak{m}_\alpha (\cdot, \mathbf{e}_\alpha)\, \mathbf{e}_\alpha, \tag{10}$$

where

$$(\cdot, \mathbf{e}_\alpha)\, \mathbf{e}_\alpha = (\exp(i(j-k)\,\theta_\alpha))_{j,k=0}^{N-1}$$

are $N \times N$ matrices of unit rank, so that Q_N transforms a $N \times 1$ column vector $\mathbf{x} = (x_j)_{j=0}^{N-1}$ into

$$Q_N^{(d)}\mathbf{x} = \sum_{\alpha=1}^{s} \mathfrak{m}_\alpha (\mathbf{x}, \mathbf{e}_\alpha)\, \mathbf{e}_\alpha, \tag{11}$$

where $(\cdot,\cdot)$ is the scalar product in the linear space $\mathbb{C}_\mathbb{N}$ of $N \times 1$ column vectors :

$$(\mathbf{x},\mathbf{y}) = \sum_{j=0}^{N-1} x_j \overline{y}_j,\ \mathbf{x} = (x_j)_{j=0}^{N-1},\ \mathbf{y} = (y_j)_{j=0}^{N-1}.$$

Notice that the vectors $\{\mathbf{e}_\alpha\}$ are linearly independent. Indeed, suppose that there is a set of complex numbers $\{z_\alpha\}$ such that

$$\sum_{\alpha=1}^{s} z_\alpha \mathbf{e}_\alpha = 0. \tag{12}$$

Due to (12), the numbers z_α satisfy the homogeneous system

$$\sum_{\alpha=1}^{s} \exp(ik\theta_\alpha)\, z_\alpha = 0, \quad k = 0,1,...,N-1.$$

But the determinant of this system is the Van der Monde determinant, which vanishes if and only if among the numbers $\{\exp(ik\theta_\alpha)\}$ there are equal ones, which is excluded here. Hence all $z_\alpha = 0$.

Let λ be a non-zero eigenvalue of $Q_N^{(d)}$ and $\mathbf{h}_\lambda$ be a corresponding non-zero eigenvector:

$$\sum_{\alpha=1}^{s} \mathfrak{m}_\alpha (\mathbf{h}_\lambda, \mathbf{e}_\alpha)\, \mathbf{e}_\alpha = \lambda \mathbf{h}_\lambda. \tag{13}$$

By (13), $\mathbf{h}_\lambda$ admits the representation:

$$\mathbf{h}_\lambda = \sum_{\alpha=1}^{s} z_\alpha \mathbf{e}_\alpha,$$

where z_α are some complex numbers, not all of which are equal to zero. Put

$$\eta_\alpha = \sqrt{\mathfrak{m}_\alpha}\, (\mathbf{h}_\lambda, \mathbf{e}_\alpha).$$

By virtue of (13), not all numbers $\eta_\alpha = 0$. Taking the scalar products of both sides of (13) with all vectors $\sqrt{\mathfrak{m}_\alpha}\mathbf{e}_\alpha$, we obtain the following homogeneous system for η_α:

$$\sum_{\alpha'=1}^{s} \sqrt{\mathfrak{m}_\alpha \mathfrak{m}_{\alpha'}}\,(\mathbf{e}_{\alpha'},\mathbf{e}_\alpha)\,\eta_{\alpha'} = \lambda\,\eta_\alpha. \tag{14}$$

Thus, the non-zero eigenvalues of $Q_N^{(d)}$ coincide, with account of their multiplicities, with the eigenvalues of the $s \times s$ Hermitian positive definite matrix

$$A_N = \left(\sqrt{\mathfrak{m}_\alpha \mathfrak{m}_{\alpha'}}\,(\mathbf{e}_{\alpha'},\mathbf{e}_\alpha)\right)_{\alpha,\alpha'=1}^{s}. \tag{15}$$

Notice that by definition of the vectors $\mathbf{e}_\alpha$, we have

$$(\mathbf{e}_{\alpha'},\mathbf{e}_\alpha) = \frac{\exp\left(iN\left(\theta_{\alpha'}-\theta_\alpha\right)\right)-1}{\exp\left(i\left(\theta_{\alpha'}-\theta_\alpha\right)\right)-1}, \quad \alpha' \neq \alpha, \quad (\mathbf{e}_\alpha,\mathbf{e}_\alpha) = N. \tag{16}$$

Hence, the matrix A_N is the sum of the diagonal matrix

$$A_{1,N} := \left(N\mathfrak{m}_\alpha \delta_{\alpha\alpha'}\right)_{\alpha,\alpha'=1}^{s}$$

and the Hermitian matrix $A_{2,N}$ with zero diagonal elements and non-diagonal elements $\sqrt{\mathfrak{m}_\alpha \mathfrak{m}_{\alpha'}}\,(\mathbf{e}_{\alpha'},\mathbf{e}_{,\alpha})$, $\alpha \neq \alpha'$. By (16) the non-diagonal elements of $A_{2,N}$ are uniformly bounded:

$$|\sqrt{\mathfrak{m}_\alpha \mathfrak{m}_{\alpha'}}\,(\mathbf{e}_{\alpha'},\mathbf{e}_\alpha)| \leqslant 2\left(\max_{\alpha'\neq\alpha}|\theta_{\alpha'}-\theta_\alpha|^{-1}\right)\left(\max_\alpha \mathfrak{m}_\alpha\right),$$

and, hence,

$$\|A_{2,N}\| \leqslant 2\,(s-1)\left(\max_{\alpha'\neq\alpha}|\theta_{\alpha'}-\theta_\alpha|^{-1}\right)\left(\max_\alpha \mathfrak{m}_\alpha\right).$$

Therefore,

$$\begin{aligned}\left[N-2\,(s-1)\left(\max_{\alpha'\neq\alpha}|\theta_{\alpha'}-\theta_\alpha|^{-1}\right)\right]\left(\max_\alpha \mathfrak{m}_\alpha\right) &\leqslant \|A_{1,N}\|-\|A_{2,N}\| \leqslant \|A_N\| \\ \leqslant \|A_{1,N}\|+\|A_{2,N}\| &\leqslant \left[N+2\,(s-1)\left(\max_{\alpha'\neq\alpha}|\theta_{\alpha'}-\theta_\alpha|^{-1}\right)\right]\left(\max_\alpha \mathfrak{m}_\alpha\right).\end{aligned}$$

We see that

$$\lambda_m^{(d)}(N) = \|A_N\| \underset{N\to\infty}{=} N\cdot\max_\alpha \mathfrak{m}_\alpha + O(1). \tag{17}$$

To prove the relation (9) for a non-decreasing step function $\sigma_d(\theta)$, $\sigma_d(\pi)-\sigma_d(-\pi) \leqslant 1$, having infinitely many points of jump, we take a small $\varepsilon > 0$ and split $\sigma_d(\theta)$ into a sum $\sigma_{1d}(\theta)+\sigma_{2d}(\theta)$ of non-decreasing step functions $\sigma_{1,d}(\theta)$ and $\sigma_{2,d}(\theta)$, where, as before, $\sigma_{1d}(\theta)$ has a finite number of jump points and $\sigma_{2d}(\theta)$ is such that

$$\int_{-\pi}^{\pi} d\sigma_{2,d}(\theta) < \varepsilon < \max_\alpha \mathfrak{m}_\alpha.$$

With respect to this split, we represent the Toeplitz matrix $Q_N^{(d)}$ as the sum $Q_N^{(1,d)} + Q_N^{(2,d)}$ of non-negative Toeplitz matrices generated by $\sigma_{1,d}(\theta)$ and $\sigma_{2,d}(\theta)$, respectively. Notice that by construction

$$\left\| Q_N^{(2,d)} \right\| \leqslant \mathrm{Tr} Q_N^{(2,d)} < N\varepsilon. \tag{18}$$

Besides,

$$\lambda_m^{(1,d)}(N) = \left\| Q_N^{(1,d)} \right\| \leqslant \lambda_m^{(d)}(N) \leqslant \lambda_m^{(1,d)}(N) + \left\| Q_N^{(2,d)} \right\|.$$

Applying the estimate (17) to $Q_N^{(1,d)}$ and taking into account the inequality (18) for $N \to \infty$ yields

$$N \cdot \max_\alpha \mathfrak{m}_\alpha + O(1) = \lambda_m^{(1,d)}(N) \leqslant \lambda_m^{(d)}(N) \leqslant N \cdot \left(\max_\alpha \mathfrak{m}_\alpha + \varepsilon \right) + O(1).$$

Finally,

$$\max_\alpha \mathfrak{m}_\alpha \leqslant \varliminf_{N\to\infty} \frac{\lambda_m^{(d)}(N)}{N} \leqslant \varlimsup_{N\to\infty} \frac{\lambda_m^{(d)}(N)}{N} \leqslant \max_\alpha \mathfrak{m}_\alpha + \varepsilon,$$

where $\varepsilon > 0$ can be taken arbitrarily small. $\square$

REMARK 1. The number $\|Q_N\| = \lambda_m(N)$ is in general only numerically available. However, there exist simple estimates in terms of the entries of the matrix Q_N. For example, let

$$(S_N b)(\theta) := \sum_{|j| \leqslant N-1} b_j e^{ij\theta}, \quad (F_N b)(\theta) := \sum_{|j| \leqslant N-1} \left(1 - \frac{|j|}{N}\right) b_j e^{ij\theta}. \tag{19}$$

Then

$$|(F_N b)(1)| \leqslant \|(F_N b)\|_\infty \leqslant \|Q_N\| \leqslant \|S_N b\|_\infty \leqslant \|S_N b\|_W \ , \tag{20}$$

where $\|c\|_\infty := \max_\theta |c(\theta)|$ is the L^∞ norm $\|c\|_W := \sum_j |c(j)|$ stands for the norm in the Wiener algebra. The upper bounds in (20) are well-known and for the lower bounds see [1], p. 122. It follows in particular that *the condition*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{p=0}^{N-1} |b(p)| = 0 \tag{21}$$

is sufficient for the smoothness of the process and that the condition

$$\lim_{N\to\infty} \frac{1}{N} \sum_{p=0}^{N-1} \left(1 - \frac{|p|}{N}\right) b(p) = 0 \tag{22}$$

is necessary for the process to be smooth.

4. Extension to continuous time processes

Real signals are, certainly, continuous time processes, $\xi(t)$. The correlation function $b(t)$ of a process having a finite second moment $\langle \xi^2(t) \rangle$ is a Hermitian positive function. As such, it admits the representation

$$b(t) = \int_{-\infty}^{\infty} \exp(i\lambda t)\, d\vartheta(\lambda), \tag{23}$$

where $\vartheta(\lambda)$ is a bounded non-decreasing function on the real axis. Like for the discrete time processes, $\vartheta(\lambda)$ can be represented, in general, as the sum

$$\vartheta(\lambda) = \vartheta_c(\lambda) + \vartheta_d(\lambda)$$

of a non-decreasing continuous function $\vartheta_c(\lambda)$ and a non-decreasing step function $\vartheta_d(\lambda)$, and we call the process *smooth* if $\vartheta(\lambda)$ is continuous and *non-smooth* otherwise. To investigate the non-smoothness characteristics of a continuous time process, we consider instead of the Toeplitz matrices Q_N, the set of non-negative integral operators

$$(B_T f)(t) = \int_0^T b(t-s) f(s)\, ds, \quad 0 < T < \infty, \tag{24}$$

in the Hilbert spaces $\mathrm{L}^2(0,T)$. Since $b(t)$ is a continuous function, all these operators are nuclear and their nuclear and Hilbert-Schmidt norms $\|B_T\|_1$ and $\|B_T\|_2$ are given by the expressions

$$\begin{aligned} \|B_T\|_1 &= Tb(0) = T \int_{-\infty}^{\infty} d\vartheta(\lambda), \\ \|B_T\|_2 &= \sqrt{2T \int_0^T \left(1 - \frac{t}{T}\right) |b(t)|^2\, dt} \\ &= \sqrt{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{4}{(\lambda-\lambda')^2} \sin^2 \frac{(\lambda-\lambda')T}{2} d\vartheta(\lambda')\, d\vartheta(\lambda)}. \end{aligned} \tag{25}$$

Let us denote, as before, by $\{\mathfrak{m}_\alpha\}$ the set of jumps of $\vartheta(\lambda)$. Using (25) and arguments similar to those employed in the proofs of Theorems 2 and 4, we obtain the following criterion of stability of a continuous time process $\xi(t)$.

THEOREM 5. *A stationary continuous time process $\xi(t)$ is smooth if and only if its correlation function $b(t)$ satisfies*

$$\lim_{T\to\infty} \frac{2}{T} \int_0^T \left(1 - \frac{t}{T}\right) |b(t)|^2\, dt = 0.$$

Otherwise,

$$\lim_{T\to\infty}\frac{2\pi}{T}\int_0^T\left(1-\frac{t}{T}\right)|b(t)|^2\,dt=\sum_\alpha \mathfrak{m}_\alpha^2.$$

We want to point out that Theorem 5 follows directly from Wiener's theorem [6] (see also [7], Chapter XIV) according to which *for the Fourier transform* $b(t)$ *of any function of bounded variation* $\vartheta(\lambda)$ *on the real axis with the set of jumps* $\{\mathfrak{m}_\alpha\}$ *the equality holds*

$$\lim_{T\to\infty}\frac{\pi}{T}\int_{-T}^T|b(t)|^2\,dt=\sum_\alpha \mathfrak{m}_\alpha^2.$$

In particular, if $\vartheta(\lambda)$ *is continuous, then*

$$\lim_{T\to\infty}\frac{\pi}{T}\int_{-T}^T|b(t)|^2\,dt=0.$$

THEOREM 6. *A stationary continuous time process* $\xi(t)$ *is smooth if and only if the operator norms* $\|B_T\|$ *of integral operators (24), where* $b(t)$ *is the correlation function of the process, are such that*

$$\lim_{T\to\infty}\frac{1}{T}\|B_T\|=0.$$

Otherwise,

$$\lim_{T\to\infty}\frac{1}{T}\|B_T\|=\max_\alpha \mathfrak{m}_\alpha. \tag{26}$$

REMARK 2. For the norm of the integral operator B_T the following estimate:

$$\|B_T\|\leqslant 2\int_0^T|b(t)|\,dt$$

is valid. As it stems from (26), *the relation*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T|b(t)|\,dt=0$$

guarantees the smoothness of the process $\xi(t)$.

The proof of [1], Theorem 5.10, for the lower bound in (20) can be modified to yield the estimates

$$\|B_T\|\geqslant\sup_{x\in\mathbb{R}}\left|\int_{-T}^T\left(1-\frac{|t|}{T}\right)b(t)\,e^{ixt}\,dt\right|\geqslant\left|\int_{-T}^T\left(1-\frac{|t|}{T}\right)b(t)\,dt\right|,$$

which shows that *the condition*

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)b(t)\,dt=0 \tag{27}$$

is necessary for the smoothness of the process.

5. Application to processing of random signals

5.1. Detection of quasi-periodic components in a random signal

The existence of jumps of the spectral distribution function $\vartheta(\lambda)$ of a stationary process is, in general, a sign of appearance of undamped oscillation components in a signal and the points of discontinuity of $\vartheta(\lambda)$ are either the frequencies of such components themselves or directly related to them. Notice that, due to physical reasons, the measurement of $\xi(t)$ is possible only at discrete moments of time with a step Δ. If t in (23) is an integer multiple of Δ, then it is evident that

$$\begin{aligned} b(t) &= \int_{-\Omega}^{\Omega}\exp(it\theta)\,d\sigma(\theta), \quad \Omega=\frac{\pi}{\Delta}, \\ \sigma(\theta) &= \sum_{n=-\infty}^{\infty}\left[\vartheta(\theta+2n\Omega)-\vartheta(2n\Omega-\Omega)\right], \quad -\Omega\leqslant\theta<\Omega. \end{aligned} \tag{28}$$

The function $\sigma(\theta)$ is bounded and non-decreasing in the interval $[-\Omega,\Omega]$. If $\vartheta(\lambda)$ loses its continuity at the points $\lambda_1,\lambda_2,\ldots$, then $\sigma(\theta)$ has a non-void set of discontinuity points

$$\left\{\theta_j'=\left(\mathbf{E}\left(\frac{\lambda_j}{2\Omega}+\frac{1}{2}\right)-\frac{1}{2}\right)2\Omega\right\}\subset[-\Omega,\Omega], \tag{29}$$

where $\mathbf{E}(x)$ is the fractional part of the number x. (In general, $\pm\theta_{j_1}'$ coincides with every $\pm\theta_{j_2}'$ such that $\lambda_{j_1}-\lambda_{j_2}$ is a multiple of 2Ω.) Therefore, in general, the jump of $\sigma(\theta)$ at a point θ_j' is the sum of the jumps of $\vartheta(\lambda)$ at all co-images of θ_j' under the mapping (29).) Taking Δ as the time measurement unit, we return to the representation (1) of $b(t)$ for integer t. Thus, the spectral distribution function $\sigma(\theta)$ inherits all discontinuities of $\vartheta(\lambda)$ from the interval $[-\Omega,\Omega]$ and also may get new ones at the points (calculated according to (29)) related to the discontinuity points of $\vartheta(\lambda)$ outside this interval. We see that the spectral distribution function for the discrete time process obtained in such a way from a continuous time process, has a non-trivial component σ_d if and only if the corresponding spectral distribution function of the initial discrete time process has non-zero jumps on some set of points. In other words, the values of a random continuous time process measured at discrete moments of time form a smooth discrete time process if and only if the initial process is smooth.

The correlation function of the discrete time process delivers not only the described gauge of non-smoothness of the process but also the following tool for the detection of the points $\{\pm\theta_\alpha\}$, which are the discontinuity points of $\sigma(\theta)$. Put

$$\Theta_N(\theta) = \sum_{k=-N+1}^{N-1} \left(1 - \frac{|k|}{N}\right) b(k) \exp ik\theta = \int_{-\pi}^{\pi} \frac{\sin^2 \frac{1}{2} N(\theta - \theta')}{N \sin^2 \frac{1}{2}(\theta - \theta')} d\sigma(\theta'). \tag{30}$$

It is not difficult to see that

$$\lim_{N\to\infty} \frac{1}{N} \Theta_N(\theta) = \sigma(\theta + 0) - \sigma(\theta - 0). \tag{31}$$

Further, take a sufficiently large N and split the interval $[-\pi, \pi]$ into equal segments of longitude δ such that $N\delta < 1$. Let $\sigma(\theta)$ have a jump $\mathfrak{m}_\alpha$ within the interval $(l\delta, (l+1)\delta)$. Since

$$\frac{2}{\pi}|x| \leqslant |\sin x| \leqslant |x|, \quad 0 \leqslant |x| \leqslant \frac{\pi}{2}$$

then, for $|\theta - l\delta| \sim \delta$, we have

$$\Theta_N(\theta) \geqslant \int_{l\delta}^{(l+1)\delta} \frac{\sin^2 \frac{1}{2} N(\theta - \theta')}{N \sin^2 \frac{1}{2}(\theta - \theta')} d\sigma(\theta') \geqslant \frac{4}{\pi^2} N \int_{l\delta}^{(l+1)\delta} d\sigma(\theta') \geqslant \frac{\pi^2}{6} N \mathfrak{m}_\alpha.$$

On the other hand, $\Theta_N(\theta) = O\left(\frac{1}{N}\right)$ for θ at fixed distance $\delta > 0$ from the growth points of σ. Besides, if the continuous part $\sigma_c(\theta)$ of $\sigma(\theta)$ is absolutely continuous and satisfies the condition $\sigma_c' \in \mathrm{L}^p(-\pi, \pi)$, $1 < p < \infty$, then applying the Hölder inequality yields

$$\begin{aligned}
\Theta_N^c(\theta) &= \int_{-\pi}^{\pi} \frac{\sin^2 \frac{1}{2} N(\theta - \theta')}{N \sin^2 \frac{1}{2}(\theta - \theta')} \sigma_c'(\theta')\, d\theta' \\
&\leqslant \frac{1}{N} \left\{ \int_{-\pi}^{\pi} \frac{\sin^{2q} \frac{1}{2} N\theta}{\sin^{2q} \frac{1}{2}\theta} d\theta \right\}^{\frac{1}{q}} \left\{ \int_{-\pi}^{\pi} \sigma'^p(\theta) d\theta \right\}^{\frac{1}{p}} \\
&\leqslant C_q N^{\frac{1}{p}} \left\| \sigma_c' \right\|_p,
\end{aligned} \tag{32}$$

where

$$C_q = \frac{\pi^2}{4} \left\{ \int_0^\infty \frac{\sin^{2q} x}{x^{2q}} dx \right\}^{\frac{1}{q}}.$$

Therefore $\Theta_N(\theta)$ can be at most $O\left(N^{\frac{1}{p}}\right)$ at points remote from the jumps of $\sigma(\theta)$. This suggests that for a real signal the appearance of one or more pronounced peaks on the graph of $\frac{1}{N}\Theta_N(\theta)$, which do not disappear with increasing N, indicates that the related process is non-smooth.

The assertion of Theorems 2 and 4 can be used for the detection of symptoms of emerging non-smoothness of a random process, which can be considered as stationary for long time intervals. The method consists in the construction of the correlation function of the process from a piece of its time series from the beginning of observation to a rather far off moment of time Υ in the future. Set, as usually,

$$\mathfrak{m} = \tfrac{1}{\Upsilon} \sum_{p=0}^{\Upsilon} \xi(p), \; b(k) = \tfrac{1}{\Upsilon-k} \sum_{p=0}^{\Upsilon-k} \xi(p+k)\,\xi(p) - \mathfrak{m}^2$$

and compute, for a sufficiently large $N < \Upsilon$, the numbers

$$\tfrac{1}{N} \sum_{p=0}^{N-1} \left| \tfrac{b(p)}{b(0)} \right|, \; \tfrac{1}{N^2} \|Q_N\|_2^2 = \tfrac{1}{N} + \tfrac{2}{N} \sum_{k=1}^{N-1} \left(1 - \tfrac{k}{N}\right) \tfrac{b^2(k)}{b^2(0)} \tag{33}$$

or the numbers

$$\frac{1}{T} \int_0^T \left(1 - \frac{t}{T}\right) |b(t)|^2 \, dt, \quad \frac{1}{T} \int_0^T |b(t)| \, dt$$

for a continuous time process. An explicit tendency of any of these numbers to be bounded, for increasing N, from below by certain positive numbers, is a serious evidence of the process non-smoothness.

The following example demonstrates that the manifestation of such a tendency begins the sooner in N the larger the contribution of the oscillating components generated by $d\sigma_d(\theta)$ into $b(0)$.

Let the correlation function of a stationary random process be given by the expressions:

$$\begin{aligned} &b(0) = 1; \\ &b(k) = \sum_{\alpha=1}^{s} \mathfrak{m}_\alpha e^{ik\theta_\alpha}, \quad k = \pm 1, \pm 2, ..., \end{aligned} \tag{34}$$

with $1 \leqslant s < \infty$, $0 < \sum_{\alpha=1}^{s} \mathfrak{m}_\alpha < 1$ and $0 < \theta_1, ..., \theta_s < \pi$.

The spectral distribution function $\sigma(\theta)$ of such a process is the sum of the spectral distribution function $\sigma_c(\theta)$ of the "white noise",

$$d\sigma_c(\theta) = \frac{p}{2\pi} d\theta, \quad p = 1 - \sum_{\alpha=1}^{s} \mathfrak{m}_\alpha,$$

and the step function $\sigma_d(\theta)$, the jump points of which are $\{\theta_\alpha\}$, and

$$\sigma_d(\theta_\alpha + 0) - \sigma_d(\theta_\alpha - 0) = \mathfrak{m}_\alpha.$$

In this special case

$$\frac{1}{N^2} \|Q_N\|_2^2 = 2 \sum_{\alpha=1}^{s} \mathfrak{m}_\alpha^2 + \frac{p(2-p)}{N} + \frac{1}{N^2} \sum_{\alpha' \neq \alpha} \mathfrak{m}_\alpha \mathfrak{m}_{\alpha'} \frac{\sin^2 \frac{1}{2} N(\theta_\alpha - \theta_{\alpha'})}{\sin^2 \frac{1}{2}(\theta_\alpha - \theta_{\alpha'})}. \tag{35}$$

Hence, the first term on the right hand side of (35) becomes dominant for

$$N > p\,(2-p)\left(\sum_{\alpha=1}^{s} \mathfrak{m}_{\alpha}^{2}\right)^{-1}.$$

5.2. Numerical results

Let us apply the latter results to the investigation of real signals obtained from the Forsmark 1&2 boiling water reactor (BWR) [4].

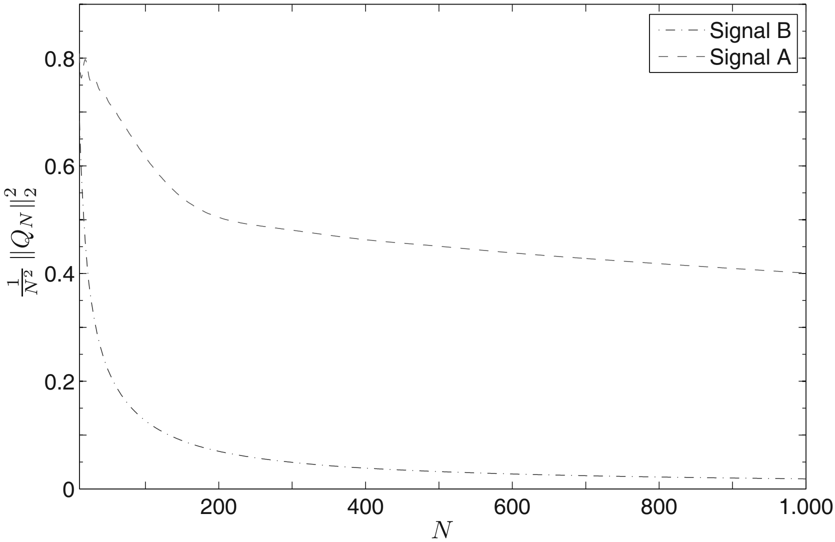

Figure 1: *Sequences of* $\frac{1}{N^2}\|Q_N\|_2^2$ *for the signals* A *(dashed line) and* B *(dash-dotted line).*

In Fig. 1 we display results for the Hilbert-Schmidt norms of Toeplitz matrices constructed for two different shots of real signals obtained in the Forsmark BWR: A for a near unstable mode and B for a quiet mode. The sampling time interval of these signals equals $0.08\,\mathrm{s}$, and both of them consist of 4209 points. We observe the difference between a more unstable mode, A and a more stable one, B. As expected, for A, the sequence $\|Q_N\|_2^2$ tends to zero slower.

To determine the points of discontinuity of $\sigma(\theta)$ we have also considered the function $\frac{1}{N}\Theta_N(\theta)$ for different values of N. Fig. 2 and Fig. 3 demonstrate different behavior of $\frac{1}{N}\Theta_N(\theta)$ for different modes A and B with N growing. In both cases the number of segments into which the interval $[-\pi,\pi]$ was split, was equal to 3000. The peaks of the signal correspond to possible discontinuities of the spectral distribution functions. It can be observed that for B the main peak of the function $\frac{1}{N}\Theta_N(\theta)$ located at $\theta = 0.27\,\mathrm{rad}$ $(f = 0.53\,\mathrm{Hz})$ tends to zero with increasing N rather rapidly, while for the corresponding peak of A $\frac{1}{N}\Theta_N(\theta)$ at $\theta = 0.24\,\mathrm{rad}$ $(f = 0.48\,\mathrm{Hz})$ this is not evident.

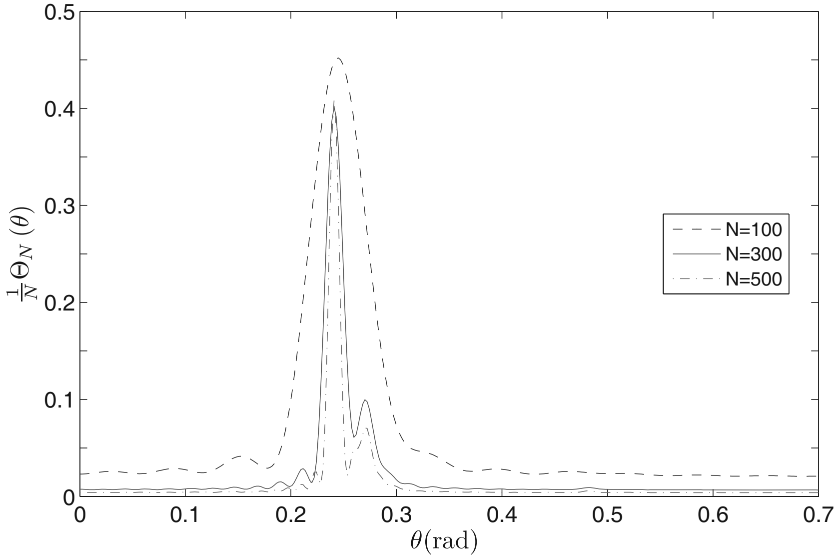

Figure 2: $\frac{1}{N}\Theta_N(\theta)$ *for the signal* A.

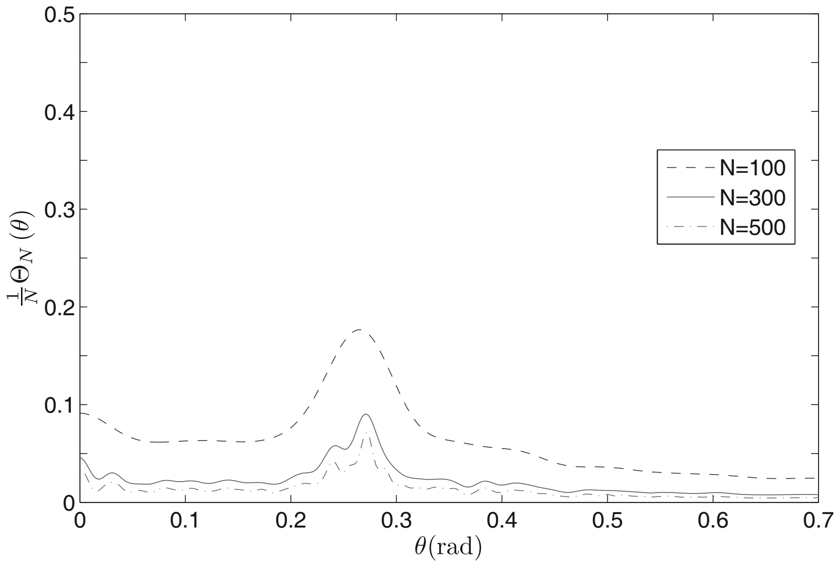

Figure 3: $\frac{1}{N}\Theta_N(\theta)$ *for the signal* B.

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(Received April 7, 2013)

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