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A stability of thin shells in view of the initial geometrical imperfections

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Abstract

In the report the non-linear deformations and stability of thin shells are considered in view of initial geometrical imperfections. The energy balance low for shell deformation includes usually three components: strain energy depending on viscid and inviscid forces, work of external forces, kinetic energy.

At quasistatic loading for steady branches it is possible to neglect components depending on viscous and dynamic forces. Frequently and the transient process is considered as static at which these components are neglected also, but in this case it is necessary to follow along a static curve of deformation, to select independent parameters of a solution as external load becomes unknown, and to search for methods of a detour of singular points. As a result the solution of a nonlinear problem of shell stability becomes very complicated.

It is possible to go on "medium" path - to neglect the kinetic energy by leaving the component connected with viscid forces. This approach is a basis of a method of an additional viscosity (Yakushev [1-3]), which allows on a uniform algorithm to discover steady prebuckling and postbuckling states, to determine the upper and lower critical loads in view of initial geometrical imperfections and nonlinear properties of a material. Such approach gives a number of computing advantages, as the solution depends only on one parameter - time, there is no necessity to search for methods of the detour of singular points.

Keywords: shell stability, nonlinear deformation, initial geometrical imperfection, added-viscosity method, finite element method.

1. Introduction

Shells of different forms are widely used in modern technologies. They give an opportunity to design light weight components required in aerospace, automobile industry, civil structures etc. One of the major steps of solving engineering problems for shells is to investigate their stability. Nowdays designers can use not only heuristical and experimental methods but also advanced mathematical modelling algorithms/techniques for this purpose.

However, to accurately predict the critical loads, it is essential to carry out nonlinear analysis.

With the growing complexity of shell structures there is an important need for having a reliable accuracy of prediction of their critical loads. Many modern finite element packages use a special treatment for this purpose. But most of them are based on linear theory of stability. In this case the structural stability analysis is governed by a system of linear uniform equations and eigen-value problem. Critical load is given by the lowest eigen value. But sometimes a real critical load can differ from the theoretical one by twice or more. And in this case the theoretical results are corrected on special tables which are based on many experiments. Unfortunately this method is not suitable for many shells because there is no experimental data for analogous constructions.

Experimental investigations of stability of shells is too expensive and the information obtained is also not sufficient. Analytical methods make too many simplifications and hence their applications are very limited. Only numerical simulation can provide acceptable solutions at affordable cost. To approach the theoretical and experimental results it is necessary to use methods of nonlinear analysis.

2. Additional viscosity technique

Several methods are now available for investigation of sub-critical and transcritical behavior of nonlinear shells. They are depended on nonlinear equations, which are used in the investigation. Generally, the nonlinear equations must take inertia forces and viscosity into account. Then the solution depends uniquely on time, and the existence of transient process is a stability criterion. This is the most valid approach, since it uses the equations of motion. However, certain difficulties are encountered in solving them.

As a result, the most thoroughly developed methods are those based on the use of static equations, among them one is the parameter-continuation method (Grigolyuk *et al.* [4]). But it is necessary in this approach to follow the variation of the solution along the deformation curve, and the parts of the curve that correspond to unstable states may not be excluded from consideration.

To a certain degree, the advantage of both approaches can be used by introducing an additional viscosity into static equation (Yakushev [1-3, 5-10]). As a result, the solution depends uniquely on only one parameter - time. Numerical solution of equations is simpler than the dynamics of the shell is considered, and algorithms are quite universal. For a constant external load, the added-viscosity technique reduces to an iterative process that converges well, even around critical loads, something that cannot always be said of other iterative schemes.

The additional viscosity may be added either in the relations between the strain and stress deviators - rheological viscosity (see section 3) or in the equilibrium equations in the form of additional external forces proportional to speeds of the displacement - external viscosity (see section 5).

In the first case a viscosity should be added in a way that instant strain changes are avoided in the correlation's obtained (Yakushev [1]). Otherwise, the inertia forces will have to be considered to ensure a continuous temporal description of the transition from the pre-critical to post-critical state under the stability loss. Strain changes will be defined at all points in time and the inertia forces will be ignored only if the rheological equations do not contain a time stress derivative (Yakushev ([1]). The simplest creep model that meets this requirement is the Focht body :

$$s_{ij} = 2G(\eta \dot{e}_{ij} + e_{ij}) \quad (1)$$

Where e_{ij} and s_{ij} are the strain and stress deviators respectively, η is a constant, *G* is shear modulus. We can take dimensionless time $T = t/\eta$ and get rid of value η.

The introduction of additional viscosity in the relations between the strain and stress deviators reduces the problem at each time-step to solving of a linear system of elliptical-type equations with coefficients of a linear system of surface coordinates. This is especially convenient when such numerical methods as finite differences or finite elements are used. We have to solve a nonlinear system of algebraic equations:

$$\mathbf{K_0 Q} + \mathbf{K_1} = 0 \quad (2)$$

where **Q** - a column of unknowns, $\mathbf{K_0}$ and $\mathbf{K_1}$ - matrices depended on **Q** and external load respectively.

The convergence of the iterations is determined by the form of matrix $\mathbf{K_0}$ [1]. Applications of rheological viscosity to the study of the shell stability showed good results. A high convergence of iterations process is achieved near and at the critical loads. This procedure does not cause any specific difficulties and high convergence is obtained at even higher loads with zero initial value (Yakushev [1-3, 5-10]).

The loading should be gradual and not continuous and the solutions should be computed for some external load values to a prescribed level, i.e., till the velocities of displacement become lower than the given value. Step-by-step load change allows determination of pre and post-buckling states and critical loads.

3. Ffinite element formulation.

To solve the problem, an algorithm based on finite element formulation and using rheological viscosity was developed (Yakushev [7]). It is capable of analyzing static nonlinear deformation of shallow shells based on Timoshenko's hypothesis. It is possible to model geometrically nonlinear deformation and stability of shells to determine the upper and lower critical loads, pre and post-buckling states.

A computational model is created using a developed preprocessor utility, with initial information about geometry of shell, material properties and external load. A 12-noded curved triangular element is used (Yakushev [7]). It is based on Timoshenko's theory and describes shallow shells. The element is considered in Gauss coordinates situated on the surface of the shell. There are five unknowns in this element namely, displacement normal to the shell surface, two displacements in the plane of surface and two angles or rotation of inclination of line original to the normal surface. The element has a third order approximation of displacement normal to the surface and second order for tangent displacement and angles. Hence, there will not be any locking problem for the element.

Approach based on added-viscosity technique is useful in examination of space shell structures, compared to parameter continuation method. Application of method of rheological viscosity in stability problem of shallow shells, taking into account of geometric nonlinearity, is considered. A triangular curvilinear lagrangian element (Figure 1) is used in finite element formulation. Based on the nonlinear theory of shallow shells, expressions for membrane, bending and shear deformations are recorded as follows:

$$\varepsilon_i = u_{i,i} + k_i w + \frac{1}{2}(w_{,i})^2; \quad \chi_i = \theta_{i,i}; \quad \tau_i = w_{,i} + \theta_i; \quad (i = 1,2) \tag{3}$$

$$\varepsilon_{12} = u_{1,2} + u_{2,1} + w_{,1} w_{,2}; \quad \chi_{12} = \theta_{1,2} + \theta_{2,1} \tag{4}$$

There are five unknowns in this element. w is displacement normal to the shell surface,

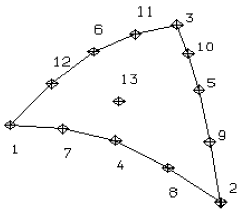

Figure 1: 12-node Triangle Element.

u_1, u_2 are two displacements in the plane of surface and θ_1, θ_2 are two angles or rotation of inclination of line original to the surface. Vectors for generalized deformations $\mathbf{E}$, and for internal forces and moments and shear forces $\mathbf{E}$, are obtained as,

$$\mathbf{E} = [\varepsilon_1, \varepsilon_2, \varepsilon_{12}, \chi_1, \chi_2, \chi_{12}, \tau_1, \tau_2]^T \tag{5}$$

$$\mathbf{F} = [N_1, N_2, N_{12}, M_1, M_2, M_{12}, Q_1, Q_2] \tag{6}$$

For orthotropic material matrix of elastic constants is obtained as,

$$\mathbf{D}_0 = \frac{1}{1-\nu_{12}\nu_{21}} \begin{bmatrix} E_1 & E_1\nu_{21} & 0 \\ E_2\nu_{12} & E_2 & 0 \\ 0 & 0 & G_{12}(1-\nu_{12}\nu_{21}) \end{bmatrix} \tag{7}$$

$$\mathbf{D}_1 = \begin{bmatrix} G_{13} & 0 \\ 0 & G_{23} \end{bmatrix} \tag{8}$$

Form a generalized matrix of elastic constants of 8 x 8 size as,

$$\mathbf{D} = \begin{bmatrix} h\mathbf{D}_0 & 0 & 0 \\ 0 & \frac{h^3}{12}\mathbf{D}_0 & 0 \\ 0 & 0 & \frac{5h}{6}\mathbf{D}_0 \end{bmatrix} \tag{9}$$

A relationship between general force vector and deformation vector is given as,

$$\mathbf{F} = \mathbf{D}(\eta \dot{\mathbf{E}} + \mathbf{E}) \qquad (10)$$

Let us define general displacement vector $\mathbf{U}$, distributed $\mathbf{R}$ and concentrated loads $\mathbf{R}^c$ as,

$$\mathbf{U} = [u_1, u_2, w, \theta_1, \theta_2]^T \qquad (11)$$

$$\mathbf{R} = [R_1, R_2, R_z, 0, 0]^T, \quad \mathbf{R}^c = [R_1^c, R_2^c, R_z^c, 0,] \qquad (12)$$

From Lagrange principle,

$$\int_S \delta \mathbf{E}^T \mathbf{F} ds - \int_S \delta \mathbf{U}^T \mathbf{R} ds - \sum \delta \mathbf{U}^T \mathbf{R}^c = 0 \qquad (13)$$

S is a region or integration. The summation is conducted on all finite element nodes. The generalized displacements and curvatures in finite elements are obtained as,

$$u_i = \sum_{k=1}^{6} N_k^2 (L_1, L_2) u_i^k, \quad \theta_i = \sum_{k=1}^{6} N_k^2 (L_1, L_2) \theta_i^k, \quad (i = 1, 2)$$

$$w = \sum_j [N_j^3 (L_1, L_2) + \alpha_j N_{13}^3 (L_1, L_2)] w_j \qquad (14)$$

$$\alpha_j = -\frac{1}{6}, \quad j = 1 \div 3; \quad \alpha_j = \frac{1}{4}, \quad j = 7 \div 12$$

Thickness h is linear function of coordinates:

$$h = \sum_{k=1}^{3} N_k^1 (L_1, L_2) h^k \qquad (15)$$

In a similar way expressions for curvature k_1, k_2 and component of surface load are written. Coordinates x_1 and x_2 are quadratic functions of coordinates:

$$x_i = \sum_{k=1}^{6} N_k^2 (L_1, L_2) x_i^k \qquad (16)$$

Where $N_k^1(L_1, L_2)$, $N_k^2(L_1, L_2)$ and $N_k^3(L_1, L_2)$ - two-dimensional basic functions from L coordinates.

In such a way as described above, for nodes 1, 2, 3 there are five functions u_1, u_2, w, θ_1, θ_2, for nodes 4, 5, 6 there are four functions u_1, u_2, θ_1, θ_2, but for nodes 7-12 there is only single sagging function w. Thus the element has 12 nodes and 33 degree of freedom.

4. Result and discussion.

The stability analysis of a typical cylindrical panel with an initial local dent under uniform pressure is carried out. To investigate the influence of initial imperfections on the critical pressure, a cycle of calculations was carried out with introduction of an additional normal displacement representing a local dent. The mathematical expectation and a root-mean-square deviation of critical pressure in the dependence on the magnitude of a root-mean-square deviation of the dent depth were found. The probability density function has some points of breaks in curve of the second kind, which are due to presence of minima in the mutual dependence of the critical load and the dent depth [8].

In Figure 2 the initial and different deformed forms at a postbuckling state are shown. Closer examination an of the stability of the panel is presented in [8].

5. Stability of shallow shells.

An algorithm for solving of nonlinear problems of deformation and stability of shallow shells is developed in view of initial geometrical imperfections in a shape and in a contour of the shell. For this purpose the dynamic equations involving damping are used. w is a normal displacement, w_0 - a initial shape function, Φ - a stress function. The external normal pressure is defined by q, t is a time. x and y - spatial coordinates, h - a thickness, $L(w,\Phi)$ - a nonlinear operator.

$$\mu\frac{\partial^2 w}{\partial t^2}+\gamma\frac{\partial w}{\partial t}=-\nabla_k^2\Phi+\frac{D}{h}\nabla^4(w-w_0)+\frac{q}{h}+L(w,\Phi),$$
$$\frac{1}{E}\nabla^4\Phi=-\nabla_k^2(w-w_0)-\frac{1}{2}\left[L(w,w)-L(w_0,w_0)\right]. \tag{17}$$

The solution of the problem is as follows in several steps: 1. Natural frequencies ω_{mn}, eigenfunctions for normal displacement $\tilde{W}_{mn}(x,y)$ and stress function $\tilde{F}_{mn}(x,y)$ for own oscillations of the ideal shell shape and contour are found. m and n - numbers of oscillation mode. 2. w and Φ are decomposed in series on earlier retrieved eigenfunctions with unknown coefficients $W_{mn}(t)$ and $\Phi_{mn}(t)$. 3. The initial imperfections are decomposed in series under own eigenfunctions with known coefficients W_{mn}^0. 4. The initial shape defect and its derivative near the contour are taken into account by adding in the series of additional term $W_b^0(x,y)$. We can apply these equations to obtain:

$$w(x,y,t)=w_0(x,y)+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}W_{mn}(t)\tilde{W}_{mn},$$
$$\Phi(x,y,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Phi_{mn}(t)\tilde{F}_{mn}, \tag{18}$$
$$w_0(x,y)=W_b^0(x,y)+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}W_{mn}^0\tilde{W}_{mn}.$$

5. The obtained series are substituted in the equations (17). 6. $\tilde{W}_{mn}(x,y)$ offer property of an orthogonality with weighting function $\rho(x,y)$. For the stress functions $\tilde{F}_{mn}(x,y)$ we select the special function $\psi_{kl}(x,y)$, which is orthogonal to $\nabla^4 F_{mn}(x,y)$. These properties note as follows:

$$\int_{\Omega} \tilde{W}_{mn}(x,y)\tilde{W}_{kl}(x,y)\rho(x,y)dxdy = 0,$$
$$\int_{\Omega} \nabla^4 F_{mn}(x,y)\,\psi_{kl}(x,y)\rho(x,y)dxdy = 0, \quad (m \neq k, \quad m \neq l). \tag{19}$$

The second relations in (19) is very important, because of it simplifies the derivation of an iteration scheme for solving of the equations (17).

Due to these properties it is possible to receive a set of equations for normal displacement in which in the left part there are the derivatives of the coefficient only for one component $W_{mn}(t)$. Moreover for each component of the stress function $\Phi_{mn}(t)$ we have the separate equation:

$$\mu\alpha_{mn}\frac{d^2W_{mn}}{dt^2} + \gamma\beta_{mn}\frac{dW_{mn}}{dt} = E^1_{mn}(q,\bar{W},\bar{W}^0,\bar{\Phi}) + L^1_{mn}(\bar{W},\bar{\Phi}),$$
$$\Phi_{mn} = E^2_{mn}(\bar{W},\bar{W}^0) + L^2_{mn}(\bar{W},\bar{W}^0). \tag{20}$$

$E^1_{mn}(q,\bar{W},\bar{W}^0,\bar{\Phi})$ и $E^2_{mn}(\bar{W},\bar{W}^0)$ - linear parts, $L^1_{mn}(\bar{W},\bar{\Phi})$ и $L^2_{mn}(\bar{W},\bar{W}^0)$ - nonlinear parts of the equations.

However, the integration of this system encounters difficulties related to its stiffness, which results from the fact that the frequencies ω_{mn} increase sharply with increasing numbers m and n. So the special finite difference scheme for a solution on time is used. It is grounded on the fact, that the rigidity of a system is connected to the linear part of the equations in main. Therefore the linear part is approximated under the implicit scheme, and nonlinear on explicit.

5. Numerical results.

Let us consider shortly stability of imperfect spherical domes (Yakushev [2,3,9]). Emphasis was placed on the agreement with the experimental results by Yamada S. *et al*. [11].

The distribution of initial geometrical imperfection for a specimen named in this paper as C98 is shown in Figure. 3. Left side is the experimental data (Yamada S. *et al*. [11]), right side is a result of our approximation. For obviousness the vertical scale is enlarged in 800 times in comparison with two remaining directions. The specimen had the radius of curvature *R*=1.81*m*, the base radius *a*=0.17*m*, the thickness $h = 0.97 \cdot 10^{-3}\,m$, geometrical parameter $b = 7.29$, Young's modulus $E = 3 \cdot 10^9\,Pa$ and Poisson's ratio respectively. $\nu = 0.36$, upper critical pressure $799\,Pa$ (dimensionless value $q^{cru} = 0.75$), lower critical pressure $192\,Pa$ ($q^{crl} = 0.18$).

Figure 4 shows by dash-and-dot line the nonlinear association between dimensionless pressure q and volume V_r :

$$V_r = \frac{1}{V_0} \int_0^{2\pi} \int_0^b \left[W(\theta, r, t) - W^0(\theta, r) \right] r \, dr \, d\theta,$$
$$V_0 = 2\pi\eta \left(\frac{H}{h} \right)^2 \left(1 - \frac{H}{3R} \right) \approx 2\pi\eta \left(\frac{H}{h} \right)^2 = \frac{\pi}{2} b^4. \tag{21}$$

It was gained as a result of the step-by-step changes of pressure q. For each value q iterative process was conducted till a moment of convergence. The pressure was varied with little step near upper $q^{cru} = 0.731$ and lower critical $q^{crl} = 0.191$ pressures for more calculation accuracy of these values. After the transition to the stable postbuckling state (points 1-2-3-4-5) the pressure was decreased and the shell went through 5-7-8-9-10-11-12 to the point 13 corresponding to stable prebuckling state. In all cases the horizontal parts of the curves correspond to stability loss of the shell. It was observed between points 1-5 (upper critical pressure), 7-8 at $q = 0.248$, 9-13 at lover critical pressure. The shell forms are shown in Figure 5. The numbers near them correspond to the numbers in Figure 4. The portion of the curve from 0 to 1 accords with the prebuckling stable states, in Figure 5 they are closely spaced.

The simulation when the volume V_r step by step changed was carried out also, in this case the pressure q was determined from the decision. The pressure versus volume is shown in Figure 4 by a continuous line with circles, which show values of volume V_r for which calculation was carried out. When pressure has reached the critical value $q^{cru} = 0.731$ the stability loss took place, and the shell has passed from a state 1 to 14.

Figure 6 shows the distributions of the normal displacement for different states: picture 0 corresponds to initial state at $q = 0$, when $W = W^0$; 1 – stable prebuckling state at $q = 0.5$; 2 - stable prebuckling state at $q = 0.730$; 4 – start of the stability loss at $q = 0.731$. Here for obviousness the vertical scale is enlarged in 116 times in comparison with two remaining directions.

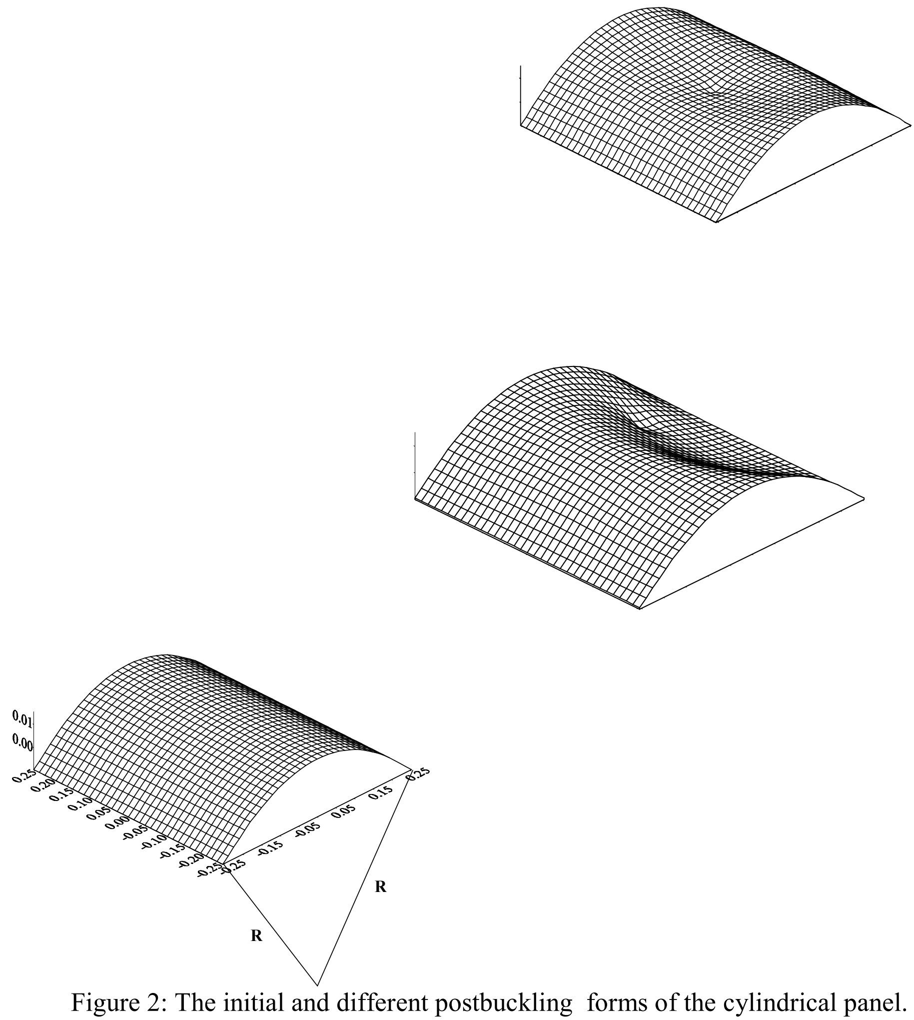

Figure 2: The initial and different postbuckling forms of the cylindrical panel.

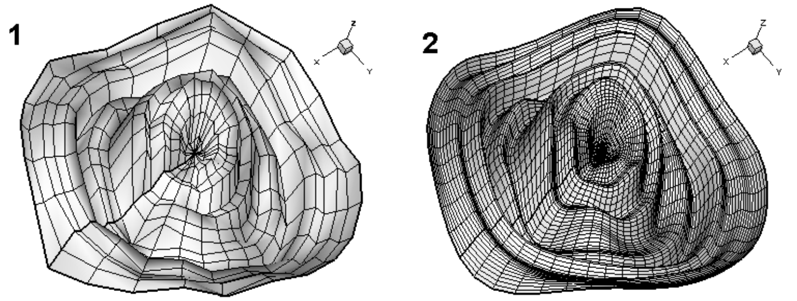

Figure 3: Comparison of distribution of initial imperfections (1- experiments, 2-approximation).

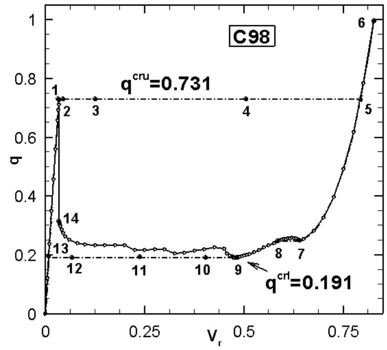

Figure 4: Pressure q versus volume V_r for the specimen named in [1] as C98.

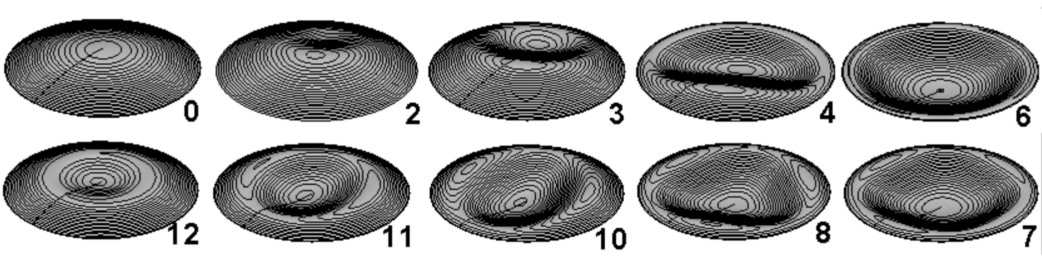

Figure 5: The dome forms for different pressures and the transient from the pre-buckling to stable post-buckling state and back

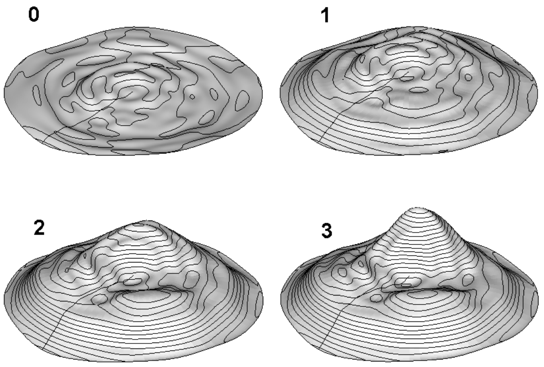

Figure 6: The distribution of the normal displacement W for different pressures

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